Exotic Galilean Symmetry and Non-Commutative Mechanics*

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Abstract. Some aspects of the "exotic" particle, associated with the two-parameter central extension of the planar Galilei group are reviewed. A fundamental property is that it has non-commuting position coordinates. Other and generalized non-commutative models are also discussed. Minimal as well as anomalous coupling to an external electromagnetic field is presented. Supersymmetric extension is also considered. Exotic Galilean symmetry is also found in Moyal field theory. Similar equations arise for a semiclassical Bloch electron, used to explain the anomalous/spin/optical Hall effects.

Key words: noncommutative spaces; Galilean symmetry; dynamical systems; quantum field theory

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1 Introduction: "exotic" Galilean symmetry

A curious property of massive non-relativistic quantum systems is that Galilean boosts only act up-to phase, so that only the 1-parameter central extension of the Galilei group acts unitarily [1]. True representations only arise for massless particles.

Do further extension parameters exist? It is well-known that central extensions are associated with non-trivial Lie algebra cohomology [2, 3], and Bargmann [4] has shown that, in $d \geq 3$ space dimensions, the Galilei group only admits a 1-parameter central extension, identified with the physical mass, m. Lévy-Leblond [5] has recognized, however, that, owing to the Abelian nature of planar rotations, the *planar* Galilei group admits a second central extension. The cohomology is two-dimensional, and is parametrized by two constants, namely by the mass and a second, "exotic", parameter κ. The second extension is highlighted by the non-commutativity of Galilean boost generators,

$$[K_1, K_2] = i\kappa.$$

This fact has long been considered, however, a mere mathematical curiosity, as planar physics itself has been viewed as a toy.

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Around 1995 the situation started to change, though, with the construction of physical models carrying such an "exotic" structure [6, 7, 8]. [8] uses an acceleration-dependent Lagrangian, while that in [9], is obtained following Souriau's method [3].

These models have the distinctive feature that the Poisson bracket of the planar coordinates does not vanish,

$$\{x_1, x_2\} = -\frac{\kappa}{m^2} \equiv \theta, \tag{1.1}$$

and provide us with two-dimensional examples of *non-commutative mechanics* [10, 11, 12, 13, 14, 15][1].

What is the physical origin of exotic mechanics? *What is the quantum mechanical counterpart?* An answer has been found soon after its introduction: it is a sort of "non-relativistic shadow" of (fractional) spin [16, 17, 18]. Our particles can be interpreted therefore, as nonrelativistic anyons [19, 20].

The supersymmetric extension of the theory is outlined in Section 9.

All these examples have been taken from one-particle mechanics; exotic Galilean symmetry can be found, however, also in field theory [21], as explained in Section 10.

Remarkably, similar structures were considered, independently and around the same time, in condensed matter physics, namely for the Bloch electron [22]: it was argued that the semiclassical dynamics should involve a "Berry term", which induces "anomalous" velocity similar to the one in the "exotic" model [9].

These, 3-space dimensional, models are also non-commutative, but the parameter θ is now promoted to a vector-valued function of the quasi-momentum: $\vec{\Theta} = \vec{\Theta}(\vec{k})$. Exotic Galilean symmetry, strictly linked to two space dimensions, is lost. However, a rich Poisson structure and an intricate interplay with external magnetic fields can be studied. Further developments include the Anomalous [23], the Spin [24] and the Optical [25, 26, 27, 28] Hall effects.

This review completes and extends those in [29, 30, 31].

2 Exotic model, constructed by Souriau's method

Models associated with a given symmetry group can be conveniently constructed using Souriau's method: the classical phase spaces of "elementary systems" correspond to coadjoint orbits of their symmetry groups [3]. This was precisely the way followed in [6, 9] to produce an "elementary" classical system carrying Lévy-Leblond's "exotic" Galilean symmetry. Translated from Souriau's to more standard terms, the model has an "exotic" symplectic form and a free Hamiltonian,

$$\Omega_0 = dp_i \wedge dx_i + \frac{1}{2}\theta\epsilon_{ij} dp_i \wedge dp_j,$$
$$H_0 = \frac{\vec{p}^{\,2}}{2m}.$$

The associated free motions follow the usual straight lines, described by the equations

$$m\dot{x}_i = p_i - m\theta\epsilon_{ij}\dot{p}_j, \qquad \dot{p}_i = 0.$$

The "exotic" structure only enters the conserved quantities, namely the boost and the angular momentum,

$$j = \epsilon_{ij} x_i p_j + \frac{\theta}{2}\vec{p}^{\,2}, \qquad K_i = -mx_i + p_i t - m\theta\epsilon_{ij} p_j. \tag{2.1}$$

[1]Our conventions are as follows. Symbols with arrows denote vectors, and those in boldface are tensors. The position vector $\vec{r}$ has coordinates x_i.

The "exotic" structure behaves hence roughly as spin: it contributes to some conserved quantities, but the new terms are separately conserved. The new structure does not seem to lead to any new physics.

The situation changes dramatically, though, if the particle is coupled to a gauge field. Applying Souriau's coupling prescription [3] yields indeed

$$\Omega = \Omega_0 + eB\,dq_1 \wedge dq_2, \qquad H = H_0 + eV, \tag{2.2}$$

where B is the magnetic field assumed to be perpendicular to the plane, and V is the electric potential. For simplicity, both B and V are assumed to be time-independent.

The associated Poisson bracket then automatically satisfies the Jacobi identity. The resulting equations of motion read

$$m^*\dot{x}_i = p_i - em\theta\,\epsilon_{ij}E_j, \qquad \dot{p}_i = eE_i + eB\,\epsilon_{ij}\dot{x}_j, \tag{2.3}$$

where the parameter $\theta = k/m^2$ measures the non-commutativity of x_1 and x_2, and we have introduced the *effective mass*

$$m^* = m(1 - e\theta B). \tag{2.4}$$

The novel features, crucial for physical applications, are twofold: Firstly, the relation between velocity and momentum contains an "anomalous velocity term" $-em\theta\epsilon_{ij}E_j$, so that $\dot{x}_i$ and p_i are not in general parallel. The second one is the interplay between the exotic structure and the magnetic field, yielding the effective mass m^* in (2.4).

Equations (2.3) do not derive from a configuration-space Lagrangian (but see Section 3). The 1-order "phase" (in fact "evolution space" [3]) formalism works, however, so that the equations of motion (2.3) come from varying the action defined by integrating the "Cartan" 1-form [3],

$$\lambda = (p_i - A_i)dx_i - \frac{\vec{p}^{\,2}}{2m}dt + \frac{\theta}{2}\epsilon_{ij}p_i dp_j$$

along the lifted world-line $\widetilde{\gamma}$ in evolution space $T^*\mathbb{R}^2 \times \mathbb{R}$. The exterior derivative, $\sigma = d\lambda$, of the Cartan form λ provides us with a closed "Lagrange–Souriau" 2-form, which, however, cannot be separated canonically into a "symplectic" and a "Hamiltonian part" [3]. Thus more general procedures have to be adopted to build such a system and clarify their Hamiltonian structure. These aspects will be discussed in detail in the following Sections 3, 4 and 5. Here we limit ourselves to notice that, in fact, when $m^* \neq 0$, (2.3) is a Hamiltonian system,

$$\dot{\xi} = \{H, \xi^\alpha\}, \qquad \xi = (x_j, p_i),$$

with Poisson brackets

$$\{x_1, x_2\} = \frac{m}{m^*}\theta, \qquad \{x_i, p_j\} = \frac{m}{m^*}\delta_{ij}, \qquad \{p_1, p_2\} = \frac{m}{m^*}eB. \tag{2.5}$$

A remarkable property is that for *vanishing effective mass* $m^* = 0$, i.e., when the magnetic field takes the critical value

$$B = \frac{1}{e\theta},$$

the system becomes singular. Then "Faddeev–Jackiw" (alias symplectic) reduction yields an essentially two-dimensional, simple system, reminiscent of "Chern–Simons mechanics" [32]. The symplectic plane plays, simultaneously, the role of both configuration and phase space. The only motions are those which follow a generalized Hall law.

Quantization of the reduced system yields, moreover, the "Laughlin" wave functions [33], which are the ground states in the Fractional Quantum Hall Effect (FQHE).

The relations (2.5) diverge as $m^* \to 0$, but after reduction we get, cf. (1.1),

$$\{x_1, x_2\} = \frac{1}{eB} = \theta.$$

The coordinates are, hence, non-commuting, and their commutator is determined by the magnetic field, fine-tuned to the parameter θ.

Relation to another non-commutative mechanics. The exotic relations (2.5) are similar to those proposed (later) in [10],

$$\{x_i, x_j\} = \theta\epsilon_{ij}, \qquad \{x_i, p_j\} = \delta_{ij}, \qquad \{p_1, p_2\} = eB, \tag{2.6}$$

which look indeed simpler. Using the standard Hamiltonian $H = \frac{p^2}{2m} + eV(x)$, the associated equations of motion read

$$mx_i' = p_i - em\theta\epsilon_{ij}E_j, \qquad p_i' = eB\epsilon_{ij}\frac{p_j}{m} + eE_i, \tag{2.7}$$

where we noted "time" by T and $(\cdot)' = \frac{d}{dT}$.

A short calculation reveals, however, that

$$\big\{x_i, \{p_1, p_2\}\big\}_{\rm cycl} = e\theta\,\epsilon_{ij}\partial_j B,$$

so that *the Jacobi identity is only satisfied if B is a constant.* In other words, the system (2.6) is *only consistent* for a constant magnetic field – which is an *unphysical condition* in general[2].

The model (2.6) has another strange feature. Let us indeed assume that the magnetic field is radially symmetric, $B = B(r)$. One would then expect to have conserved angular momentum. For constant B, applying Noether's theorem to an infinitesimal rotation $\delta\xi_i = \epsilon_{ij}\xi_j$ yields indeed $\delta\xi_i = -\{j^{\rm NP}, \xi_i\}$, with

$$j^{\rm NP} = \frac{1}{1 - e\theta B}\underbrace{\left(\vec{x}\times\vec{p} + \frac{\theta}{2}\vec{p}^{\,2} + \frac{eB}{2}\vec{x}^{\,2}\right)}_{j}. \tag{2.8}$$

This differs from the standard expression by the pre-factor $(1 - e\theta B)^{-1}$. But what is even worse is that, for a radial but non constant magnetic field, (2.8) is *not conserved*:

$$\frac{dj^{\rm NP}}{dT} = \frac{e\theta j}{(1 - e\theta B)^2}\partial_i B x_i',$$

while j in (2.1) is still conserved as it should.

Can the theory defined by (2.6) be extended to an arbitrary B?

Let us first assume that $B =$ const. s.t. $m^* \neq 0$, and *let us redefine the time*[3], as

$$T \to t = (1 - e\theta B)T \quad \Rightarrow \quad \frac{d}{dT} = (1 - e\theta B)\frac{d}{dt}. \tag{2.9}$$

Then equations (2.7) are carried into the exotic equations, (2.3). When $B =$ const. s.t. $m^* \neq 0$, the two theories are therefore equivalent.

Remarkably, the *time redefinition* (2.9) actually *extends the previous theory*, since it carries it into the "exotic model", for which *the Jacobi identity holds for any, not necessarily constant B.* Thus, the transformation (2.9) (which is singular for $eB\theta = 1$), removes the unphysical restriction to constant magnetic fields: (2.9) regularizes the system (2.6).

[2]For another role of the Jacobi identity in non-commutative mechanics, see [34].

[3]This was suggested to us by G. Marmo (private communication).

3 Acceleration-dependent Lagrangian in configuration space

An independent and rather different approach was followed in [8].

We start again with a particle characterized by the two central charges

$$m \qquad \text{and} \qquad \kappa = -m^2\theta$$

of the exotic Galilei group. These charges appear in the following Lie-brackets (represented by Poisson-brackets (PBs)) between the translation generators P_i and the boost generators K_i

$$\{P_i, K_j\} = m\delta_{ij}, \qquad \{K_i, K_j\} = -m^2\theta\epsilon_{ij}. \tag{3.1}$$

In order to find a configuration space Lagrangian whose Noether charges for boosts satisfy (3.1) we must add the second time derivative of the coordinates $\ddot{x}_i$ to the usual variables x_i and $\dot{x}_i$. As shown in [8] the most general one-particle Lagrangian, which is at most linearly dependent on $\ddot{x}_i$, leading to the Euler–Lagrange equations of motion which are covariant w.r.t. the planar Galilei group, is given, up to gauge transformations, by

$$\mathcal{L} = \frac{m}{2}\dot{x}_i^2 + \frac{m^2\theta}{2}\epsilon_{ij}\dot{x}_i\ddot{x}_j. \tag{3.2}$$

Introducing the Lagrange multipliers p_i and adding $p_i(\dot{x}_i - y_i)$ to (3.2) yields

$$\mathcal{L} = p_i\dot{x}_i + \frac{m^2\theta}{2}\epsilon_{ij}y_i\dot{y}_j - H(\vec{y}, \vec{p}) \tag{3.3}$$

with

$$H(\vec{y}, \vec{p}) = y_i p_i - \frac{m}{2}y_i^2.$$

(3.3) describes a constrained system, because we have

$$\frac{\partial\mathcal{L}}{\partial\dot{y}_i} = -\frac{m^2\theta}{2}\epsilon_{ij}y_j.$$

Therefore the PBs, obtained by means of the Faddeev–Jackiw procedure, take a non-standard form

$$\{x_i, p_j\} = \delta_{ij}, \qquad \{y_i, y_j\} = -\frac{1}{m^2\theta}\epsilon_{ij}. \tag{3.4}$$

All other PBs vanish.

For the conserved boost generator we obtain

$$K_i = -mx_i + p_i t - m^2\theta\epsilon_{ij}y_j$$

and therefore, due to (3.4), the PB resp. commutator of two boosts is nonvanishing

$$\{K_i, K_j\} = -m^2\theta\epsilon_{ij}. \tag{3.5}$$

The Lagrangian (3.3) shows that the phase space is 6-dimensional. In order to split off two internal degrees of freedom, we have to look for a Galilean invariant decomposition of the 6-dim phase space into two dynamically independent parts: a 4-dim external and a 2-dim internal part. This decomposition is achieved by the transformation [8, 19] $(\vec{x}, \vec{p}, \vec{y}) \to (\vec{X}, \vec{p}, \vec{Q})$ with

$$y_i = \frac{p_i}{m} + \frac{Q_i}{m\theta} \qquad \text{and} \qquad x_i = X_i - \epsilon_{ij}Q_j, \tag{3.6}$$

leading to the following decomposition of the Lagrangian (3.3)

$$\mathcal{L} = \mathcal{L}_{\rm ext} + \mathcal{L}_{\rm int}$$

with

$$\mathcal{L}_{\rm ext} = p_i\dot{X}_i + \frac{\theta}{2}\epsilon_{ij}p_i\dot{p}_j - \frac{p_i^2}{2m} \qquad \text{and} \qquad \mathcal{L}_{\rm int} = \frac{1}{2m\theta^2}Q_i^2 + \frac{1}{2\theta}\epsilon_{ij}Q_i\dot{Q}_j. \tag{3.7}$$

From (3.6) and the PBs (3.4) it now follows that the new coordinates X_i are noncommutative

$$\{X_i, X_j\} = \theta\epsilon_{ij}. \tag{3.8}$$

The remaining nonvanishing PBs are

$$\{X_i, p_j\} = \delta_{ij}, \qquad \{Q_i, Q_j\} = -\theta\epsilon_{ij}.$$

Conclusion. The particle Lagrangian (3.2) containing $\ddot{x}_i$ leads to a nonvanishing commutator of two boosts. But in order to obtain noncommutative coordinates we are forced to decompose the 6-dim phase space in a Galilean invariant manner into two dynamically independent 4-dim external and 2-dim internal phase spaces.

Relation of the DH and LSZ models. The relation of the "minimal" and the configuration-space models of DH [9] and of L.S.Z. [8] respectively, has been studied in [19]. Introducing the coordinates X_i, Q_i, and p_i on 6-dimensional phase space according to (3.6) allows us to present the symplectic structure and the Hamiltonian associated with (3.2) as

$$\Omega = dp_i \wedge dX_i + \frac{\theta}{2}\varepsilon_{ij}dp_i \wedge dp_j + \frac{1}{2\theta}\varepsilon_{ij}dQ_i \wedge dQ_j,$$
$$H = \frac{\vec{p}^{\,2}}{2m} - \frac{1}{2m\theta^2}\vec{Q}^2.$$

Thus, the model of L.S.Z. is decomposed into the DH theory, augmented with a two-dimensional internal space, and a negative zero point of the Hamiltonian. Note that the "external" and "internal" phase spaces are "almost" independent: the only effect of internal motion is indeed through the length of the internal vector, $|\vec{Q}|$.

Generalization of (3.2). If we add to (3.2) a term $f(\ddot{x}_i^2)$, the obtained Lagrangian is the most general one involving, in a Galilean quasi-invariant manner, the variables x_i, $\dot{x}_i$ and $\ddot{x}_i$.

Then one can show

i) the PB of the two boosts (3.5) will not change,

ii) the new 8-dim phase space may be decomposed again in a Galilean invariant manner into two dynamically independent 4-dim parts, an external and an internal one.

Commutative – versus noncommutative plane.

The subalgebra of the Galilean algebra containing only translations and boosts is given in the cases of, respectively, their one- or two-fold central extensions by

one-fold centrally extended	two-fold centrally extended
$\{p_i, K_j\} = m\delta_{ij}$,	$\{p'_i, K'_j\} = m\delta_{ij}$,
$\{p_i, p_j\} = 0$,	$\{p'_i, p'_j\} = 0$,
$\{K_i, K_j\} = 0$,	$\{K'_i, K'_j\} = -m^2\theta\epsilon_{ij}$.

Obviously both are related by the transformations

$$K'_i = K_i - \frac{m\theta}{2}\epsilon_{ij}p_j, \qquad p'_i = p_i.$$

To this corresponds the following point transformation between noncommutative coordinates X_i and commutative ones q_i

$$X_i = q_i - \frac{\theta}{2}\epsilon_{ij}p_j$$

as can be read off immediately from the form of $\mathcal{L}_{\text{ext}}$ in (3.7).

Now the question arises: *What to use in physics, the commutative or the non-commutative plane?*

Answer. For free particles both possibilities are equivalent. But in the case of a nontrivial interaction one has to use the commutative (noncommutative) plane, if a local potential or gauge interaction is given in terms of q_i (X_i).

It is worth mentioning that the acceleration-dependent model presented in this Section can be related to radiation damping [35].

4 General form of noncommutative mechanics

Up to now noncommutativity has been described by a constant θ in the PB (3.8). But it is possible to get θ as a function of $\vec{X}$ and $\vec{p}$ if one considers external Lagrangians more general than (3.7).

To do this consider a very general class of Lagrangians given by

$$\mathcal{L} = p_i\dot{X}_i + \tilde{A}_i(\vec{X},\vec{p})\dot{p}_i - H(\vec{p},\vec{X}) \tag{4.1}$$

leading to the PBs

$$\{X_i, X_j\} \sim \epsilon_{ij}\tilde{B}, \qquad \tilde{B} = \epsilon_{k\ell}\partial_{p_k}\tilde{A}_\ell(\vec{X},\vec{p})$$

with

$$\{P_i, P_j\} = 0.$$

We dispense with the reproduction of the more complicated form of the PBs for the phase space coordinates (X_i, p_j). Again by the point transformation

$$X_i \to q_i = X_i - \tilde{A}_i(\vec{X},\vec{p})$$

we obtain commuting coordinates q_i as follows from

$$p_i\dot{X}_i + \tilde{A}_i\dot{p}_i = p_i\dot{q}_i + \frac{d}{dt}(\tilde{A}_i p_i).$$

<u>Examples:</u>

i)

$$\tilde{A}_i = f(p^2)(\vec{X}\cdot\vec{p})p_i \tag{4.2}$$

leading to the PBs of the phase space variables

$$\{X_i, X_j\} = \frac{f(p^2)}{1-p^2f(p^2)}\epsilon_{ij}L, \qquad L = \epsilon_{k\ell}X_k p_\ell, \tag{4.3}$$

$$\{X_i, p_j\} = \delta_{ij} + \frac{f(p^2)}{1-p^2f(p^2)}p_i p_j. \tag{4.4}$$

A particular example is given by

$$f(p^2) = \frac{\theta}{1+p^2\theta} \qquad \text{and therefore} \qquad \frac{f}{1-p^2 f} = \theta.$$

This gives exactly Snyder's NC-algebra, presented in 1947 [36].

Another case, defined by

$$f(p^2) = \frac{2}{p^2}, \tag{4.5}$$

can be related to a deformed Galilei algebra (to be discussed in the next section).

ii)

$$\tilde{A}_i = \tilde{A}_i(\vec{p})$$

leading to the PBs

$$\{X_i, X_j\} = \epsilon_{ij}\tilde{B}(\vec{p}), \qquad \{X_i, p_j\} = \delta_{ij}.$$

$\tilde{B}$ is the Berry curvature for the semiclassical dynamics of electrons in condensed matter to be discussed in Section 11.

We may generalize (4.1) to the most general 1st-order Lagrangian

$$\mathcal{L} = (p_i + A_i(\vec{X}, \vec{p}))\dot{X}_i + \tilde{A}_i(\vec{X}, \vec{p})\dot{p}_i - H(\vec{p}, \vec{X}).$$

Here $A_i(\vec{X})$ describes standard electromagnetic interaction (cp. Section 6, and Section 11 for the 3 dimensional case, respectively). A particular case of a $\vec{p}$-dependent A_i has been considered in [37].

5 Lagrangian realization of the $\tilde{k}$-deformed Galilei algebra as a symmetry algebra

In 1991 Lukierski, Nowicki, Ruegg and Tolstoy invented the k-deformed Poincaré algebra [38], which later found applications, e.g., in Quantum Gravity [39]. By rescaling the Poincaré generators and the deformation parameter k, the corresponding nonrelativisitic limit, the $\tilde{k}$-deformed Galilei algebra, has been derived by Giller et al. [40] and, in a different basis, by Azcarraga et al. [41]. In this Section we will describe a Lagrangian realization of the latter.

Again we look at the classical Lagrangian (4.1) specified by (4.2) and (4.5) together with the following choice of the Hamiltonian

$$H = \tilde{k}\ln(p^2/2). \tag{5.1}$$

According to (4.3), (4.4) we obtain the PBs

$$\{X_i, X_j\} = -\frac{2}{p^2}\epsilon_{ij}L \qquad \text{and} \qquad \{X_i, p_j\} = \delta_{ij} - \frac{2}{p^2}p_i p_j$$

which lead, together with the Hamiltonian (5.1), to the equations of motion

$$\dot{p}_i = 0 \qquad \text{and} \qquad \dot{X}_i = -\frac{2\tilde{k}}{p^2}p_i.$$

Then, we may define the "pseudo-boosts" K_i

$$K_i = p_i t + \frac{p^2}{2\tilde{k}} X_i$$

which are conserved. They satisfy, together with p_i and H, the PB-algebra

$$\{K_i, p_j\} = \frac{\delta_{ij}}{2\tilde{k}} p^2 - \frac{p_i p_j}{\tilde{k}}, \qquad \{K_i, H\} = -p_i, \qquad \{K_i, K_j\} = 0. \tag{5.2}$$

Together with the standard algebra of translations (represented by p_i and H) and rotations (represented by L) the relations (5.2), build the $\tilde{k}$-deformed Galilei algebra derived in [41].

The limit $\tilde{k} \to \infty$ leads to a divergent Hamiltonian (5.1). Therefore, the $\tilde{k}$-deformation does not have a standard "no-deformation limit".

6 Physical origin of the exotic structure

A free relativistic "elementary" particle in the plane corresponds to a unitary representation of the planar Lorentz group $O(2,1)$ [17]. These representations are in turn associated with the coadjoint orbits of $SO(2,1)$, endowed with their canonical symplectic structures, interpreted by Souriau as classical phase spaces [3]. Applied to the planar Lorentz group, the procedure yields the *relativistic* model [17, 18]

$$\Omega_{\mathrm{rel}} = dp_\alpha \wedge dx^\alpha + \frac{s}{2} \epsilon^{\alpha\beta\gamma} \frac{p_\alpha dp_\beta \wedge dp_\gamma}{(p^2)^{3/2}},$$
$$H_{\mathrm{rel}} = \frac{1}{2m}\big(p^2 - m^2 c^2\big).$$

The p-dependent contribution looks like a "magnetic monopole in momentum space" (cf. (11.10) below).

As pointed out by Jackiw and Nair [16], the free exotic model can be recovered considering a tricky non-relativistic limit , namely

$$s/c^2 \to \kappa = m^2\theta.$$

The two-form $\Omega_{\mathrm{rel}}\big|_{H_{\mathrm{rel}}=0}$ goes indeed over into the exotic symplectic form. Intuitively, the exotic structure can be viewed as a *"non-relativistic shadow" of relativistic spin.*

The exotic Galilei group can itself be derived from the planar Poincaré group by "Jackiw–Nair" contraction [16]. One starts with the planar Lorentz generators,

$$\{J^\alpha, J^\beta\} = \epsilon^{\alpha\beta\gamma} J_\gamma.$$

For the classical system

$$J_\mu = \epsilon_{\mu\nu\rho} x^\nu p^\rho + s \frac{p_\mu}{\sqrt{p^2}}.$$

A non-relativistic boost is the "JN" limit of a suitable Lorentz transformation,

$$\frac{1}{c} \epsilon_{ij} J^j \to m x_i - p_i t + m\theta \epsilon_{ij} p_j = -K_i,$$

and the exotic relation is recovered,

$$\{K_1, K_2\} = J_0/c^2 \to \frac{s}{c^2} = \kappa.$$

The angular momentum is in turn

$$J_0 = \vec{x} \times \vec{p} + s + \frac{s}{m^2c^2}\vec{p}^2 \to \vec{x} \times \vec{p} + \frac{\theta}{2}\vec{p}^2 = j.$$

whereas the divergent term $s = \kappa c^2$ has to be removed by hand.

It is worth mentioning that the "Jackiw–Nair limit" of a relativistic particle with torsion [42] provides us with the L.S.Z. model [19], and a similar procedure yields the so-called "Newton–Hooke" system [43]. Applied to the infinite-component Majorana-type anyon equations [17, 18] yields, furthermore, a first-order infinite-component "Lévy-Leblond type" system with exotic Galilean symmetry [20].

About anyons constructed from orbits, see also [44].

7 Anomalous coupling of anyons

It has been suggested [45] that a classical, *relativistic* anyon in an electromagnetic field should be described by the equations

$$\begin{aligned} &m\frac{dx^\alpha}{d\tau} = p^\alpha && \text{(velocity-momentum)},\\ &\frac{dp^\alpha}{d\tau} = \frac{e}{m}F^{\alpha\beta}p_\beta && \text{(Lorentz equation)}, \end{aligned} \tag{7.1}$$

$\alpha, \beta, \ldots = 0, 1, 2$ and τ denotes the proper time. These equations are Hamiltonian, with symplectic form and Hamilton's function

$$\begin{aligned} &\Omega = \Omega_{\rm rel} + \frac{1}{2}eF_{\alpha\beta}dx^\alpha \wedge dx^\beta,\\ &H = H_{\rm rel} + \frac{es}{2m\sqrt{p^2}}\epsilon_{\alpha\beta\gamma}F^{\alpha\beta}p^\gamma, \end{aligned} \tag{7.2}$$

respectively. Let us observe that the second, non-minimal term in the Hamiltonian is *dictated by the required form of the velocity relation* in (7.1).

The model of [45] has gyromagnetic ratio $g = 2$, and some theoreticians have long believed [45, 46] that this is indeed the "correct" g value of anyons. Experimental evidence shows, however, that in the Fractional Quantum Hall Effect, for example, the measured value of g is approximately *zero* [47]. Is it possible to construct an "anomalous" model with $g \neq 2$? The answer is affirmative [48], as we now explain.

Planar spin has to satisfy the relation $S_{\alpha\beta}p^\beta = 0$. The spin tensor has, therefore, the form

$$S_{\alpha\beta} = \frac{s}{\sqrt{p^2}}\epsilon_{\alpha\beta\gamma}p^\gamma.$$

Introducing the shorthand $-F_{\alpha\beta}S^{\alpha\beta} = F \cdot S$, the Hamiltonian (7.2) is presented as

$$H^{\rm CNP} = \frac{1}{2m}\left(p^2 - M^2c^2\right) \qquad \text{where} \qquad M^2 = m^2 + \frac{e}{c^2}F \cdot S. \tag{7.3}$$

Let us observe that the "mass" M depends here on spin-field coupling. Our clue for generalizing this model has been the formula put forward by Duval [49, 50]: let us posit, instead of (7.3), the mass formula

$$M^2 = m^2 + \frac{g}{2}\frac{e}{c^2}F \cdot S,$$

where g is an arbitrary real constant. Then *consistent equations of motion* are obtained for any g, namely

$$D\frac{dx^\alpha}{d\tau} = G\frac{p^\alpha}{M} + (g-2)\frac{es}{4M^2}\epsilon^{\alpha\beta\gamma}F_{\beta\gamma}, \tag{7.4}$$

$$\frac{dp^\alpha}{d\tau} = \frac{e}{m}F^{\alpha\beta}p_\beta, \tag{7.5}$$

where the coefficients denote the complicated, field-dependent expressions

$$D = 1 + \frac{eF\cdot S}{2M^2c^2}, \qquad G = 1 + \frac{g}{2}\frac{eF\cdot S}{2M^2c^2}.$$

Choosing $g = 2$, the generalized model plainly reduces to equation (7.1), proposed in [45]. We stress, however, that *no physical principle requires that the second, "anomalous" term should vanish* in (7.4). $g = 2$ is *not* a physical necessity therefore: a perfectly consistent model is obtained for any g, as it has been advocated a long time ago [49, 50].

Non-relativistic anyon with anomalous coupling. We can now consider the "Jackiw–Nair" non-relativistic limit of the above relativistic model. This provides us, for any g, with the Lorentz equation (7.5), supplemented with

$$(M_gD)\dot{x}_i = Gp_i - \left(1 - \frac{g}{2}\right)eM_g\theta\epsilon_{ij}E_j,$$

where

$$M_g = m(\sqrt{1 - g\theta eB}), \qquad D = \big(1 - (g+1)\theta eB\big), \qquad G = \big(1 - (3g/2))\theta eB\big).$$

• It is a most important fact that, for *any* $g \neq 2$, the only consistent motions follow a generalized Hall law, whenever the field takes *either* of the critical values

$$B = \frac{1}{1+g}\frac{1}{e\theta} \qquad \text{or} \qquad \frac{2}{3g}\frac{1}{e\theta}.$$

One can indeed show that, for any $g \neq 2$, the models can be transformed into each other by a suitable redefinition. For $g = 0$ the equations become identically satisfied. See [48] for details.

• In particular, for $g = 0$ the minimal exotic model of [9] is recovered. The latter is, hence, *not* the NR limit of the model of [45] (7.1) [which has $g = 2$, as said]. *The experimental evidence* [47] *is, hence, a strong argument in favor of the minimal model of* [9].

• $g = 2$ is the only case when the velocity and the momentum are parallel. This is, however, *not* required by any first principle.

Having an anomalous velocity relation seems to be unusual in high-energy physics; it is, however, a well accepted requirement in condensed matter physics, as explained in Section 11.

Let us mention that relativistic anyons can be described, at the field theoretical level, by infinite-component fields of the Majorana–Dirac type [18]. Coupling them to an external gauge field is a major unsolved problem. Partial results can be obtained in the non-relativistic case [51].

8 Two ways of introducing electromagnetic interactions

In this section we will show that Souriau's coupling prescription (2.2) is not the only possibility to introduce electromagnetic (e.m.) interaction into the Lagrangian $\mathcal{L}_{\text{ext}}$ (3.7).

In the commutative case we have the principle of minimal e.m. coupling

$$p_i\dot{X}_i - \frac{p_i^2}{2m} \to (p_i + eA_i(\vec{X},t))\dot{X}_i - \frac{p_i^2}{2m} + eA_0(\vec{X},t),$$

called the minimal additon rule, which is equivalent, due to the point transformation $p_i \to p_i - eA_i$, to the minimal substitution rule [52],

$$p_i\dot{X}_i - \frac{p_i^2}{2m} \to p_i\dot{X}_i - \frac{(p_i - eA_i)^2}{2m} + eA_0(\vec{X}, t).$$

In the noncommutative case the equivalence of minimal addition and minimal substitution rule is not valid. Therefore we have to consider two different ways of introducing the minimal e.m. coupling:

Minimal addition (Duval–Horvathy [9], called *DH-model*)

$$\mathcal{L} \to \mathcal{L}_{\text{e.m.}} = \mathcal{L} + e(A_i\dot{X}_i + A_0), \tag{8.1}$$

which, as usual, is quasi-invariant w.r.t. standard gauge transformations

$$A_\mu(\vec{X}, t) \to A_\mu(\vec{X}, t) + \partial_\mu\Lambda(\vec{X}, t).$$

Obviously the minimal addition rule (8.1) is equivalent to Souriau's prescription (2.2).

Minimal substitution (Lukierski–Stichel–Zakrzewski [52], called *L.S.Z. model*)[4]

$$H = \frac{p_i^2}{2m} \to H_{\text{e.m.}} = \frac{(p_i - e\hat{A}_i)^2}{2m} - e\hat{A}_0.$$

The corresponding Lagrangian is quasi-invariant w.r.t. generalized gauge transformations, given in infinitesimal form by

$$\delta\hat{A}_\mu(\vec{X}, t) = \hat{A}'_\mu(\vec{X} + \delta\vec{X}, t) - \hat{A}_\mu(\vec{X}, t) = \partial_\mu\Lambda(\vec{X}, t),$$

with

$$\delta X_i = -e\theta\epsilon_{ij}\partial_j\Lambda \tag{8.2}$$

and supplemented by

$$\delta p_i = e\partial_i\Lambda.$$

Note that the coordinate transformations (8.2) are area preserving.

It turns out that both models are related to each other by a noncanonical transformation of phase space variables supplemented by a classical Seiberg–Witten transformation of the corresponding gauge potentials:

If we denote the phase space variables and potentials for

– the DH-model by $(\vec{\eta}, \vec{\mathcal{P}}, A_\mu)$,

– the L.S.Z.-model by $(\vec{X}, \vec{p}, \hat{A}_\mu)$,

then we find the relations

$$\eta_i(\vec{X}, t) = X_i + e\theta\epsilon_{ij}\hat{A}_j(\vec{X}, t),$$
$$\mathcal{P}_i = p_i - e\hat{A}_i(\vec{X}, t)$$

with the corresponding field strengths related by

$$\hat{F}_{\mu\nu}(\vec{X}, t) = \frac{F_{\mu\nu}(\vec{\eta}, t)}{1 - e\theta B(\vec{\eta}, t)}. \tag{8.3}$$

[4] In this model the gauge fields carry a "hat" in order to distinguish them from the corresponding quantities in the DH-model.

The Seiberg–Witten transformation between the resp. gauge fields is more involved, and will not be reproduced here (for details cp. [52]).

These results lead to an interesting by-product: Consider the PBs of coordinates in both models, given by

$$\{\eta_i,\eta_j\} = \frac{\theta\,\epsilon_{ij}}{1-e\theta B(\vec{\eta},t)} \qquad \text{and} \qquad \{X_i,X_j\} = \theta\,\epsilon_{ij}. \tag{8.4}$$

Then the foregoing results implicitly give the coordinate transformation between a model with a constant noncommutativity parameter $m^2\theta$ and one with arbitrary coordinate-dependent non-commutativity function $m^2\theta(\vec{X},t)$ (this result has been rediscovered in [53]).

Now the question arises, which of both models has to be used for physical applications? Let us look at one example, the Quantum Hall effect. As already shown in Section 2 in the case of the DH-model [9] the Hall law,

$$\dot{X}_i = \epsilon_{ij}\frac{E_j}{B}, \tag{8.5}$$

is valid at the critical magnetic field

$$B_{\rm crit} = \big(e\theta\big)^{-1}.$$

Then it follows from the field transformation law (8.3) that, for the L.S.Z.-model, the Hall law is valid in the limit of large e.m. fields. In order to see this in more detail we have to consider the equations of motion for the L.S.Z.-model formulated in terms of the gauge-invariant phase space variables $\vec{\eta}$ and $\vec{\mathcal{P}}$. For that, we use the equations of motion (2.3) for the DH model written in terms of η_i and $\mathcal{P}_i$, transform the e.m. fields according to (8.3) and we obtain ($e=1$, $m=1$)

$$\begin{aligned} \dot{\eta}_i &= (1+\theta\hat{B})\mathcal{P}_i - \theta\epsilon_{ij}\hat{E}_j,\\ \dot{\mathcal{P}}_i &= \hat{B}\epsilon_{ij}\mathcal{P}_j + \hat{E}_i. \end{aligned} \tag{8.6}$$

For the particular case of homogeneous e.m. fields we obtain finally

$$\ddot{\eta}_i = \hat{B}\epsilon_{ij}\dot{\eta}_j + \hat{E}_i \tag{8.7}$$

leading to the Hall law (8.5) in the high field limit.

Note that (8.7) has the same functional form as in the commutative case.

Another point of view is presented in [54].

9 Supersymmetry

In the following, we supersymmetrize the e.m. coupling models treated in the last section. To do that we follow the treatment in Section 3 of [55]. For that, we consider standard $N=2$ SUSY characterized by

$$H = \frac{i}{2}\{Q,\bar{Q}\} \tag{9.1}$$

and

$$\{Q,Q\} = \{\bar{Q},\bar{Q}\} = 0. \tag{9.2}$$

In order to construct the supercharge Q, satisfying (9.1), we start with the common structure of the bosonic Hamiltonian H_b for both models ($e=1$, $m=1$)

$$H_b = \frac{1}{2}\big(\mathcal{P}_i^2 + W_i^2(\vec{X})\big) \tag{9.3}$$

with

$$\mathcal{P}_i = p_i \qquad \text{for the DH-model}$$

and

$$\mathcal{P}_i = p_i - A_i \qquad \text{for the L.S.Z.-model.}$$

Note that, in accordance with the quantized form of (9.1), the potential term in (9.3) is chosen to be positive

$$A_0 = -\frac{1}{2} W_i^2. \tag{9.4}$$

In order to add to (9.3) its fermionic superpartner, we supplement the bosonic phase space variables with fermionic coordinates $\psi_i(\bar{\psi}_i)$ satisfying canonical PBs

$$\{\psi_i, \bar{\psi}_j\} = -i\delta_{ij}.$$

Now we assume

$$Q = i(\mathcal{P}_i + iW_i)\psi_i$$

such that (9.3) is valid. But now the relations (9.2) are fulfilled only if the following two conditions are satisfied:

$$\{\mathcal{P}_i, \mathcal{P}_j\} = \{W_i, W_j\} \tag{9.5}$$

and

$$\{\mathcal{P}_i, W_j\} = \{\mathcal{P}_j, W_i\}. \tag{9.6}$$

It can be shown that (9.6) is satisfied automatically in both models, whereas (9.5) fixes the magnetic field in terms of W_i (same form for both models):

$$B = \frac{\theta}{2} \epsilon_{ij} \epsilon_{k\ell} \partial_k W_i \partial_\ell W_j. \tag{9.7}$$

The connection between B-field (9.7) and electric potential A_0 (9.4) takes a simple form in the case of rotational invariance. From

$$W_i(\vec{X}) = \partial_i W(r)$$

we obtain

$$A_0(r) = -\frac{1}{2}(W'(r))^2$$

and

$$B(r) = -\frac{\theta}{r} A_0'(r).$$

As an example, consider the harmonic oscillator. Then

$$A_0 = -\frac{\omega^2}{2} r^2$$

and we obtain a homogeneous B-field of strength

$$B = \theta\omega^2.$$

The supersymmetric extension of the DH model, and of anyons, have also been studied in [56] and in [57], respectively.

10 Galilean symmetry in Moyal field theory

As we mentioned already, the physical explanation of the Fractional Quantum Hall Effect (FQHE) relies on the dynamics of quasiparticles which carry both an electric and a magnetic charge [33]. In the field theory context, these quasiparticles arise as charged vortex solutions of the coupled field equations. The phenomenologically preferred theory of Zhang et al. [58] is Galilei invariant; the Galilean boost commute for these models, though. Does there exist a field theoretical model with "exotic" Galilean symmetry? The answer is yes, if we consider Moyal field theory [14, 59]. Here one considers the usually-looking Lagrangian

$$L = i\bar{\psi}D_t\psi - \frac{1}{2}\big|\vec{D}\psi\big|^2 + \kappa\left(\frac{1}{2}\epsilon_{ij}\partial_t A_i A_j + A_t B\right)$$

but where the covariant derivative and the field strength,

$$\begin{aligned} &D_\mu\psi = \partial_\mu\psi - ieA_\mu \star \psi, \\ &F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie\big(A_\mu \star A_\nu - A_\nu \star A_\mu\big), \end{aligned}$$

respectively, involve the Moyal "star" product, associated with the parameter θ,

$$\big(f \star g\big)(x_1, x_2) = \exp\left(i\frac{\theta}{2}\big(\partial_{x_1}\partial_{y_2} - \partial_{x_2}\partial_{y_1}\big)\right) f(x_1, x_2)g(y_1, y_2)\Big|_{\vec{x}=\vec{y}}.$$

Here the matter field ψ is in the fundamental representation of the gauge group $U(1)_*$ i.e., A_μ acts from the left. The associated field equations look formally as in the commutative case,

$$\begin{aligned} &iD_t\psi + \frac{1}{2}\vec{D}^2\psi = 0, \\ &\kappa E_i - e\epsilon_{ik}{j^l}_k = 0, \\ &\kappa B + e\rho^l = 0, \end{aligned} \tag{10.1}$$

where $B = \epsilon_{ij}F_{ij}$, $E_i = F_{i0}$. Note, however, that ρ^l and $\vec{j}^{\,l}$ denote here the *left density* and *left current*, respectively,

$$\rho^l = \psi \star \bar{\psi}, \qquad \vec{j}^{\,l} = \frac{1}{2i}\left(\vec{D}\psi \star \bar{\psi} - \psi \star (\overline{\vec{D}\psi})\right).$$

These theories admit static, finite-energy vortex solutions [59] which generalize those found before in ordinary CS theory [58, 60].

Are these theories Galilean invariant? At first sight, the answer seems to be negative, and it has been indeed a widely shared view that Moyal field theory is inconsistent with Galilean symmetry. The situation is more subtle, however. The conventional infinitesimal implementation of a Galilean boost,

$$\delta^0 B = -t\vec{b}\cdot\vec{\nabla}B \qquad \text{but} \qquad \delta^0\rho^l = -\frac{\theta}{2}\vec{b}\times\vec{\nabla}\rho^l - \vec{b}\cdot\vec{\nabla}\rho^l.$$

is indeed broken, as the Gauss constraint (10.1) is not preserved. Galilean symmetry can be restored taking into account the Moyal structure [21], namely considering the *antifundamental representation*

$$\delta^r\psi = \psi \star (i\vec{b}\cdot\vec{x}) - t\vec{b}\cdot\vec{\nabla}\psi = (i\vec{b}\cdot\vec{x})\psi + \frac{\theta}{2}\vec{b}\times\vec{\nabla}\psi - t\vec{b}\cdot\vec{\nabla}\psi.$$

Observing that

$$\delta^r\psi = \delta^0\psi + \frac{\theta}{2}\vec{b}\times\vec{\nabla}\psi$$

we find that the θ-terms cancel in $\delta^r\rho^l$, leaving us with the homogeneous transformation law

$$\delta^r\rho^l = -t\vec{b}\cdot\vec{\nabla}\rho^l.$$

Putting $\delta^r A_\mu = \delta^0 A_\mu$, so that $\delta^r B = \delta^0 B$, the Gauss constraint (10.1) is right-invariant, as are all the remaining equations. The associated boost generator, calculated using the Noether theorem, reads

$$\vec{K}^r = t\vec{P} - \int \vec{x}\rho^r d^2\vec{x}, \tag{10.2}$$

where

$$P_i = \int \frac{1}{2i}\big(\bar\psi\partial_i\psi - (\overline{\partial_i\psi})\psi\big)d^2\vec{x} - \frac{\kappa}{2}\int \epsilon_{jk}A_k\partial_i A_j d^2\vec{x}$$

is the conserved momentum. The conservation of (10.2) can also be checked directly, using the continuity equation satisfied by the right density, $\rho^r = \bar\psi\star\psi$. At last, the boost components have the exotic commutation relation

$$\big\{K_i, K_j\big\} = \epsilon_{ij}k, \qquad k \equiv -\theta\int|\psi|^2 d^2x.$$

Let us note, in conclusion, that Galilean symmetry as established here makes it possible to produce moving vortices by boosting the static solutions constructed in [59], see [61].

11 Noncommutativity in 3 dimensions: the semiclassical Bloch electron

11.1 The semiclassical model

Around the same time and with no relation to the above developments, a very similar theory has arisen in condensed matter physics. For instance, applying a Berry-phase argument to a Bloch electron in a lattice, the standard semiclassical equations [62] are modified by new terms [22], generating purely quantum effects on the mean values of the electron's position and quasi-momentum $\vec{r}$ and $\vec{p}$, respectively, which add to the force due to the momentum gradient of the energy band dispersion relation $\epsilon_n(\vec{p})$ and to the external (for instance, Lorentz) forces. The semiclassical approach allows several applications and generalizations, both from the physical [22, 23, 24, 25, 13], and the mathematical [29, 63, 64, 65] side.

The clue is that the semiclassical model fits perfectly into Souriau's general framework [3] presented above. One starts with a "microscopic" Hamiltonian operator $\hat{H}\big[\hat{\vec{r}},\hat{\vec{p}},f(\hat{\vec{r}},t)\big]$ for a particle (electron) for a periodic potential, which is adiabatically (in space-time) modified by a perturbation f (possibly an external field). The position/momentum operators $\hat{\vec{r}}$ and $\hat{\vec{p}}$ satisfy the Heisenberg algebra, as usual. Moreover, for any constant f, the Hamiltonian $\hat{H}$ reduces to the usual one for a periodic crystal lattice.

The adiabatic features of the perturbation f are expressed by the inequalities $l_{\rm latt} \ll l_{wp} \ll l_{\rm mod}$, among the lattice constant length $l_{\rm latt}$, the wave-packet dispersion length l_{wp} and the modulation wave-length $l_{\rm mod}$. Furthermore, the characteristic time scale $\hbar/\Delta E_{\rm gap}$ must be much smaller than the typical time-scale of variations of f.

The first order truncation of the Hamiltonian around the instantaneous mean position $\vec{r}_c$,

$$\hat{H}\left[\hat{\vec{r}}, \hat{\vec{p}}, f(\hat{\vec{r}}, t)\right] = \hat{H}_{(\vec{r}_c,t)} + \hat{W}_{(\vec{r}_c,t)},$$
$$\hat{W}_{(\vec{r}_c,t)} = \frac{1}{2}\left[\partial_f \hat{H}\, \nabla_{\vec{r}_c} f(\vec{r}_c, t) \cdot (\hat{\vec{r}} - \vec{r}_c) + \text{h.c.}\right],$$

defines a quasi-static Hamiltonian $\hat{H}_{(\vec{r}_c,t)}$, depending on the "slow" parameters $c = (\vec{r}_c, t)$. $\hat{H}_{(\vec{r}_c,t)}$ is periodic under $\vec{a}-$translations, and its eigenstates are Bloch the waves. The latter are defined, for any fixed time t and $\vec{r}_c$, by

$$\hat{H}_{(\vec{r}_c,t)} |\psi^{n,\vec{q}}_{(\vec{r}_c,t)}\rangle = E^{n,\vec{q}}_{(\vec{r}_c,t)} |\psi^{n,\vec{q}}_{(\vec{r}_c,t)}\rangle, \qquad \langle \psi^{n,\vec{q}}_{(\vec{r}_c,t)} | \psi^{n',\vec{q}'}_{(\vec{r}_c,t)}\rangle = \delta_{n,n'}\delta\left(\vec{q}-\vec{q}'\right),$$
$$\langle \vec{r} | \psi^{n,\vec{q}}_{(\vec{r}_c,t)}\rangle = e^{i\vec{q}\cdot\vec{r}} u^{n,\vec{q}}_{(\vec{r}_c,t)}\left(\vec{r}\right), \qquad u^{n,\vec{q}}_{(\vec{r}_c,t)}\left(\vec{r}+\vec{a}\right) = u^{n,\vec{q}}_{(\vec{r}_c,t)}\left(\vec{r}\right),$$

where the energy eigenvalues $E^{n,\vec{q}}_{(\vec{r}_c,t)}$ are labeled by the band index, n, and by the quasi-momentum $\vec{q}$, restricted to the first Brillouin zone (IBZ). We assume that the time evolution of $\vec{r}_c$ closely follows the one obtained by the exact integration of the Schrödinger equation and that the eigenvalues $E^{n,\vec{q}}_{(\vec{r}_c,t)}$ form well separated bands, and that band jumping is forbidden. The label n will be dropped in what follows.

A classical result by Karplus and Luttinger [66] says that

$$\langle \psi^{\vec{q}}_{(\vec{r}_c,t)} |\, \hat{\vec{r}}\, | \psi^{\vec{q}'}_{(\vec{r}_c,t)}\rangle = \left[i\nabla_{\vec{q}} + \langle u^{\vec{q}}_{(\vec{r}_c,t)}\left(\vec{r}\right) | i\nabla_{\vec{q}}\, u^{\vec{q}}_{(\vec{r}_c,t)}\left(\vec{r}\right)\rangle_{\text{cell}}\right]\delta\left(\vec{q}'-\vec{q}\right). \tag{11.1}$$

That is, the momentum representation of $\hat{\vec{r}}$ is

$$\hat{\vec{r}} = i\nabla_{\vec{q}} + \vec{\mathcal{A}}\left(\vec{r}_c, \vec{q}, t\right), \qquad \vec{\mathcal{A}} = \langle u^{\vec{q}}_{(\vec{r}_c,t)} | i\nabla_{\vec{q}}\, u^{\vec{q}}_{(\vec{r}_c,t)}\rangle_{\text{cell}},$$

where $\langle\cdot|\cdot\rangle_{\text{cell}}$ is the restriction of the scalar product to the unit cell with periodic boundary conditions, and with normalization factor $(2\pi)^3/V_{\text{cell}}$. Then the quantity $\vec{\mathcal{A}}(\vec{r}_c, \vec{q}, t)$ is interpreted as a $U(1)$ *Berry connection*, whose curvature appears in the commutation relations for the position operator components,

$$[\hat{r}_j, \hat{r}_l] = i\,\epsilon_{jl}\,\partial_{q_j}\mathcal{A}_l(\vec{r}_c, \vec{q}, t) = \Theta_{jl}\left(\vec{r}_c, \vec{q}, t\right), \tag{11.2}$$

which converts the dynamics of an ordinary particle in a periodic background potential into a quantum mechanical system in a *non-commutative configuration space* [14]. The antisymmetric tensor $\mathbf{\Theta} = (\Theta_{ij})$ generalizes in fact the scalar parameter θ of the planar non-commutative theory.

Its effects cannot be disregarded for the semiclassical motion of a wave-packet

$$|\widetilde{\Psi}\left[\vec{r}_c\left(t\right), \vec{q}_c\left(t\right)\right]\rangle = \int_{\text{IBZ}} \Phi\left(\vec{q}, t\right) |\psi^{\vec{q}}_{(\vec{r}_c,t)}\rangle\, d\vec{q},$$

built by superimposing one-band Bloch waves with a normalized amplitude $\Phi\left(\vec{q}, t\right)$. In fact, under the assumptions of small momentum dispersion, $\Delta_q \ll 2\pi/l_{\text{latt}}$, it can be proved that the mean packet-position is

$$\vec{r}_c\left(t\right) = \langle\widetilde{\Psi}|\,\hat{\vec{r}}\,|\widetilde{\Psi}\rangle \approx -\nabla_{\vec{q}_c} \arg\left[\Phi\left(\vec{q}_c, t\right)\right] + \vec{\mathcal{A}}\left(\vec{r}_c, \vec{q}_c, t\right),$$

where the mean quasi-momentum,

$$\vec{q}_c\left(t\right) = \int_{IBZ} \vec{q}\, |\Phi\left(\vec{q}, t\right)|^2\, d\vec{q}, \tag{11.3}$$

has been introduced. Then, the semiclassical description of the wave-packet is reduced to that of a particle – like system in the $(\vec{r}_c, \vec{q}_c)$ "phase space", the dynamics of which is obtained by minimizing the Schrödinger field action

$$S = \int_{t_1}^{t_2} \left\{ \frac{i}{2} \frac{\langle \Psi | \frac{d\Psi}{dt} \rangle - \langle \frac{d\Psi}{dt} | \Psi \rangle}{\langle \Psi | \Psi \rangle} - \frac{\langle \Psi | \hat{H} | \Psi \rangle}{\langle \Psi | \Psi \rangle} \right\} dt,$$

where $(\vec{r}_c(t), \vec{q}_c(t))$ parametrize the wave-function [67]. This leads to an "approximate Lagrangian" for a point-like classical particle of the form (4.1), namely to

$$\begin{aligned} L_{\rm app} = \dot{\vec{r}}_c \cdot \left(\vec{q}_c + \overrightarrow{\mathcal{R}}\,(\vec{r}_c, \vec{q}_c, t) \right) + \dot{\vec{q}}_c \cdot \vec{\mathcal{A}}\,(\vec{r}_c, \vec{q}_c, t) + \mathcal{T}\,(\vec{r}_c, \vec{q}_c, t) \\ - \mathcal{E}\,(\vec{r}_c, \vec{q}_c, t) - \Delta\mathcal{E}\,(\vec{r}_c, \vec{q}_c, t)\,, \end{aligned} \tag{11.4}$$

where

$$\begin{aligned} &\mathcal{T}\,(\vec{r}_c, \vec{q}_c, t) = \langle u^{\vec{q}_c}_{(\vec{r}_c,t)} | i\partial_t\, u^{\vec{q}_c}_{(\vec{r}_c,t)} \rangle_{\rm cell}, \\ &\overrightarrow{\mathcal{R}}\,(\vec{r}_c, \vec{q}_c, t) = \langle u^{\vec{q}_c}_{(\vec{r}_c,t)} | i\nabla_{\vec{r}_c}\, u^{\vec{q}_c}_{(\vec{r}_c,t)} \rangle_{\rm cell}, \\ &\mathcal{E} = \langle \widetilde{\Psi} | \hat{H}_{(\vec{r}_c,t)} | \widetilde{\Psi} \rangle, \qquad \Delta\mathcal{E} = \langle \widetilde{\Psi} | \hat{W}_{(\vec{r}_c,t)} | \widetilde{\Psi} \rangle. \end{aligned}$$

Together with $\vec{\mathcal{A}}$, the scalar $\mathcal{T}$ and the vector field $\overrightarrow{\mathcal{R}}$ provide us with the complete Berry connection on the entire "environmental parameter space" $(\vec{r}_c, \vec{q}_c, t)$. The quantity $\mathcal{E}$ expresses the potential energy felt by the wave packet in the periodic environment and $\Delta\mathcal{E}$ comes from the adiabatic perturbations.

For slowly changing electromagnetic potentials $\left(\vec{A}\,(\vec{r}, t)\,, V_{el}\,(\vec{r}, t)\,\right)$, the rather involved expressions above take an elegant form [22], – but the Bloch eigenfunctions get a gauge-dependent phase modification $\approx e\vec{A}(\vec{r}_c, t) \cdot \vec{r}$. In fact, a change of phase has no influence on the Berry connection because of (11.1), so one can introduce the gauge invariant kinetic momentum

$$\vec{k}_c = \vec{q}_c - e\vec{A}(\vec{r}_c, t)$$

and set again

$$\begin{aligned} &\vec{\mathcal{A}}\,(\vec{r}_c, \vec{q}_c, t) = \vec{\mathcal{A}}\,(\vec{k}_c), \\ &\overrightarrow{\mathcal{R}} \simeq -e\,\nabla_{\vec{r}_c}\big(\vec{A}\,(\vec{r}_c, t) \cdot \vec{r} \big)\Big|_{\vec{r}=\vec{r}_c}, \qquad \mathcal{T} \simeq -e\,\partial_t \vec{A}\,(\vec{r}_c, t) \cdot \vec{r}_c, \\ &\mathcal{E} = \mathcal{E}_0(\vec{k}_c) + e\,V_{el}(\vec{r}_c, t), \qquad \Delta\mathcal{E} = -\vec{M}(\vec{k}_c, t) \cdot \vec{B}(\vec{r}_c, t), \end{aligned}$$

where

$$\vec{M}(\vec{k}_c, t) = -\frac{e}{2m} \langle \widetilde{\Psi} | \hat{\vec{L}} | \widetilde{\Psi} \rangle$$

is the mean magnetic moment of the wave-packet. $\vec{B}(\vec{r}_c, t)$ and $\vec{E}(\vec{r}_c, t)$ are defined as usual from the mean values of the potentials. Dropping the label c, and putting

$$\Theta_i = \tfrac{1}{2}\epsilon_{ijk}\Theta_{jk},$$

the generalized semiclassical equations of motion are

$$\begin{aligned} &\dot{\vec{r}} = \nabla_{\vec{k}}\big[\mathcal{E}_0(\vec{k}) - \vec{M}(\vec{k}, t) \cdot \vec{B}(\vec{r}, t) \big] - \dot{\vec{k}} \times \vec{\Theta}(\vec{k}), \\ &\dot{\vec{k}} = -e\big(\dot{\vec{r}} \times \vec{B}(\vec{r}, t) + \vec{E}(\vec{r}, t) \big) + \nabla_{\vec{r}}\big(\vec{M}(\vec{k}, t) \cdot \vec{B}(\vec{r}, t) \big), \end{aligned} \tag{11.5}$$

further confirming the idea of the non-commutativity parameter, now a function of momentum-space variables, is in fact a Berry phase effect.

Notice, here that the semiclassical procedure has consistently "averaged" on the gauge degrees of freedom at local scales of order $\sim l_{wp}$, but the final model still possesses the same gauge invariant character as a point-like particle interacting with an external e.m. field.

Then, for the electronic wave-packet semiclassically described by (11.5), one can adapt the symplectic techniques described in the previous sections and it can be used for a Hamiltonian formulation.

11.2 Hamiltonian structure

Comparing the system (11.5) with the previous ones in (2.3) or (8.6), one recognizes a general common structure. The nice group-geometrical symmetry properties of the 2D Galilei group, which partially motivated the present research, are broken in general. However, the unifying framework for such differential systems is provided by the same ideology adopted in Sections 2 and 6, i.e. writing them as the kernel of a postulated anti-symmetric, closed, constant-rank Lagrange–Souriau 2-form σ of the form

$$\begin{aligned}\sigma = {} & [(1-Q_i)\,dq_i - eE_i dt] \wedge (dr_i - g_i dt) \\ & + \frac{1}{2} e\epsilon_{ijk} B_k dr_i \wedge dr_j + \frac{1}{2}\epsilon_{ijk}\Theta_k dq_i \wedge dq_j + Q_0 \epsilon_{ij} dr_i \wedge dq_j,\end{aligned} \tag{11.6}$$

where the Souriau's prescription to explicitly include the electromagnetic contributions has been used. The vector fields $\vec{g}$, $\vec{\Theta}$, $\vec{Q}$ and the scalar functions Q_0 may depend on all independent variables $(\vec{r}, \vec{q}, t)$. Notice that (11.6) only contains "forces", i.e. gauge invariant quantities. Moreover, the so called "Maxwell principle" [3], i.e. the closure relation $d\sigma = 0$, implies a set of integrability conditions for functions involved, which reduce to the usual Maxwell equations for $(\vec{E}, \vec{B})$, when the new extra fields are set to constants. Even in this case, and in 2 space dimensions, the resulting equations are non-trivial, coinciding for instance with (2.3) after the identifications $r_i \to x_i$, $g_i \to q_i/m$, $\theta_3 \to \theta$ and $Q_i \equiv 0$.

We note (like in Section 2) that a model defined by the 2-form (11.6) may not possess a globally defined configuration space Lagrangian. This makes the value of the semiclassical Lagrangian (11.4) questionable. If it is assumed valid at least locally, the physical meaning of the coefficients appearing in (11.6) can be deduced, via exterior derivative, from the Cartan 1-form

$$\lambda = \left(\vec{q} + \vec{\mathcal{R}}\right) \cdot d\vec{r} + \vec{\mathcal{A}} \cdot d\vec{q} + (\mathcal{T} - \mathcal{E} - \Delta\mathcal{E})\, dt.$$

Thus, the most general equations of motion deriving from (11.4) (or equivalently from (11.6)) are

$$\begin{aligned}&(1+\boldsymbol{\Xi})\,\dot{\vec{r}} + \Theta\dot{\vec{q}} = \nabla_{\vec{q}}\,[\mathcal{E} + \Delta\mathcal{E} - \mathcal{T}] + \partial_t \vec{\mathcal{A}}, \\ &X\dot{\vec{r}} + (1+\boldsymbol{\Xi})\,\dot{\vec{q}} = -\nabla_{\vec{r}}\,[\mathcal{E} + \Delta\mathcal{E} - \mathcal{T}] - \partial_t \vec{\mathcal{R}},\end{aligned} \tag{11.7}$$

where the antisymmetric matrices $\boldsymbol{\Xi} = (\Xi_{ij})$ and $\mathbf{X} = (X_{ij})$ have elements

$$\boldsymbol{\Xi}_{ij} = \partial_{r_i}\mathcal{A}_j - \partial_{q_j}\mathcal{R}_i, \qquad X_{ij} = \partial_{r_i}\mathcal{R}_j - \partial_{r_j}\mathcal{R}_i. \tag{11.8}$$

The dynamical system (11.7) is defined on the tangent manifold of the configuration space, endowed with generalized coordinates $\vec{\xi} = (\vec{r}, \vec{q})$. But, when $\partial_t \vec{\mathcal{A}} = \partial_t \vec{\mathcal{R}} \equiv 0$, the rearrangement $\sigma = \omega - dH \wedge dt$ of the terms in (11.6) is possible, introducing the symplectic 2-form

$$\omega = (\delta_{i,j} + \Xi_{ij})\, dr_i \wedge dq_j + \frac{1}{2}\left[X_{ij} dq_i \wedge dq_j - \Theta_{ij} dr_i \wedge dr_j\right]$$

and the Hamiltonian function $\mathcal{H} = \mathcal{E} + \Delta\mathcal{E} - \mathcal{T}$.

Actually, the closure of σ implies that, $d\omega = 0$, for ω. Equivalently, the set of differential constraints

$$\begin{aligned} &\varepsilon_{ijk}\partial_{q_i}\Theta_{jk} = 0, && \varepsilon_{ijk}\partial_{r_i}X_{jk} = 0, \\ &\partial_{q_j}\Xi_{ij} = -\partial_{r_j}\Theta_{ij}, && \partial_{r_j}\Xi_{ij} = \partial_{q_j}X_{ij}, \\ &(1-\delta_{hk})\,\varepsilon_{kij}\partial_{q_k}\Xi_{ij} = \varepsilon_{hij}\partial_{r_h}\Theta_{ij}, \qquad && (1-\delta_{hk})\,\varepsilon_{kij}\partial_{r_k}\Xi_{ij} = -\varepsilon_{hij}\partial_{q_h}X_{ij}, \end{aligned} \tag{11.9}$$

which, however, are automatically satisfied, because of the antisymmetry and the differentiability properties of the tensors $\mathbf{\Theta}$, $\mathbf{\Xi}$ and $\mathbf{X}$ defined in (11.2) and (11.8). Thus, for non degenerate $\omega = \omega_{\alpha\beta}d\xi_\alpha \wedge d\xi_\beta$, Poisson brackets,

$$\{f, g\} = \omega^{\alpha\beta}\partial_\alpha f \partial_\beta g$$

can be defined for any pair of functions $f(\vec{\xi})$ and $g(\vec{\xi})$, where $\omega^{\alpha\gamma}\omega_{\gamma\beta} = \delta^\alpha_\beta$ is the inverse of the symplectic matrix [3, 68]. Thus, the equations (11.7) take the usual Hamiltonian form $\dot{\xi}_\alpha = \{\xi_\alpha, \mathcal{H}\}$. In the present case $(\omega_{\alpha\beta})$ is a real symplectic 6×6 matrix, which is non degenerate when

$$\sqrt{\det(\omega_{\alpha\beta})} = 1 - \frac{1}{2}\mathrm{Tr}\left(\mathbf{\Xi}^2 + \mathbf{X}\left(\mathbf{1} + 2\mathbf{\Xi}\right)\mathbf{\Theta}\right) \neq 0.$$

Such a factor generalizes the denominators present in the Poisson brackets (2.5), (4.3) or (8.4). Moreover, it crucially appears in the expression of the invariant phase-space volume, ensuring the validity of the Liouville theorem [69, 64].

As special example, we deal with only momentum (gauge invariant) dependent Berry curvature $\vec{\Theta}\,(\vec{q})$ which is to be divergence-free according to the first equation in (11.9). That condition can be satisfied, except in one point, e.g., by a *monopole in $\vec{q}$-space*,

$$\vec{\Theta} = g\frac{\vec{q}}{q^3}, \tag{11.10}$$

which is indeed the only possibility consistent with the spherical symmetry and the canonical relations $\{x_i, q_j\} = \delta_{ij}$ [65]. The expression (11.10) appears to be consistent, at least qualitatively, with the data reported in [23] and in Spin Hall Effects [24].

In absence of a magnetic field and taking, for simplicity, the energy band $\epsilon_n(\vec{q})$ to be parabolic, the equations (11.5) for become

$$\dot{\vec{r}} = \vec{q} + \frac{eg}{q^3}\vec{E}\times\vec{q}, \qquad \dot{\vec{q}} = -e\vec{E}.$$

The anomalous term shifts the velocity and deviates, hence, the particle's trajectory perpendicularly to the electric field, just like in the anomalous Hall effect, see [23].

A similar pattern arises in optics [25, 26, 27, 28]: to first order in the gradient of the refractive index n, spinning light is approximately described by the equations

$$\dot{\vec{r}} \approx \vec{p} - \frac{s}{\omega}\,\mathrm{grad}\left(\frac{1}{n}\right)\times\vec{p}, \qquad \dot{\vec{p}} \approx -n^3\omega^2\mathrm{grad}\left(\frac{1}{n}\right),$$

where s denotes the photon's spin. In the first relation we recognize, once again, an anomalous velocity relation of the type (11.5). The new term makes the light's trajectory deviate from that predicted in ordinary geometrical optics, giving rise to the "optical Magnus effect" [25]. A manifestation of this is the displacement of the light ray perpendicularly to the plane of incidence at the interface of two media with different refraction index: this is the "Optical Hall Effect" [26, 27, 28].

Another nice illustration is provided by the non-commutative Kepler problem [13]. Choosing the non-commutative vector $\vec{\Theta}$ in the vertical direction,

$$\Theta_i = \theta\delta_{iz}$$

the 3D problem reduces to the "exotic" model presented in Section 2. Then the authors of [13] show that, for the Kepler potential $V \propto r^{-1}$ the perturbation due to non-commutativity induces the precession of the perihelion point of planetary orbit.

As yet another example, we would like to mention the recent work [70], in which it is shown that a particle with "monopole-type" noncommutativity (11.10) admits a conserved Runge–Lenz vector, namely

$$\vec{K} = \vec{r} \times \vec{J} - \alpha\frac{\vec{q}}{q},$$

provided the Hamiltonian is

$$H = \frac{\vec{r}^{\,2}}{2} + \frac{g^2}{2q^2} + \frac{\alpha}{q}.$$

Note that $\vec{q}$ here is the momentum: the "monopole" is in "dual space". Let us observe that this expression is reminiscent of the Chern–Simons mechanics" [32] in that it has no mass term. The associated the equations of motion read,

$$\dot{\vec{r}} = -\left(\frac{g^2}{q^4} + \frac{\alpha}{q^3}\right)\vec{q} + g\frac{\vec{q}\times\vec{r}}{q^3}, \qquad \dot{\vec{q}} = -\vec{r}.$$

The Kepler-type dynamical symmetry then allows one to show that the classical motions follow (arcs of) oblique ellipses [70].

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