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C-Integrability Test for Discrete Equations via Multiple Scale Expansions

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Abstract. In this paper we are extending the well known integrability theorems obtained by multiple scale techniques to the case of linearizable difference equations. As an example we apply the theory to the case of a differential-difference dispersive equation of the Burgers hierarchy which via a discrete Hopf–Cole transformation reduces to a linear differential difference equation. In this case the equation satisfies the A_1, A_2 and A_3 linearizability conditions. We then consider its discretization. To get a dispersive equation we substitute the *time* derivative by its symmetric discretization. When we apply to this nonlinear partial difference equation the multiple scale expansion we find out that the lowest order non-secularity condition is given by a non-integrable nonlinear Schrödinger equation. Thus showing that this discretized Burgers equation is neither linearizable not integrable.

Key words: linearizable discrete equations; linearizability theorem; multiple scale expansion; obstructions to linearizability; discrete Burgers

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1 Introduction

Calogero in 1991 [3] introduced the notion of S and C integrable equations to denote those nonlinear partial differential equations which are solvable through an inverse **S**pectral transform or linearizable through a **C**hange of variables. Using the multiple scale reductive technique he was able to show that the Nonlinear Schrödinger Equation (NLSE)

$$\mathrm{i}\partial_t f + \partial_{xx} f = \rho_2 |f|^2 f, \qquad f = f(x,t), \tag{1.1}$$

appears as a universal equation governing the evolution of slowly varying packets of quasi-monochromatic waves in weakly nonlinear media featuring dispersion. Calogero and Eckhaus then showed that a necessary condition for the S-integrability of a dispersive nonlinear partial differential equation is that its multiple scale expansion around a slowly varying packet of quasi-monochromatic wave should provide at the lowest order in the perturbation parameter an integrable NLSE. Then, by a paradox, they showed that a C-integrable equation must reduce to a linear equation or another C-integrable equation as the Eckhaus equation [4, 22].

In the case of discrete equations it has been shown [26, 1, 16, 18, 17, 19, 10, 11, 12] that a similar situation is also true. One presents the equivalent of the Calogero–Eckhaus theorem stating that a necessary condition for a nonlinear dispersive partial difference equation to be S-integrable is that the lowest order multiple scale expansion on $\mathcal{C}^{(\infty)}$ functions give rise to integrable NLSE. The nonlinear dispersive partial difference equation will be C-integrable if its multiple scale expansion on $\mathcal{C}^{(\infty)}$ functions will give rise at the lowest order to a linear or a C-integrable differential equation.

By going to a higher order in the expansion, the multiple scale techniques give more stringent conditions which have been used to find new S-integrable Partial Differential Equations (PDE's) and to prove the integrability of new nonlinear equations [8, 9, 15]. Probably the most important example of such nonlinear PDE is the Degasperis–Procesi equation [7]. Up to our knowledge higher order expansions for nonlinear linearizable equations have not been considered in details.

The purpose of this paper is to show that the integrability theorem, stated in [11], can be extended to the case of linearizable difference equations, providing a way to discriminate between S-integrable, C-integrable and non–integrable lattice equations. The continuous (and thus discrete) higher order C-integrability conditions are, up to our knowledge, presented here for the first time. We apply here the resulting linearizability conditions to a differential-difference dispersive nonlinear equation of the discrete Burgers hierarchy [21] and its difference-difference analogue.

In Section 2 we present the differential-difference linearizable nonlinear dispersive Burgers and its partial difference analogue and discuss the tools necessary to carry out the multiple scale C-integrability test. In Section 3 we apply them to the two equations previously introduced, leaving to the Appendix all details of the calculation, and present in Section 4 some conclusive remarks.

2 Multiple scale perturbation reduction of Burgers equations

The Burgers equation, the simplest nonlinear equation for the study of gas dynamics with heat conduction and viscous effect, was introduced by Burgers in 1948 [2, 27]. Explicit solutions of the Cauchy problem on the infinite line for the Burgers equation may be obtained by the Hopf–Cole transform, introduced independently by Hopf and Cole in 1950 [5, 14]. This transformation linearize the equation and the solution of the linearized equation provide solutions of the Burgers equation.

Bruschi, Levi and Ragnisco [21] extended the Hopf–Cole transformation to construct hierarchies of linearizable nonlinear matrix PDE's, nonlinear differential-difference equations and difference-difference equations.

The simplest differential–difference nonlinear dispersive equation of the Burgers hierarchy is given by

$$\partial_t u_n(t) = \frac{1}{2h}\left\{[1+hu_n(t)]\,[u_{n+1}(t)-u_n(t)] - \frac{u_{n-1}(t)-u_n(t)}{1+hu_{n-1}(t)}\right\}, \tag{2.1}$$

where the function $u_n(t)$ and the lattice parameter h are all supposed to be real. Equation (2.1) has a nonlinear dispersion relation $\omega = -\frac{\sin(\kappa h)}{h}$.

When $h \to 0$ and $n \to \infty$ in such a way that $x = nh$ is finite, the Burgers equation (2.1) reduces to the one dimensional wave equation $\partial_t u - \partial_x u = \mathcal{O}(h^3)$.

Through the discrete Cole–Hopf transformation

$$u_n(t) = \frac{\phi_{n+1}(t) - \phi_n(t)}{h\phi_n(t)} \tag{2.2}$$

equation (2.1) linearizes to the discrete linear wave equation

$$\partial_t \phi_n(t) = \frac{\phi_{n+1}(t) - \phi_{n-1}(t)}{2h}. \tag{2.3}$$

The transformation (2.2) can be inverted and gives

$$\phi_n = \phi_{n_0} \prod_{j=n_0}^{j=n-1} (1+hu_j), \qquad n \geq n_0 + 1, \tag{2.4a}$$

$$\phi_n = \frac{\phi_{n_0}}{\prod_{j=n}^{j=n_0-1} (1+hu_j)}, \qquad n \le n_0 - 1, \tag{2.4b}$$

where $\phi_{n_0} = \phi_{n_0}(t)$ is the function ϕ_n calculated at a given initial point $n = n_0$. When $u_n(t)$ satisfies equation (2.1), the function $\phi_n(t)$ will satisfy the discrete wave equation (2.3) if ϕ_{n_0} satisfies the ordinary differential equation

$$\dot{\phi}_n - \frac{1}{2h}\left[1 + hu_n - \frac{1}{(1+hu_{n-1})}\right]\phi_n\Big|_{n=n_0} = 0,$$

whose solution is given by

$$\phi_{n_0}(t) = \phi_{n_0}(t_0)\exp\left\{\frac{1}{2h}\int_{t_0}^{t}\left[1 + hu_n - \frac{1}{(1+hu_{n-1})}\right]\Big|_{n=n_0} dt'\right\},$$

and t_0 is an initial time. If $\lim_{n\to-\infty} u_n(t) = u_{-\infty}$ is finite equations (2.4) reduce to

$$\phi_n = \alpha\,(t)\,(1+hu_{-\infty})^n \prod_{\gamma=-\infty}^{\gamma=n-1}\left(\frac{1+hu_\gamma}{1+hu_{-\infty}}\right),$$

where $\alpha\,(t)$ is a t-dependent function,

$$\alpha(t) = \alpha_0 \exp\left\{\frac{1}{2}\left(1 + \frac{1}{1+hu_{-\infty}}\right)u_{-\infty}t\right\}.$$

A C-integrable discretization of equation (2.1) is given by the partial difference equation

$$\frac{u_{n,m+1}-u_{n,m}}{\sigma} = \frac{1}{2h}\left[(1+hu_{n,m})(u_{n+1,m}-u_{n,m+1}) - \frac{u_{n-1,m}-u_{n,m+1}}{1+hu_{n-1,m}}\right], \tag{2.5}$$

where σ is the constant lattice parameter in the time variable. Equation (2.5) is dissipative as it has a complex dispersion relation $\omega = \frac{i}{\sigma}\ln\left[1 + i\frac{\sigma}{h}\sin(\kappa h)\right]$. As we are *not* able to construct a dispersive counterpart of equation (2.1) we consider a straightforward discretization of equation (2.1)

$$\frac{u_{n,m+1}-u_{n,m-1}}{2\sigma} = \frac{1}{2h}\left[(1+hu_{n,m})\,(u_{n+1,m}-u_{n,m}) - \frac{(u_{n-1,m}-u_{n,m})}{1+hu_{n-1,m}}\right], \tag{2.6}$$

whose nonlinear dispersion relation is

$$\sin(\omega\sigma) = -\frac{\sin(\kappa h)\,\sigma}{h}.$$

In the remaining part of this section we will present the tools necessary to carry out the multiple scale expansion of equations (2.1), (2.6) and construct the conditions which they must satisfy to be C-integrable equation of j order, with $j = 1, 2, 3$, i.e. such that asymptotically they reduce to a linear equation up to terms respectively of the third, fourth and fifth order in the perturbation parameter. These conditions up to our knowledge have been presented for the first time by Dr. Scimiterna in his PhD Thesis [25] and are published here for the first time.

2.1 Expansion of real dispersive partial difference equations

For completeness we briefly illustrate here all the ingredients of the reductive perturbative technique necessary to treat difference equations, as presented in [10, 11, 25].

2.1.1 From shifts to derivatives

Let us consider a function $u_n : \mathbb{Z} \to \mathbb{R}$ depending on an index $n \in \mathbb{Z}$ and let us suppose that:

- The dependence of u_n on n is realized through the *slow variable* $n_1 \doteq \epsilon n \in \mathbb{R}$, $\epsilon \in \mathbb{R}$, $0 < \epsilon \ll 1$, that is to say $u_n \doteq u(n_1)$.
- The function $u(n_1) \in \mathcal{C}^{(\infty)}(\mathcal{D})$, where $\mathcal{D} \in \mathbb{R}$ is a region containing the point n_1.

Under these hypotheses one can write the action of the shift operator T_n such that $T_n u_n \doteq u_{n+1} = u(n_1 + \epsilon)$ as the following (formal) series

$$\begin{aligned} T_n u(n_1) &= u(n_1) + \epsilon u_{,n_1}(n_1) + \frac{\epsilon^2}{2} u_{,2\,n_1}(n_1) + \cdots + \frac{\epsilon^k}{k!} u_{,k\,n_1}(n_1) + \cdots \\ &= \sum_{k=0}^{+\infty} \frac{\epsilon^k}{k!} u_{,k\,n_1}(n_1), \end{aligned} \tag{2.7}$$

where $u_{,k\,n_1}(n_1) \doteq d^k u(n_1)/dn_1^k \doteq d_{n_1}^k u(n_1)$, being d_{n_1} the total derivative operator. The last expression suggests the following formal expansion for the differential operator T_n:

$$T_n = \sum_{k=0}^{+\infty} \frac{\epsilon^k}{k!}\, d_{n_1}^k \doteq e^{\epsilon d_{n_1}}$$

valid only when the series in equation (2.7) is converging. So we must require that the radius of convergence of the series starting at n_1 is wide enough to include as an *inner point* at least the point $n_1 + \epsilon$.

Let us introduce more complicated dependencies of u_n on n. For example one can assume a simultaneous dependence on the *fast variable* n and on the slow variable n_1, i.e. $u_n \doteq u(n, n_1)$. The action of the *total* shift operator T_n will now be given by $T_n u_n \doteq u_{n+1} = u(n+1, n_1 + \epsilon)$ so that we can write $T_n \doteq \mathcal{T}_n^{(1)} \mathcal{T}_{n_1}^{(\epsilon)}$, where the *partial* shift operators $\mathcal{T}_n^{(1)}$ and $\mathcal{T}_{n_1}^{(\epsilon)}$ are defined respectively by

$$\mathcal{T}_n^{(1)} u(n, n_1) = u(n+1, n_1) = \sum_{k=0}^{\infty} \frac{1}{k!} \partial_n^k u(n, n_1) = e^{\partial_n} u(n, n_1),$$

and

$$\mathcal{T}_{n_1}^{(\epsilon)} u(n, n_1) = u(n, n_1 + \epsilon) = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \partial_{n_1}^k u(n, n_1) = e^{\epsilon \partial_{n_1}} u(n, n_1). \tag{2.8}$$

The dependence of u_n on n can be easily extended to the case of one fast variable n and K slow variables $n_j \doteq \epsilon_j n$, $\epsilon_j \in \mathbb{R}$, $1 \le j \le K$ each of them being defined by its own parameter ϵ_j. The action of the total shift operator T_n will now be given in terms of the partial shifts $\mathcal{T}_n^{(1)}$, $\mathcal{T}_{n_j}^{(\epsilon)}$, as $T_n \doteq \mathcal{T}_n^{(1)} \prod_{j=1}^{K} \mathcal{T}_{n_j}^{(\epsilon_j)}$.

Let us now consider a nonlinear partial difference equation

$$F\left[\{u_{n+k,m+j}\}_{k=\left(-\mathcal{N}^{(-)},\mathcal{N}^{(+)}\right)}^{j=\left(-\mathcal{K}^{(-)},\mathcal{K}^{(+)}\right)}\right] = 0, \qquad \left(\mathcal{N}^{(\pm)}, \mathcal{K}^{(\pm)}\right) \ge 0, \tag{2.9}$$

for a function $u_{n,m} : \mathbb{Z}^2 \to \mathbb{R}$ which now depends on two indexes n and $m \in \mathbb{Z}$ which we will term respectively as discrete *space* and *time* indices. Equation (2.9) contains m and n-shifts, respectively in the intervals $(m - \mathcal{K}^{(-)}, m + \mathcal{K}^{(+)})$ and $(n - \mathcal{N}^{(-)}, n + \mathcal{N}^{(+)})$. Under some obvious

Table 1. The operators $\mathcal{A}_n^{(j)}$, $\mathcal{B}_m^{(j)}$ and $\mathcal{C}_{n,m}^{(j)}$ appearing in equations (2.10).

	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$
$\mathcal{A}_n^{(j)}$	1	$N_1\partial_{n_1}$	$\frac{N_1^2}{2}\partial_{n_1}^2$	$\frac{N_1^3}{6}\partial_{n_1}^3$	$\frac{N_1^4}{24}\partial_{n_1}^4$
$\mathcal{B}_m^{(j)}$	1	$M_1\partial_{m_1}$	$\frac{M_1^2}{2}\partial_{m_1}^2+M_2\partial_{m_2}$	$\frac{M_1^3}{6}\partial_{m_1}^3+$ $+M_1M_2\partial_{m_1}\partial_{m_2}+$ $+M_3\partial_{m_3}$	$\frac{M_1^4}{24}\partial_{m_1}^4+\frac{M_1^2M_2}{2}\partial_{m_1}^2\partial_{m_2}+$ $+\frac{M_2^2}{2}\partial_{m_2}^2+M_1M_3\partial_{m_1}\partial_{m_3}+M_4\partial_{m_4}$
$\mathcal{C}_{n,m}^{(j)}$	1	$\mathcal{A}_n^{(1)}+\mathcal{B}_m^{(1)}$	$\mathcal{A}_n^{(2)}+\mathcal{B}_m^{(2)}+$ $+N_1M_1\partial_{n_1}\partial_{m_1}$	$\mathcal{A}_n^{(3)}+\mathcal{B}_m^{(3)}+$ $+N_1M_2\partial_{n_1}\partial_{m_2}+$ $+\frac{M_1N_1^2}{2}\partial_{n_1}^2\partial_{m_1}+$ $+\frac{N_1M_1^2}{2}\partial_{n_1}\partial_{m_1}^2$	$\mathcal{A}_n^{(4)}+\mathcal{B}_m^{(4)}+$ $+\frac{M_1^3N_1}{6}\partial_{m_1}^3\partial_{n_1}+\frac{N_1^3M_1}{6}\partial_{n_1}^3\partial_{m_1}+$ $+N_1M_1M_2\partial_{n_1}\partial_{m_1}\partial_{m_2}+\frac{N_1^2M_2}{2}\partial_{n_1}^2\partial_{m_2}+$ $+\frac{N_1^2M_1^2}{4}\partial_{n_1}^2\partial_{m_1}^2+N_1M_3\partial_{n_1}\partial_{m_3}$

hypothesis on the $\mathcal{C}^{(\infty)}$ property of the function $u_{n,m}$ and on the radius of convergence of its Taylor expansion for all shifts in the indices n and m involved in the difference equation (2.9), we can write a series representation of $u_{n+k,m+j}$ around $u_{n,m}$. We choose the slow variables as $n_k \doteq \epsilon_{n_k} n$, $m_j \doteq \epsilon_{m_j} m$ with

$$\epsilon_{n_k} \doteq N_k \epsilon^k, \qquad 1 \le k \le K_n, \qquad \epsilon_{m_j} \doteq M_j \epsilon^j, \qquad 1 \le j \le K_m,$$

where the various constants N_k, M_j and ϵ are all real numbers. In this presentation we assume $K_n = 1$ and $K_m = K$ (eventually $K = +\infty$) so that

$$T_n = \mathcal{T}_n^{(1)} \mathcal{T}_{n_1}^{(\epsilon_{n_1})} = \mathcal{T}_n^{(1)} \sum_{j=0}^{+\infty} \epsilon^j \mathcal{A}_n^{(j)}, \tag{2.10a}$$

$$T_m = \mathcal{T}_m^{(1)} \prod_{j=1}^{K} \mathcal{T}_{m_j}^{(\epsilon_{m_j})} = \mathcal{T}_m^{(1)} \sum_{j=0}^{+\infty} \epsilon^j \mathcal{B}_m^{(j)}, \tag{2.10b}$$

$$T_n T_m = \mathcal{T}_n^{(1)} \mathcal{T}_m^{(1)} \mathcal{T}_{n_1}^{(\epsilon_{n_1})} \prod_{j=1}^{K} \mathcal{T}_{m_j}^{(\epsilon_{m_j})} = \mathcal{T}_n^{(1)} \mathcal{T}_m^{(1)} \sum_{j=0}^{+\infty} \epsilon^j \mathcal{C}_{n,m}^{(j)}, \tag{2.10c}$$

where the operators $\mathcal{A}_n^{(j)}$, $\mathcal{B}_m^{(j)}$, and $\mathcal{C}_{n,m}^{(j)}$ are given in Table 1. Inserting the explicit expressions (2.10) of the shift operators into equation (2.9), this turns into a PDE of infinite order. So we will assume for the function $u_{n,m} = u(n, m, n_1, \{m_j\}_{j=1}^{K}, \epsilon)$ a double expansion in harmonics and in the perturbative parameter ϵ

$$u_{n,m} = \sum_{\gamma=1}^{+\infty} \sum_{\theta=-\gamma}^{\gamma} \epsilon^{\gamma} u_{\gamma}^{(\theta)} (n_1, m_j, j \ge 1)\, e^{i\theta(\kappa h n - \omega(\kappa)\sigma m)}, \tag{2.11}$$

with $u_{\gamma}^{(-\theta)} (n_1, m_j, j \ge 1) = \bar{u}_{\gamma}^{(\theta)} (n_1, m_j, j \ge 1)$, where by a bar we denote the complex conjugate, in order to ensure the reality of $u_{n,m}$. The index γ is chosen ≥ 1 so that the nonlinear terms of equation (2.9) enter as a perturbation in the multiple scale expansion. For simplicity we will set $N_1 = M_j = 1$, $j \ge 1$. Moreover we will assume that the functions $u_{\gamma}^{(\theta)}$ satisfy the asymptotic conditions $\lim_{n_1 \to \pm\infty} u_{\gamma}^{(\theta)} = 0$, $\forall\, \gamma$ and θ to provide a meaningful expansion.

2.1.2 From derivatives to shifts

The multiple scale approach discussed above reduces a given partial difference equation into a partial differential equation for the amplitudes $u_{\gamma}^{(\theta)}$ contained in the definition (2.11).

We can rewrite the so obtained partial differential equation as a partial difference equation inverting the expansion of the partial shift operator in term of partial derivatives (2.8). From (2.8) we have

$$\partial_{n_1} = \frac{1}{\epsilon} \ln \mathcal{T}_{n_1}^{(\epsilon)} = \frac{1}{\epsilon} \ln\left(1 + \epsilon \Delta_{n_1}^{(+)}\right) \doteq \sum_{k=1}^{+\infty} \frac{(-\epsilon)^{k-1}}{k} \left[\Delta_{n_1}^{(+)}\right]^k, \tag{2.12}$$

where $\Delta_{n_1}^{(+)} \doteq (\mathcal{T}_{n_1}^{(\epsilon)} - 1)/\epsilon$ is the first *forward* difference operator with respect to the slow-variable n_1. This is just one of the possible inversion formulae for the operator $\mathcal{T}_{n_1}^{(\epsilon)}$. For example an expression similar to equation (2.12) can be written for the first *backward* difference operator $\Delta_{n_1}^{(-)} \doteq \left(1 - \left[\mathcal{T}_{n_1}^{(\epsilon)}\right]^{-1}\right)/\epsilon$. For the first *symmetric* difference operator $\Delta_{n_1}^{(s)} \doteq \left(\mathcal{T}_{n_1}^{(\epsilon)} - \left[\mathcal{T}_{n_1}^{(\epsilon)}\right]^{-1}\right)/2\epsilon$ we get

$$\partial_{n_1} = \sinh^{-1} \epsilon \Delta_{n_1}^{(s)} \doteq \sum_{k=1}^{+\infty} \frac{P_{k-1}(0)\epsilon^k}{k} \left[\Delta_{n_1}^{(s)}\right]^k,$$

where $P_k(0)$ is the k-th *Legendre* polynomial evaluated at $x = 0$.

Only when we impose that the function u_n is a slow-varying function of order ℓ in the variable n_1, i.e. that $\Delta_{n_1}^{\ell+1} u_n = 0$, we can see that the ∂_{n_1} operator, which is given by formal series containing in general infinite powers of the Δ_{n_1}, reduces to polynomial of order at most ℓ. In [17], choosing $\ell = 2$ for the indexes n_1 and m_1 and $\ell = 1$ for m_2, it was shown that the integrable *lattice potential KdV* equation [20] reduces to a completely discrete and local nonlinear Schrödinger equation which has been proved to be not integrable by singularity confinement and algebraic entropy [13, 23]. Consequently, if one passes from derivatives to shifts, one ends up in general with a *nonlocal* partial difference equation in the slow variables n_κ and m_δ.

2.2 The orders beyond the Schrödinger equation and the *C*-integrability conditions

The multiple scale expansion of an equation of the Burgers hierarchy on functions of infinite order will thus give rise to PDE's. So a multiple scale integrability test will require that a dispersive equation like equation (2.1) is C-integrable if its multiple scale expansion will go into the hierarchy of the Schrödinger equation

$$i\partial_t \psi + \partial_x^2 \psi = 0.$$

To be able to verify the C-integrability we need to consider in principle all the orders beyond the Schrödinger equation. This in general will not be possible but already a few orders beyond the Schrödinger equation might be sufficient to verify if the equation is linearizable or not. In the case of S-integrable nonlinear PDE's the first attempt to go beyond the NLSE order has been presented by Degasperis, Manakov and Santini in [8]. These authors, starting from an S-integrable model, through a combination of an asymptotic functional analysis and spectral methods, succeeded in removing all the secular terms from the reduced equations order by order. Their results could be summarized as follows:

1. The number of slow-time variables required for the amplitudes $u_j^{(\theta)}$ appearing in (2.11) coincides with the number of nonvanishing coefficients of the Taylor expansion of the dispersion relation, $\omega_j(\kappa) = \frac{1}{j!} \frac{d^j \omega(k)}{dk^j}$.
2. The amplitude $u_1^{(1)}$ evolves at the slow-times m_s, $s \geq 2$ according to the s-th equation of the NLS hierarchy.

3. The amplitudes of the higher perturbations of the first harmonic $u_j^{(1)}$, $j \geq 2$ evolve, taking into account some *asymptotic boundary conditions*, at the slow-times m_s, $s \geq 2$ according to certain *linear, nonhomogeneous* equations.

Then they concluded that the cancellation at each stage of the perturbation process of all the secular terms is a sufficient condition to uniquely fix the evolution equations followed by every $u_j^{(1)}$, $j \geq 1$ for each slow-time m_s. Point 2 implies that a hierarchy of integrable equations provide for a function u always compatible evolutions, i.e. the equations in its hierarchy are *generalized symmetries* of each other. In this way this procedure provides *necessary and sufficient* conditions to get secularity-free reduced equations [8].

We apply the present procedure to the case of C-integrable partial difference equations. Following Degasperis and Procesi [9] we state the following theorem:

Theorem 1. *If equation* (2.9) *is C-integrable then, after a multiple scale expansion, the functions $u_j^{(1)}$, $j \geq 1$ satisfy the equations*

$$\partial_{m_s} u_1^{(1)} - (-i)^{s-1} B_s \partial_{n_1}^s u_1^{(1)} \doteq M_s u_1^{(1)} = 0, \tag{2.13a}$$

$$M_s u_j^{(1)} = f_s(j), \tag{2.13b}$$

$\forall\, j$, $s \geq 2$, where $B_s \partial_{n_1}^s u_1^{(j)}$ is the s-th $flow$ in the linear Schrödinger hierarchy and B_s are real constants. All the other $u_j^{(\kappa)}$, $\kappa \geq 2$ are expressed as differential monomials of $u_r^{(1)}$, $r \leq j-1$.

In equation (2.13b) $f_s(j)$ is a nonhomogeneous *nonlinear* forcing written in term of differential monomials of $u_r^{(1)}$, $r \leq j$. From Theorem 1 it follows that a nonlinear partial difference equation is said to be C-integrable if its asymptotic multiple scale expansion is given by a uniform asymptotic series whose leading harmonic $u^{(1)}$ possesses an infinity of generalized symmetries evolving at different times and given by commuting linear equations. Equations (2.13) are a *necessary* condition for C-integrability.

It is worthwhile to stress here the non completely obvious fact that, in contrast to the first order wave equation, $\partial_t u = \partial_x u$, all the symmetries of the Schrödinger equation commuting with it are given by the equations (2.13a) and only by them. This implies that all the equations appearing in the multiple scale expansion for a C-integrable equation are uniquely defined.

It is obvious that the operators M_s defined in equation (2.13a) commute among themselves. However the compatibility of equations (2.13b) is not always guaranteed but is subject to some compatibility conditions among their r.h.s. terms $f_s(j)$. Once we fix the index $j \geq 2$ in the set of equations (2.13b), this commutativity condition implies the *compatibility* conditions

$$M_s f_{s'}(j) = M_{s'} f_s(j), \qquad \forall\, s, s' \geq 2, \tag{2.14}$$

where, as $f_s(j)$ and $f_{s'}(j)$ are functions of the different perturbations $u_r^{(1)}$ of the fundamental harmonic up to degree $j-1$, the time derivatives ∂_{m_s}, $\partial_{m_{s'}}$ appearing respectively in M_s and $M_{s'}$ have to be eliminated using the evolution equations (2.13) up to the index $j-1$. These last commutativity conditions turn out to be a *linearizability test*.

Following [8] we conjecture that the relations (2.13) are a *sufficient* condition for the C-integrability or that the C-integrability is a *necessary* condition to have a multiple scale expansion where equations (2.13) are satisfied. To characterize the functions $f_s(j)$ we introduce the following definitions:

Definition 1. A differential monomial $\mathcal{M}\big[u_j^{(1)}\big]$, $j \geq 1$ in the functions $u_j^{(1)}$, its complex conjugate and its n_1-derivatives is a monomial of "gauge" 1 if it possesses the transformation property

$$\mathcal{M}\big[\tilde{u}_j^{(1)}\big] = e^{i\theta} \mathcal{M}\big[u_j^{(1)}\big], \qquad \text{when} \quad \tilde{u}_j^{(1)} \doteq e^{i\theta} u_j^{(1)}.$$

Definition 2. A finite dimensional vector space $\mathcal{P}_\nu$, $\nu \geq 2$ is the set of all differential polynomials of gauge 1 in the functions $u_j^{(1)}$, $j \geq 1$, their complex conjugates and their n_1-derivatives such that their total order in ϵ is ν, i.e.

$$\mathrm{order}\big(\partial_{n_1}^\mu u_j^{(1)}\big) = \mathrm{order}\big(\partial_{n_1}^\mu \bar{u}_j^{(1)}\big) = \mu + j = \nu, \qquad \mu \geq 0.$$

Definition 3. $\mathcal{P}_\nu(\mu)$, $\mu \geq 1$ and $\nu \geq 2$ is the subspace of $\mathcal{P}_\nu$ whose elements are differential polynomials of gauge 1 in the functions $u_j^{(1)}$, their complex conjugates and their n_1-derivatives such that their total order is ν and $1 \leq j \leq \mu$.

From Definition 3 it follows that $\mathcal{P}_\nu = \mathcal{P}_\nu(\nu-2)$. Moreover in general $f_s(j) \in \mathcal{P}_{j+s}(j-1)$ where j, $s \geq 2$. The basis monomials of the spaces $\mathcal{P}_\nu(\mu)$ in which we can express the functions $f_s(j)$ can be found, for example, in [25].

Proposition 1. *If for each fixed $j \geq 2$ the equation* (2.14) *with $s = 2$ and $s' = 3$, namely*

$$M_2 f_3(j) = M_3 f_2(j), \tag{2.15}$$

is satisfied, then there exist unique differential polynomials $f_s(j)$ $\forall\, s \geq 4$ such that the flows $M_s u_j^{(1)} = f_s(j)$ commute for any $s \geq 2$ [24, 6].

Hence among the relations (2.14) only those with $s = 2$ and $s' = 3$ have to be tested.

Proposition 2. *The homogeneous equation $M_s u = 0$ has no solution u in the vector space $\mathcal{P}_\mu$, i.e.* $\mathrm{Ker}\,(M_s) \cap \mathcal{P}_\mu = \varnothing$.

Consequently the multiple scale expansion (2.13) is *secularity-free.* This does not mean that, in solving equation (2.13b), we have to set to zero all the contributions to the solution coming from the homogeneous equation but only that part of it which is present in the forcing terms. Finally:

Definition 4. If the relations (2.14) are satisfied up to the index j, $j \geq 2$, we say that our equation is asymptotically C-integrable of degree j or A_j C-integrable.

2.2.1 Integrability conditions for the Schrödinger hierarchy

Here we present the conditions for the asymptotic C-integrability of order k or A_k C-integrability conditions with $k = 1, 2, 3$. To simplify the notation, we will use for $u_j^{(1)}$ the concise form $u(j)$.

The A_1 C-integrability condition is given by the absence of the coefficient ρ_2 of the nonlinear term in the NLSE (1.1).

The A_2 integrability conditions are obtained choosing $j = 2$ in the compatibility conditions (2.14) with $s = 2$ and $s' = 3$ as in (2.15). In this case we have that $f_2(2) \in \mathcal{P}_4(1)$ and $f_3(2) \in \mathcal{P}_5(1)$ where $\mathcal{P}_4(1)$ contains 2 different differential monomials and $\mathcal{P}_5(1)$ contains 5 different differential monomials, so that $f_2(2)$ and $f_3(2)$ will be respectively identified by 2 and 5 complex constants

$$\begin{aligned} f_2(2) &\doteq a u_{,n_1}(1)|u(1)|^2 + b\bar{u}_{,n_1}(1)u(1)^2, \qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.16)\\ f_3(2) &\doteq \alpha|u(1)|^4 u(1) + \beta|u_{,n_1}(1)|^2 u(1) + \gamma u_{,n_1}(1)^2\bar{u}(1) + \delta\bar{u}_{,2n_1}(1)u(1)^2 + \varepsilon|u(1)|^2 u_{,2n_1}(1). \end{aligned}$$

In this way, eliminating from equation (2.15) the derivatives of $u(1)$ with respect to the slow-times m_2 and m_3 using the evolutions (2.13a) with $s = 2, 3$ and equating term by term, we obtain that the A_2 C-integrability conditions gives no constraints on the coefficients a and b

appearing in $f_2(2)$. The expression of the coefficients α, β, γ, δ, ε appearing in $f_3(2)$ in terms of a and b are

$$\alpha = 0, \qquad \beta = -\frac{3iB_3 b}{B_2}, \qquad \gamma = -\frac{3iB_3 a}{2B_2}, \qquad \delta = 0, \qquad \varepsilon = \gamma.$$

The A_3 C-integrability conditions are derived in a similar way setting $j = 3$ in equation (2.15). In this case we have that $f_2(3) \in \mathcal{P}_5(2)$ and $f_3(3) \in \mathcal{P}_6(2)$ where $\mathcal{P}_5(2)$ contains 12 different differential monomials and $\mathcal{P}_6(2)$ contains 26 different differential monomials, so that $f_2(3)$ and $f_3(3)$ will be respectively identified by 12 and 26 complex constants

$$\begin{aligned} f_2(3) \doteq{} & \tau_1 |u(1)|^4 u(1) + \tau_2 |u_{,n_1}(1)|^2 u(1) + \tau_3 |u(1)|^2 u_{,2n_1}(1) + \tau_4 \bar{u}_{,2n_1}(1) u(1)^2 \\ & + \tau_5 u_{,n_1}(1)^2 \bar{u}(1) + \tau_6 u_{,n_1}(2) |u(1)|^2 + \tau_7 \bar{u}_{,n_1}(2) u(1)^2 + \tau_8 u(2)^2 \bar{u}(1) \\ & + \tau_9 |u(2)|^2 u(1) + \tau_{10} u(2) u_{,n_1}(1) \bar{u}(1) + \tau_{11} u(2) \bar{u}_{,n_1}(1) u(1) + \tau_{12} \bar{u}(2) u_{,n_1}(1) u(1), \\ f_3(3) \doteq{} & \gamma_1 |u(1)|^4 u_{,n_1}(1) + \gamma_2 |u(1)|^2 u(1)^2 \bar{u}_{,n_1}(1) + \gamma_3 |u(1)|^2 u_{,3n_1}(1) + \gamma_4 u(1)^2 \bar{u}_{,3n_1}(1) \\ & + \gamma_5 |u_{,n_1}(1)|^2 u_{,n_1}(1) + \gamma_6 \bar{u}_{,2n_1}(1) u_{,n_1}(1) u(1) + \gamma_7 u_{,2n_1}(1) \bar{u}_{,n_1}(1) u(1) \\ & + \gamma_8 u_{,2n_1}(1) u_{,n_1}(1) \bar{u}(1) + \gamma_9 |u(1)|^4 u(2) + \gamma_{10} |u(1)|^2 u(1)^2 \bar{u}(2) + \gamma_{11} \bar{u}_{,n_1}(1) u(2)^2 \\ & + \gamma_{12} u_{,n_1}(1) |u(2)|^2 + \gamma_{13} |u_{,n_1}(1)|^2 u(2) + \gamma_{14} |u(2)|^2 u(2) + \gamma_{15} u_{,n_1}(1)^2 \bar{u}(2) \\ & + \gamma_{16} |u(1)|^2 u_{,2n_1}(2) + \gamma_{17} u(1)^2 \bar{u}_{,2n_1}(2) + \gamma_{18} u(2) \bar{u}_{,2n_1}(1) u(1) \\ & + \gamma_{19} u(2) u_{,2n_1}(1) \bar{u}(1) + \gamma_{20} \bar{u}(2) u_{,2n_1}(1) u(1) + \gamma_{21} u(2) u_{,n_1}(2) \bar{u}(1) \\ & + \gamma_{22} \bar{u}(2) u_{,n_1}(2) u(1) + \gamma_{23} u_{,n_1}(2) u_{,n_1}(1) \bar{u}(1) + \gamma_{24} u_{,n_1}(2) \bar{u}_{,n_1}(1) u(1) \\ & + \gamma_{25} \bar{u}_{,n_1}(2) u_{,n_1}(1) u(1) + \gamma_{26} \bar{u}_{,n_1}(2) u(2) u(1). \end{aligned} \tag{2.17}$$

Let us eliminate from equation (2.15) with $j = 3$ the derivatives of $u(1)$ with respect to the slow-times m_2 and m_3 using the evolutions (2.13a) with $s = 2, 3$ and the same derivatives of $u(2)$ using the evolutions (2.13b) with $s = 2, 3$. Equating the remaining terms term by term, the A_3 C-integrability conditions turn out to be:

$$\begin{aligned} & \tau_1 = -\frac{i}{4B_2} \left[b\left(\tau_{11} - 2\tau_6\right) + \bar{a}\tau_7 \right], \qquad \bar{b}\tau_7 = \frac{1}{2}\left(b - a\right)\left(\tau_{11} + \tau_{10} - \tau_6\right) + \bar{a}\tau_7, \\ & a\tau_8 = b\tau_8 = 0, \qquad a\tau_9 = b\tau_9 = 0, \qquad \bar{a}\tau_{12} = a\left(\tau_{10} - \tau_{11}\right) + b\tau_6 + \bar{a}\tau_7, \\ & \left(\bar{b} - \bar{a}\right)\tau_{12} = \left(b - a\right)\tau_{10}. \end{aligned} \tag{2.18}$$

Sometimes a and b turn out to be both real. In this case the conditions given in equations (2.18) becomes:

$$\begin{aligned} & R_1 = \frac{1}{4B_2}\left[b\left(I_{11} - 2I_6\right) + aI_7 \right], \qquad I_1 = -\frac{1}{4B_2}\left[b\left(R_{11} - 2R_6\right) + aR_7 \right], \\ & \left(b - a\right)\left(R_{11} + R_{10} - R_6 - 2R_7\right) = 0, \qquad \left(b - a\right)\left(I_{11} + I_{10} - I_6 - 2I_7\right) = 0, \\ & \left(b - a\right)R_8 = 0, \qquad \left(b - a\right)I_8 = 0, \qquad \left(b - a\right)R_9 = 0, \qquad \left(b - a\right)I_9 = 0, \\ & a\left(R_{12} + R_{11} - R_{10} - R_7\right) = bR_6, \qquad a\left(I_{12} + I_{11} - I_{10} - I_7\right) = bI_6, \\ & \left(b - a\right)\left(R_{12} - R_{10}\right) = 0, \qquad \left(b - a\right)\left(I_{12} - I_{10}\right) = 0, \end{aligned} \tag{2.19}$$

where $\tau_i = R_i + iI_i$ for $i = 1, \ldots, 12$. The expressions of the γ_j as functions of the τ_i are:

$$\begin{aligned} & \gamma_1 = \frac{3B_3}{4B_2^2}\left(a\tau_6 - 4iB_2\tau_1 + \bar{b}\tau_{12}\right), \qquad \gamma_2 = \frac{3B_3}{4B_2^2}\left(b\tau_6 + \bar{a}\tau_7\right), \\ & \gamma_3 = -\frac{3iB_3\tau_3}{2B_2}, \qquad \gamma_4 = 0, \quad \gamma_5 = -\frac{3iB_3\tau_2}{2B_2}, \qquad \gamma_6 = -\frac{3iB_3\tau_4}{B_2}, \end{aligned}$$

$$\gamma_7 = \gamma_5, \qquad \gamma_8 = \gamma_3 - \frac{3iB_3\tau_5}{B_2}, \qquad \gamma_9 = \gamma_{10} = \gamma_{11} = 0,$$
$$\gamma_{12} = -\frac{3iB_3\tau_9}{2B_2}, \qquad \gamma_{13} = -\frac{3iB_3\tau_{11}}{2B_2}, \qquad \gamma_{14} = 0, \qquad \gamma_{15} = -\frac{3iB_3\tau_{12}}{2B_2},$$
$$\gamma_{16} = -\frac{3iB_3\tau_6}{2B_2}, \qquad \gamma_{17} = \gamma_{18} = 0, \qquad \gamma_{19} = -\frac{3iB_3\tau_{10}}{2B_2}, \qquad \gamma_{20} = \gamma_{15},$$
$$\gamma_{21} = -\frac{3iB_3\tau_8}{B_2}, \qquad \gamma_{22} = \gamma_{12}, \qquad \gamma_{23} = \gamma_{16} + \gamma_{19}, \qquad \gamma_{24} = \gamma_{13},$$
$$\gamma_{25} = -\frac{3iB_3\tau_7}{B_2}, \qquad \gamma_{26} = 0.$$

The conditions given in equations (2.18), (2.19) appear to be new. Their importance resides in the fact that a C-integrable equation must satisfy those conditions.

3 Linearizability of the equations of the Burgers hierarchy

Taking into account the results of the previous section we can carry out the multiple scale expansion of the equations of the Burgers hierarchy. To do so we substitute the definition (2.11) into equations (2.1), (2.6) and write down the coefficients of the various harmonics θ and of the various orders j of ϵ. When we deal with the differential-difference equation (2.1), we have to make the substitutions $\sigma m \to t$, $\sigma m_i \to t_i$. This transformation implies that in this case the corresponding coefficients ρ_2 and B_2 will turn out to be σ-independent. In Appendix we present all relevant equations and here we just present their results.

Proposition 3. *The differential-difference equation* (2.1) *of the Burgers hierarchy satisfies the* A_1 (*and consequently also the* A_2) *and also the* A_3 C*-integrability conditions.*

Proposition 4. *The partial difference Burgers-like equation* (2.6) *reduces for* $j = 3$ *and* $\alpha = 1$ *to a NLSE with a nonlinear complex coefficient* ρ_2 *given by equation* (A.9c)*. Thus the equation is neither* S*-integrable nor* C*-integrable.*

4 Conclusions

In the present paper we have presented all the steps necessary to apply the perturbative multiple scale expansion to dispersive nonlinear differential-difference or partial difference equations which may be linearizable. These passages involve the representation of the lattice variables in terms of an infinite set of derivative with respect to the lattice index and the analysis of the higher order of the perturbation which give rise to a set of compatible higher order linear PDE's belonging to the hierarchy of the Schrödinger equation. The compatibility of these equations give rise to a linearizability test. We applied the so obtained test to the case of a differential-difference dispersive Burgers equation and its discretization. It turns out that the Burgers is linearizable (as it should be) but its discretization is neither S-integrable nor C-integrable. So, effectively this procedure is able to distinguish between linearizable and non-linearizable equations.

A Appendix

Let us now start performing a multiple scale analysis of the partial difference equation (2.6). We present here the equations we get at the various orders of ϵ and for the different harmonics θ.

- *Order* ϵ *and* $\theta = 0$: In this case the resulting equation is automatically satisfied.

- *Order* ϵ *and* $\theta = 1$: If one requires that $u_1^{(1)} \neq 0$, one obtains the dispersion relation
$$\sin(\omega\sigma) = -\frac{\sin(\kappa h)\,\sigma}{h}. \tag{A.1}$$
- *Order* ϵ^2 *and* $\theta = 0$: We obtain the evolution
$$\partial_{m_1} u_1^{(0)} - \frac{\sigma}{h}\partial_{n_1} u_1^{(0)} = 0, \tag{A.2}$$
which implies that $u_1^{(0)}$ depends on the variable $\rho \doteq h n_1 + \sigma m_1$.
- *Order* ϵ^2 *and* $\theta = 1$: Taking into account the dispersion relation (A.1), we have
$$\partial_{m_1} u_1^{(1)} - \frac{\sigma}{\cos(\omega\sigma)}\left[\frac{\cos(\kappa h)}{h}\partial_{n_1} u_1^{(1)} - 2\sin^2\left(\frac{\kappa h}{2}\right) u_1^{(0)} u_1^{(1)}\right] = 0, \tag{A.3}$$
which implies that $u_1^{(1)}$ has the form
$$u_1^{(1)} = g(\xi, m_j, j \geq 2)\exp\left\{\delta \int_{\rho_0}^{\rho} u_1^{(0)}(\rho')\,d\rho'\right\}, \qquad \delta \doteq \frac{2\sin^2(\kappa h/2)}{[\cos(\kappa h) - \cos(\omega\sigma)]}, \tag{A.4}$$
where $\xi \doteq h n_1 + \frac{\cos(\kappa h)}{\cos(\omega\sigma)}\sigma m_1$, g is an arbitrary function of its arguments going to zero as $\xi \to \pm\infty$ and ρ_0, by a proper redefinition of g, can always be chosen to be a zero of $u_1^{(0)}$ as when $\rho \to \pm\infty$, $u_1^{(0)} \to 0$ so that there will exists at least one zero.
- *Order* ϵ^2 *and* $\theta = 2$: Taking into account the dispersion relation (A.1), we have
$$u_2^{(2)} = \frac{\left(e^{-i\kappa h} - 1\right) h}{2[\cos(\kappa h) - \cos(\omega\sigma)]}\left[u_1^{(1)}\right]^2. \tag{A.5}$$
- *Order* ϵ^3 *and* $\theta = 0$: We have
$$\partial_{m_1} u_2^{(0)} - \frac{\sigma}{h}\partial_{n_1} u_2^{(0)} = -\partial_{m_2} u_1^{(0)} - 2\sigma\sin^2(\kappa h/2)\left[\partial_{n_1} + 2h u_1^{(0)}\right]\left|u_1^{(1)}\right|^2. \tag{A.6}$$
By equation (A.2), the term $\partial_{m_2} u_1^{(0)}$ is a solution of the left hand side of equation (A.6), hence it is a secular term. As a consequence we have to require that
$$\partial_{m_1} u_2^{(0)} - \frac{\sigma}{h}\partial_{n_1} u_2^{(0)} = -2\sigma\sin^2(\kappa h/2)\left[\partial_{n_1} + 2h u_1^{(0)}\right]\left|u_1^{(1)}\right|^2,$$
$$\partial_{m_2} u_1^{(0)} = 0.$$
Solving equation (A) taking into account equation (A.4), we obtain
$$u_2^{(0)} = f(\rho, m_j, j \geq 2) - h\delta\cos(\omega\sigma)\left[|u_1^{(1)}|^2 + 2(1+\delta)\, u_1^{(0)} \int_{\xi_0}^{\xi} |u_1^{(1)}|^2 d\xi'\right], \tag{A.7}$$
where f is an arbitrary function of its arguments going to zero as $\rho \to \pm\infty$ and ξ_0 is an arbitrary value of the variable ξ. Let us restrict ourselves for simplicity to the case where there is no dependence at all on ρ. If one wants that the harmonic $u_1^{(0)}$ depends on ξ and not on ρ, from equation (A.2) one has that $\partial_\xi u_1^{(0)} = 0$, so that $u_1^{(0)}$ depends on the slow variables m_j, $j \geq 2$ only. Similarly we have that $\partial_\xi f = 0$. In this case, in order to satisfy the asymptotic conditions $\lim\limits_{\xi \to \pm\infty} u_\gamma^{(0)} = 0$, $\gamma = 1, 2$, one has to take $u_1^{(0)} = f = 0$ (unless we take the fully continuous limit $h \to 0$, $h n_1 \doteq x_1$, $\sigma \to 0$, $\sigma m_1 \doteq t_1$ in which $\rho \to \xi$). Equation (A.7) then becomes
$$u_2^{(0)} = -h\delta\cos(\omega\sigma)\left|u_1^{(1)}\right|^2. \tag{A.8}$$

- *Order* ϵ^3 *and* $\theta = 1$: Taking into account the dispersion relation (A.1) and the equations $u_1^{(0)} = 0$, (A.3), (A.5), (A.8), we have

$$\partial_{m_1} u_2^{(1)} - \frac{\sigma \cos(\kappa h)}{h \cos(\omega\sigma)} \partial_{n_1} u_2^{(1)} = -\partial_{m_2} u_1^{(1)} - iB_2 \partial_\xi^2 u_1^{(1)} - i\rho_2 \big|u_1^{(1)}\big|^2 u_1^{(1)}, \tag{A.9a}$$

$$B_2 \doteq \frac{h\sigma \left(h^2 - \sigma^2\right) \sin(\kappa h)}{2\left[\sigma^2 \sin^2(\kappa h) - h^2\right] \cos(\omega\sigma)}, \tag{A.9b}$$

$$\begin{aligned} \rho_2 \doteq & -\frac{2h\sigma \sin^2(\omega\sigma/2)\left[2\cos(\kappa h/2) - \cos(3\kappa h/2)\right] \sin(\kappa h/2)}{\left[\cos(\kappa h) - \cos(\omega\sigma)\right] \cos(\omega\sigma)} \\ & - \frac{2ih\sigma \sin^2(\omega\sigma/2) \sin(\kappa h/2) \sin(\kappa h/2)}{\left[\cos(\kappa h) - \cos(\omega\sigma)\right] \cos(\omega\sigma)}. \end{aligned} \tag{A.9c}$$

As a consequence of equation (A.3) with $u_1^{(0)} = 0$, the right hand side of equation (A.9a) is secular. Hence we have to require that

$$\partial_{m_1} u_2^{(1)} - \frac{\sigma \cos(\kappa h)}{h \cos(\omega\sigma)} \partial_{n_1} u_2^{(1)} = 0, \tag{A.10a}$$

$$i\partial_{m_2} u_1^{(1)} = B_2 \partial_\xi^2 u_1^{(1)} + \rho_2 \big|u_1^{(1)}\big|^2 u_1^{(1)}. \tag{A.10b}$$

Equation (A.10a) implies that $u_2^{(1)}$ also depends on ξ while equation (A.10b) is a nonintegrable nonlinear Schrödinger equation, as from the definition (A.9c) we can see that ρ_2 is a complex coefficient. So we can conclude that equation (2.6) is not A_1-integrable.

Let us perform the multiple scale reduction of the Burgers equation (2.1). Equation (2.1) can be always obtained as a semicontinuous limit of equation (2.6) defining the slow times $t_j \doteq \sigma m_j$, $j \geq 1$. In such a way we can use in the present calculation the results presented up above.

- *Order* ϵ *and* $\theta = 0$: In this case the resulting equation is automatically satisfied.
- *Order* ϵ *and* $\theta = 1$: Taking the semi continuous limit of equation (A.1), one obtains the dispersion relation

$$\omega = -\frac{\sin(\kappa h)}{h}. \tag{A.11}$$

- *Order* ϵ^2 *and* $\theta = 0$: Taking the semi continuous limit of equation (A.2), one obtains

$$\partial_{t_1} u_1^{(0)} - \frac{1}{h} \partial_{n_1} u_1^{(0)} = 0, \tag{A.12}$$

which implies that $u_1^{(0)}$ depends on the variable $\rho \doteq hn_1 + t_1$.

- *Order* ϵ^2 *and* $\alpha = 1$: Taking the semi continuous limit of equation (A.3), one obtains

$$\partial_{t_1} u_1^{(1)} - \frac{\cos(\kappa h)}{h} \partial_{n_1} u_1^{(1)} + 2\sin^2(\kappa h/2)\, u_1^{(0)} u_1^{(1)} = 0, \tag{A.13}$$

which implies that $u_1^{(1)}$ has the form

$$u_1^{(1)} = g^{(1)}(\xi, t_j, j \geq 2) \exp\left\{-\int_{\rho_0}^{\rho} u_1^{(0)}(\rho')\, d\rho'\right\}, \tag{A.14}$$

where $\xi \doteq hn_1 + \cos(\kappa h)\, t_1$, $g^{(1)}$ is an arbitrary function of its arguments going to zero as $\xi \to \pm\infty$ and ρ_0, by a proper redefinition of g, can always be chosen to be a zero of $u_1^{(0)}$.

- *Order* ϵ^2 *and* $\theta = 2$: Taking the semi continuous limit of equation (A.5), one obtains

$$u_2^{(2)} = \frac{h}{1 - e^{i\kappa h}} u_1^{(1)2}. \tag{A.15}$$

- *Order* ϵ^3 *and* $\theta = 0$: Taking the semi continuous limit of equation (A.6), one obtains

$$\partial_{t_1} u_2^{(0)} - \frac{1}{h}\partial_{n_1} u_2^{(0)} = -\partial_{t_2} u_1^{(0)} - 2\sin^2(\kappa h/2)\left[\partial_{n_1} + 2hu_1^{(0)}\right]\left|u_1^{(1)}\right|^2. \tag{A.16}$$

By equation (A.12), the term $\partial_{t_2} u_1^{(0)}$ is a solution of the left hand side of equation (A.16), hence it is a secular term. As a consequence we have to require that

$$\partial_{t_1} u_2^{(0)} - \frac{1}{h}\partial_{n_1} u_2^{(0)} = -2\sin^2(\kappa h/2)\left[\partial_{n_1} + 2hu_1^{(0)}\right]\left|u_1^{(1)}\right|^2, \tag{A.17}$$
$$\partial_{t_2} u_1^{(0)} = 0.$$

Solving equation (A.17) taking into account equation (A.14), we obtain

$$u_2^{(0)} = f(\rho, t_j, j \geq 2) + h\left|u_1^{(1)}\right|^2, \tag{A.18}$$

where f is an arbitrary function of its arguments going to zero $\rho \to \pm\infty$.

- *Order* ϵ^3 *and* $\theta = 1$: For simplicity, from now on we require no dependence on ρ[1] so that, in order to satisfy the asymptotic conditions, it necessarily follows that

$$u_1^{(0)} = f = 0. \tag{A.19}$$

Taking the semi continuous limit of equations (A.9a), (A.9b), one obtains

$$\partial_{t_1} u_2^{(1)} - \frac{\cos(\kappa h)}{h}\partial_{n_1} u_2^{(1)} = -\partial_{t_2} u_1^{(1)} - iB_2\partial_\xi^2 u_1^{(1)}, \qquad B_2 \doteq -\frac{h\sin(\kappa h)}{2}. \tag{A.20}$$

As a consequence of equation (A.13) (with $u_1^{(0)} = 0$), the right hand side of equation (A.20) is secular. Hence we have to require that

$$\partial_{t_1} u_2^{(1)} - \frac{\cos(\kappa h)}{h}\partial_{n_1} u_2^{(1)} = 0, \tag{A.21a}$$
$$i\partial_{t_2} u_1^{(1)} - B_2\partial_\xi^2 u_1^{(1)} = 0. \tag{A.21b}$$

Equation (A.21a) implies that $u_2^{(1)}$ depends also on ξ while, contrary to equation (A.10b), equation (A.21b) now is a linear Schrödinger equation, reflecting the C-integrability of equation (2.1).

- *Order* ϵ^3 *and* $\alpha = 2$: Taking into account the dispersion relation (A.11), the fact that $u_1^{(0)} = 0$ and the equations (A.13), (A.15), we have

$$u_3^{(2)} = h\left[\frac{2}{1 - e^{i\kappa h}} u_2^{(1)} - \frac{h}{4\sin^2(\kappa h/2)}\partial_\xi u_1^{(1)}\right] u_1^{(1)}. \tag{A.22}$$

[1] If $\partial_\rho u_1^{(0)} \neq 0$, $\partial_\rho f \neq 0$, we have:

$$u_2^{(1)} \doteq g^{(2)}(n_1, t_j, j \geq 1)\exp\left\{-\int_{\rho_0}^{\rho} u_1^{(0)}(\rho')\, d\rho'\right\}, \qquad i\partial_{t_2} g^{(1)} = B_2\partial_\xi^2 g^{(1)},$$

$$g^{(2)}/g^{(1)} = p(\xi, t_j, j \geq 2) + h\left[1 + \frac{i}{2}\cot\left(\frac{\kappa h}{2}\right)\right] u_1^{(0)} + \frac{h}{2}\int_{\rho_0}^{\rho} u_1^{(0)2} d\rho' - \int_{\rho_0}^{\rho} f(\rho')\, d\rho',$$

with p arbitrary function of its arguments going to zero as $\xi \to \pm\infty$.

- *Order* ϵ^3 *and* $\theta = 3$: Taking into account the dispersion relation (A.11) and equation (A.15), we obtain

$$u_3^{(3)} = \left(\frac{h}{1-e^{i\kappa h}}\right)^2 \left[u_1^{(1)}\right]^3. \tag{A.23}$$

- *Order* ϵ^4 *and* $\theta = 0$: Taking into account equations (A.15), (A.18), (A.19), (A.21b), we get

$$\partial_{t_1} u_3^{(0)} - \frac{1}{h}\partial_{n_1} u_3^{(0)} = h\partial_\xi \left[\frac{h}{2}\partial_\xi \left|u_1^{(1)}\right|^2 - 2\sin^2(\kappa h/2)\left(u_2^{(1)}u_1^{(-1)} + u_2^{(-1)}u_1^{(1)}\right)\right]. \tag{A.24}$$

Solving equation (A.24), we obtain

$$u_3^{(0)} = \tau(\rho, t_j, j \geq 2) + h\left(u_2^{(1)}u_1^{(-1)} + u_2^{(-1)}u_1^{(1)}\right) - \frac{h^2}{4\sin^2(\kappa h/2)}\partial_\xi \left|u_1^{(1)}\right|^2, \tag{A.25}$$

where τ is an arbitrary function of its arguments going to zero as $\rho \to \pm\infty$. As usually, if we don't want any dependence at all from ρ but only on ξ, in order to satisfy the asymptotic conditions $\lim_{\xi\to\pm\infty} u_3^{(0)} = 0$, we have to take

$$\tau = 0 \tag{A.26}$$

(unless we take the fully continuous limit).

- *Order* ϵ^4 *and* $\theta = 1$: Taking into account the dispersion relation (A.11) and equations (A.15), (A.18), (A.19), (A.22), (A.25), (A.26), we get

$$\begin{aligned} &\partial_{t_1} u_3^{(1)} - \frac{\cos(\kappa h)}{h}\partial_{n_1} u_3^{(1)} = -\partial_{t_3} u_1^{(1)} - \partial_{t_2} u_2^{(1)} - B_3\partial_\xi^3 u_1^{(1)} - iB_2\partial_\xi^2 u_2^{(1)}, \\ &B_3 \doteq -\frac{h^2\cos(\kappa h)}{6}. \end{aligned} \tag{A.27}$$

As a consequence of the equations (A.13), (A.21a) and as $u_1^{(0)} = 0$, the right hand side of equation (A.27) is secular, so that

$$\partial_{t_1} u_3^{(1)} - \frac{\cos(\kappa h)}{h}\partial_{n_1} u_3^{(1)} = 0, \tag{A.28a}$$

$$\partial_{t_2} u_2^{(1)} + iB_2\partial_\xi^2 u_2^{(1)} = -\partial_{t_3} u_1^{(1)} - B_3\partial_\xi^3 u_1^{(1)}. \tag{A.28b}$$

The first relation implies that $u_3^{(1)}$ also depends on ξ while the second one, as a consequence of equation (A.21b), implies that the right hand side is secular, so that

$$i\partial_{t_2} u_2^{(1)} - B_2\partial_\xi^2 u_2^{(1)} = 0, \tag{A.29a}$$

$$\partial_{t_3} u_1^{(1)} + B_3\partial_\xi^3 u_1^{(1)} = 0. \tag{A.29b}$$

Equation (A.29a), as one can see from the definition (2.16), has a forcing term $f_2(2)$ with coefficients $a = b = 0$.

- *Order* ϵ^4 *and* $\theta = 2$: Taking into account the dispersion relation (A.11) and the equations (A.13), (A.15), (A.18), (A.19), (A.21), (A.22), (A.23), we get

$$\begin{aligned} u_4^{(2)} = &-h^3\left[\frac{1}{\left(1-e^{i\kappa h}\right)^2}u_1^{(1)}|u_1^{(1)}|^2 + \frac{i\cos(\kappa h/2)}{8\sin^3(\kappa h/2)}\partial_\xi^2 u_1^{(1)}\right]u_1^{(1)} \\ &- \frac{h^2}{4\sin^2(\kappa h/2)}\partial_\xi\left(u_1^{(1)}u_2^{(1)}\right) + \frac{h}{1-e^{i\kappa h}}\left(u_2^{(1)2} + 2u_1^{(1)}u_3^{(1)}\right). \end{aligned} \tag{A.30}$$

- *Order* ϵ^4 *and* $\theta = 3$: Taking into account the dispersion relation (A.11) and the equations (A.13), (A.15), (A.19), (A.22), (A.23), we get

$$u_4^{(3)} = \left(\frac{h}{1-e^{i\kappa h}}u_1^{(1)}\right)^2 \left(3u_2^{(1)} + \frac{2he^{i\kappa h}}{1-e^{i\kappa h}}\partial_\xi u_1^{(1)}\right).$$

- *Order* ϵ^4 *and* $\theta = 4$: Taking into account the dispersion relation (A.11) and the equations (A.15), (A.23), we get

$$u_4^{(4)} = \left(\frac{h}{1-e^{i\kappa h}}\right)^3 [u_1^{(1)}]^4.$$

- *Order* ϵ^5 *and* $\theta = 0$: Taking into account equations (A.15), (A.18), (A.19), (A.21b), (A.22), (A.26), (A.25), (A.28), we get

$$\begin{aligned}\partial_{t_1} u_4^{(0)} - \frac{1}{h}\partial_{n_1} u_4^{(0)} &= h\partial_\xi\Big\{ -2\sin^2(\kappa h/2)\big(u_1^{(1)}u_3^{(-1)} + u_1^{(-1)}u_3^{(1)} + |u_2^{(1)}|^2\big) \\ &+ \left[4\sin^2(\kappa h/2) - 1\right]\frac{h^2}{2}\big|u_1^{(1)}\big|^4 + \frac{h}{2}\partial_\xi\big(u_1^{(1)}u_2^{(-1)} + u_1^{(-1)}u_2^{(1)}\big) \\ &+ \frac{ih^2}{4}\cot(\kappa h/2)\,\partial_\xi\big(u_1^{(-1)}\partial_\xi u_1^{(1)} - u_1^{(1)}\partial_\xi u_1^{(-1)}\big)\Big\}.\end{aligned} \tag{A.31}$$

Solving equation (A.31), we obtain

$$\begin{aligned}u_4^{(0)} &= \Theta(\rho, t_j, j \geq 2) + h\big(u_1^{(1)}u_3^{(-1)} + u_1^{(-1)}u_3^{(1)} + |u_2^{(1)}|^2\big) \\ &+ \frac{h^2}{4\sin^2(\kappa h/2)}\Big\{\partial_\xi\big(u_1^{(1)}u_2^{(-1)} + u_1^{(-1)}u_2^{(1)}\big) + \left[4\sin^2(\kappa h/2) - 1\right]h\big|u_1^{(1)}\big|^4\Big\} \\ &- \frac{ih^3\cos(\kappa h/2)}{8\sin^3(\kappa h/2)}\partial_\xi\big(u_1^{(-1)}\partial_\xi u_1^{(1)} - u_1^{(1)}\partial_\xi u_1^{(-1)}\big),\end{aligned} \tag{A.32}$$

where Θ is an arbitrary function of its arguments going to zero as $\rho \to \pm\infty$. As usually, if we don't want any dependence from ρ but only on ξ, in order to satisfy the asymptotic conditions $\lim_{\xi\to\pm\infty} u_4^{(0)} = 0$, we have to take

$$\Theta = 0$$

(unless we take the fully continuous limit).

- *Order* ϵ^5 *and* $\theta = 1$: Taking into account the dispersion relation (A.11) and equations $u_1^{(0)} = f = \tau = \Theta = 0$, together with the equations (A.15), (A.18), (A.22), (A.23), (A.25), (A.30), (A.32), we get

$$\begin{aligned}\partial_{t_1} u_4^{(1)} - \frac{\cos(\kappa h)}{h}\partial_{n_1} u_4^{(1)} &= -\partial_{t_4} u_1^{(1)} - \partial_{t_3} u_2^{(1)} - \partial_{t_2} u_3^{(1)} - iB_2\partial_\xi^2 u_3^{(1)} - B_3\partial_\xi^3 u_2^{(1)} \\ &+ iB_4\partial_\xi^4 u_1^{(1)} + \zeta\left[u_1^{(1)}\big|\partial_{n_1} u_1^{(1)}\big|^2 + u_1^{(-1)}\big(\partial_{n_1} u_1^{(1)}\big)^2 + u_1^{(1)2}\partial_{n_1}^2 u_1^{(-1)}\right],\end{aligned} \tag{A.33}$$

$$B_4 \doteq \frac{h^3\sin(\kappa h)}{24}, \qquad \zeta \doteq \frac{h\left[1 + \cos(\kappa h) + 3i\sin(\kappa h)\right]}{e^{i\kappa h} - 1}.$$

As a consequence of the equations (A.13), (A.21a), (A.28a) and of $u_1^{(0)} = 0$, the right hand side of equation (A.33) is secular, so that

$$\partial_{t_1} u_4^{(1)} - \frac{\cos(\kappa h)}{h}\partial_{n_1} u_4^{(1)} = 0,$$

$$\partial_{t_2} u_3^{(1)} + iB_2 \partial_\xi^2 u_3^{(1)} = -\partial_{t_4} u_1^{(1)} - \partial_{t_3} u_2^{(1)} - B_3 \partial_\xi^3 u_2^{(1)} \\ + iB_4 \partial_\xi^4 u_1^{(1)} + \zeta \left[u_1^{(1)} \left| \partial_{n_1} u_1^{(1)} \right|^2 + u_1^{(-1)} \left(\partial_{n_1} u_1^{(1)} \right)^2 + u_1^{(1)2} \partial_{n_1}^2 u_1^{(-1)} \right]. \qquad \text{(A.34)}$$

The first relation tells us that $u_4^{(1)}$ depends on ξ too while in the second one, as a consequence of equations (A.21b), (A.29a), the first our terms in the right hand side of equation (A.34) are secular, so that

$$\partial_{t_2} u_3^{(1)} + iB_2 \partial_\xi^2 u_3^{(1)} = \zeta \left[u_1^{(1)} \left| \partial_{n_1} u_1^{(1)} \right|^2 + u_1^{(-1)} \left(\partial_{n_1} u_1^{(1)} \right)^2 + u_1^{(1)2} \partial_{n_1}^2 u_1^{(-1)} \right], \qquad \text{(A.35a)}$$

$$\partial_{t_3} u_2^{(1)} + B_3 \partial_\xi^3 u_2^{(1)} = -\partial_{t_4} u_1^{(1)} + iB_4 \partial_\xi^4 u_1^{(1)}. \qquad \text{(A.35b)}$$

As a consequence of equation (A.29b), the right hand side of equation (A.35b) is secular so that

$$\partial_{t_3} u_2^{(1)} + B_3 \partial_\xi^3 u_2^{(1)} = 0, \qquad \partial_{t_4} u_1^{(1)} - iB_4 \partial_\xi^4 u_1^{(1)} = 0.$$

Equation (A.35a), as one can see from the definition (2.17) and taking into account that $a = b = 0$, has a forcing term $f_2(3)$ that respects all the A_3 C-integrability conditions (2.18).

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