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Hopf Maps, Lowest Landau Level, and Fuzzy Spheres*

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Abstract. This paper is a review of monopoles, lowest Landau level, fuzzy spheres, and their mutual relations. The Hopf maps of division algebras provide a prototype relation between monopoles and fuzzy spheres. Generalization of complex numbers to Clifford algebra is exactly analogous to generalization of fuzzy two-spheres to higher dimensional fuzzy spheres. Higher dimensional fuzzy spheres have an interesting hierarchical structure made of "compounds" of lower dimensional spheres. We give a physical interpretation for such particular structure of fuzzy spheres by utilizing Landau models in generic even dimensions. With Grassmann algebra, we also introduce a graded version of the Hopf map, and discuss its relation to fuzzy supersphere in context of supersymmetric Landau model.

Key words: division algebra; Clifford algebra; Grassmann algebra; Hopf map; non-Abelian monopole; Landau model; fuzzy geometry

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Contents

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1 Introduction

Fuzzy two-sphere introduced by Madore [1] was a typical and important fuzzy manifold where particular notions of fuzzy geometry have been cultivated. In subsequent explorations of fuzzy geometry, higher dimensional and supersymmetric generalizations of the fuzzy spheres were launched by Grosse et al. [2, 3, 4]. Furthermore, as well recognized now, string theory provides a natural set-up for non-commutative geometry and non-anti-commutative geometry [5, 6, 7]. In particular, classical solutions of matrix models with Chern–Simons term are identified with fuzzy two-spheres [8], fuzzy four-spheres [9] and fuzzy superspheres [10]. Fuzzy manifolds also naturally appear in the context of intersections of D-branes [11, 12, 13]. In addition to applications to physics, the fuzzy spheres themselves have intriguing mathematical structures. As Ho and Ramgoolam showed in [14] and subsequently in [15] Kimura investigated, the commutative limit of (even-dimensional) fuzzy sphere takes the particular form[1]:

$$S_F^{2k} \simeq SO(2k+1)/U(k).$$

From this coset representation, one may find, though S_F^{2k} is called $2k$-dimensional fuzzy sphere, its genuine dimension is *not* $2k$ but $k(k+1)$. Furthermore, the fuzzy spheres can be expressed as the lower dimensional fuzzy sphere-fibration over the sphere

$$S_F^{2k} \approx S^{2k} \times S_F^{2k-2}.$$

Thus, S_F^{2k} has "extra-dimensions" coming from the lower dimensional fuzzy sphere S_F^{2k-2}. One may wonder why fuzzy spheres have such extra-dimensions. A mathematical explanation may go as follows. Fuzzification is performed by replacing Poisson bracket with commutator on a manifold. Then, to fuzzificate a manifold, the manifold has to have a symplectic structure to be capable to define Poisson bracket. However, unfortunately, higher dimensional spheres S^{2k} ($k \geq 2$) do not accommodate symplectic structure, and their fuzzification is not straightforward. A possible resolution is to adopt a minimally extended symplectic manifold with spherical symmetry. The coset $SO(2k+1)/U(k)$ suffices for the requirement.

[1] The odd-dimensional fuzzy spheres can be given by $S_F^{2k-1} \simeq SO(2k)/(U(1)\otimes U(k-1))$ [16, 17]. For instance, $S_F^3 \approx S^2 \times S^2$.

Since detail mathematical treatments of fuzzy geometry have already been found in the excellent reviews [18, 19, 20, 21, 22, 23], in this paper, we mainly focus on a physical approach for understanding of fuzzy spheres, brought by the developments of higher dimensional quantum Hall effect (see as a review [24] and references therein). The lowest Landau level (LLL) provides a nice way for physical understanding of fuzzy geometry (see for instance [25]). As we shall see in the context, generalization of the fuzzy two-spheres is closely related to the generalization of the complex numbers in 19th century. Quaternions discovered by W. Hamilton were the first generalization of complex numbers [26]. Soon after the discovery, the other division algebra known as octonions was also found (see for instance [27]). Interestingly, the division algebras are closely related to topological maps from sphere to sphere in different dimensions, i.e. the Hopf maps [28, 29]. While the division algebras consist of complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$ and octonions $\mathbb{O}$ (except for real numbers $\mathbb{R}$), there is another generalization of complex numbers and quaternions; the celebrated Clifford algebra invented by W. Clifford [30]. The generalization of complex numbers to Clifford algebra is exactly analogous to generalization of fuzzy two-sphere to its higher dimensional cousins. Particular geometry of fuzzy spheres directly reflects features of Clifford algebra. With non-Abelian monopoles in generic even dimensional space, we explain the particular geometry of fuzzy spheres in view of the lowest Landau level physics. There is another important algebra invented by H. Grassmann [31]. Though less well known compared to the original three Hopf maps, there also exists a graded version of the (1st) Hopf map [32]. We also discuss its relation to fuzzy supersphere.

The organization is as follows. In Section 2, we introduce the division algebras and the Hopf maps. In Section 3, with the explicit construction of the 1st Hopf map, we analyze the Landau problem on a two-sphere and discuss its relations to fuzzy two-sphere S_F^2. The graded version of the Hopf map and its relation to fuzzy superspheres are investigated in Section 4. In Section 5, we extend the former discussions to the 2nd Hopf map and fuzzy four-sphere S_F^4. In Section 6, we consider the 3rd Hopf map and the corresponding fuzzy manifolds. In Section 7, we generalize the observations to even higher dimensional fuzzy spheres based on Clifford algebra. Section 8 is devoted to summary and discussions. In Appendix A, for completeness, we introduce the 0th Hopf map and related "Landau problem" on a circle. In Appendix B, the $SU(k+1)$ Landau model and fuzzy $\mathbb{C}P^k$ manifold are surveyed.

2 Hopf maps and division algebras

As the division algebras consist of only three algebras, there exist three corresponding Hopf maps, 1st, 2nd and 3rd. The three Hopf maps

$$\begin{array}{ccccccc} & & S^3 & \xrightarrow{S^1} & S^2 & & (1\text{st}) \\ & S^7 & \longrightarrow & S^4 & & & (2\text{nd}) \\ S^{15} & \longrightarrow & S^8 & & & & (3\text{rd}) \end{array}$$

are closely related bundle structures of $U(1)$, $SU(2)$, and $SO(8)$ monopoles [33, 34, 35]. Interestingly, the Hopf maps exhibit a hierarchical structure. Each of the Hopf maps can be understood as a map from a circle in 2D division algebra space to corresponding projective space:

$$\begin{array}{ccccccc} & & S^1_{\mathbb{C}} & \xrightarrow{S^1} & \mathbb{C}P^1 & & (1\text{st}) \\ & S^1_{\mathbb{H}} & \longrightarrow & \mathbb{H}P^1 & & & (2\text{nd}) \\ S^1_{\mathbb{O}} & \longrightarrow & \mathbb{O}P^1 & & & & (3\text{rd}) \end{array}$$

For instance, in the 1st Hopf map, the total space $S^1_{\mathbb{C}}$ represents a circle in 2D complex space, i.e. S^3, and the basespace $\mathbb{C}P^1$ denotes the complex projective space equivalent to S^2. For the 2nd and 3rd Hopf maps, same interpretations hold by replacing complex numbers $\mathbb{C}$ with quaternions $\mathbb{H}$, and octonions $\mathbb{O}$, respectively. For later convenience, we summarize the basic properties of quaternions and octonions. The quaternion basis elements, 1, q_1, q_2, q_3, are defined so as to satisfy the algebra [26]

$$q_1^2 = q_2^2 = q_3^2 = q_1 q_2 q_3 = -1, \qquad q_i q_j = -q_j q_i \quad (i \neq j),$$

or equivalently,

$$\{q_i, q_j\} = -2\delta_{ij}, \qquad [q_i, q_j] = 2\epsilon_{ijk} q_k,$$

where ϵ_{ijk} is Levi-Civita antisymmetric tensor with $\epsilon_{123} = 1$. Thus, quaternion algebra is non-commutative. As is well known, the quaternion algebra is satisfied by the Pauli matrices with the identification, $q_i = -i\sigma_i$. (We will revisit this point in Section 5.) An arbitrary quaternion is expanded by the quaternion basis elements:

$$q = r_0 1 + \sum_{i=1}^{3} r_i q_i,$$

where r_0, r_i are real expansion coefficients, and the conjugation of q is given by

$$q^* = r_0 1 - \sum_{i=1}^{3} r_i q_i.$$

The norm of quaternion, $||q||$, is given by

$$||q|| = \sqrt{q^* q} = \sqrt{q q^*} = \sqrt{{r_0}^2 + \sum_{i=1}^{3} {r_i}^2}.$$

It is noted that q and q^* are commutative. The normalized quaternionic space corresponds to S^3, and the total manifold of the 2nd Hopf map, S^7, is expressed as the quaternionic circle, $S^1_{\mathbb{H}}$:

$$q^* q + q'^* q' = r_0^2 + \sum_{i=1}^{3} r_i^2 + {r'_0}^2 + \sum_{i=1}^{3} {r'_i}^2 = 1.$$

Similarly, the octonion basis elements, $1, e_1, e_2, \ldots, e_7$, are defined so as to satisfy the algebras

$$\{e_I, e_J\} = -2\delta_{IJ}, \qquad [e_I, e_J] = 2 f_{IJK} e_K, \tag{2.1}$$

where $I, J, K = 1, 2, \ldots, 7$. δ_{IJ} denotes Kronecker delta symbol and f_{IJK} does the antisymmetric structure constants of octonions (see Table 1).

The octonions do not respect the associativity as well as the commutativity. (The non-associativity can be read from Table 1, for instance, $(e_1 e_2) e_4 = e_7 = -e_1 (e_2 e_4)$.) Due to their non-associativity, octonions cannot be represented by matrices unlike quaternions. However, the conjugation and magnitude of octonion can be similarly defined as those of quaternion, simply replacing the role of the imaginary quaternion basis elements q_i with the imaginary octonion basis elements e_I. An arbitrary octonion is given by

$$o = r_0 1 + \sum_{I=1}^{7} r_I e_I,$$

with real expansion coefficients r_0, r_I, and its conjugation is

$$o^* = r_0 1 - \sum_{I=1}^{7} r_I e_I.$$

Table 1. The structure constants of the octonion algebra. For instance, $f_{145} = 1$ can be read from $e_1 e_4 = e_5$. The octonion structure constants are given by $f_{123} = f_{145} = f_{176} = f_{246} = f_{527} = f_{374} = f_{365} = 1$, and other non-zero octonion structure constants are obtained by the cyclic permutation of the indices.

	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	$-e_4$	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

The norm of octonion, $||o||$, is

$$||o|| = \sqrt{o^* o} = \sqrt{o o^*} = \sqrt{{r_0}^2 + \sum_{I=1}^{7} {r_I}^2}.$$

Like the case of quaternion, o and o^* are commutative. The normalized octonion $o^* o = 1$ represents S^7, and the total manifold of the 3rd Hopf map, S^{15}, is given by the octonionic circle $S^1_{\mathbb{O}}$,

$$o^* o + o'^* o' = r_0^2 + \sum_{I=1}^{7} r_I^2 + {r'_0}^2 + \sum_{I=1}^{7} {r'_I}^2 = 1.$$

One may wonder there might exist even higher dimensional generalizations. Indeed, following to the Cayley–Dickson construction [27], it is possible to construct new species of numbers. Next to the octonions, sedenions consisting of 16 basis elements can be constructed. However, the sedenions do not even respect the alternativity, and hence the multiplication law of norms does not hold: $||x||\,||y|| \neq ||x \cdot y||$. Then, the usual concept of "length" does not even exist in sedenions, and a sphere cannot be defined with sedenions and hence the corresponding Hopf maps either. Consequently, there only exit three division algebras and corresponding three Hopf maps.

3 1st Hopf map and fuzzy two-sphere

In this section, we give a realization of the 1st Hopf map

$$S^3 \xrightarrow{S^1} S^2,$$

and discuss basic procedure of fuzzification of sphere in LLL.

3.1 1st Hopf map and $U(1)$ monopole

The 1st Hopf map can explicitly be constructed as follows. We first introduce a normalized complex two-spinor

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

satisfying $\phi^\dagger\phi = 1$. Thus ϕ, which we call the 1st Hopf spinor, represents the coordinate on the total space S^3, and plays a primary role in the fuzzification of sphere as we shall see below. With the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.1}$$

the first Hopf map is realized as

$$\phi \to x_i = \phi^\dagger \sigma_i \phi. \tag{3.2}$$

It is straightforward to check that x_i satisfy the condition for S^2:

$$x_i x_i = (\phi^\dagger\phi)^2 = 1. \tag{3.3}$$

Thus, (3.2) demonstrates the 1st Hopf map. The analytic form of the Hopf spinor except for the south pole is given by

$$\phi = \frac{1}{\sqrt{2(1+x_3)}} \begin{pmatrix} 1+x_3 \\ x_1 + i x_2 \end{pmatrix}, \tag{3.4}$$

and the corresponding fibre-connection is derived as

$$A = dx_i A_i = -i\phi^\dagger d\phi, \tag{3.5}$$

where

$$A_i = -\frac{1}{2(1+x_3)} \epsilon_{ij3} x_j.$$

The curvature is given by

$$F_{ij} = \partial_i A_j - \partial_j A_i = \frac{1}{2}\epsilon_{ijk} x_k, \tag{3.6}$$

which corresponds to the field strength of Dirac monopole with minimum charge (see for instance [36]). The analytic form of the Hopf spinor except for the north pole is given by

$$\phi' = \frac{1}{\sqrt{2(1-x_3)}} \begin{pmatrix} x_1 - i x_2 \\ 1 - x_3 \end{pmatrix}.$$

The corresponding gauge field is

$$A' = -i\phi'^\dagger d\phi' = dx_i A'_i,$$

where

$$A'_i = \frac{1}{2(1-x_3)} \epsilon_{ij3} x_j.$$

The field strength $F'_{ij} = \partial_i A'_j - \partial_j A'_i$ is same as F_{ij} (3.6), which suggests the two expressions of the Hopf spinor are related by gauge transformation. Indeed,

$$\phi' = \phi \cdot g = g \cdot \phi,$$

where g is a $U(1)$ gauge group element

$$g = e^{-i\chi} = \frac{1}{\sqrt{1 - {x_3}^2}} (x_1 - i x_2).$$

Here, the gauge parameter χ is given by $\tan(\chi) = \frac{x_2}{x_1}$. The $U(1)$ phase factor is canceled in the map (3.2), and there always exists such $U(1)$ gauge degree of freedom in expression of the Hopf spinor. With the formula

$$-ig^*dg = \frac{1}{1-{x_3}^2}\epsilon_{ij3}x_i dx_j,$$

the above gauge fields are represented as

$$A = \frac{i}{2}(1-x_3)g^*dg, \qquad A' = -\frac{i}{2}(1+x_3)g^*dg,$$

and related by the $U(1)$ gauge transformation

$$A' = A - ig^*dg.$$

The non-trivial bundle structure of $U(1)$ monopole on S^2 is guaranteed by the homotopy theorem

$$\pi_1(U(1)) \simeq \mathbb{Z},$$

specified by the 1st Chern number

$$c_1 = \frac{1}{4\pi}\int_{S^2} F.$$

3.2 $SO(3)$ Landau model

Here, we consider a Landau model on a two-sphere in $U(1)$ monopole background. In 3D space, the Landau Hamiltonian is given by

$$H = -\frac{1}{2M}D_i^2 = -\frac{1}{2M}\frac{\partial^2}{\partial r^2} - \frac{1}{Mr}\frac{\partial}{\partial r} + \frac{1}{2Mr^2}\Lambda_i^2, \tag{3.7}$$

where $D_i = \partial_i + iA_i$, $r = \sqrt{x_i x_i}$, and $i = 1,2,3$ are summed over. Λ_i is the covariant angular momentum $\Lambda_i = -i\epsilon_{ijk}x_j D_k$. The monopole gauge field has the form

$$A_i = -\frac{I}{2r(r+x_3)}\epsilon_{ij3}x_j,$$

and the corresponding field strength is

$$F_i = \epsilon_{ijk}\partial_j A_k = \frac{I}{2r^3}x_i.$$

The covariant angular momentum Λ_i does not satisfy the $SU(2)$ algebra, but satisfies

$$[\Lambda_i, \Lambda_j] = i\epsilon_{ijk}\big(\Lambda_k - r^2 F_k\big).$$

The conserved $SU(2)$ angular momentum is constructed as

$$L_i = \Lambda_i + r^2 F_i,$$

which satisfies the genuine $SU(2)$ algebra

$$[L_i, L_j] = i\epsilon_{ijk}L_k.$$

On a two-sphere, the Hamiltonian (3.7) is reduced to the $SO(3)$ Landau model [37]

$$H = \frac{1}{2MR^2}\Lambda_i^2, \tag{3.8}$$

where R is the radius of two-sphere. The $SO(3)$ Landau Hamiltonian can be rewritten as

$$H = \frac{1}{2MR^2}\left(L_i^2 - \frac{I^2}{4}\right), \tag{3.9}$$

where we used the orthogonality between the covariant angular momentum and the field strength; $\Lambda_i F_i = F_i \Lambda_i = 0$. Thus, the Hamiltonian is represented by the $SU(2)$ Casimir operator, and the eigenvalue problem is boiled down to the problem of obtaining the irreducible representation of $SU(2)$. Such irreducible representations are given by monopole harmonics [38]. The $SU(2)$ Casimir operator takes the eigenvalues $L_i^2 = j(j+1)$ with $j = \frac{I}{2} + n$ ($n = 0, 1, 2, \ldots$). (The minimum of j is not zero but a finite value $I/2$ due to the existence of the field angular momentum of monopole.) Then, the eigenenergies are derived as

$$E_n = \frac{1}{2MR^2}\left(n^2 + n(I+1) + \frac{I}{2}\right). \tag{3.10}$$

In the thermodynamic limit, $R, I \to \infty$, with $B = I/2R^2$ fixed, (3.10) reproduces the usual Landau levels on a plane:

$$E_n \to \omega\left(n + \frac{1}{2}\right),$$

where $\omega = B/M$ is the cyclotron frequency. The degeneracy of the nth Landau level is given by

$$d(n) = 2l + 1 = 2n + I + 1. \tag{3.11}$$

In particular, in the LLL ($n = 0$), the degeneracy is

$$d_{\mathrm{LLL}} = I + 1.$$

The monopole harmonics in LLL is simply constructed by taking symmetric products of the components of the 1st Hopf spinor

$$\phi_{\mathrm{LLL}}^{(m_1,m_2)} = \sqrt{\frac{I!}{m_1! m_2!}}\,\phi_1^{m_1}\phi_2^{m_2}, \tag{3.12}$$

where $m_1 + m_2 = I$ ($m_1, m_2 \geq 0$). In the LLL, the kinetic term of the covariant angular momentum is quenched, and the $SU(2)$ total angular momentum is reduced to the monopole field strength

$$L_i \to r^2 F_i = \frac{I}{2R} x_i.$$

Then in LLL, the coordinates x_i are regarded as the operator

$$X_i = \alpha L_i,$$

which satisfies the definition algebra of fuzzy two-sphere

$$[X_i, X_j] = i\alpha\epsilon_{ijk} X_k, \tag{3.13}$$

with $\alpha = 2R/I$.

We reconsider the LLL physics with Lagrange formalism. In Lagrange formalism, importance of the Hopf map becomes more transparent. The present one-particle Lagrangian on a two-sphere is given by

$$L = \frac{m}{2}\dot{x}_i\dot{x}_i + \dot{x}_i A_i,$$

with the constraint

$$x_i x_i = R^2. \tag{3.14}$$

In the LLL, the Lagrangian is represented only by the interaction term

$$L_{\text{LLL}} = \dot{x}_i A_i.$$

From (3.5), the LLL Lagrangian can be simply rewritten as

$$L_{\text{LLL}} = -iI\phi^\dagger \frac{d}{dt}\phi, \tag{3.15}$$

and the constraint (3.14) is

$$\phi^\dagger \phi = 1. \tag{3.16}$$

It is noted that, in the LLL, the kinetic term drops and only the first order time derivative term survives. The Lagrangian and the constraint can be represented in terms of the Hopf spinor. Usually, in the LLL, the quantization is preformed by regarding the Hopf spinor as fundamental quantity, and the canonical quantization condition is imposed not on the original coordinate on two-sphere, but on the Hopf spinor. We follow such quantization procedure to fuzzificate two-sphere. From (3.15), the conjugate momentum is derived as $\pi = -iI\phi^*$, and the canonical quantization is given by

$$[\phi_\alpha, \phi^*_\beta] = -\frac{1}{I}\delta_{\alpha\beta}. \tag{3.17}$$

This quantization may remind the quantization procedure of the spinor field theory, but readers should not be confused: The present quantization is for one-particle mechanics, and the spinor is quantized as "boson" (3.17). Then, the complex conjugation is regarded as the derivative

$$\phi^* = \frac{1}{I}\frac{\partial}{\partial\phi}, \tag{3.18}$$

and the constraint (3.16) is considered as a condition on the LLL basis

$$\phi^t \frac{\partial}{\partial\phi}\phi_{\text{LLL}} = I\phi_{\text{LLL}}.$$

The previously derived LLL basis (3.12) indeed satisfies the condition.

By inserting the expression (3.18) to the Hopf map, we find that x_i are regarded as coordinates on fuzzy two-sphere

$$X_i = R\phi^\dagger \sigma_i \phi = \frac{\alpha}{2}\phi^t \sigma_i \frac{\partial}{\partial\phi},$$

which satisfies the algebra (3.13). Thus, also in the Lagrange formalism, we arrive at the fuzzy two-sphere algebra in LLL. The crucial role of the Hopf spinor is transparent in the Lagrange formalism: The Hopf spinor is first fuzzificated (3.17), and subsequently the coordinates on two-sphere are fuzzificated. This is the basic fuzzification mechanics of sphere in the context of the Hopf map.

3.3 Fuzzy two-sphere

The fuzzy two-sphere is a fuzzy manifold whose coordinates satisfy the $SU(2)$ algebraic relation [1]

$$[\hat{X}_i, \hat{X}_j] = i\alpha\epsilon_{ijk}\hat{X}_k.$$

The magnitude of fuzzy sphere is specified by the dimension of corresponding $SU(2)$ irreducible representation. A convenient way to deal with the irreducible representation is to adopt the Schwinger boson formalism, in which the $SU(2)$ operators are given by

$$\hat{X}_i = \frac{\alpha}{2}\hat{\phi}^\dagger\sigma_i\hat{\phi}.$$

Here, $\hat{\phi} = (\hat{\phi}_1, \hat{\phi}_2)^t$ stands for a Schwinger boson operator that satisfy

$$[\hat{\phi}_\alpha, \hat{\phi}_\beta^\dagger] = \delta_{\alpha\beta},$$

with $\alpha, \beta = 1, 2$. Square of the radius of a fuzzy two-sphere reads as

$$\hat{X}_i\hat{X}_i = \frac{\alpha^2}{4}(\hat{\phi}^\dagger\hat{\phi})(\hat{\phi}^\dagger\hat{\phi} + 2). \tag{3.19}$$

The commutative expression (3.3) and the fuzzy expression (3.19) merely differ by the "ground state energy". Thus, the radius of the fuzzy sphere is

$$R_I = \frac{\alpha}{2}\sqrt{I(I+2)},$$

where I is the integer eigenvalue of the number operator $\hat{I} \equiv \hat{\phi}^\dagger\hat{\phi} = \hat{\phi}_1^\dagger\hat{\phi}_1 + \hat{\phi}_2^\dagger\hat{\phi}_2$. In the classical limit $I \to \infty$, the eigenvalue reproduces the radius of commutative sphere

$$R_I = \frac{\alpha}{2}\sqrt{I(I+2)} \to \frac{\alpha}{2}I = R.$$

The corresponding $SU(2)$ irreducible representation is constructed as

$$|m_1, m_2\rangle = \frac{1}{\sqrt{m_1!m_2!}}\hat{\phi}_1^{m_1}\hat{\phi}_2^{m_2}|0\rangle, \tag{3.20}$$

where $m_1 + m_2 = I$ $(m_1, m_2 \geq 0)$, and the degeneracy is given by $d(I) = I + 1$. Apparently, there is one-to-one correspondence between the LLL monopole harmonics (3.12) and the states on fuzzy sphere (3.20). Their "difference" is superficial, coming from the corresponding representations: Schwinger boson representation for fuzzy two-sphere, while the $SU(2)$ coherent representation for LLL physics. This is the basic observation of equivalence between fuzzy geometry and LLL physics.

4 Graded Hopf map and fuzzy supersphere

Before proceeding to the 2nd Hopf map, in this section, we discuss how the relations between the Hopf map and fuzzy sphere are generalized with introducing the Grassmann numbers, mainly based on Hasebe and Kimura [39]. The Grassmann numbers, η_a, are anticommuting numbers [31]

$$\eta_a\eta_b = -\eta_b\eta_a. \tag{4.1}$$

In particular, $\eta_a^2 = 0$ (no sum for a). With Grassmann numbers, the graded Hopf map was introduced by Landi et al. [32, 40, 41],

$$S^{3|2} \xrightarrow{S^1} S^{2|2}, \tag{4.2}$$

where the number on the left hand side of the slash stands for the number of bosonic (Grassmann even) coordinates and the right hand side of the slash does for the number of fermionic (Grassmann odd) coordinates. For instance, $S^{2|2}$ signifies a supersphere with two bosonic and two fermionic coordinates. The bosonic part of the graded Hopf map (4.2) is equivalent to the 1st Hopf map.

4.1 Graded Hopf map and supermonopole

As the 1st Hopf map was realized by sandwiching Pauli matrices between two-component normalized spinors, the graded Hopf map is realized by doing $UOSp(1|2)$ matrices between three-component (super)spinors. First, let us begin with the introduction of basic properties of $UOSp(1|2)$ algebra [42, 43, 44]. The $UOSp(1|2)$ algebra consists of three bosonic generators L_i $(i = 1, 2, 3)$ and two fermionic generators L_α $(\alpha = \theta_1, \theta_2)$ that satisfy

$$[L_i, L_j] = i\epsilon_{ijk}L_k, \qquad [L_i, L_\alpha] = \frac{1}{2}(\sigma_i)_{\beta\alpha}L_\beta, \qquad \{L_\alpha, L_\beta\} = \frac{1}{2}(\epsilon\sigma_i)_{\alpha\beta}L_i, \tag{4.3}$$

where $\epsilon = i\sigma_2$. As realized in the first algebra of (4.3), the $UOSp(1|2)$ algebra contains the $SU(2)$ as its subalgebra. L_i transforms as $SU(2)$ vector and L_α does as $SU(2)$ spinor. The Casimir operator for $UOSp(1|2)$ is constructed as

$$C = L_iL_i + \epsilon_{\alpha\beta}L_\alpha L_\beta,$$

and its eigenvalues are given by $L(L+\frac{1}{2})$ with $L = 0, 1/2, 1, 3/2, 2, \ldots$. The dimension of the corresponding irreducible representation is $4L + 1$. The fundamental representation matrix is given by the following 3×3 matrices

$$l_i = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}, \qquad l_\alpha = \frac{1}{2}\begin{pmatrix} 0 & \tau_\alpha \\ -(\epsilon\tau_\alpha)^t & 0 \end{pmatrix},$$

where σ_i are the Pauli matrices, $\tau_1 = (1,0)^t$, and $\tau_2 = (0,1)^t$. With l_i and l_α, the graded Hopf map (4.2) is realized as

$$\varphi \ \rightarrow \ x_i = 2\varphi^\ddagger l_i\varphi, \qquad \theta_\alpha = 2\varphi^\ddagger l_\alpha\varphi, \tag{4.4}$$

where φ is a normalized three-component superspinor which we call the Hopf superspinor:

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \eta \end{pmatrix}. \tag{4.5}$$

Here, the first two components φ_1 and φ_2 are Grassmann even quantities, while the last component η is Grassmann odd quantity. The superadjoint $\ddagger$ is defined as[2]

$$\varphi^\ddagger = (\varphi_1^*, \varphi_2^*, -\eta^*),$$

[2] The symbol $*$ represents the pseudo-conjugation that acts as $(\eta^*)^* = -\eta$, $(\eta_1\eta_2)^* = \eta_1^*\eta_2^*$ for Grassmann odd quantities η_1 and η_2. See [45] for more details.

and φ represents coordinates on $S^{3|2}$, subject to the normalization condition

$$\varphi^\ddagger\varphi = \varphi_1^*\varphi_1 + \varphi_2^*\varphi_2 - \eta^*\eta = 1.$$

x_i and θ_α given by (4.4) are coordinates on supersphere $S^{2|2}$, since the definition of supersphere is satisfied:

$$x_i x_i + \epsilon_{\alpha\beta}\theta_\alpha\theta_\beta = (\varphi^\ddagger\varphi)^2 = 1.$$

Except for the south-pole of $S^{2|2}$, the Hopf superspinor has an analytic form

$$\varphi = \frac{1}{\sqrt{2(1+x_3)}}\begin{pmatrix} (1+x_3)\left(1-\frac{1}{4(1+x_3)}\theta\epsilon\theta\right) \\ (x_1+ix_2)\left(1+\frac{1}{4(1+x_3)}\theta\epsilon\theta\right) \\ (1+x_3)\theta_1 + (x_1+ix_2)\theta_2 \end{pmatrix}.$$

The corresponding connection is derived as

$$A = -i\varphi^\ddagger d\varphi = dx_i A_i + d\theta_\alpha A_\alpha,$$

with

$$A_i = -\frac{1}{2(1+x_3)}\epsilon_{ij3}x_j\left(1+\frac{2+x_3}{2(1+x_3)}\theta\epsilon\theta\right), \qquad A_\alpha = -\frac{1}{2}i(x_i\sigma_i\epsilon\theta)_\alpha. \tag{4.6}$$

A_i and A_α are the super gauge field of supermonopole. The field strength is evaluated by the formula

$$F = dA = \frac{1}{2}dx_i\wedge dx_j F_{ij} + dx_i\wedge d\theta_\alpha F_{i\alpha} - \frac{1}{2}d\theta_\alpha\wedge d\theta_\beta F_{\alpha\beta},$$

where

$$\begin{aligned} F_{ij} &= \partial_i A_j - \partial_j A_i = \frac{1}{2}\epsilon_{ij3}x_j\left(1+\frac{3}{2}\theta\epsilon\theta\right), \\ F_{i\alpha} &= \partial_i A_\alpha - \partial_\alpha A_i = -i\frac{1}{2}(\theta\sigma_j\epsilon)_\alpha(\delta_{ij}-3x_ix_j), \\ F_{\alpha\beta} &= \partial_\alpha A_\beta + \partial_\beta A_\alpha = -ix_i(\sigma_i\epsilon)_{\alpha\beta}\left(1+\frac{3}{2}\theta\epsilon\theta\right). \end{aligned} \tag{4.7}$$

The analytic form of the Hopf superspinor except for the north pole is given by

$$\varphi' = \frac{1}{\sqrt{2(1-x_3)}}\begin{pmatrix} (x_1-ix_2)\left(1+\frac{1}{4(1-x_3)}\theta\epsilon\theta\right) \\ (1-x_3)\left(1-\frac{1}{4(1-x_3)}\theta\epsilon\theta\right) \\ (x_1-ix_2)\theta_1 + (1-x_3)\theta_2 \end{pmatrix},$$

and the corresponding gauge field is obtained as

$$A'_i = \frac{1}{2(1-x_3)}\epsilon_{ij3}x_j\left(1+\frac{2-x_3}{2(1-x_3)}\theta\epsilon\theta\right), \qquad A'_\alpha = -i\frac{1}{2}(x_i\sigma_i\epsilon\theta)_\alpha = A_\alpha. \tag{4.8}$$

The field strength $F' = dA'$ is same as (4.7), suggesting the two expressions of the Hopf superspinor are related by the transformation

$$\varphi' = \varphi \cdot g = g \cdot \varphi.$$

Here, g denotes $\mathcal{U}(1)$ gauge group element given by

$$g = e^{-i\chi} = \frac{x_1 - ix_2}{\sqrt{1-x_3^2}} \left(1 + \frac{1}{2(1-x_3^2)}\theta\epsilon\theta\right),$$

with χ given by $\tan\chi = \frac{x_2}{x_1}$. The $\mathcal{U}(1)$ group element yields

$$-ig^* dg = \frac{1}{1-x_3^2}\epsilon_{ij3}x_i dx_j \left(1 + \frac{1}{1-x_3^2}\theta\epsilon\theta\right),$$

and A (4.6) and A' (4.8) are expressed as

$$A = i\frac{1}{2}\left(1 - x_3(1 + \frac{1}{2}\theta\epsilon\theta)\right) g^* dg + d\theta_\alpha A_\alpha,$$
$$A' = -i\frac{1}{2}\left(1 + x_3(1 + \frac{1}{2}\theta\epsilon\theta)\right) g^* dg + d\theta_\alpha A_\alpha.$$

Then, obviously the gauge fields are related as

$$A' = A - ig^* dg,$$

and then

$$F' = F.$$

4.2 $UOSp(1|2)$ Landau model

Next, we consider Landau problem on a supersphere in supermonopole background [39]. The supermonopole gauge field is given by

$$A_i = -\frac{I}{2r(r+x_3)}\epsilon_{ij3}x_j \left(1 + \frac{2r+x_3}{2r^2(r+x_3)}\theta\epsilon\theta\right), \qquad A_\alpha = -\frac{I}{2r^3} i(x_i\sigma_i\epsilon\theta)_\alpha,$$

where $\frac{I}{2}$ (I denotes an integer) is a magnetic charge of the supermonopole. Generalizing the $SO(3)$ Landau Hamiltonian (3.8) to $UOSp(1|2)$ form, we have

$$H = \frac{1}{2MR^2}(\Lambda_i^2 + \epsilon_{\alpha\beta}\Lambda_\alpha\Lambda_\beta), \tag{4.9}$$

where Λ_i and Λ_α are the bosonic and fermionic components of covariant angular momentum:

$$\Lambda_i = -i\epsilon_{ijk}x_j D_k + \frac{1}{2}\theta_\alpha(\sigma_i)_{\alpha\beta}D_\beta, \qquad \Lambda_\alpha = \frac{1}{2}(\epsilon\sigma_i)_{\alpha\beta}x_i D_\beta - \frac{1}{2}\theta_\beta(\sigma_i)_{\beta\alpha}D_i,$$

with $D_i = \partial_i + iA_i$ and $D_\alpha = \partial_\alpha + iA_\alpha$. Λ_i and Λ_α obey the following graded commutation relations

$$[\Lambda_i, \Lambda_j] = i\epsilon_{ijk}(\Lambda_k - r^2 F_k), \qquad [\Lambda_i, \Lambda_\alpha] = \frac{1}{2}(\sigma_i)_{\beta\alpha}(\Lambda_\beta - r^2 F_\beta),$$
$$\{\Lambda_\alpha, \Lambda_\beta\} = \frac{1}{2}(\epsilon\sigma_i)_{\alpha\beta}(\Lambda_i - r^2 F_i),$$

where F_i and F_α are

$$F_i = \frac{I}{2r^3}x_i, \qquad F_\alpha = \frac{I}{2r^3}\theta_\alpha.$$

Just as in the case of the $SO(3)$ Landau model, the covariant angular momentum is not a conserved quantity, in the sense $[\Lambda_i, H] \neq 0$, $[\Lambda_\alpha, H] \neq 0$. Conserved angular momentum is constructed as

$$L_i = \Lambda_i + r^2 F_i, \qquad L_\alpha = \Lambda_\alpha + r^2 F_\alpha.$$

It is straightforward to see L_i and L_α satisfy the $UOSp(1|2)$ algebra (4.3). Since the covariant angular momentum and the field strength are orthogonal in the supersymmetric sense: $\Lambda_i F_i + \epsilon_{\alpha\beta}\Lambda_\alpha F_\beta = F_i\Lambda_i + \epsilon_{\alpha\beta}F_\alpha\Lambda_\beta = 0$, the $UOSp(1|2)$ Landau Hamiltonian (4.9) can be rewritten as

$$H = \frac{1}{2MR^2}\left(L_i^2 + \epsilon_{\alpha\beta}L_\alpha L_\beta - \frac{I^2}{4}\right).$$

With the Casimir index $J = \frac{I}{2} + n$, the energy eigenvalue is derived as

$$E_n = \frac{1}{2MR^2}\left(J\left(J+\frac{1}{2}\right) - \frac{I^2}{4}\right)\Big|_{J=n+\frac{I}{2}} = \frac{1}{2MR^2}\left(n\left(n+\frac{1}{2}\right) + I\left(n+\frac{I}{4}\right)\right),$$

and the degeneracy in the nth Landau level is

$$d_n = 4J+1|_{J=n+\frac{I}{2}} = 4n + 2I + 1.$$

In particular in the LLL ($n=0$), the degeneracy becomes

$$d_{\mathrm{LLL}} = 2I + 1. \tag{4.10}$$

The eigenstates of the $UOSp(1|2)$ Hamiltonian are referred to the supermonopole harmonics, and the LLL eigenstates are constructed by taking symmetric products of the components of the Hopf superspinor (4.5):

$$\varphi_{\mathrm{LLL}}^{B\ (m_1,m_2)} = \sqrt{\frac{I!}{m_1!m_2!}}\varphi_1^{m_1}\varphi_2^{m_2}, \qquad \varphi_{\mathrm{LLL}}^{F\ (n_1,n_2)} = \sqrt{\frac{I!}{n_1!n_2!}}\varphi_1^{n_1}\varphi_2^{n_2}\eta, \tag{4.11}$$

where $m_1 + m_2 = n_1 + n_2 + 1 = I$ ($m_1, m_2, n_1, n_2 \geq 0$). The total number of $\varphi_{\mathrm{LLL}}^{B\ (m_1,m_2)}$ and $\varphi_{\mathrm{LLL}}^{F\ (n_1,n_2)}$ is $(I+1) + (I) = 2I+1$, which coincides with (4.10). In the LLL, the $UOSp(1|2)$ angular momentum is reduced to

$$L_i \ \rightarrow\ R^2 F_i = \frac{I}{2R}x_i, \qquad L_\alpha \ \rightarrow\ R^2 F_\alpha = \frac{I}{2R}\theta_\alpha,$$

and coordinates on supersphere are identified with the $UOSp(1|2)$ operators

$$X_i = \alpha L_i, \qquad \Theta_\alpha = \alpha L_\alpha,$$

which satisfy the algebra defining fuzzy supersphere:

$$[X_i, X_j] = i\alpha\epsilon_{ijk}X_k, \qquad [X_i, \Theta_\alpha] = \frac{\alpha}{2}(\sigma_i)_{\beta\alpha}\Theta_\beta, \qquad \{\Theta_\alpha, \Theta_\beta\} = \frac{\alpha}{2}(\epsilon\sigma_i)_{\alpha\beta}X_i. \tag{4.12}$$

With Lagrange formalism, we reconsider the LLL physics. The present one-particle Lagrangian on a supersphere is given by

$$L = \frac{M}{2}(\dot{x}_i^2 + \epsilon_{\alpha\beta}\dot{\theta}_\alpha\dot{\theta}_\beta) + A_i\dot{x}_i + \dot{\theta}_\alpha A_\alpha$$

with the constraint

$$x_i x_i + \epsilon_{\alpha\beta}\theta_\alpha\theta_\beta = R^2.$$

In the LLL, the kinetic term is quenched, and the Lagrangian takes the form of

$$L_{\text{LLL}} = \dot{x}_i A_i + \dot{\theta}_\alpha A_\alpha = -iI\varphi^\ddagger \frac{d}{dt}\varphi, \tag{4.13}$$

with the constraint

$$\varphi^\ddagger\varphi = 1. \tag{4.14}$$

From the LLL Lagrangian (4.13), the canonical momentum of φ is derived as $\pi = iI\varphi^*$, and from the commutation relation between φ and π, the complex conjugation is quantized as

$$\varphi^* = \frac{1}{I}\frac{\partial}{\partial\varphi}. \tag{4.15}$$

After quantization, the normalization condition (4.14) is imposed on the LLL basis:

$$\varphi^t \frac{\partial}{\partial\varphi}\varphi_{\text{LLL}} = I\varphi_{\text{LLL}}.$$

One may confirm the LLL basis (4.11) satisfies the condition. By inserting (4.15) to the graded Hopf map (4.4), we obtain

$$X_i = \alpha\varphi^t l_i \frac{\partial}{\partial\varphi}, \qquad \Theta_\alpha = \alpha\varphi^t l_\alpha \frac{\partial}{\partial\varphi}.$$

Apparently, they satisfy the algebra of fuzzy supersphere (4.12). Thus, in the case of the fuzzy two-sphere, the appearance of fuzzy supersphere in LLL is naturally understood in the context of the graded Hopf map.

4.3 Fuzzy supersphere

Fuzzy supersphere is constructed by taking symmetric representation of the $UOSp(1|2)$ group [3, 4, 46, 22]. We first introduce a superspinor extension of the Schwinger operator

$$\hat{\varphi} = \begin{pmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \\ \hat{\eta} \end{pmatrix},$$

where $\hat{\varphi}_1$ and $\hat{\varphi}_2$ are Schwinger boson operators, and $\hat{\eta}$ is a fermion operator: $[\hat{\varphi}_i, \hat{\varphi}_j^\dagger] = \delta_{ij}$, $[\hat{\varphi}_i, \hat{\eta}] = 0$ and $\{\hat{\eta}, \hat{\eta}^\dagger\} = 1$. With such Schwinger superoperator, coordinates on fuzzy supersphere are constructed as

$$\hat{X}_i = \alpha\hat{\varphi}^\dagger l_i \hat{\varphi}, \qquad \hat{\Theta}_\alpha = \alpha\hat{\varphi}^\dagger l_\alpha \hat{\varphi}.$$

Square of the radius of fuzzy supersphere is given by

$$\hat{X}_i\hat{X}_i + \epsilon_{\alpha\beta}\hat{\Theta}_\alpha\hat{\Theta}_\beta = \frac{\alpha^2}{4}(\hat{\varphi}^\dagger\hat{\varphi})(\hat{\varphi}^\dagger\hat{\varphi} + 1),$$

and then the radius is specified by the integer eigenvalue I of the number operator $\hat{I} \equiv \hat{\varphi}^\dagger\hat{\varphi} = \hat{\varphi}_1^\dagger\hat{\varphi}_1 + \hat{\varphi}_2^\dagger\hat{\varphi}_2 + \hat{\eta}^\dagger\hat{\eta}$ as

$$R_I = \frac{\alpha}{2}\sqrt{I(I+1)}.$$

The $UOSp(1|2)$ symmetric irreducible representation is explicitly constructed as

$$|m_1, m_2\rangle = \frac{1}{\sqrt{m_1!m_2!}}(\hat{\varphi}_1^\dagger)^{m_1}(\hat{\varphi}_2^\dagger)^{m_2}|0\rangle, \qquad |n_1, n_2\rangle = \frac{1}{\sqrt{n_1!n_2!}}(\hat{\varphi}_1^\dagger)^{n_1}(\hat{\varphi}_2^\dagger)^{n_2}\hat{\eta}^\dagger|0\rangle,$$

with $m_1 + m_2 = n_1 + n_2 + 1 = I$ $(m_1, m_2, n_1, n_2 \geq 0)$. Thus, the total number of the states constructing fuzzy supersphere is $d = (I+1)+(I) = 2I+1$. There is one-to-one correspondence between the supermonopole harmonics in LLL and the states on fuzzy supersphere. Especially, the Hopf superspinor corresponds to the superspin coherent state of Schwinger superoperator.

5 2nd Hopf map and fuzzy four-sphere

In this section, we discuss relations between the 2nd Hopf map

$$S^7 \xrightarrow{S^3} S^4$$

and fuzzy four-sphere. Though the 2nd and 3rd Hopf maps were first introduced by Hopf [29], we follow the realization given by Zhang and Hu [47] in the following discussions.

5.1 2nd Hopf map and $SU(2)$ monopole

Realization of the 2nd Hopf map is easily performed by replacing the imaginary unit with the imaginary quaternions, q_1, q_2, q_3. The Pauli matrices (3.1) are promoted to the following quaternionic Pauli matrices

$$\gamma_1 = \begin{pmatrix} 0 & -q_1 \\ q_1 & 0 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & -q_2 \\ q_2 & 0 \end{pmatrix}, \qquad \gamma_3 = \begin{pmatrix} 0 & -q_3 \\ q_3 & 0 \end{pmatrix},$$
$$\gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.1}$$

γ_i $(i = 1, 2, 3)$ correspond to σ_2, γ_4 to σ_1, and γ_5 to σ_3, respectively. With such quaternionic "Pauli matrices", the 2nd Hopf map is realized as

$$t \;\rightarrow\; \psi^\dagger \gamma_a \psi = x_a, \tag{5.2}$$

where $a = 1, 2, \ldots, 5$, and ψ is a two-component *quaternionic* (quaternion-valued) spinor satisfying the normalization condition $\psi^\dagger\psi = 1$. As in the 1st Hopf map, x_a given by the map (5.2) automatically satisfies the condition $x_a x_a = (\psi^\dagger\psi)^2 = 1$. Like the 1st Hopf spinor, the quaternionic Hopf spinor has an analytic form, except for the south pole, as

$$\psi = \frac{1}{\sqrt{2(1+x_5)}}\begin{pmatrix} 1 + x_5 \\ x_4 + q_i x_i \end{pmatrix},$$

and, except for the north pole, as

$$\psi' = \frac{1}{\sqrt{2(1-x_5)}}\begin{pmatrix} x_4 - q_i x_i \\ 1 - x_5 \end{pmatrix},$$

where $i = 1, 2, 3$ are summed over. These two expressions are related by the transformation

$$\psi' = \psi \cdot g = g \cdot \psi,$$

where g is a quaternionic $U(1)$ group element given by

$$g = e^{-q_i \chi_i} = \cos(\chi) - q_i \frac{\chi_i}{\chi}\sin(\chi) = \frac{1}{\sqrt{1 - x_5^2}}(x_4 - q_i x_i).$$

Here, χ_i and its magnitude $\chi = \sqrt{\chi_i^2}$ are determined by $\frac{\chi_i}{\chi}\tan(\chi) = \frac{1}{x_4}x_i$. It is possible to pursue the discussions with use of quaternions, but for later convenience, we utilize the Pauli matrix representation of the imaginary quaternions:

$$q_1 = -i\sigma_1, \qquad q_2 = -i\sigma_2, \qquad q_3 = -i\sigma_3.$$

The quaternionic $U(1)$ group is usually denoted as $U(1,\mathbb{H})$, and as obvious from the above identification, $U(1,\mathbb{H})$ is isomorphic to $SU(2)$. The quaternionic Pauli matrices (5.1) are now represented by the following $SO(5)$ gamma matrices:

$$\gamma_1 = \begin{pmatrix} 0 & i\sigma_1 \\ -i\sigma_1 & 0 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, \qquad \gamma_3 = \begin{pmatrix} 0 & i\sigma_3 \\ -i\sigma_3 & 0 \end{pmatrix},$$
$$\gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which satisfy $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$ $(a, b = 1, 2, 3, 4, 5)$. Corresponding to the quaternionic Hopf spinor, we introduce a $SO(5)$ four-component spinor (the 2nd Hopf spinor)

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \tag{5.3}$$

subject to the constraint

$$\psi^\dagger\psi = 1.$$

With ψ, the 2nd Hopf map (5.2) is rephrased as

$$\psi \to x_a = \psi^\dagger\gamma_a\psi. \tag{5.4}$$

It is easy to check that x_a satisfies the condition of S^4:

$$x_a x_a = (\psi^\dagger\psi)^2 = 1.$$

An analytic form of the 2nd Hopf spinor, except for the south pole, is given by

$$\psi = \frac{1}{\sqrt{2(1+x_5)}}\begin{pmatrix} (1+x_5)\phi \\ (x_4 - i\sigma_i x_i)\phi \end{pmatrix}, \tag{5.5}$$

where ϕ is the 1st Hopf spinor representing S^3-fibre. The connection is derived as

$$A = -i\psi^\dagger d\psi = \phi^\dagger dx_a A_a \phi \tag{5.6}$$

where

$$A_\mu = -\frac{1}{2(1+x_5)}\eta^+_{\mu\nu i}x_\nu\sigma_i, \qquad A_5 = 0. \tag{5.7}$$

They are the $SU(2)$ gauge field of Yang monopole [34]. Here, $\eta^+_{\mu\nu i}$ signifies the 'tHooft eta-symbol of instanton [48]:

$$\eta^+_{\mu\nu i} = \epsilon_{\mu\nu i4} + \delta_{\mu i}\delta_{\nu 4} - \delta_{\mu 4}\delta_{\nu i}.$$

The corresponding field strength $F = dA + iA\wedge A = \frac{1}{2}dx_a\wedge dx_b F_{ab}$:

$$F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b],$$

is evaluated as

$$F_{\mu\nu} = -x_\mu A_\nu + x_\nu A_\mu + \frac{1}{2}\eta^{+}_{\mu\nu i}\sigma_i, \qquad F_{\mu 5} = -F_{5\mu} = (1+x_5)A_\mu. \tag{5.8}$$

Another analytic form of the 2nd Hopf spinor, except for the north pole, is given by

$$\psi' = \frac{1}{\sqrt{2(1-x_5)}}\begin{pmatrix}(x_4 + ix_i\sigma_i)\phi \\ (1-x_5)\phi\end{pmatrix}.$$

The corresponding connection is calculated as

$$A'_\mu = -\frac{1}{2(1-x_5)}\eta^{-}_{\mu\nu i}x_\nu\sigma_i, \qquad A'_5 = 0, \tag{5.9}$$

where

$$\eta^{-}_{\mu\nu i} = \epsilon_{\mu\nu i4} - \delta_{\mu i}\delta_{\nu 4} + \delta_{\mu 4}\delta_{\nu i},$$

and the field strength is

$$F'_{\mu\nu} = x_\mu A'_\nu - x_\nu A'_\mu + \frac{1}{2}\eta^{-}_{\mu\nu i}\sigma_i, \qquad F'_{\mu 5} = -F'_{5\mu} = (1-x_5)A'_\mu.$$

As is well known, the 'tHooft eta-symbol satisfies the self (anti-self) dual relation

$$\eta^{\pm}_{\mu\nu i} = \pm\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\eta^{\pm}_{\rho\sigma i}.$$

The two expressions, ψ and ψ', are related by the $SU(2)$ transition function

$$g = \frac{1}{\sqrt{1-{x_5}^2}}(x_4 + ix_i\sigma_i),$$

which yields

$$-ig^\dagger dg = -\frac{1}{1-{x_5}^2}\eta^{-}_{\mu\nu i}x_\nu\sigma_i dx_\mu, \qquad -idgg^\dagger = \frac{1}{1-{x_5}^2}\eta^{+}_{\mu\nu i}x_\nu\sigma_i dx_\mu,$$

and the gauge fields (5.7) and (5.9) are concisely represented as

$$A = \frac{1-x_5}{2}idgg^\dagger, \qquad A' = -\frac{1+x_5}{2}ig^\dagger dg.$$

Then, A and A' are related as

$$A' = g^\dagger A g - ig^\dagger dg,$$

and their field strengths are also

$$F' = g^\dagger F g.$$

This manifests the non-trivial topology of the $SU(2)$ bundle on a four-sphere. In the Language of the homotopy theorem, the non-trivial topology of the $SU(2)$ bundle is expressed by

$$\pi_3(SU(2)) \simeq \mathbb{Z},$$

which is specified by the 2nd Chern number

$$c_2 = \frac{1}{2(2\pi)^2}\int_{S^4}\mathrm{tr}\,(F\wedge F).$$

5.2 $SO(5)$ Landau model

We next explore the Landau problem in 5D space [47]. The Landau Hamiltonian is given by

$$H = -\frac{1}{2N}D_a^2 = -\frac{1}{2M}\frac{\partial^2}{\partial r^2} - \frac{2}{Mr}\frac{\partial}{\partial r} + \frac{1}{2Mr^2}\sum_{a<b}\Lambda_{ab}^2, \tag{5.10}$$

where $D_a = \partial_a + iA_a$, $r = \sqrt{x_a x_a}$, and Λ_{ab} are the $SO(5)$ covariant angular momentum

$$\Lambda_{ab} = -ix_a D_b + ix_b D_a.$$

As in the previous 3D case, Λ_{ab} do not satisfy a closed algebra, but satisfy

$$\begin{aligned}[\Lambda_{ab}, \Lambda_{cd}] &= i(\delta_{ac}\Lambda_{bd} + \delta_{bd}\Lambda_{ac} - \delta_{bc}\Lambda_{ad} - \delta_{ad}\Lambda_{bc}) \\ &\quad - i(x_a x_c F_{bd} + x_b x_d F_{ac} - x_b x_c F_{ad} - x_a x_d F_{bc}),\end{aligned}$$

where F_{ab} are given by (5.8). The $SO(5)$ conserved angular momentum is constructed as

$$L_{ab} = \Lambda_{ab} + r^2 F_{ab}.$$

On a four-sphere, the Hamiltonian (5.10) is reduced to the $SO(5)$ Landau Hamiltonian

$$H = \frac{1}{2MR^2}\sum_{a<b}\Lambda_{ab}^2,$$

which is rewritten as

$$H = \frac{1}{2MR^2}\sum_{a<b}\left(L_{ab}^2 - R^4 F_{ab}^2\right) = \frac{1}{2MR^2}\sum_{a<b}\left(L_{ab}^2 - \frac{1}{2}I(I+4)\right),$$

where the orthogonality, $\Lambda_{ab}F_{ab} = F_{ab}\Lambda_{ab} = 0$, was used. Thus, the energy eigenvalue problem of the Hamiltonian is again boiled down to the problem of obtaining irreducible representation of the $SO(5)$ Casimir. Since the $SO(5)$ group has two Casimirs and the irreducible representations are specified by two indices $[\lambda_1, \lambda_2]$.[3] With the identification $[\lambda_1, \lambda_2] = [n + \frac{I}{2}, \frac{I}{2}]$, the energy eigenvalues are expressed as

$$E_n = \frac{1}{2MR^2}\big(n^2 + n(I+3) + I\big). \tag{5.11}$$

The integer I ($I = 0, 1, 2, \dots$) specifies the $SU(2)$ representation of the gauge field, while n ($n = 0, 1, 2, \dots$) does the Landau level. The states in the LLL ($n = 0$) correspond to the fully symmetric spinor representation of $SO(5)$, $[\frac{I}{2}, \frac{I}{2}]$. The degeneracy in the nth Landau level is derived as

$$d_n = \frac{1}{3!}(n+1)(I+1)(n+I+2)(2n+I+3). \tag{5.12}$$

The eigenstates of the Hamiltonian are given by the $SU(2)$ monopole harmonics [50]. In particular, the $SU(2)$ monopole harmonics in the LLL are simply constructed by taking the symmetric products of the components of the 2nd Hopf spinor (5.3)

$$\psi_{\mathrm{LLL}}^{(m_1,m_2,m_3,m_4)} = \sqrt{\frac{I!}{m_1!m_2!m_3!m_4!}}\psi_1^{m_1}\psi_2^{m_2}\psi_3^{m_3}\psi_4^{m_4} \tag{5.13}$$

[3] We follow the notation in [49].

with $m_1+m_2+m_3+m_4=I$ $(m_1,m_2,m_3,m_4\geq 0)$. Then, the LLL degeneracy is given by

$$d_{\rm LLL}=\frac{1}{3!}(I+3)(I+2)(I+1),$$

which actually coincides with $d_{n=0}$ of (5.12). Up to now, everything is parallel with the $SO(3)$ Landau model, but emergence of fuzzy four-sphere in LLL is not transparent unlike the fuzzy two-sphere case.

With Lagrange formalism, we revisit LLL physics of the $SO(5)$ Landau model. The present one-particle Lagrangian on a four-sphere is given by

$$L=\frac{M}{2}\dot{x}_a\dot{x}_a+\dot{x}_aA_a, \tag{5.14}$$

with a constraint

$$x_ax_a=R^2. \tag{5.15}$$

In the LLL, the interaction term survives to yield

$$L_{\rm LLL}=\dot{x}_aA_a, \tag{5.16}$$

and the relation (5.6) implies

$$L_{\rm LLL}=-i\psi^\dagger\frac{d}{dt}\psi, \tag{5.17}$$

and also the constraint (5.15) is rewritten as

$$\psi^\dagger\psi=1. \tag{5.18}$$

Interestingly, in the LLL, the original $SO(5)$ symmetry of the Lagrangian (5.14) is enhanced to the $SU(4)$ symmetry: the rotational symmetry of the 2nd Hopf spinor. We treat the 2nd spinor as the fundamental variable and apply the quantization condition. After quantization, the complex conjugate spinor is regarded as the derivative $\psi^*=\frac{1}{I}\frac{\partial}{\partial\psi}$, and the normalization condition (5.18) is translated to the LLL condition

$$\psi^t\frac{d}{d\psi}\psi_{\rm LLL}=I\psi_{\rm LLL}. \tag{5.19}$$

The LLL states (5.13) indeed satisfy the condition (5.19). Here, we comment on the origin of the $SU(4)$ symmetry and its relation to the 2nd Hopf map. In the LLL, the 2nd Hopf spinor plays a primary role, and the total manifold S^7 naturally appears in LLL. Projecting out the $U(1)$ phase from S^7, we obtain the structure of

$$\mathbb{C}P^3\simeq S^7/S^1.$$

This suggests physical equivalence between the LLL of $SO(5)$ Landau model and that of $SU(4)$ Landau model on $\mathbb{C}P^3$. (Detail discussions on physical equivalence between two Lagrangians (5.16) and (5.17) are found in [51], and see Appendix B.2 also.) The appearance of $\mathbb{C}P^3$ can also be understood as follows. As mentioned in Introduction, S^4 is not a Kähler manifold that accommodates symplectic structure. The "minimally extended" symplectic manifold of S^4 is $\mathbb{C}P^3$, which is given by the coset $SU(4)/U(3)$, and then the $SU(4)$ structure naturally appears. Such observation is completely consistent with the mathematical expression of the fuzzy four-sphere, since

$$S_F^4\simeq SO(5)/U(2)\simeq SU(4)/U(3)\simeq\mathbb{C}P^3,$$

where we used $SO(6)/SO(5) \simeq S^5 \simeq U(3)/U(2)$ and $SO(6) \simeq SU(4)$. By inserting the derivative expression of the complex 2nd Hopf spinor to the 2nd Hopf map (5.4), we find that the coordinate on S^4 is expressed by the following operator

$$X_a = \alpha \psi^t \gamma_a \frac{\partial}{\partial \psi}.$$

As we shall see in the next subsection, the $SU(4)$ structure also appears in the enhanced algebra of X_a.

5.3 Fuzzy four-sphere

The fuzzy four-sphere is constructed by taking a fully symmetric representation of the $SO(5)$ spinor [2, 9, 14, 15, 52, 53, 54]. As in the fuzzy two-sphere case, the Schwinger boson formalism is useful to construct coordinates on fuzzy four-sphere

$$\hat{X}_a = \frac{\alpha}{2} \hat{\psi}^\dagger \gamma_a \hat{\psi}, \tag{5.20}$$

where $\hat{\psi}$ is a four-component Schwinger boson operator satisfying $[\hat{\psi}_\alpha, \hat{\psi}^\dagger_\beta] = \delta_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3, 4)$. The commutations relation of X_a gives

$$[\hat{X}_a, \hat{X}_b] = \frac{\alpha}{2} \hat{X}_{ab}$$

where $\hat{X}_{ab}$ is the $SO(5)$ generator of the form

$$\hat{X}_{ab} = \frac{\alpha}{2} \hat{\psi}^\dagger \sigma_{ab} \hat{\psi} \tag{5.21}$$

with $\sigma_{ab} = -\frac{i}{4}[\gamma_a, \gamma_b]$. It is important to notice, unlike the case of fuzzy two-sphere, the fuzzy coordinates do *not* satisfy a closed algebra by themselves but yield the $SO(5)$ generators. With the $SO(5)$ generators $\hat{X}_{ab}$, the fuzzy coordinates satisfy the following closed algebra,

$$[\hat{X}_a, \hat{X}_b] = \frac{\alpha}{2} \hat{X}_{ab}, \qquad [\hat{X}_a, \hat{X}_{bc}] = -i\frac{\alpha}{2}(\delta_{ab}\hat{X}_c - \delta_{ac}\hat{X}_b),$$
$$[\hat{X}_{ab}, \hat{X}_{cd}] = i\frac{\alpha}{2}(\delta_{ac}\hat{X}_{bd} - \delta_{ad}\hat{X}_{bc} + \delta_{bc}\hat{X}_{ad} - \delta_{bd}\hat{X}_{ac}).$$

By identifying $\hat{X}_{a6} = \frac{1}{2}\hat{X}_a$ and $\hat{X}_{ab} = \hat{X}_{ab}$, we find the above algebra is concisely expressed by the $SO(6)$ algebra,

$$[\hat{X}_{AB}, \hat{X}_{CD}] = i\frac{\alpha}{2}(\delta_{AC}\hat{X}_{BD} - \delta_{AD}\hat{X}_{BC} + \delta_{BC}\hat{X}_{AD} - \delta_{BD}\hat{X}_{AC}),$$

where $A, B = 1, 2, \ldots, 6$. Thus, the algebra defining fuzzy four-sphere is $SO(6) \simeq SU(4)$. We encountered the $SU(4)$ structure again, and the fuzzy manifold naturally defined by $SU(4)$ algebra is fuzzy $\mathbb{C}P^3$ (see Appendix B.1). The enhanced $SU(4)$ algebra with extra X_{ab} coordinates accounts for the existence of extra fuzzy-dimensions [14, 15]. Interestingly, $\mathbb{C}P^3$ is locally expressed as the two-sphere fibration over the four-sphere:

$$\mathbb{C}P^3 \approx S^4 \times S^2,$$

since $\mathbb{C}P^3 \simeq S^7/S^1 \approx S^4 \times S^3/S^1$. The "extra dimension" of S^4_F can be understood as two-sphere fibration over the four-sphere.

Square of the radius of fuzzy four-sphere is calculated as

$$\hat{X}_a \hat{X}_a = \frac{\alpha^2}{4}(\hat{\psi}^\dagger \hat{\psi})(\hat{\psi}^\dagger \hat{\psi} + 4),$$

and, the radius is

$$R_I = \frac{\alpha}{2}\sqrt{I(I+4)},$$

where I signifies an integer eigenvalue of the number operator $\hat{I} \equiv \hat{\psi}^\dagger\hat{\psi} = \hat{\psi}_1^\dagger\hat{\psi}_1 + \hat{\psi}_2^\dagger\hat{\psi}_2 + \hat{\psi}_3^\dagger\hat{\psi}_3 + \hat{\psi}_4^\dagger\hat{\psi}_4$. Similarly, the $SO(5)$ Casimir operator is calculated as

$$\sum_{a<b} \hat{X}_{ab}^2 = \frac{\alpha^2}{8}(\hat{\psi}^\dagger\hat{\psi})(\hat{\psi}^\dagger\hat{\psi}+4),$$

which yields the eigenvalue $\frac{\alpha^2}{8}I(I+4)$, and the corresponding fully symmetric representation is constructed as

$$|m_1, m_2, m_3, m_4\rangle = \frac{1}{\sqrt{m_1!m_2!m_3!m_4!}}(\hat{\psi}_1^\dagger)^{m_1}(\hat{\psi}_2^\dagger)^{m_2}(\hat{\psi}_3^\dagger)^{m_3}(\hat{\psi}_4^\dagger)^{m_4}|0\rangle,$$

with $m_1+m_2+m_3+m_4 = I$ $(m_1, m_2, m_3, m_4 \geq 0)$. The dimension for the symmetric representation reads as $\frac{(I+3)!}{I!3!}$, which is equal to that of fuzzy $\mathbb{C}P^3$, suggesting equivalence between fuzzy four-sphere and fuzzy $\mathbb{C}P^3$.

6 3rd Hopf map and fuzzy manifolds

Here, we consider realization of the 3rd Hopf map

$$S^{15} \xrightarrow{S^7} S^8,$$

and corresponding fuzzy manifolds. Unlike the previous cases, there are two kinds of fuzzy manifolds; $S_F^8 \simeq SO(9)/U(4)$ and $\mathbb{C}P_F^7 \simeq SU(8)/U(7)$, depending on the choice of irreducible representation of $SO(9)$. The contents in this section are mainly based on Bernevig et al. [55].

6.1 3rd Hopf map and $SO(8)$ monopole

The 1st and 2nd Hopf maps were realized by sandwiching Pauli and quaternionic Pauli matrices by spinors. One may expect that such realization can be applied to the 3rd Hopf map. However, it is not so straightforward, since octonions cannot be represented by matrices due to their non-associative property. To begin with, we construct Majorana representation of the $SO(9)$ gamma matrix with the octonion structure constants (Table 1). With $e_0 = 1$, the octonion algebra (2.1) is expressed as

$$e_I e_J = -\delta_{IJ}e_0 + f_{IJK}e_K, \qquad \text{or} \qquad e_P e_Q = f_{PQR}e_R,$$

where $P, Q, R = 0, 1, \ldots, 7$. With use of f_{PQR}, the $SO(7)$ gamma matrices $-i\lambda_I$ $(I = 1, 2, \ldots, 7)$ are constructed as

$$(\lambda_I)_{PQ} = -f_{IPQ},$$

or

$$\lambda_1 = -i\begin{pmatrix} \sigma_2 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_2 & 0 \\ 0 & 0 & 0 & -\sigma_2 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -\sigma_3 & 0 & 0 \\ \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1_2 \\ 0 & 0 & 1_2 & 0 \end{pmatrix},$$

$$\lambda_3 = \begin{pmatrix} 0 & -\sigma_1 & 0 & 0 \\ \sigma_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\sigma_2 \\ 0 & 0 & i\sigma_2 & 0 \end{pmatrix}, \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & -\sigma_3 & 0 \\ 0 & 0 & 0 & 1_2 \\ \sigma_3 & 0 & 0 & 0 \\ 0 & -1_2 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -\sigma_1 & 0 \\ 0 & 0 & 0 & i\sigma_2 \\ \sigma_1 & 0 & 0 & 0 \\ 0 & -i\sigma_2 & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 & -1_2 \\ 0 & 0 & -\sigma_3 & 0 \\ 0 & \sigma_3 & 0 & 0 \\ 1_2 & 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 & -i\sigma_2 \\ 0 & 0 & -\sigma_1 & 0 \\ 0 & \sigma_1 & 0 & 0 \\ -i\sigma_2 & 0 & 0 & 0 \end{pmatrix}.$$

They are real antisymmetric matrices that satisfy

$$\{\lambda_I, \lambda_J\} = -2\delta_{IJ}.$$

With $\lambda_0 \equiv 1_8$, λ_0 and λ_I $(I = 1, 2, \ldots, 7)$ are regarded as the $SO(8)$ "Weyl +" gamma matrices. Utilizing λ_0 and λ_I, the $SO(9)$ gamma matrices Γ_A are constructed as

$$\Gamma_I = i\lambda_I \otimes \sigma_2, \qquad \Gamma_8 = 1_8 \otimes \sigma_1, \qquad \Gamma_9 = 1_8 \otimes \sigma_3,$$

or

$$\Gamma_I = \begin{pmatrix} 0 & \lambda_I \\ -\lambda_I & 0 \end{pmatrix}, \qquad \Gamma_8 = \begin{pmatrix} 0 & 1_8 \\ 1_8 & 0 \end{pmatrix}, \qquad \Gamma_9 = \begin{pmatrix} 1_8 & 0 \\ 0 & -1_8 \end{pmatrix}.$$

Again, they are real symmetric matrices that satisfy

$$\{\Gamma_A, \Gamma_B\} = 2\delta_{AB},$$

where $A, B, C = 1, 2, \ldots, 9$. The octonion structure constants appear in the off-diagonal elements of Γ_I. The $SO(9)$ generators are constructed as

$$\Sigma_{AB} = -i\frac{1}{4}[\Gamma_A, \Gamma_B], \tag{6.1}$$

or more explicitly

$$\Sigma_{IJ} = \begin{pmatrix} \sigma_{IJ} & 0 \\ 0 & \sigma_{IJ} \end{pmatrix}, \qquad \Sigma_{I8} = -\frac{i}{2}\begin{pmatrix} \lambda_I & 0 \\ 0 & -\lambda_I \end{pmatrix},$$
$$\Sigma_{I9} = \frac{i}{2}\begin{pmatrix} 0 & \lambda_I \\ \lambda_I & 0 \end{pmatrix}, \qquad \Sigma_{89} = -i\frac{1}{2}\begin{pmatrix} 0 & -1_8 \\ 1_8 & 0 \end{pmatrix},$$

where σ_{IJ} are the $SO(7)$ generators

$$\sigma_{IJ} = i\frac{1}{4}[\lambda_I, \lambda_J].$$

Since Γ_A are real matrices, the corresponding $SO(9)$ generators (6.1) are purely imaginary matrices; $\Sigma^*_{AB} = -\Sigma_{AB}$. Thus, the present representation is indeed the Majorana representation, in which the charge conjugation matrix is given by unit matrix, and the $SO(9)$ Majorana spinor is simply represented by (16-component) real spinor. The 3rd Hopf spinor is introduced as $SO(9)$ Majorana spinor subject to the normalization condition

$$\Psi^t \Psi = 1, \tag{6.2}$$

and the 3rd Hopf spinor is regarded as the coordinate of S^{15}. By sandwiching Γ_A between the 3rd Hopf spinors, we now realize the 3rd Hopf map as

$$\Psi \to x_A = \Psi^t \Gamma_A \Psi. \tag{6.3}$$

x_A in (6.3) are coordinates on S^8, since

$$\sum_{A=1,2,\dots,9} x_A x_A = (\Psi^t \Psi)^2 = 1.$$

An analytic form of Ψ, except for the south pole, is represented as

$$\Psi = \frac{1}{\sqrt{2(1+x_9)}} \begin{pmatrix} (1+x_9)\Phi \\ (x_8 - \lambda_I x_I)\Phi \end{pmatrix},$$

where Φ is a $SO(7)$ real 8-component spinor subject to the constraint

$$\Phi^t \Phi = 1,$$

representing the S^7-fibre. Then, Φ has the same degrees of freedom of the 2nd Hopf spinor ψ. We may assign $\Phi = (\mathrm{Re}\,\psi, \mathrm{Im}\,\psi)^t$, and the 3rd Hopf spinor is expressed as

$$\Psi = \frac{1}{\sqrt{2(1+x_9)}} \begin{pmatrix} (1+x_9) \begin{pmatrix} \mathrm{Re}\,\psi \\ \mathrm{Im}\,\psi \end{pmatrix} \\ (x_8 - \lambda_I x_I) \begin{pmatrix} \mathrm{Re}\,\psi \\ \mathrm{Im}\,\psi \end{pmatrix} \end{pmatrix}.$$

Naively anticipated connection $A = -i\Psi^t d\Psi$ vanishes due to the Majorana property of Ψ, however, defining

$$\boldsymbol{\Psi} = \frac{1}{\sqrt{2(1+x_9)}} \begin{pmatrix} (1+x_9)1_8 \\ x_8 1_8 - \lambda_I x_I \end{pmatrix}, \tag{6.4}$$

the connection of S^7-fibre is evaluated as

$$A = -i\boldsymbol{\Psi}^t d\boldsymbol{\Psi} = dx_A A_A,$$

where $A_A = (A_M, A_9)$ $(M = 1, 2, \cdots, 8)$ are

$$A_M = -\frac{1}{1+x_9}\sigma_{MN} x_N, \qquad A_9 = 0, \tag{6.5}$$

with $M = 1, 2, \dots, 8$. Here, σ_{MN} are $SO(8)$ "Weyl +" generators given by

$$\sigma_{IJ} = i\frac{1}{4}[\lambda_I, \lambda_J], \qquad \sigma_{I8} = -\sigma_{8I} = -i\frac{1}{2}\lambda_I.$$

(σ_{IJ} and σ_{I8} are pure imaginary antisymmetric matrices.) The field strength $F_{AB} = \partial_A A_B - \partial_B A_A + i[A_A, A_B]$ is also evaluated as

$$F_{MN} = -x_M A_N + x_N A_M + \sigma_{MN}, \qquad F_{M9} = -F_{9M} = (1+x_9)A_M,$$

which represent the $SO(8)$ monopole gauge field [35]. Similarly, except for the north pole, the 3rd Hopf spinor has an analytic form

$$\boldsymbol{\Psi}' = \frac{1}{\sqrt{2(1-x_9)}} \begin{pmatrix} x_8 1_8 + \lambda_I x_I \\ (1-x_9)1_8 \end{pmatrix}, \tag{6.6}$$

and the connection is

$$A'_M = -\frac{1}{1-x_9}\bar{\sigma}_{MN}x_N, \qquad A_9 = 0, \tag{6.7}$$

where

$$\bar{\sigma}_{IJ} = \sigma_{IJ}, \qquad \bar{\sigma}_{I8} = -\bar{\sigma}_{8I} = \sigma_{I8}.$$

The corresponding field strength is derived as

$$F'_{MN} = x_M A'_N - x_N A'_M + \bar{\sigma}_{MN}, \qquad F'_{M9} = -F'_{9M} = (1-x_9)A'_M.$$

Here, the $SO(8)$ generators σ_{MN} ($\bar{\sigma}_{MN}$) satisfy a generalized self (anti-self) dual relation:

$$\sigma_{MN} = \frac{4}{6!}\epsilon_{MNPQABCD}\sigma_{PQ}\sigma_{AB}\sigma_{CD}, \qquad \bar{\sigma}_{MN} = -\frac{4}{6!}\epsilon_{MNPQABCD}\bar{\sigma}_{PQ}\bar{\sigma}_{AB}\bar{\sigma}_{CD}.$$

The two expressions (6.4) and (6.6) are related by

$$\mathbf{\Psi}' = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \cdot \mathbf{\Psi},$$

where g signifies an $SO(8)$ group element

$$g = \frac{1}{\sqrt{1-x_9^2}}(x_8 + \lambda_I x_I),$$

which yields

$$-ig^t dg = -\frac{2}{1-{x_9}^2}\Sigma^-_{MN}x_N dx_M, \qquad -igdg^t = \frac{2}{1-{x_9}^2}\Sigma^+_{MN}x_N dx_M.$$

Then, the gauge fields, (6.5) and (6.7), are concisely represented as

$$A = i\frac{1}{2}(1-x_9)dgg^t, \qquad A' = -i\frac{1}{2}(1+x_9)g^t dg,$$

and are related by

$$A' = g^t A g - ig^t dg.$$

Similarly, their field strengths are

$$F' = g^t F g.$$

The non-trivial topological structure of the $SO(8)$ monopole bundle is guaranteed by the homotopy theorem

$$\pi_7(SO(8)) \simeq \mathbb{Z} \oplus \mathbb{Z},$$

which is specified by the Euler number

$$e = \int_{S^8} \mathrm{tr}\,(F \wedge F \wedge F \wedge F).$$

Table 2. In the 1st and 2nd Hopf maps, the fuzzy manifolds are uniquely determined, since the corresponding fuzzy spheres and complex projective spaces are equivalent, i.e. $S_F^2 \simeq \mathbb{C}P_F^1$ and $S_F^4 \simeq \mathbb{C}P_F^3$. Meanwhile, in the 3rd Hopf map, two corresponding fuzzy manifolds, $S_F^8 \simeq SO(9)/U(4)$ and $\mathbb{C}P_F^7 \simeq S^{15}/S^1$, are not equivalent.

Division algebra	Complex numbers	Quaternions	Octonions
Hopf maps	1st	2nd	3rd
Fuzzy sphere	$S_F^2 \simeq SO(3)/U(1)$	$S_F^4 \simeq SO(5)/U(2)$	$S_F^8 \simeq SO(9)/U(4)$
Fuzzy $\mathbb{C}P^n$	$\mathbb{C}P_F^1 \simeq S^3/S^1$	$\mathbb{C}P_F^3 \simeq S^7/S^1$	$\mathbb{C}P_F^7 \simeq S^{15}/S^1$

6.2 Fuzzy $\mathbb{C}P^7$ and fuzzy S^8

In the realization of the 3rd Hopf map, we utilized the real (Majorana) spinor. The fuzzification procedure in the previous sections can *not* be straightforwardly applied to the present case, since we do not have the complex conjugate spinor to be identified with derivative. However, with 16 real components of the 3rd Hopf spinor, we can construct an 8-component normalized complex spinor to be identified with coordinates on $\mathbb{C}P^7$. Then, in the present case, there exist two different types of fuzzy manifolds, depending on the choice of the irreducible representation of $SO(9)$. The first one is the above mentioned fuzzy $\mathbb{C}P^7 \simeq S^{15}/S^1$ specified by the vector representation of $SO(9)$, while the other is the fuzzy eight-sphere $S_F^8 \simeq SO(9)/U(4)$ specified by the spinor representation. Both two fuzzy manifolds are reasonable generalizations of the previous low dimensional fuzzy spheres and fuzzy complex projective spaces (see Table 2).

7 Beyond Hopf maps: even higher dimensional generalization

We have reviewed the construction of fuzzy manifolds based on the Hopf maps. Since the Hopf maps are only three kinds, the corresponding fuzzy manifolds are also limited. However, from the results of the Hopf maps, one may naturally infer two possible generalizations of the fuzzy manifolds, one of which is a series of fuzzy spheres:

$$S_F^{2k} \simeq SO(2k+1)/U(k),$$

and the other is that of fuzzy complex projective spaces:

$$\mathbb{C}P_F^k \simeq SU(k+1)/U(k) \simeq S^{2k+1}/S^1.$$

In this section, we first introduce mathematics of fuzzy spheres S_F^{2k} in arbitrary even dimensions, and next provide their physical interpretations, mainly based on Hasebe and Kimura [56] (see also Fabinger [57] and Meng [58]). Fuzzy $\mathbb{C}P^k$ manifolds and their corresponding Landau models are discussed in Appendix B.

7.1 Clifford algebra: another generalization of complex numbers

It may be worthwhile to begin with the story of generalization of the complex numbers to Clifford algebra. Generalization from fuzzy two-sphere to its higher dimensional cousins are quite analogous to the generalization of complex numbers. As discussed in Section 2, Cayley–Dickson construction provides one systematic way to construct new numbers by duplicating the original numbers, but this construction method has a fatal problem: If we utilize the method, the resulting algebra loses a nice property of numbers one by one. For instance, in the construction of quaternions, the commutativity of the complex numbers was lost. In the construction of octonions, even the associativity of quaternions was abandoned. Consequently, generalization of complex numbers ends up with the octonions, and there are only three division algebras (except for real numbers). Clifford found another generalization of complex numbers, based on Hamilton's quaternions and Grassmann algebras, known as Clifford algebra. The Clifford algebra, Cliff_n, consists of 2^n basis elements $\{1, e_a, e_a e_b, e_a e_b e_b, \dots, e_1 e_2 e_3 \cdots e_n\}$ ($a \neq b, a \neq b \neq c, \dots$), and its algebraic structure is determined by the relation[4]

$$\{e_a, e_b\} = 2\delta_{ab}. \tag{7.1}$$

The division algebras except for the octonions are realized as special cases of Clifford algebra, i.e. $\mathbb{R} = \mathrm{Cliff}_0$, $\mathbb{C} = \mathrm{Cliff}_1$, and $\mathbb{H} = \mathrm{Cliff}_2$. The Clifford algebra can also be regarded as the "quantized" Grassmann algebra (compare (7.1) with $\{\eta_a, \eta_b\} = 0$ (4.1)). Though the division property is in general lost, the Clifford algebras always maintain the nice associative property, and are represented by gamma matrices that satisfy

$$\{\gamma_a, \gamma_b\} = 2\delta_{ab}.$$

Importantly, there are analogous geometrical properties between the division algebras and the Clifford algebras: In the division algebras, new numbers are constructed by the Cayley–Dickson construction. Similarly, higher dimensional gamma matrices are constructed by the lower dimensional gamma matrices. Specifically, $SO(2k-1)$ gamma matrices $\gamma_i^{(2k-1)}$ ($i = 1, 2, \dots, 2k-1$) are provided, $SO(2k+1)$ gamma matrices $\gamma_a^{(2k+1)}$ ($a = 1, 2, \dots, 2k+1$) can be constructed as

$$\gamma_i^{(2k+1)} = \begin{pmatrix} 0 & i\gamma_i^{(2k-1)} \\ -i\gamma_i^{(2k-1)} & 0 \end{pmatrix}, \qquad \gamma_{2k}^{(2k+1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_{2k+1}^{(2k+1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.2}$$

Thus, in any higher dimensional gamma matrices are constructed by repeating the above procedure from the $SO(3)$ gamma matrices, $\gamma_1^{(3)} = -\sigma_2$, $\gamma_2^{(3)} = \sigma_1$, $\gamma_3^{(3)} = \sigma_3$. As the Hopf maps exhibit the hierarchical structure stemming from the Cayley–Dickson construction, the geometry of higher dimensional fuzzy spheres reflects the iterative construction structure of the gamma matrices as we shall see below.

7.2 Mathematical aspects of fuzzy sphere

We first introduce mathematical construction of fuzzy spheres [2, 9, 14, 15, 52, 53, 54]. Coordinates on fuzzy $2k$-sphere are constructed by the $SO(2k+1)$ gamma matrices in the fully symmetric representation $[\frac{I}{2}, \frac{I}{2}, \dots, \frac{I}{2}]$:

$$X_a = \frac{\alpha}{2}(\gamma_a \otimes 1 \otimes \cdots \otimes 1 + \gamma_a \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \gamma_a)_{\mathrm{sym}},$$

where γ_a ($a = 1, 2, \dots, 2k+1$) are the $SO(2k+1)$ gamma matrices in the fundamental representation, and the number of the tensor product is I. Square of the radius of fuzzy $2k$-sphere is given by

$$\sum_{a=1}^{2k+1} X_a X_a = \frac{\alpha^2}{4} I(I+2k).$$

Similarly, in the symmetric representation, the eigenvalue of the $SO(2k+1)$ Casimir is

$$\sum_{a<b}^{2k+1} X_{ab}^2 = \frac{\alpha^2}{16} kI(I+2k),$$

[4] Though we treat Clifford algebras with Euclidean signature, Clifford algebras can be generally defined with indefinite signature.

where $X_{ab} = -i\frac{1}{4}[X_a, X_b]$ are the $SO(2k+1)$ generators. (Detail calculation techniques for symmetric representation can be found in [21].) Thus, the index I of the symmetric representation determines the magnitude of the radius of fuzzy sphere. As the dimension of the symmetric representation becomes "larger", the corresponding fuzzy sphere becomes "larger".

As in the case of fuzzy-four sphere, the fuzzy coordinates X_a do not satisfy a closed algebra by themselves, but X_a and X_{ab} satisfy the following enlarged algebra

$$[X_a, X_b] = 2i\alpha X_{ab}, \qquad [X_a, X_{bc}] = -i\frac{\alpha}{2}(\delta_{ab}X_c - \delta_{ac}X_b),$$
$$[X_{ab}, X_{cd}] = i\frac{\alpha}{2}(\delta_{ac}X_{bd} - \delta_{ad}X_{bc} + \delta_{bc}X_{ad} - \delta_{bd}X_{ac}).$$

With identification $X_{a,2k+2} = -X_{2k+2,a} = \frac{1}{2}X_a$ and $X_{ab} = X_{ab}$, the above algebra is found to be equivalent to the $SO(2k+2)$ algebra

$$[X_{AB}, X_{CD}] = i\frac{\alpha}{2}(\delta_{AC}X_{BD} - \delta_{AD}X_{BC} + \delta_{BC}X_{AD} - \delta_{BD}X_{AC}),$$

where $A, B, C, D = 1, 2, \ldots, 2k+2$. Thus, the algebra of fuzzy sphere S_F^{2k+1} is $SO(2k+2)$ [14], and S_F^{2k} is expressed as

$$S_F^{2k} \simeq SO(2k+2)/U(k+1) \simeq SO(2k+1)/U(k),$$

where we used the relation $SO(2k+2)/SO(2k+1) \simeq S^{2k+1} \simeq U(k+1)/U(k)$. The above coset representation can also be expressed as

$$S_F^{2k} \approx SO(2k+1)/SO(2k) \times SO(2k-1)/U(k-1) \approx S^{2k} \times S_F^{2k-2}. \tag{7.3}$$

Fuzzy $2k$-sphere is constructed not only by the operators X_a but also the "extra" operators X_{ab}, and the very existence of X_{ab} brings the extra fuzzy-space S_F^{2k-2} over S^{2k}. From (7.3), one may find the fuzzy sphere is expressed by the hierarchical fibrations of lower dimensional spheres

$$S_F^{2k} \approx S^{2k} \times S^{2k-2} \times \cdots \times S^4 \times S^2, \tag{7.4}$$

which reflects the iterative construction of gamma matrices from lower dimensions.

7.3 Hopf spinor matrix and $SO(2k)$ monopole

To obtain monopole bundles in generic even dimensions, we "extend" the Hopf maps. First, we define "Hopf spinor matrix" of the form

$$\mathbf{\Psi} = \frac{1}{\sqrt{2(1+x_{2k+1})}}\begin{pmatrix}(1+x_{2k+1})\mathbf{1}\\(x_{2k}\mathbf{1} - i\gamma_i x_i)\end{pmatrix}, \tag{7.5}$$

where $\mathbf{1}$ stands for $2^k \times 2^k$ unit matrix, γ_i $(i = 1, 2, \ldots, 2k-1)$ are $SO(2k-1)$ gamma matrices, and x_a $(a = 1, 2, \ldots, 2k+1)$ are coordinate on $2k$-sphere satisfying $\sum_{a=1}^{2k+1} x_a x_a = 1$. The Hopf spinor matrix is a $2^{k+1} \times 2^k$ matrix that satisfies

$$x_a\mathbf{1} = \mathbf{\Psi}^\dagger \gamma_a \mathbf{\Psi},$$

where γ_a are $SO(2k+1)$ gamma matrices (7.2). The corresponding monopole gauge field is evaluated as

$$A = dx_a A_a = -i\mathbf{\Psi}^\dagger d\mathbf{\Psi},$$

where

$$A_\mu = -\frac{1}{2(1+x_{2k+1})}\Sigma^+_{\mu\nu}x_\nu, \qquad A_{2k+1} = 0. \tag{7.6}$$

$\Sigma^+_{\mu\nu}$ $(\mu,\nu = 1,2,\ldots,2k)$ are $SO(2k)$ generators given by

$$\Sigma^+_{ij} = -i\frac{1}{4}[\gamma_i,\gamma_j], \qquad \Sigma^+_{i2k} = -\Sigma^+_{2ki} = \frac{1}{2}\gamma_i.$$

The field strength $F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b]$ is calculated to yield

$$F_{\mu\nu} = -x_\mu A_\nu + x_\nu A_\mu + \Sigma^+_{\mu\nu}, \qquad F_{\mu,2k+1} = -F_{2k+1,\mu} = (1+x_9)A_\mu.$$

These are the $SO(2k)$ non-Abelian monopole gauge field strength [59, 60, 61]. Another representation of the Hopf spinor matrix is introduced as

$$\mathbf{\Psi}' = \frac{1}{\sqrt{2(1-x_{2k+1})}}\begin{pmatrix}(x_{2k}\mathbf{1}+i\gamma_i x_i)\\(1-x_{2k+1})\mathbf{1}\end{pmatrix}, \tag{7.7}$$

and the corresponding gauge field, $A' = -i\mathbf{\Psi}^\dagger d\mathbf{\Psi}'$, reads as

$$A'_\mu = -\frac{1}{2(1-x_{2k+1})}\Sigma^-_{\mu\nu}x_\nu, \qquad A'_{2k+1} = 0, \tag{7.8}$$

where

$$\Sigma^-_{ij} = \Sigma^+_{ij}, \qquad \Sigma^-_{i,2k} = -\Sigma^+_{2k,i}.$$

The gauge field strength F'_{ab} is

$$F'_{\mu\nu} = x_\mu A'_\nu - x_\nu A'_\mu + \Sigma^-_{\mu\nu}, \qquad F'_{\mu,2k+1} = -F'_{2k+1,\mu} = (1-x_9)A'_\mu.$$

The $SO(2k)$ generators, $\Sigma^+_{\mu\nu}$ and $\Sigma^-_{\mu\nu}$, satisfy the generalized self and anti-self dual relations, respectively:

$$\Sigma^\pm_{\mu_1\mu_2} = \pm\frac{2^{k-1}}{(2k-2)!}\epsilon_{\mu_1\mu_2\mu_3\mu_4\cdots\mu_{2k-1}\mu_{2k}}\Sigma^\pm_{\mu_3\mu_4}\cdots\Sigma^\pm_{\mu_{2k-1}\mu_{2k}}.$$

The two Hopf spinor matrices (7.5) and (7.7) are related by the transformation

$$\mathbf{\Psi}' = \begin{pmatrix} g & 0\\ 0 & g\end{pmatrix}\cdot\mathbf{\Psi},$$

where g is an $SO(2k)$ group element

$$g = \frac{1}{\sqrt{1-x_{2k+1}^2}}(x_{2k} + i\gamma_i x_i).$$

With g, the gauge fields (7.6) and (7.8) are concisely represented as

$$A = i\frac{1}{2}(1-x_{2k+1})dgg^\dagger, \qquad A' = -i\frac{1}{2}(1+x_{2k+1})g^\dagger dg,$$

where

$$-ig^\dagger dg = -\frac{2}{1-x_{2k+1}^2}\Sigma^-_{\mu\nu}x_\nu dx_\mu, \qquad -igdg^\dagger = \frac{2}{1-x_{2k+1}^2}\Sigma^+_{\mu\nu}x_\nu dx_\mu.$$

Then, the two gauge fields are related as

$$A' = g^\dagger A g - i g^\dagger dg,$$

and the field strengths are

$$F' = g^\dagger F g.$$

The homotopy theorem guarantees the non-trivial topology of the $SO(2k)$ bundle fibration over S^{2k}:

$$\pi_{2k-1}(SO(2k)) = \mathbb{Z},$$

which is specified by the Euler number

$$e = \int_{S^{2k}} \mathrm{tr}\left(F^k\right).$$

7.4 $SO(2k+1)$ Landau model

In generic d-dimensional space, Landau Hamiltonian is given by [56]

$$H = -\frac{1}{2M} D_a^2 = -\frac{1}{2M}\frac{\partial^2}{\partial r^2} - (d-1)\frac{1}{2Mr}\frac{\partial}{\partial r} + \frac{1}{2Mr^2}\sum_{a<b}\Lambda_{ab}^2, \tag{7.9}$$

where $D_a = \partial_a + iA_a$ (A_a is the $SO(2k)$ monopole gauge field), and $\Lambda_{ab} = -ix_a D_b + ix_b D_a$. The covariant momentum Λ_{ab} does not satisfy the $SO(2k+1)$ algebra but satisfies

$$\begin{aligned}[\Lambda_{ab}, \Lambda_{cd}] &= i(\delta_{ac}\Lambda_{bd} + \delta_{bd}\Lambda_{ac} - \delta_{bc}\Lambda_{ad} - \delta_{ad}\Lambda_{bc}) \\ &\quad - i(x_a x_c F_{bd} + x_b x_d F_{ac} - x_b x_c F_{ad} - x_a x_d F_{bc}).\end{aligned}$$

The conserved $SO(2k+1)$ angular momentum is constructed as

$$L_{ab} = \Lambda_{ab} + r^2 F_{ab},$$

which satisfies the $SO(2k+1)$ algebra

$$[L_{ab}, L_{cd}] = i(\delta_{ac}L_{bd} + \delta_{bd}L_{ac} - \delta_{bc}L_{ad} - \delta_{ad}L_{bc}),$$

and generates the $SO(2k+1)$ transformations, for instance

$$\begin{aligned}[L_{ab}, \Lambda_{cd}] &= i(\delta_{ac}\Lambda_{bd} + \delta_{bd}\Lambda_{ac} - \delta_{bc}\Lambda_{ad} - \delta_{ad}\Lambda_{bc}), \\ [L_{ab}, F_{cd}] &= i(\delta_{ac}F_{bd} + \delta_{bd}F_{ac} - \delta_{bc}F_{ad} - \delta_{ad}F_{bc}).\end{aligned}$$

On $2k$-sphere, the Landau Hamiltonian (7.9) is reduced to $SO(2k+1)$ Landau Hamiltonian

$$H = \frac{1}{2MR^2}\sum_{a<b}\Lambda_{ab}^2.$$

With the orthogonality $\Lambda_{ab}F_{ab} = F_{ab}\Lambda_{ab}$, the Hamiltonian is rewritten as

$$H = \frac{1}{2MR^2}\left(\sum_{a<b} L_{ab}^2 - \sum_{a<b} F_{ab}^2\right) = \frac{1}{2MR^2}\left(\sum_{a<b} L_{ab}^2 - \sum_{\mu<\nu}\Sigma_{\mu\nu}^2\right),$$

Table 3. The fuzzy $2k$-sphere is physically realized in the LLL of the $SO(2k+1)$ Landau model. Previously encountered monopoles are understood as the special cases of $SO(2k)$ monopoles, for instance $U(1) \simeq SO(2)$, $SU(2)(\otimes SU(2)) \simeq SO(4)$. Note also $SU(4) \simeq SO(6)$. The holonomy groups of spheres are equal to the corresponding monopole gauge groups.

Fuzzy sphere	Corresponding original sphere	Monopole gauge group
$S_F^2 \simeq SO(3)/U(1)$	$S^2 \simeq SO(3)/SO(2)$	$U(1)$
$S_F^4 \simeq SO(5)/U(2)$	$S^4 \simeq SO(5)/SO(4)$	$SU(2)$
$S_F^6 \simeq SO(7)/U(3)$	$S^6 \simeq SO(7)/SO(6)$	$SU(4)$
$S_F^8 \simeq SO(9)/U(4)$	$S^8 \simeq SO(9)/SO(8)$	$SO(8)$
$\vdots$	$\vdots$	$\vdots$
$S_F^{2k} \simeq SO(2k+1)/U(k)$	$S^{2k} \simeq SO(2k+1)/SO(2k)$	$SO(2k)$

where $\sum_{a<b} F_{ab}^2 = \sum_{\mu<\nu} \Sigma_{\mu\nu}^2$ was used. As in the previous Landau models, we take the fully symmetric spinor representation $(I/2) \equiv [\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}]$ for the $SO(2k)$ Casimir $\sum_{\mu<\nu} \Sigma_{\mu\nu}^2$, and the irreducible representation $(n, I/2) \equiv [n+\frac{I}{2}, \frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}]$ for the $SO(2k+1)$ Casimir $\sum_{a<b} L_{ab}^2$, with the Landau level index n. In such a representation, the energy eigenvalues are given by

$$E_n = \frac{1}{2MR^2}(C_{2k+1}(n, I/2) - C_{2k}(I/2)),$$

where $C_{2k+1}(n, I/2)$ is the $SO(2k+1)$ Casimir eigenvalue for $(n, I/2)$:

$$C_{2k+1}(n, I/2) = n^2 + n(I + 2k - 1) + \frac{k}{4} I(I + 2k),$$

and $C_{2k}(I/2)$ is the $SO(2k)$ Casimir eigenvalue for $(I/2)$:

$$C_{2k}(I/2) = \sum_{\mu<\nu} \Sigma_{\mu\nu}^2 = \frac{1}{4} Ik(I + 2k - 2).$$

Consequently, the energy eigenvalues of the $SO(2k+1)$ Landau model are derived as

$$E_n = \frac{1}{2MR^2}(n^2 + n(I + 2k - 1) + \frac{1}{2} Ik).$$

In the thermodynamic limit: $R, I \to \infty$ with I/R^2 fixed, the Landau levels are reduced to

$$E_n \to \frac{I}{2MR^2}(n + \frac{1}{2}k).$$

The LLL energy, $E_{\text{LLL}} = \frac{I}{4MR^2}k$, depends on the spacial dimension $2k$. In the thermodynamic limit, S^{2k} is reduced to $2k$-dimensional plane, and the zero-point energy $\frac{B}{2M} = \frac{I}{4MR^2}$ coming from each 2-dimensional plane amounts to E_{LLL}.

As discussed above, coordinates on fuzzy $2k$-sphere are given by the $SO(2k+1)$ gamma matrices in the symmetric spinor representation. Similarly, the LLL basis of the $SO(2k+1)$ Landau model realizes such a symmetric spinor representation. Thus, the LLL of $SO(2k+1)$ Landau model provides a physical set-up for $2k$-dimensional fuzzy sphere (see Table 3).

7.5 Dimensional hierarchy

Here, we give a physical interpretation of the hierarchical geometry of higher dimensional fuzzy spheres (7.4). From the formula of the irreducible representation of $SO(2k+1)$ [49], the degeneracy in nth LL is given by

$$d(n) = \frac{2n+I+2k-1}{(2k-1)!!} \frac{(n+k-1)!}{n!(k-1)!} \prod_{i=1}^{k-1}(I+2i-1) \prod_{i=2}^{k} \frac{n+I+2k-i}{2k-i} \prod_{l=1}^{k-2} \prod_{i=l+2}^{k} \frac{I+2k-i-l}{Ik-i-l}.$$

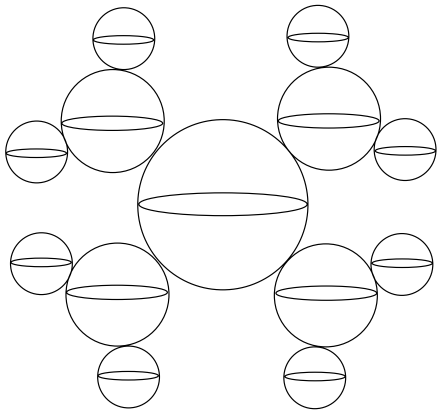

Figure 1. Lower dimensional spheres gather spherically to form a higher dimensional fuzzy sphere.

Table 4. Correspondence between algebras, monopoles, and fuzzy spheres.

Algebras	Bundle structure	Monopoles	Fuzzy manifolds
Division algebra	Canonical bundle	$U(1)$, $SU(2)$, $SO(8)$	Fuzzy 2,4,8-spheres
Grassmann algebra	Graded canonical bundle	Supermonopole	Fuzzy supersphere
Clifford algebra	Spinor bundle	$SO(2k)$	Higher d. fuzzy spheres

In the LLL, the representation is reduced to the fully symmetric spinor representation $[\frac{I}{2},\frac{I}{2},\ldots,\frac{I}{2}]$, and the degeneracy becomes

$$d_{\mathrm{LLL}} = \frac{(I+2k-1)!!}{(2k-1)!!(I-1)!!}\prod_{l=1}^{k-1}\frac{(I+2l)!l!}{(I+l)!(2l)!} \longrightarrow I\cdot I^2\cdot I^3\cdots I^{k-1}\cdot I^k = I^{\frac{1}{2}k(k+1)}. \quad (7.10)$$

The last expression implies a nice intuitive picture of the hierarchical geometry of fuzzy spheres. Each of the $SO(2k)$ monopole fluxes on S^{2k} occupies an area $\ell_B^{2k} = (1/B)^k = (2R^2/I)^k$ with the magnetic field $B = (2\pi I)/(4\pi R^2)$, and the number of fluxes on S^{2k} is $\sim R^{2k}/\ell_B^{2k} \sim I^k$. Besides, the monopole flux itself is represented by the generators of the non-Abelian $SO(2k)$ group, and is regarded as a $(2k-2)$-dimensional fuzzy sphere. Again, the $(2k-2)$-dimensional fuzzy sphere is interpreted as a $(2k-2)$-dimensional sphere in $SO(2k-2)$ monopole background, and then, on S^{2k-2}, there are $SO(2k-2)$ fluxes each of which occupies the area $\ell_B^{2k-2} = (1/B)^{k-1} = (2R^2/I)^{k-1}$. Therefore, the number of $SO(2k-2)$ fluxes on S^{2k-2} is $\sim R^{2k-2}/\ell_B^{2k-2} \sim I^{k-1}$. Similarly, the $SO(2k-2)$ non-Abelian flux is given by the generators of the $SO(2k-2)$ group, and regarded as a fuzzy S^{2k-4}. Thus, on S^{2k}, we have I^k S^{2k-2}, on which I^{k-1} S^{2k-4}, on which I^{k-2} S^{2k-6}, ... By this iteration, we obtain the formula (7.10). Inversely, we can view this mechanism from low dimensions: Lower dimensional spheres gather spherically to form a higher dimensional sphere, and such iterative process amounts to construct a higher dimensional fuzzy sphere. The dimensional hierarchy is depicted in Fig. 1.

8 Summary and discussions

We reviewed the close relations between monopoles, LLL, and fuzzy spheres. The fuzzy $2k$-sphere is physically realized in the LLL of the $SO(2k+1)$ Landau model. In the generalization of fuzzy spheres, three classical algebras; division algebra, Grassmann algebra, and Clifford algebra, played crucial roles. They brought the basic structures of monopole bundles and fuzzy spheres (Table 4). In particular, the hierarchical geometry of fuzzy spheres is a direct manifestation of their gamma matrix construction: As higher dimensional gamma matrices are constructed by lower dimensional gamma matrices, higher dimensional fuzzy sphere is constructed by lower dimensional spheres. Such dimensional hierarchy can be physically understood in the context of higher dimensional Landau model. It should be mentioned that, in [62], such interpretation was successfully applied to dual description of higher dimensional fuzzy spheres in string theory. We also emphasize the importance of the Hopf map in realizing fuzzy sphere. In the fuzzification of spheres, the total manifolds (Hopf spinor spaces) of the Hopf maps played a fundamental role: The total manifolds were firstly fuzzificated and as "a consequence" the basemanifolds (spheres) are fuzzificated. Interestingly, this fuzzification mechanism coincides with the philosophy of twistor theory (see [63] and references therein).

There are various works related to the present paper: Even restricted to the recent ones, supersymmetric extensions of the Landau model [64, 65, 66, 67, 68, 69], supersymmetric fuzzy manifolds [70, 71], generalizations of the Hopf maps [72, 73, 74, 75], supersymmetric quantum mechanics [76, 77, 78, 79, 80, 81, 82], and applications to string theory [83, 84, 85, 86].

Finally, we comment on applications to many-body physics. The correspondence between fuzzy geometry and LLL physics argued in the paper was at one-particle level observation. Interestingly, there even exists correspondence at many-body level: Many-body groundstate wavefunction of quantum Hall effect (Laughlin wavefunction) is mathematically analogous to an antiferromagnetic ground state (AKLT state) [87, 88]. Accompanied with the higher dimensional and supersymmetric generalizations of the quantum Hall effect[5], their formalism has begun to be applied to the construction of antiferromagnetic quantum spin states with higher symmetries [88, 93].

A 0th Hopf map and $SO(2)$ "Landau model"

Real numbers are the "0th" member of the division algebra. For completeness, we introduce the 0th Hopf map

$$S^1 \xrightarrow{Z_2} S^1,$$

and the corresponding Landau model. The 0th Hopf map is realized by identifying "opposite points" on a circle, and is simply visualized as the geometry of Möbius strip whose basemanifold is S^1 and transition function is Z_2. Unlike the other Hopf maps, the dimension of the basemanifold is odd and the structure group is a discrete group.

A.1 Realization of the 0th Hopf map

With the coordinate on a circle, $w = (w_1, w_2)^t$ (a real two-component spinor subject to $w^t w = w_1^2 + w_2^2 = 1$), the 0th Hopf map is realized as

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \rightarrow x_1 = w^t \sigma_3 w, \qquad x_2 = w^t \sigma_1 w. \tag{A.1}$$

Here, σ_1 and σ_3 are the real Pauli matrices. x_1 and x_2 are invariant under the transformation $(w_1, w_2) \rightarrow -(w_1, w_2)$, and hence (x_1, x_2) is Z_2 projection of (w_1, w_2). From (A.1), one may find $x_1^2 + x_2^2 = (w^t w)^2 = 1$. Inverting the map, w can be expressed as

$$w = \frac{1}{\sqrt{2(1+x_1)}} \begin{pmatrix} 1 + x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\dfrac{\theta}{2} \\ \sin\dfrac{\theta}{2} \end{pmatrix},$$

[5] Interested readers are expected to consult the review [24], and [89, 90, 91, 92, 63] for more recent works.

where x_1 and x_2 are parameterized as $x_1 = \cos\theta$ and $y = \sin\theta$. The corresponding connection vanishes: $A = -iw^t dw = 0$. Meanwhile, using a $U(1)$ element $\omega = w_1 + iw_2 = e^{i\frac{\theta}{2}}$, the 0th Hopf map is restated as

$$\omega \to x_1 + ix_2 = \omega^2,$$

and the connection, $A = -i\omega^* d\omega = iw^t \sigma_2 dw = dxA_x + dyA_y$, is given by

$$A_x = -\frac{1}{2}y, \qquad A_y = \frac{1}{2}x \qquad \text{or} \qquad A_\theta = -\frac{1}{2}.$$

It is straightforward to see the field strength, $B = \partial_x A_y - \partial_y A_x$, represents a solenoid-like magnetic field at the origin:

$$B = \pi\delta(x, y).$$

A.2 $SO(2)$ "Landau model"

Next, we introduce "Landau model" on a circle in the presence of magnetic fluxes [94]

$$B = I\pi\delta(x, y).$$

Here, I is the number of magnetic fluxes, and takes an integer value. The corresponding gauge field is given by

$$A_x = -\frac{I}{2r^2}y, \qquad A_y = \frac{I}{2r^2}x,$$

where $r^2 = x^2 + y^2$. Since the magnetic fluxes are at the origin, the classical motion of a charged particle on a circle is not affected by the existence of the magnetic fluxes. In quantum mechanics, however, the result is different. With the covariant derivatives $D_x = \partial_x + iA_x$ and $D_y = \partial_y + iA_y$, Landau Hamiltonian on 2D plane is given by

$$H = -\frac{1}{2M}\left(D_x^2 + D_y^2\right) = -\frac{1}{2M}\frac{\partial^2}{\partial r^2} + \frac{1}{2Mr^2}\Lambda^2,$$

where Λ is the covariant angular momentum

$$\Lambda = -ixD_y + iyD_x = -i\frac{\partial}{\partial\theta} + A_\theta,$$

with $A_\theta = \frac{I}{2}$. On a circle with radius R, the Hamiltonian is reduced to $SO(2)$ "Landau Hamiltonian"

$$H = \frac{1}{2MR^2}\Lambda^2 = -\frac{1}{2MR^2}\left(\frac{\partial}{\partial\theta} + \frac{i}{2}I\right)^2.$$

This is a one-dimensional quantum mechanical Hamiltonian easily solved. In higher dimensions, the covariant angular momentum is not a conserved quantity, but the present case it is, as simply verified $[\Lambda, H] = 0$. Since the magnetic field angular momentum does not exist on the circle, the particle angular momentum itself is conserved. Imposing the periodic boundary condition

$$u(\theta = 2\pi) = u(\theta = 0),$$

the eigenvalue problem is classified to two cases: even I and odd I. For even I, the energy eigenvalue are given by

$$E_n = \frac{1}{2MR^2}n^2,$$

where $n = 0, 1, 2, 3, \ldots$. The eigenstates are

$$u(\theta)_{\pm n} = \frac{1}{\sqrt{2\pi}} e^{i\left(\pm n - \frac{I}{2}\right)\theta}.$$

Except for the lowest energy level $n = 0$, every excited energy level is two-fold degenerate. The physical origin of the double degeneracy comes from the left and the right movers on the circle. The energy levels are identical to those of free particle on a circle. Thus, for even I, the magnetic flux does not affect the energy spectrum of the system. Meanwhile, for odd I, the energy eigenvalues become

$$E'_n = \frac{1}{2MR^2}\left(n + \frac{1}{2}\right)^2, \tag{A.2}$$

and the corresponding eigenstates are

$$u'(\theta)_{\pm n} = \frac{1}{\sqrt{2\pi}} e^{i\left(\pm\left(n+\frac{1}{2}\right) - \frac{I}{2}\right)\theta}.$$

All the Landau levels are doubly degenerate even for $n = 0$. The energy spectra (A.2) are different from those of the free particle on a circle, reflecting the particular role of the gauge field in quantum mechanics. It is also noted that the number of the magnetic fluxes I has nothing to do with the degeneracy in Landau levels unlike the other Landau models (see equation (3.11) for instance).

B Generalized 1st Hopf map and $SU(k+1)$ Landau model

B.1 $SU(k+1)$ Landau model and fuzzy complex projective space

The projection from sphere to complex projective space,

$$S^{2k+1} \xrightarrow{S^1} \mathbb{C}P^k,$$

is a straightforward generalization of the 1st Hopf map. (Note $\mathbb{C}P^1 \simeq S^2$.) In [95], Karabali and Nair introduced $SU(k+1)$ Landau model on $\mathbb{C}P^k$ in $U(1)$ monopole background. Since the $SU(k+1)$ Landau models are reviewed in [20, 24], we survey the main results. The $SU(k+1)$ Landau model Hamiltonian is given by

$$H = \frac{1}{2MR^2}\left(C_{SU(k+1)} - \frac{k}{2(k+1)} I^2\right), \tag{B.1}$$

where $C_{SU(k+1)}$ represents the Casimir operator of the $SU(k+1)$ group. One may notice analogies between (B.1) and the spherical Landau model Hamiltonians (see for instance (3.9)). The $SU(k+1)$ group has k Casimirs, and hence its irreducible representations are specified with Young tableau index, $[\lambda_1, \lambda_2, \ldots, \lambda_k]$. In the $SU(k+1)$ Landau model, Young tableau index is chosen as $[\lambda_1, \lambda_2, \ldots, \lambda_k] = [p+q, q, \ldots, q]$. When $q = 0$, the Young tableau index becomes $[p, 0, \ldots, 0]$ representing the fully symmetric representation of the $SU(k+1)$ spinor. Meanwhile, when $p = 0$, the index becomes $[q, q, \ldots, q]$ representing the fully symmetric representation of the $SU(k+1)$ complex spinor. The "difference" between the fundamental and complex representations is specified by $I = p - q$, which corresponds to the $U(1)$ monopole charge. The Landau level index can be taken as $q = n$, since, in LLL ($n = 0$), the irreducible representation is reduced to the fully symmetric spinor representation. Thus, the $SU(k+1)$ Young tableau index $(n, I) \equiv [2n+I, n, \ldots, n]$ specifies the states of the nth Landau level in $U(1)$ monopole background with magnetic charge $\sqrt{\frac{k}{2(k+1)}}I$. Then, the energy eigenvalues are derived as

$$E_n = \frac{1}{2MR^2}\left(C_{SU(k+1)}(n, I) - \frac{k}{2(k+1)}I^2\right) = \frac{1}{2MR^2}\left(n(n+k) + I\left(n + \frac{k}{2}\right)\right) \quad \text{(B.2)}$$

and the corresponding degeneracies are

$$d(n, I) = \frac{1}{k!(k-1)!}(2n+I+k)\frac{(I+n+k-1)!(n+k-1)!}{(I+n)!n!}. \quad \text{(B.3)}$$

The k dependence of the LLL energy $E_{\text{LLL}} = \frac{I}{4MR^2}k$ is simply understood by remembering that $\mathbb{C}P^k$ consists of k complex planes each of which equally provides zero-point energy contribution $\frac{B}{2M} = \frac{I}{4MR^2}$ to LLL energy. The LLL basis elements are constructed by the fully symmetric products of the $SU(k+1)$ Hopf spinor or the Perelomov coherent state of $SU(k+1)$ [96]

$$\psi = \frac{1}{\sqrt{1 + z_1^* z_1 + z_2^* z_2 + \cdots + z_k^* z_k}}\begin{pmatrix} 1 \\ z_1 \\ z_2 \\ \vdots \\ z_k \end{pmatrix}, \quad \text{(B.4)}$$

as

$$\psi_{\text{LLL}}^{(m_1, m_2, \ldots, m_k)} = \frac{1}{\sqrt{1 + z_1^* z_1 + z_2^* z_2 + \cdots + z_k^* z_k}^{\,I}} 1^{m_1} {z_1}^{m_2} {z_2}^{m_3} \cdots {z_k}^{m_{k+1}}, \quad \text{(B.5)}$$

where $m_1 + m_2 + \cdots + m_{k+1} = I$ $(m_1, m_2, \ldots, m_k \geq 0)$. Thus, the degeneracy in the LLL is $d_{\text{LLL}} = \frac{(I+k)!}{I!k!}$ that coincides with $d(0, I)$ of (B.3). With the $SU(k+1)$ Hopf spinor (B.4), the LLL Lagrangian is given by

$$L_{\text{LLL}} = -iI\psi^\dagger \frac{d}{dt}\psi.$$

After quantization, the normalization constraint $\psi^\dagger\psi = 1$ is imposed on the LLL states:

$$\psi^t \frac{d}{d\psi}\psi_{\text{LLL}} = I\psi_{\text{LLL}}.$$

Indeed, the LLL states (B.5) satisfy the above condition.

Fuzzy $\mathbb{C}P^k$ is constructed by taking the fully symmetric representation of $SU(k+1)$ [97, 98, 99, 22]. With the $SU(k+1)$ extended Schwinger boson operator

$$\hat{\psi} = \begin{pmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \\ \vdots \\ \hat{\psi}_{k+1} \end{pmatrix}$$

satisfying $[\hat{\psi}_\alpha, \hat{\psi}_\beta] = \delta_{\alpha\beta}$ $(\alpha, \beta = 1, 2, \ldots, k+1)$, coordinates on fuzzy $\mathbb{C}P^k$ are constructed as

$$\hat{T}_i = \alpha\hat{\psi}^\dagger t_i \hat{\psi}$$

where t_i ($i = 1, 2, \ldots, k^2+2k$) stand for the fundamental representation matrix of the $SU(k+1)$ generators, normalized as $\mathrm{tr}\,(t_i t_j) = \frac{1}{2}\delta_{ij}$. Square of the radius of fuzzy $\mathbb{C}P^k$ is given by the $SU(k+1)$ Casimir operator:

$$\sum_{i=1}^{k^2+2k} \hat{T}_i \hat{T}_i = \frac{k}{2(k+1)}\alpha^2 I(I+k+1). \tag{B.6}$$

In particular for fuzzy $\mathbb{C}P^1$ ($k=1$), equation (B.6) reproduces the result of fuzzy two-sphere (3.19):

$$\sum_{i=1}^{3} \hat{T}_i \hat{T}_i = \frac{\alpha^2}{4} I(I+2).$$

Similarly for fuzzy $\mathbb{C}P^3$ ($k=3$), equation (B.6) becomes

$$\sum_{i=1}^{15} \hat{T}_i \hat{T}_i = \frac{3}{8}\alpha^2 I(I+4),$$

which is equal to the Casimir of fuzzy four-sphere

$$\frac{1}{4}\sum_{a=1}^{5} \hat{X}_a \hat{X}_a + \sum_{a<b} \hat{X}_{ab}\hat{X}_{ab} = \frac{3}{8}\alpha^2 I(I+4),$$

where $\hat{X}_a$ and $\hat{X}_{ab}$ are given by equations (5.20) and (5.21), respectively.

The symmetric representation is constructed as

$$|m_1, m_2, \ldots, m_k\rangle_{sym.} = \frac{1}{\sqrt{m_1! m_2! \cdots m_{k+1}!}} (\hat{\psi}_1^\dagger)^{m_1} (\hat{\psi}_2^\dagger)^{m_2} \cdots (\hat{\psi}_{k+1}^\dagger)^{m_{k+1}} |0\rangle,$$

where $m_1 + m_2 + \cdots + m_{k+1} = I$ ($m_1, m_2, \ldots, m_{k+1} \geq 0$). The number of the states consisting of fuzzy $\mathbb{C}P^k$ is $d(0,I) = \frac{(I+k)!}{k!I!}$. From the above discussion, the correspondences between the $SU(k+1)$ Landau model and the $\mathbb{C}P^k$ fuzzy manifold is apparent.

B.2 Relations to spherical Landau models

As easily verified, the $SU(2)$ Landau model is equivalent to the $SO(3)$ Landau model. From the formulae (B.2) and (B.3), the energy eigenvalues and degeneracies of the $SU(2)$ Landau Hamiltonian are derived as

$$E_n = \frac{1}{2MR^2}\left(n(n+1) + I\left(n + \frac{1}{2}\right)\right), \qquad d_n = 2n + I + 1.$$

These indeed reproduce the results of the $SO(3)$ Landau model, (3.10) and (3.11). Moreover, the $SU(2)$ coherent state

$$\psi = \frac{1}{\sqrt{1+z^* z}}\begin{pmatrix} 1 \\ z \end{pmatrix},$$

corresponds to the 1st Hopf spinor (3.4) by the relation

$$z = \frac{\phi_2}{\phi_1} = \frac{x_1 + i x_2}{1 + x_3},$$

and the $SU(2)$ LLL basis elements (B.5) for $k=1$ are equal to equation (3.12).

There also exists correspondence between the $SU(4)$ and $SO(5)$ Landau models in the LLL. Again from equations (B.2) and (B.3), the energy eigenvalues and degeneracies of the $SU(4)$ Landau model are read as

$$E_n = \frac{1}{2MR^2}\left(n(n+3) + I\left(n+\frac{3}{2}\right)\right),$$
$$d_n = \frac{1}{12}(n+1)(n+2)(I+n+1)(I+n+2)(I+2n+3). \tag{B.7}$$

The $SU(4)$ Landau level energy is different from that of the $SO(5)$ Landau model (5.11), only by the total energy shift $\frac{1}{4MR^2}I$. This discrepancy is understood by a simple geometrical argument [95]: Since $\mathbb{C}P^3 \approx S^4 \times S^2$, the $\mathbb{C}P^3$ is regarded as S^2-fibration over S^4, and the zero-point energy from the extra S^2-space, $\frac{B}{2M} = \frac{1}{4MR^2}I$, gives rise to the difference. Generally, the degeneracy of Landau levels of the $SU(4)$ Landau model (B.7) is different from that of the $SO(5)$ Landau model (5.12), but in the LLL, both quantities coincide to yield $d_{\mathrm{LLL}} = \frac{1}{6}(I+1)(I+2)(I+3)$. This manifests the equivalence between the $SO(5)$ and $SU(4)$ Landau models in LLL. By comparing the 2nd Hopf spinor (5.5) with the $SU(4)$ coherent state

$$\psi = \frac{1}{\sqrt{1+z_1^*z_1+z_2^*z_2+z_3^*z_3}}\begin{pmatrix}1\\ z_1\\ z_2\\ z_3\end{pmatrix},$$

we find the correspondence

$$z_1 = \frac{\psi_2}{\psi_1} = \frac{\phi_2}{\phi_1}, \qquad z_2 = \frac{\psi_3}{\psi_1} = \frac{1}{1+x_5}\left(x_4 - ix_3 - (ix_1+x_2)\frac{\phi_2}{\phi_1}\right),$$
$$z_3 = \frac{\psi_4}{\psi_1} = \frac{1}{1+x_5}\left(-ix_1 + x_2 + (x_4+ix_3)\frac{\phi_2}{\phi_1}\right),$$

and also for the LLL basis elements, (5.13) and (B.5) for $k=3$.

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