Discrete Minimal Surface Algebras*

Joakim ARNLIND † *and Jens HOPPE* ‡

† *Institut des Hautes Études Scientifiques, Le Bois-Marie, 35, Route de Chartres, 91440 Bures-sur-Yvette, France*
E-mail: *arnlind@ihes.fr*

‡ *Eidgenössische Technische Hochschule, 8093 Zürich, Switzerland (on leave of absence from Kungliga Tekniska Högskolan, 100 44 Stockholm, Sweden)*
E-mail: *hoppe@itp.phys.ethz.ch, hoppe@math.kth.se*

Received March 23, 2010, in final form May 18, 2010; Published online May 26, 2010
doi:10.3842/SIGMA.2010.042

Abstract. We consider discrete minimal surface algebras (DMSA) as generalized noncommutative analogues of minimal surfaces in higher dimensional spheres. These algebras appear naturally in membrane theory, where sequences of their representations are used as a regularization. After showing that the defining relations of the algebra are consistent, and that one can compute a basis of the enveloping algebra, we give several explicit examples of DMSAs in terms of subsets of $\mathfrak{sl}_n$ (any semi-simple Lie algebra providing a trivial example by itself). A special class of DMSAs are Yang–Mills algebras. The representation graph is introduced to study representations of DMSAs of dimension $d \leq 4$, and properties of representations are related to properties of graphs. The representation graph of a tensor product is (generically) the Cartesian product of the corresponding graphs. We provide explicit examples of irreducible representations and, for coinciding eigenvalues, classify all the unitary representations of the corresponding algebras.

Key words: noncommutative surface; minimal surface; discrete Laplace operator; graph representation; matrix regularization; membrane theory; Yang–Mills algebra

2010 Mathematics Subject Classification: 81R10; 06B15

1 Introduction

Noncommutative analogues of manifolds have been studied in many different contexts. One way of constructing such objects is to relate the Poisson structure of a manifold M to the commutator structure of a sequence of matrix algebras $\mathcal{A}_\hbar$ (parametrized by $\hbar > 0$), where the dimension of the matrices increases as $\hbar \to 0$. Namely, for some set of values of the parameter $\hbar$, one defines a map $T^\hbar : C^\infty(M) \to \mathcal{A}_\hbar$ such that

$$\lim_{\hbar\to 0}\left|\left|T^\hbar\big(\{f,g\}\big) - \frac{1}{i\hbar}\big[T^\hbar(f), T^\hbar(g)\big]\right|\right| = 0 \tag{1}$$

for all $f, g \in C^\infty(M)$.

For surfaces, the map $T^\hbar$ has been constructed in several different ways. One rather concrete approach is to consider the surface Σ as embedded in an ambient manifold M with embedding coordinates $x_1(\sigma_1,\sigma_2), \dots, x_d(\sigma_1,\sigma_2)$. If the Poisson brackets satisfy

$$\{x_i, x_j\} = p_{ij}(x_1,\dots,x_d)$$

*This paper is a contribution to the Special Issue "Noncommutative Spaces and Fields". The full collection is available at http://www.emis.de/journals/SIGMA/noncommutative.html where $p_{ij}(x_1,\dots,x_d)$ are polynomials, then one defines an algebra of non-commuting variables $X_1,\dots,X_d$ such that the following relations hold

$$\big[X_i,X_j\big]=i\hbar\Psi(p_{ij})(X_1,\dots,X_d)$$

where Ψ is an ordering map, mapping commutative polynomials to non-commutative ones (such that when composed with the projection back to commutative polynomials one gets the identity map). Thus, for any pair of polynomials p, q it holds that

$$\Psi(\{p,q\})-\frac{1}{i\hbar}[\Psi(p),\Psi(q)]=O(\hbar),$$

and by considering representations of the above defined algebra one obtains a sequence of matrix algebras and maps $T^\hbar$ such that relation (1) holds for all polynomial functions (see [1] for details). It is natural to demand that a notion of noncommutative analogues of manifolds should have some features that can be traced back to the geometry of the original manifold. For surfaces, the genus is the obvious invariant, and one can show that the above procedure gives rise to algebras whose representation theory encodes geometric data [3].

For a surface Σ embedded in $\mathbb{R}^d$, the Laplace–Beltrami operator on Σ acting on the embedding coordinates $x_1(\sigma_1,\sigma_2),\dots,x_d(\sigma_1,\sigma_2)$ can be written as

$$\Delta(x_i)=\sum_{j=1}^{d}\left\{\left\{x_i,x_j\right\},x_j\right\}$$

where $\left\{f,h\right\}(\sigma_1,\sigma_2)=\frac{1}{\sqrt{g}}(\partial_1 f\partial_2 h-\partial_1 h\partial_2 f)$ and g is the determinant of the induced metric on Σ. With this notation, minimal surfaces in S^{d-1} can be found by constructing embedding coordinates such that

$$\sum_{j=1}^{d}\left\{\left\{x_i,x_j\right\},x_j\right\}=-2x_i$$

subject to the constraint $\sum_{j=1}^d x_j^2=1$. In the above spirit of replacing Poisson brackets by commutators, corresponding noncommutative minimal surfaces are defined by the relations

$$\hbar^2\sum_{j=1}^{d}\big[[X_i,X_j],X_j\big]=2X_i. \tag{2}$$

Another example where equations like (2) arise is in the context of a physical theory of "Membranes" [12]. The equations of motion for a membrane moving in $d+1$ dimensional Minkowski space (with a particular choice of coordinates) can be written as

$$\partial_t^2 x_i=\sum_{j=1}^{d}\left\{\left\{x_i,x_j\right\},x_j\right\}\qquad\text{and}\qquad\sum_{j=1}^{d}\left\{\partial_t x_j,x_j\right\}=0.$$

A regularized theory is given by d time-dependent Hermitian matrices X_i satisfying the equations

$$\partial_t^2 X_i=-\hbar^2\sum_{j=1}^{d}\big[[X_i,X_j],X_j\big]\quad\text{and}\quad\sum_{j=1}^{d}[\partial_t X_i,X_j]=0. \tag{3}$$

In the first equation, we can separate time from the matrices by making the ansatz

$$X_i(t)=\sum_{j=1}^{d}\frac{a}{\hbar}\big(e^{A(at+b)}\big)_{ij}M_j, \tag{4}$$

(see also [14, 4]) where A is a $d \times d$ antisymmetric matrix such that $A^2 = \mathrm{diag}(-\mu_1, \ldots, -\mu_d)$ with $\mu_{2i-1} = \mu_{2i}$ for $i = 1, 2, \ldots, \lfloor d/2 \rfloor$. Then X_i (as defined in (4)) solves the first equation in (3) provided

$$\sum_{j=1}^{d} \big[[M_i, M_j], M_j\big] = \mu_i M_i. \tag{5}$$

Motivated by the above examples, we set out to study algebras generated by relations (5) (for *arbitrary* μ_i's). In particular, we shall study their representation theory for $d \leq 4$.

In Section 2 we introduce Discrete Minimal Surface Algebras, and after showing that a basis of the enveloping algebra can be computed via the Diamond lemma, some properties are investigated and several examples are given. Section 3 deals with Hermitian representations of the first non-trivial algebras, for which the representation graph is introduced, and properties of representations are related to properties of graphs. Finally, we provide explicit examples of irreducible representations and their corresponding graphs and, in the case when $\mathrm{Spec}(\mathcal{A}) = \{\mu\}$, we classify all unitary representations.

2 Discrete minimal surface algebras

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. For any finite subset $\mathcal{X} = \{x_1, x_2, \ldots, x_d\} \subseteq \mathfrak{g}$ we define a linear map $\Delta_{\mathcal{X}} : \mathfrak{g} \to \mathfrak{g}$ by

$$\Delta_{\mathcal{X}}(a) = \sum_{j=1}^{d} \big[[a, x_j], x_j\big],$$

and let $\langle \mathcal{X} \rangle$ denote the vector space spanned by the elements in $\mathcal{X}$.

Definition 1 (DMSA). Let $\mathfrak{g}$ be a Lie algebra and let $\mathcal{X} = \{x_1, x_2, \ldots, x_d\}$ be a set of linearly independent elements of $\mathfrak{g}$. We call $\mathcal{A} = (\mathfrak{g}, \mathcal{X})$ a *discrete minimal surface algebra* (DMSA) (of dimension d) if there exist complex numbers $\mu_1, \ldots, \mu_d$ such that $\Delta_{\mathcal{X}}(x_i) = \mu_i x_i$ for $i = 1, \ldots, d$. The set $\{\mu_1, \mu_2, \ldots, \mu_d\}$ is called *the spectrum of* $\mathcal{A}$ and is denoted by $\mathrm{Spec}(\mathcal{A})$.

Definition 2. Two DMSAs $\mathcal{A} = (\mathfrak{g}, \mathcal{X})$ and $\mathcal{B} = (\mathfrak{g}', \mathcal{Y})$ are *isomorphic* if there exists a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ such that $\phi|_{\langle \mathcal{X} \rangle}$ is a vector space isomorphism of $\langle \mathcal{X} \rangle$ onto $\langle \mathcal{Y} \rangle$.

Note that $\mathrm{Spec}(\mathcal{A})$ is not an invariant within an isomorphism class. Let $\mathfrak{g}$ be a Lie algebra with structure constants f_{ij}^k (relative to the basis $x_1, \ldots, x_n$) and let $\mathfrak{g}'$ be a Lie algebra with structure constants $c f_{ij}^k$ (relative to the basis $y_1, \ldots, y_n$), for some non-zero complex number c. Then the two DMSAs $\mathcal{A} = (\mathfrak{g}, \mathcal{X} = \{x_1, \ldots, x_d\})$ and $\mathcal{B} = (\mathfrak{g}', \mathcal{Y} = \{y_1, \ldots, y_d\})$ will be isomorphic (through $\phi(x_i) = y_i/c$), and if $\mathrm{Spec}(\mathcal{A}) = \{\mu_1, \ldots, \mu_d\}$ then $\mathrm{Spec}(\mathcal{B}) = \{c^2\mu_1, \ldots, c^2\mu_d\}$.

Definition 3. Let $\mathcal{A} = (\mathfrak{g}, \mathcal{X})$ be a DMSA with $\mathrm{Spec}(\mathcal{A}) = \{\mu_1, \ldots, \mu_d\}$ and let $\mathbb{C}\langle \mathcal{X} \rangle$ denote the free associative algebra over $\mathbb{C}$, generated by the set $\mathcal{X} = \{x_1, x_2, \ldots, x_d\}$. The enveloping algebra of $\mathcal{A}$ is defined as the quotient $\mathfrak{U}_d(\mathcal{A}) = \mathbb{C}\langle \mathcal{X} \rangle / \langle \Delta_{\mathcal{X}}(x_i) - \mu_i x_i \rangle$, where $[a, b] = ab - ba$.

Remark 1. Relating the above enveloping algebra to minimal surfaces in S^{d-1}, one could impose the relation $x_1^2 + x_2^2 + \cdots + x_d^2 = \nu \mathbb{1}$ together with $\Delta_{\mathcal{X}}(x_i) = \mu x_i$. Combining these relations gives the equations

$$\sum_{j=1}^{d} x_j x_i x_j = \frac{1}{2}(2\nu - \mu) x_i$$

for $i = 1, \ldots, d$.

The following result shows how to compute a basis of the enveloping algebra. Note that in [5], it was proven that a class of so called inhomogeneous Yang–Mills algebras (including $\mathfrak{U}_d(\mathcal{A})$) has the Poincaré–Birkhoff–Witt property.

Proposition 1. *A basis of $\mathfrak{U}_d(\mathcal{A})$ is provided by the set of words on $\{x_1,\dots,x_d\}$ that do not contain any of the following sub-words*

$$x_d^2x_1,\ x_d^2x_2,\ \dots,\ x_d^2x_{d-1},\ x_dx_{d-1}^2.$$

Proof. In the following proof we will use the notation and terminology of [6], to which we refer for details. The defining relations of the algebra

$$\sum_{j=1}^{d}\big[[x_i,x_j],x_j\big]=\mu_ix_i$$

for $i=1,\dots,d$ will be put into the following reduction system S

$$\sigma_i=(W_i,f_i)=\left(x_d^2x_i, 2x_dx_ix_d-x_ix_d^2+\lambda_ix_i-\sum_{j=1}^{d-1}\big[[x_i,x_j],x_j\big]\right),\qquad 1\le i<d,$$

$$\sigma_d=(W_d,f_d)=\left(x_dx_{d-1}^2, 2x_{d-1}x_dx_{d-1}-x_{d-1}^2x_d+\lambda_dx_d-\sum_{j=1}^{d-2}\big[[x_d,x_j],x_j\big]\right).$$

Next, we need to define a semi-group partial ordering on words, that is compatible with S, i.e. every word in f_i should be less than W_i. Let w_1, w_2 be two words on $x_1,\dots,x_d$; if w_1 has smaller length than w_2 then we set $w_1<w_2$. If w_1 and w_2 have the same length, we set $w_1<w_2$ if w_1 precedes w_2 lexicographically, where the lexicographical ordering is induced by $x_1<x_2<\cdots<x_d$. These definitions imply that if $w_1<w_2$ then $aw_1b<aw_2b$ for any words a, b (which defines a semi-group partial ordering). It is also easy to check that this ordering is compatible with S. Does the ordering satisfy the descending chain condition? Let $w_1\ge w_2\ge w_3\ge\cdots$ be an infinite sequence of decreasing words. Clearly, since the length of w_i is a positive integer, it must eventually become constant. Thus, for all $i>N$ the length of w_i is the same. But this implies that the series eventually become constant because there is only a finite number of words preceding a given word lexicographically. Hence, the ordering satisfies the descending chain condition.

Now, we are ready to apply the Diamond lemma. If we can show that all ambiguities in S are resolvable, then a basis for the algebra is provided by the irreducible words. In this case there is only one ambiguity in the reduction system S. Namely, there are two ways of reducing the word $x_d^2x_{d-1}^2$: Either we write it as $x_d(x_dx_{d-1}^2)$ and apply σ_d or we write it as $(x_d^2x_{d-1})x_{d-1}$ and apply σ_{d-1}. Let us prove that $A\equiv x_df_d-f_{d-1}x_{d-1}=0$, i.e. the ambiguity is resolvable

$$\begin{aligned}A&=-x_{d-1}\overbrace{x_d^2x_{d-1}}^{\to f_{d-1}}+\overbrace{x_dx_{d-1}^2}^{\to f_d}x_d+\lambda_{d-1}x_{d-1}^2-\lambda_dx_d^2\\&\quad-\sum_{j=1}^{d-1}\big[[x_{d-1},x_j],x_j\big]x_{d-1}+x_d\sum_{j=1}^{d-2}\big[[x_d,x_j],x_j\big]\\&=\sum_{j=1}^{d-2}\Big[x_{d-1},\big[[x_{d-1},x_j],x_j\big]\Big]+\sum_{j=1}^{d-2}\Big[x_d,\big[[x_d,x_j],x_j\big]\Big].\end{aligned}$$

Now, let us rewrite the second commutator above as follows:

$$\Big[x_d,\big[[x_d,x_j],x_j\big]\Big]=(\overbrace{x_d^2x_j}^{\to f_j})x_j-2x_dx_jx_dx_j-x_j^2x_d^2+2x_jx_dx_jx_d$$

$$= 2x_jx_dx_jx_d - x_j^2x_d^2 + \lambda_j x_j^2 - x_j \overbrace{x_d^2 x_j}^{\to f_j} - \sum_{k=1}^{d-1} [[x_j, x_k], x_k]x_j$$
$$= \sum_{k=1}^{d-1} \Big[x_j, [[x_j, x_k], x_k]\Big].$$

By introducing the sum, we obtain

$$\sum_{j=1}^{d-2} \Big[x_d, [[x_d, x_j], x_j]\Big] = -\sum_{j=1}^{d-2} \Big[x_{d-1}, [[x_{d-1}, x_j], x_j]\Big] + \sum_{j,k=1}^{d-2} \Big[x_j, [[x_j, x_k], x_k]\Big]$$
$$= -\sum_{j=1}^{d-2} \Big[x_{d-1}, [[x_{d-1}, x_j], x_j]\Big],$$

which implies that $A = 0$. From the Diamond lemma, we can now conclude that the set of all irreducible words, with respect to the reduction system S, provides a basis of the algebra. ■

Let us start by noting that any semi-simple Lie algebra $\mathfrak{g}$ is itself a DMSA. Namely, if we let $\mathcal{X} = \{x_1, x_2, \ldots, x_d\}$ be a basis of $\mathfrak{g}$ such that $K(x_i, x_j) = \delta_{ij}$ (where K denotes the Killing form), then the structure constants will be totally anti-symmetric which implies that

$$\Delta_{\mathcal{X}}(x_i) = \sum_{j=1}^{d} [[x_i, x_j], x_j] = \sum_{j,k,l=1}^{d} f_{ij}^k f_{kj}^l x_l = \sum_{l=1}^{d} K(x_i, x_l)x_l = x_i.$$

Thus, in such a basis $(\mathfrak{g}, \mathfrak{g})$ is a DMSA for any semi-simple Lie algebra $\mathfrak{g}$. On the other hand, if $\mathfrak{g}$ is nilpotent, it follows that the map $\Delta_{\mathcal{X}}$ is nilpotent. Thus, for a DMSA related to a nilpotent Lie algebra, it must hold that $\mathrm{Spec}(\mathcal{A}) = \{0\}$. Note that the class of DMSA for which $\mathrm{Spec}\,\mathcal{A} = \{0\}$ has also been studied under the name of (*Lie*) *Yang–Mills algebras* [16, 8], and their representation theory has been studied in [11].

Apart from semi-simple Lie algebras, it is also true that Clifford algebras satisfy the relations imposed in the enveloping algebra. Let $Cl_{p,q}$ be a Clifford algebra generated by $e_1, \ldots, e_d$ (with $d = p + q$), satisfying

$$\begin{aligned} &e_i^2 = 1 && \text{for } \; i = 1, \ldots, p, \\ &e_i^2 = -1 && \text{for } \; i = p+1, \ldots, p+q, \\ &e_ie_j = -e_je_i && \text{when } \; i \neq j. \end{aligned}$$

It is then easy to check that

$$\sum_{j=1}^{d} [[e_i, e_j], e_j] = \begin{cases} 4(p-q-1)e_i & \text{if } \; i \in \{1, \ldots, p\}, \\ 4(p-q+1)e_i & \text{if } \; i \in \{p+1, \ldots, p+q\}. \end{cases}$$

The linear operator $\Delta_{\mathcal{X}}$ is invariant under orthogonal transformations of the elements in $\mathcal{X}$ in the following sense:

Lemma 1. *Let $\mathcal{X} = \{x_1, x_2, \ldots, x_d\}$ and $\mathcal{X}' = \{x'_1, \ldots, x'_d\}$ be subsets of a Lie algebra $\mathfrak{g}$ such that $x'_i = \sum_{j=1}^{d} R_{ij}x_j$ for some orthogonal $d \times d$-matrix R. Then $\Delta_{\mathcal{X}}(a) = \Delta_{\mathcal{X}'}(a)$ for all $a \in \mathfrak{g}$.*

Proof. The proof is given by the following calculation:

$$\begin{aligned}\Delta_{\mathcal{X}'}(a) &= \sum_{j,k,l=1}^{d} R_{jk}R_{jl}\big[[a,x_k],x_l\big] = \sum_{k,l=1}^{d}\big(R^TR\big)_{kl}\big[[a,x_k],x_l\big]\\ &= \sum_{k,l=1}^{d}\delta_{kl}\big[[a,x_k],x_l\big] = \sum_{j=1}^{d}\big[[a,x_j],x_j\big] = \Delta_{\mathcal{X}}(a).\end{aligned}$$ ■

Remark 2. Note that it is not necessarily true that $(\mathfrak{g},\mathcal{X}')$ is a DMSA when $(\mathfrak{g},\mathcal{X})$ is a DMSA. However, if we let $\operatorname{Spec}\big((\mathfrak{g},\mathcal{X})\big) = \{\mu_1,\dots,\mu_k\}$ and $m_1,\dots,m_k$ be the multiplicities of each eigenvalue, then any block-diagonal orthogonal matrix $R = \operatorname{diag}(R_1,\dots,R_k)$ (where R_i has dimension m_i) will generate a DMSA. In other words, we can always choose to make an orthogonal transformation among those x_i that belong to the same eigenvalue.

The preceding lemma enables us to make the following observation. Let $\mathcal{X}$ be a set of linearly independent elements in a n-dimensional Lie algebra $\mathfrak{g}$, and let $\langle\mathcal{X}\rangle$ denote the linear span of the elements in $\mathcal{X}$. Furthermore, assume that $\langle\mathcal{X}\rangle$ is closed under the action of $\Delta_{\mathcal{X}}$, i.e. $\Delta_{\mathcal{X}}(x)\in\langle\mathcal{X}\rangle$ for all $x\in\langle\mathcal{X}\rangle$. Relative to a basis where $x_1,\dots,x_d$ are chosen to be the first d basis elements, the $n\times n$ matrix of $\Delta_{\mathcal{X}}$ has the block form

$$\Delta_{\mathcal{X}} = \begin{pmatrix} X_0 & A\\ 0 & B\end{pmatrix}$$

where X_0 is a $d\times d$ matrix. If X_0 is diagonalizable by an orthogonal matrix R then the elements $x_i' = \sum_{j=1}^{d} R_{ij}x_j$ will be eigenvectors of $\Delta_{\mathcal{X}'}$, since the action of $\Delta_{\mathcal{X}}$ is invariant under orthogonal transformations in $\mathcal{X}$. Thus, $(\mathfrak{g},\mathcal{X}')$ is a DMSA. In particular, if we choose an orthonormal basis of $\mathfrak{g}$, then the matrices ad_{x_i} are antisymmetric, which implies that $\Delta_{\mathcal{X}}$ is symmetric. In this case, X_0 will be diagonalizable by an orthogonal matrix.

We will now concentrate on subsets $\mathcal{X}$ of the Lie algebra $\mathfrak{sl}_n$, such that $\mathcal{A} = (\mathfrak{sl}_n,\mathcal{X})$ is a DMSA. To perform calculations the following set of conventions will be used: $\alpha_1,\dots,\alpha_{n-1}$ denote the simple roots and for every positive root α, we choose elements e_α, $e_{-\alpha}$, h_α such that

$$[h,e_\alpha] = \alpha(h)e_\alpha, \qquad [e_\alpha,e_{-\alpha}] = h_\alpha,$$

and h_α is the element of the Cartan subalgebra $\mathfrak{h}$ such that $\alpha(h) = K(h_\alpha,h)$ for all $h\in\mathfrak{h}$. For any pair of roots α,β we define the constants $N(\alpha,\beta)$ by

$$[e_\alpha,e_\beta] = N(\alpha,\beta)e_{\alpha+\beta},$$

and when $\alpha+\beta$ is not a root, we set $N(\alpha,\beta)=0$. In $\mathfrak{sl}_n$ all roots have the same length, and we denote $(\alpha,\alpha)\equiv K(h_\alpha,h_\alpha) = \alpha(h_\alpha) = l^2$. With these conventions, the constants $N(\alpha,\beta)$ satisfy the relations

$$\begin{aligned}&N(\alpha,\beta) = N(\beta,\gamma) = N(\gamma,\alpha) \qquad \text{if}\ \ \alpha+\beta+\gamma = 0,\\ &N(\alpha,\beta)N(-\alpha,-\beta) = -\frac{l^2}{2}q(p+1),\end{aligned}$$

where p, q are positive integers such that $\beta-p\alpha,\dots,\beta,\dots,\beta+q\alpha$ are roots. Furthermore, in $\mathfrak{sl}_n$ it holds that if $\beta+\alpha$ is a root then $\beta-\alpha$ is not a root and $\beta\pm2\alpha$ is never a root. Therefore, if $N(\alpha,\beta)$ is non-zero, we have that

$$N(\alpha,\beta)N(-\alpha,-\beta) = -\frac{1}{2}l^2.$$

Although the following result does not depend on it, we will for definiteness choose each $N(\alpha,\beta)$ such that $N(-\alpha,-\beta) = -N(\alpha,\beta)$.

Lemma 2. *For every positive root α in $\mathfrak{sl}_n$, we set*

$$e^+_\alpha = ic\big(e_\alpha + e_{-\alpha}\big) \qquad \text{and} \qquad e^-_\alpha = c\big(e_\alpha - e_{-\alpha}\big),$$

for an arbitrary $c \in \mathbb{R}$. Then the following holds

1. $\big[[e^+_\alpha, e^+_\beta], e^+_\beta\big] = \big[[e^+_\alpha, e^-_\beta], e^-_\beta\big] = -\frac{1}{2}c^2 l^2 e^+_\alpha$ *(when $\alpha \pm \beta$ is a root),*
2. $\big[[e^-_\alpha, e^+_\beta], e^+_\beta\big] = \big[[e^-_\alpha, e^-_\beta], e^-_\beta\big] = -\frac{1}{2}c^2 l^2 e^-_\alpha$ *(when $\alpha \pm \beta$ is a root),*
3. $\big[[e^\pm_\alpha, e^\mp_\alpha], e^\mp_\alpha\big] = -2c^2 l^2 e^\pm_\alpha$,
4. $\big[[e^\pm_\alpha, h_\beta], h_\beta\big] = (\alpha, \beta)^2 e^\pm_\alpha$,
5. $\big[[h_\alpha, e^\pm_\beta], e^\pm_\beta\big] = \mp 2c^2(\alpha, \beta) h_\beta$.

From this lemma, it is easy to construct a couple of examples.

Example 1. Let $\mathcal{X} = \{e^\pm_{\beta_1}, \ldots, e^\pm_{\beta_d}\}$ (where the signs are chosen independently) for any positive roots β_i. In this case, $\big[[x_i, x_j], x_j\big]$ is proportional to x_i for all $x_i, x_j \in \mathcal{X}$.

Example 2. Let $\mathcal{X} = \{h_{\beta_1}, \ldots, h_{\beta_k}, e^+_{\gamma_1}, e^-_{\gamma_1}, \ldots, e^+_{\gamma_l}, e^-_{\gamma_l}\}$. Now, $\big[[h_{\beta_i}, e^+_{\gamma_j}], e^+_{\gamma_j}\big]$ might not be proportional to h_{β_i}. However, since both $e^+_{\gamma_j}, e^-_{\gamma_j} \in \mathcal{X}$ this term will cancel against $\big[[h_{\beta_i}, e^-_{\gamma_j}], e^-_{\gamma_j}\big]$. Thus, $\Delta_{\mathcal{X}}(h_{\beta_i}) = 0$ for $i = 1, \ldots, k$.

Example 3. Let $\mathcal{X} = \{h_{\beta_1}, \ldots, h_{\beta_k}, e^\pm_{\beta_1}, \ldots, e^\pm_{\beta_k}\}$ (where the signs are chosen to be the same). In this case $\Delta_{\mathcal{X}}(h_{\beta_i})$ will not be proportional to h_{β_i}. However, the matrix $\Delta_{\mathcal{X}}$ will be symmetric, which implies that there exists an orthogonal $k \times k$ matrix R such that $(\mathfrak{sl}_n, \{x_1, \ldots, x_k, e^\pm_{\beta_1}, \ldots, e^\pm_{\beta_k}\})$ is a DMSA if $x_i = \sum_{j=1}^k R_{ij} h_{\beta_j}$.

When the dimension $d = 2m$ (i.e. even) and every eigenvalue in $\operatorname{Spec}(\mathcal{A})$ has an even multiplicity (which is relevant for one of the applications mentioned in the introduction) there is a convenient complexified basis provided by

$$t_i = x_{2i-1} + i x_{2i}, \qquad s_i = x_{2i-1} - i x_{2i}$$

for $i = 1, \ldots, m$. The defining relations of a DMSA can then be written as

$$2\mu_i t_i = \sum_{j=1}^m \big(\big[[t_i, s_j], t_j\big] + \big[[t_i, t_j], s_j\big]\big), \qquad 2\mu_i s_i = \sum_{j=1}^m \big(\big[[s_i, t_j], s_j\big] + \big[[s_i, s_j], t_j\big]\big).$$

The lowest dimensional non-trivial DMSA has dimension 2. In this case the algebra is generated by x and y satisfying

$$\big[[x, y], y\big] = \lambda x, \qquad \big[[y, x], x\big] = \mu y,$$

and by defining $z = -i[x, y]$ one sees that x, y and z span a 3-dimensional Lie algebra. By rescaling the elements we obtain the following result:

- $\lambda \neq 0, \mu \neq 0$: $\mathcal{A}$ is isomorphic to $\mathfrak{sl}_2$,
- $\lambda = \mu = 0$: $\mathcal{A}$ is isomorphic to the Heisenberg algebra,
- $\lambda \neq 0, \mu = 0$ or $\lambda = 0, \mu \neq 0$: $\mathcal{A}$ is isomorphic to the Lie algebra VII_1 in the Bianchi classification [7]. This algebra is defined by the relations: $[u, v] = -w$, $[v, w] = 0$ and $[w, u] = -v$.

3 Hermitian representations of $\mathfrak{U}_4(\mathcal{A})$

In general, any representation of the Lie algebra $\mathfrak{g}$ gives rise to a representation of the DMSA $(\mathfrak{g}, \mathcal{X})$. In the following, we shall however concentrate on finding *Hermitian* representations. Hermitian representations ϕ of $\mathfrak{U}_d(\mathcal{A})$ are given by $\phi(x_i) = X_i$ where $X_1, \dots, X_d$ are Hermitian matrices satisfying

$$\sum_{j=1}^{d} \big[[X_i, X_j], X_j\big] = \mu_i X_i \qquad \text{for} \ \ i = 1, \dots, d.$$

We note that, unless all μ_i are real, no Hermitian (or anti-Hermitian) representations can exist (except for the trivial one: $\phi(x_i) = 0$ for all i). Hence, from now on we will assume the spectrum to be real and all representations to be Hermitian.

Since the subalgebra (of the full matrix algebra) generated by the matrices $\{\phi(x_1), \dots, \phi(x_d)\}$ (through arbitrary products and sums) is invariant under Hermitian conjugation, the following result is immediate.

Proposition 2. *Any Hermitian representation of $\mathfrak{U}_d(\mathcal{A})$ is completely reducible.*

As DMSAs of dimension 2 are isomorphic to Lie algebras, we will start by considering the case when $d = 4$. We expect that these algebras have a rich structure of representations even for the case when $\mu_1 = \cdots = \mu_4$. Namely, since the equations defining a 4-dimensional DMSA can be thought of as discrete analogues of minimal surface equations in S^3 (see the introduction), and minimal surfaces of any genus exist in S^3 [15], we believe that there will be representations corresponding to many (if not all) of these surfaces.

Since the defining relations of $\mathfrak{U}_d(\mathcal{A})$ are expressed entirely in terms of commutators, the tensor product of Lie algebra representations, i.e.

$$\big(\phi \otimes \phi'\big)(x) = \phi(x) \otimes \mathbb{1} + \mathbb{1} \otimes \phi'(x),$$

also defines a tensor product for representations of DMSAs. In contrast to Lie algebras, the tensor product of two irreducible representations might again be irreducible (as we shall explicitly see for $\mathfrak{U}_4(\mathcal{A})$). Thus, when studying the representation theory of $\mathfrak{U}_d(\mathcal{A})$ it becomes natural to look, not only for irreducible representations, but also for *prime representations*, i.e. irreducible representations that can not be written as a tensor product of two other representations.

To each Hermitian representation ϕ of $\mathfrak{U}_4(\mathcal{A})$ we shall associate a directed graph with vertices in $\mathbb{C}$, such that the vertices of the graph are placed at the characteristic roots of $\phi(x_1 + ix_2)$. The edges of the graph are determined by the matrix $\phi(x_3 + ix_4)$ as described below. We note that this construction can be carried out for Hermitian representations of any algebra on at most four generators. In the following, we will use the notation $t_1 = x_1 + ix_2$, $t_2 = x_3 + ix_4$, $\phi(t_1) = \Lambda$ and $\phi(t_2) = T$. Let us start by recalling the directed graph of a matrix.

Definition 4. Let T be a $n \times n$ matrix and let $G = (V, E)$ be a directed graph on n vertices with vertex set $V = \{1, \dots, n\}$ and edge set $E \subseteq V \times V$. We say that G is the *directed graph of* T, and write $G = G_T$, if it holds that

$$T_{ij} \neq 0 \quad \Longleftrightarrow \quad (i, j) \in E.$$

for $i, j = 1, \dots, n$.

The idea is now to associate a graph to every representation, such that each vertex is assigned an eigenvalue of Λ and the graph itself being the directed graph of T. Needless to say, the graph of T depends on the basis chosen and therefore we will introduce a particular choice of basis in which all graphs will be refered to.

Definition 5. Let Λ and T be linear operators on $W = \mathbb{C}^n$, and let $\mathcal{B}$ be the $*$-algebra generated by Λ, $\Lambda^\dagger$, T, $T^\dagger$. Furthermore, let $W = W_1 \oplus \cdots \oplus W_m$ be a decomposition of W into irreducible subspaces with respect to $\mathcal{B}$. For each i, let $v_1^{(i)}, \ldots, v_{n_i}^{(i)}$ denote a Jordan basis for $\Lambda\big|_{W_i}$. Then $v_1^{(1)}, \ldots, v_{n_1}^{(1)}, \ldots, v_1^{(m)}, \ldots, v_{n_m}^{(m)}$ is a basis for W and is called *a Jordan basis of Λ with respect to T*.

Definition 6. Let ϕ be a n-dimensional representation of $\mathfrak{U}_4(\mathcal{A})$ and let $v_1, \ldots, v_n$ denote a Jordan basis of $\Lambda = \phi(t_1)$ with respect to $T = \phi(t_2)$. Define the matrix α by defining its matrix elements through

$$Tv_i = \sum_{j=1}^{n} \alpha_{ji} v_j,$$

i.e., α is the matrix of T in the given Jordan basis. Furthermore, let $G_\alpha = (\{1, 2, \ldots, n\}, E)$ denote the directed graph of α and let $\lambda : V \to \mathbb{C}$ be defined by $\lambda(i) = \lambda_i$, where λ_i is the eigenvalue corresponding to v_i. We set $G_\phi = (G_\alpha, \lambda)$ and call G_ϕ *a representation graph of ϕ*.

Two representation graphs $G_\phi = (\{1, \ldots, n\}, E, \lambda)$ and $G_{\phi'} = (\{1, \ldots, n\}, E', \lambda')$ are *isomorphic* if there exists a permutation $\sigma \in S_n$ such that $(i, j) \in E \Leftrightarrow \big(\sigma(i), \sigma(j)\big) \in E'$ and $\lambda = \lambda' \circ \sigma$. In particular, $(\{1, \ldots, n\}, E)$ and $(\{1, \ldots, n\}, E')$ are isomorphic as directed graphs.

Note that two representation graphs corresponding to the same representation need not be isomorphic. This could be resolved by further fixing the basis in which the directed graph of T is calculated. However, let us postpone this choice and study the properties of representation graphs that follow from the above definition.

The first property that one might wish for, is a correspondence between disconnected components of the representation graph and the irreducible components of the representation. That a connected graph corresponds to an irreducible representation follows immediately from the definition of the Jordan basis with respect to T.

Proposition 3. *Let G_ϕ be a representation graph of ϕ. If G_ϕ is connected then ϕ is irreducible.*

Proof. Let G_ϕ be a connected representation graph of ϕ. If ϕ is reducible, then G_ϕ consists of at least two components, since the matrix α is block diagonal with at least two blocks, by the construction of the Jordan basis. Hence, ϕ must be irreducible. ■

For convenience, let us introduce some terminology indicating when the matrices of a representation have certain properties.

Definition 7. Let ϕ be a representation of $\mathfrak{U}_4(\mathcal{A})$. If $\Lambda = \phi(t_1)$ is diagonalizable then ϕ is called diagonalizable. If all eigenvalues of Λ are distinct, then ϕ is called non-degenerate. If Λ is normal then ϕ is called semi-normal. If both Λ and $T = \phi(t_2)$ are normal then ϕ is called normal. If Λ and T are unitary, then ϕ is called unitary.

For semi-normal representations, the result in Proposition 3 can be strengthened to an if and only if statement.

Proposition 4. *Let ϕ be a semi-normal representation of $\mathfrak{U}_4(\mathcal{A})$ and let G_ϕ be a representation graph of ϕ. Then ϕ is irreducible if and only if G_ϕ is connected.*

Proof. Assuming that G_ϕ is connected, the first implication follows from Proposition 3. Now, assume that ϕ is an irreducible n-dimensional representation. When Λ is normal, the Jordan basis is given by a set of orthogonal vectors $\{v_1, \ldots, v_n\}$ (the eigenvectors of Λ). Therefore, any matrix P, bringing Λ to the (diagonal) Jordan normal form $P^{-1}\Lambda P$, can be written as $P = UD$

where U is a unitary matrix and D is an invertible diagonal matrix (reflecting the choice of length of the eigenvectors). In such a basis, the matrix α, representing T, and the matrix $\tilde{\alpha}$, representing $T^\dagger$ are related by $\tilde{\alpha}^\dagger = D^\dagger D\alpha(D^\dagger D)^{-1}$. Clearly, since conjugation by a diagonal invertible matrix does not change the structure of non-zero matrix elements, the graph of $\tilde{\alpha}$ is obtained from the graph of α by reversing the arrows. In particular, this implies that $G_{\tilde{\alpha}}$ and G_α have the same number of connected components. We now continue to prove that if G_α is disconnected then there exists an invariant subspace, which contradicts the assumption that ϕ is irreducible.

Assume that G_α is disconnected and let $\{i_1,\dots,i_k\}\subset V$ (with $k<n$) be the vertices corresponding to one of the components. By the construction of G_α, the vectors $v_{i_1},\dots,v_{i_k}$ span an invariant subspace for α. Since $G_{\tilde{\alpha}}$ is obtained from G_α by reversing all arrows, this is also an invariant subspace for $\tilde{\alpha}$. Moreover, since $v_{i_1},\dots,v_{i_k}$ are eigenvectors of both Λ and $\Lambda^\dagger$, the space spanned by these vectors is also an invariant subspace of Λ and $\Lambda^\dagger$. This implies that the representation is reducible, which contradicts the assumption. Hence, G_ϕ is connected. ■

When the eigenvalues of $\Lambda=\phi(t_1)$ are distinct, one can easily show that all representation graphs of ϕ are isomorphic.

Proposition 5. *Let ϕ be a non-degenerate representation. Then any Jordan basis for Λ is a Jordan basis for Λ with respect to T up to a permutation of the basis vectors. Moreover, all representation graphs of ϕ are isomorphic.*

Proof. When all eigenvalues of Λ are distinct, the only freedom in choosing a basis in which Λ is diagonal lies in the length of the eigenvectors and the ordering of the basis vectors. Hence, given two Jordan bases for Λ it is always possible to apply a permutation to obtain one basis from the other, up to a rescaling of the vectors. Furthermore, a rescaling of the basis vectors does not change the block diagonal form of a matrix. Hence, any Jordan basis of Λ is a Jordan basis of Λ with respect to T up to a permutation. In particular, this implies that any two representation graphs are related by a permutation of the vertices. ■

Proposition 5 has the consequence that if one constructs a representation ϕ, in which Λ is diagonal and has distinct eigenvalues, then the directed graph of T is the unique representation graph of ϕ.

Let us now study the representation graph of the tensor product. For directed graphs, forming the tensor product

$$\hat{T}=T\otimes\mathbb{1}+\mathbb{1}\otimes T'$$

amounts to taking the *Cartesian product* of G_T and $G_{T'}$ [17, 10]. The Cartesian product of two graphs $G=(V,E)$ and $H=(U,F)$ is defined as the graph $G'=(V\times U,E')$ such that

$$\begin{aligned}&\big((v_1,u_1),(v_2,u_2)\big)\in E' \quad\Longleftrightarrow\\ &\big\{v_1=v_2 \text{ and } (u_1,u_2)\in F\big\} \text{ or } \big\{u_1=u_2 \text{ and } (v_1,v_2)\in E\big\}.\end{aligned}$$

Now, one might ask if the Cartesian product of two representation graphs is a representation graph of the tensor product? This is not always true, but we have the following result.

Proposition 6. *Let ϕ and ϕ' be representations such that $\phi\otimes\phi'$ is a non-degenerate representation. Then ϕ and ϕ' are non-degenerate and $G_{\phi\otimes\phi'}$ is the Cartesian product of G_ϕ and $G_{\phi'}$.*

Proof. Let $\Lambda=\phi(t_1)$ and $\Lambda'=\phi'(t_1)$ and let P and Q be matrices whose column vectors are Jordan bases of Λ and Λ' with respect to $T=\phi(t_2)$ and $T'=\phi'(t_2)$. By assumption, the eigenvalues of $\hat{\Lambda}=\Lambda\otimes\mathbb{1}+\mathbb{1}\otimes\Lambda'$ are distinct, which implies that the eigenvalues of the matrix

$$M=\big(P\otimes Q\big)^{-1}\Big[\Lambda\otimes\mathbb{1}+\mathbb{1}\otimes\Lambda'\Big]\big(P\otimes Q\big)=\big(P^{-1}\Lambda P\big)\otimes\mathbb{1}+\mathbb{1}\otimes\big(Q^{-1}\Lambda' Q\big)$$

are distinct. Since $P^{-1}\Lambda P$ and $Q^{-1}\Lambda' Q$ are upper triangular (and hence has their eigenvalues on the diagonal) the matrix M will also be upper triangular. The diagonal elements of M (its eigenvalues) will be all possible sums of eigenvalues from Λ and Λ'. Since the eigenvalues of M are distinct the eigenvalues of Λ and Λ' must be distinct. This proves the first part of the statement.

Since Λ and Λ' have distinct eigenvalues, the matrices $P^{-1}\Lambda P$ and $Q^{-1}\Lambda' Q$ are in fact diagonal, which implies that the matrix M is diagonal, so $P \otimes Q$ clearly provides us with a Jordan basis for $\hat{\Lambda}$. Since $\phi\otimes\phi'$ is non-degenerate, Proposition 5 tells us that the (unique) representation graph is given by the directed graph of

$$\big(P\otimes Q\big)^{-1}\big[T\otimes\mathbb{1}+\mathbb{1}\otimes T'\big]\big(P\otimes Q\big)=\big(P^{-1}TP\big)\otimes\mathbb{1}+\mathbb{1}\otimes\big(Q^{-1}T'Q\big).$$

Now, since $P^{-1}TP$ and $Q^{-1}T'Q$ define G_ϕ and $G_{\phi'}$, we conclude that $G_{\phi\otimes\phi'}$ is given by the Cartesian product of G_ϕ and $G_{\phi'}$. ■

Let us now proceed to construct representations of $\mathfrak{U}_4(\mathcal{A})$. As noted earlier, even the case when $\mu_1=\cdots=\mu_4\neq 0$ is expected to have a rich representation theory. Therefore, we will start by concentrating on the case for which $\mu_1=\mu_2=\mu$ and $\mu_3=\mu_4=\rho$ (which is also relevant for applications, as mentioned in the introduction). In this case, representations are found by solving the matrix equations

$$\begin{aligned}
2\mu\Lambda &= \big[[\Lambda,T],T^\dagger\big]+\big[[\Lambda,T^\dagger],T\big]+\big[[\Lambda,\Lambda^\dagger],\Lambda\big],\\
2\rho T &= \big[[T,\Lambda],\Lambda^\dagger\big]+\big[[T,\Lambda^\dagger],\Lambda\big]+\big[[T,T^\dagger],T\big].
\end{aligned}$$

The action of the group $O(2)\times O(2)$ can be explicitly realized by letting $\Lambda\to e^{i\theta}\Lambda$ and $T\to e^{i\theta'}T$, which gives a new representation for any $\theta,\theta'\in\mathbb{R}$; this representation will be denoted by $\phi_{\theta\theta'}$ and is in general not equivalent to ϕ since the eigenvalues of e.g. Λ will be different. This enables us to construct new irreducible representations from a given one via the tensor product. Namely, let ϕ be a non-degenerate irreducible representation; then one can always choose θ, θ' such that $\phi\otimes\phi_{\theta\theta'}$ is a non-degenerate representation. By Proposition 6 the representation graph of $\phi\otimes\phi_{\theta\theta'}$ will be the Cartesian product of two connected graphs (the representation graphs of ϕ and $\phi_{\theta\theta'}$), which implies that it is connected [10]. Hence, it follows from Proposition 3 that $\phi\otimes\phi_{\theta\theta'}$ is irreducible.

3.1 The Fuzzy sphere

As any semi-simple Lie algebra is itself a DMSA, it follows that Hermitian representations of $\mathfrak{su}(2)$ should induce Hermitian representations of $\mathfrak{U}_4(\mathcal{A})$. Indeed, choosing Hermitian $n\times n$ matrices S_1, S_2, S_3, with non-zero elements

$$\begin{aligned}
\big(S_1\big)_{k,k+1} &= \frac{1}{2}\sqrt{k(n-k)}=\big(S_1\big)_{k+1,k}, \qquad k=1,\dots,n-1,\\
\big(S_2\big)_{k,k+1} &= -\frac{i}{2}\sqrt{k(n-k)}=-\big(S_2\big)_{k+1,k}, \qquad k=1,\dots,n-1,\\
\big(S_3\big)_{k,k} &= \frac{1}{2}(n+1-2k), \qquad k=1,\dots,n,
\end{aligned}$$

satisfying $[S_i,S_j]=i\epsilon_{ijk}S_k$, yields a representation ϕ by defining

$$\Lambda=\phi(t_1)=e^{i\theta}S_3, \qquad T=\phi(t_2)=S_1+iS_2.$$

for any $\theta\in\mathbb{R}$ (no θ' appears in $\phi(t_2)$ since it can always be removed by conjugating with a diagonal unitary matrix). One easily calculates that this is a representation of $\mathfrak{U}_4(\mathcal{A})$ with

$\mathrm{Spec}(\mathcal{A}) = \{2\}$. Note that in the special case when $\theta = 0$, in which case Λ is Hermitian, this provides a representation of $\mathfrak{U}_3(\mathcal{A})$.

This is a non-degenerate semi-normal representation, and the representation graph takes the form as in Fig. 1.

Figure 1. The representation graph of the Fuzzy sphere.

Furthermore, this representation is irreducible by Proposition 3, and its representation graph is prime with respect to the Cartesian product since any Cartesian product graph with n vertices has at least n edges (whereas the above graph has $n-1$ edges). The matrix algebras generated by these matrices have been used to construct sequences of matrix algebras (of increasing dimension) converging to the Poisson algebra of smooth functions on S^2 [12].

For increasing n, the algebras that are generated by these matrices (with an appropriate normalization) are recognized as a sequence converging to the Poisson algebra of functions on S^2 [12].

Let us for this case demonstrate the tensor product and construct the corresponding Cartesian product of the representation graphs. For simplicity, we let ϕ_2 and ϕ_3 be a two- respectively three-dimensional representation of the type described above, and set

$$\phi(t_1) = \phi_3(t_1) \otimes \mathbb{1}_2 + \mathbb{1}_3 \otimes \phi_2(t_1),$$
$$\phi(t_2) = \phi_3(t_2) \otimes \mathbb{1}_2 + \mathbb{1}_3 \otimes \phi_2(t_2).$$

If we denote the arbitrary phases by θ_2 and θ_3 the representation graph takes the form as in Fig. 2.

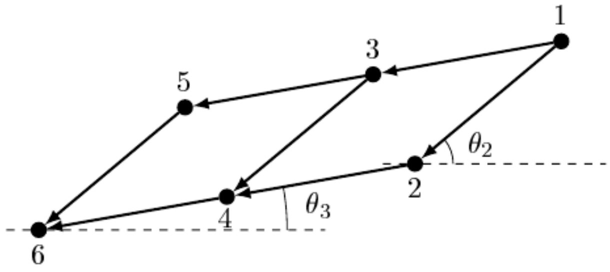

Figure 2. The representation graph of a tensor product of two Fuzzy sphere representations.

Since this representation is non-degenerate it will be irreducible by Proposition 3, and in general we obtain inequivalent representations for different choices of θ_2 and θ_3.

We note that the matrices S_1, S_2, S_3 gives rise to another representation by setting

$$\Lambda = zS_3, \tag{6}$$
$$T = w(S_1 + aS_2) \tag{7}$$

for arbitrary $z, w \in \mathbb{C}$ and $a \in \mathbb{R}$. This gives a representation of $\mathfrak{U}_4(\mathcal{A})$ with $\mathrm{Spec}(\mathcal{A}) = \{|z|^2, |w|^2(1 + a^2)\}$. The representation graph can be seen in Fig. 3.

Figure 3. The representation graph of a normal representation constructed from $\mathfrak{su}(2)$.

We conclude that this is an irreducible non-degenerate normal representation, which is not equivalent to the Fuzzy sphere, since the corresponding graphs are not isomorphic. Moreover, its representation graph is prime with respect to the Cartesian product.

3.2 The Fuzzy torus

The fuzzy torus algebra (cp. [9, 13]) is generated by the matrices g and h, with non-zero elements

$$(h)_{k,k+1} = 1, \qquad (h)_{n,1} = 1, \qquad k = 1, \dots, n-1,$$
$$(g)_{kk} = q^{k-1}, \qquad k = 1, \dots, n,$$

fulfilling the relation $hg = q \cdot gh$ with $q^n = 1$. It is known that they generate matrix sequences that converge to functions on T^2. A representation ϕ of $\mathfrak{U}_4(\mathcal{A})$, with $\mathrm{Spec}(\mathcal{A}) = \{|1-q|^2/2\}$, is obtained by setting

$$\Lambda = \phi(t_1) = e^{i\theta} g, \qquad T = \phi(t_2) = e^{i\theta'} h,$$

for any $\theta, \theta' \in \mathbb{R}$. This is an irreducible non-degenerate unitary representation, with a representation graph as in Fig. 4.

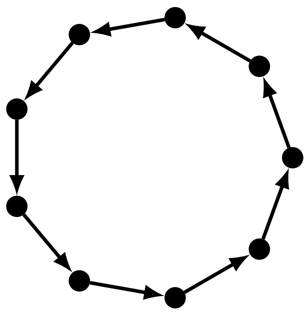

Figure 4. The representation graph of the Fuzzy torus.

Furthermore, this graph is prime with respect to the Cartesian product. Let us now show that this is essentially the only irreducible unitary representation when $\mathrm{Spec}(\mathcal{A}) = \{\mu\}$.

3.3 Unitary representations

When Λ and T are unitary and $\mathrm{Spec}(\mathcal{A}) = \{\mu\}$, the equations can be written as

$$\lambda\Lambda = T^\dagger \Lambda T + T\Lambda T^\dagger, \tag{8}$$
$$\lambda T = \Lambda^\dagger T \Lambda + \Lambda T \Lambda^\dagger, \tag{9}$$

where we have introduced $\lambda = 2 - \mu$. By multiplying the first equation from the right by T and the second equation from the left by Λ we note that $[\Lambda T, T\Lambda] = 0$. Hence, by a unitary transformation, we can always choose a basis such that $D = \Lambda T$ and $\tilde{D} = T\Lambda$ are diagonal. Let us denote the eigenvalues by

$$D = \mathrm{diag}(d_1, \dots, d_n) = \mathrm{diag}\left(e^{i\varphi_1}, \dots, e^{i\varphi_n}\right),$$
$$\tilde{D} = \mathrm{diag}(\tilde{d}_1, \dots, \tilde{d}_n) = \mathrm{diag}\left(e^{i\tilde{\varphi}_1}, \dots, e^{i\tilde{\varphi}_n}\right).$$

The equations (8) and (9) (together with $D = \Lambda T$ and $\tilde{D} = T\Lambda$) can equivalently be written as

$$\lambda\Lambda = \Lambda\tilde{D}^\dagger D + \tilde{D}\Lambda\tilde{D}^\dagger, \tag{10}$$
$$\lambda\Lambda\tilde{D}^\dagger = \tilde{D}^\dagger\Lambda + \Lambda D^\dagger, \tag{11}$$
$$\Lambda\tilde{D} = D\Lambda, \tag{12}$$
$$T = \tilde{D}\Lambda^\dagger. \tag{13}$$

Thus, given unitary Λ, D, $\tilde{D}$ satisfying (10)–(12), we obtain a solution to the original equations by defining $T = \tilde{D}\Lambda^\dagger$. Written out in components, the three first equations become

$$\Lambda_{ij}\left[\lambda - \bar{\tilde{d}}_j d_j - \tilde{d}_i \bar{\tilde{d}}_j\right] = 0, \tag{14}$$

$$\bar{\Lambda}_{ij}\left[\lambda \tilde{d}_j - \tilde{d}_i - d_j\right] = 0, \tag{15}$$

$$\Lambda_{ij}\left[\tilde{d}_j - d_i\right] = 0. \tag{16}$$

If $\Lambda_{ij} \neq 0$ then we obtain the following relations

$$\begin{pmatrix} d_j \\ \tilde{d}_j \end{pmatrix} = \begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} d_i \\ \tilde{d}_i \end{pmatrix} \equiv s\begin{pmatrix} d_i \\ \tilde{d}_i \end{pmatrix}$$

since (15) and (16) together imply (14). Now, consider the directed graph $G_\Lambda = (V, E)$ of Λ, where we have assigned the vector $\vec{x}_i = (d_i, \tilde{d}_i)$ to each vertex $i \in V$. We can restrict ourselves to connected graphs, since if G_Λ is disconnected then the representation will trivially be reducible. The above considerations tell us that whenever there is an edge $(i, j) \in E$, it must hold that $\vec{x}_j = s(\vec{x}_i)$. In particular, since D and $\tilde{D}$ are unitary matrices, the map s must take $\vec{x} \in S^1 \times S^1$ to another vector in $S^1 \times S^1$. If $\vec{x} = (e^{i\varphi}, e^{i\tilde{\varphi}})$ then this is true only if $\lambda = 0$ or

$$\lambda = 2\cos(\varphi - \tilde{\varphi}). \tag{17}$$

This observation leads to the following result.

Proposition 7. *Let $\mathcal{A}$ be a DMSA with* $\mathrm{Spec}(\mathcal{A}) = \{\mu\}$*. If $\mu < 0$ or $\mu > 4$ then there exists no unitary representation of* $\mathfrak{U}_4(\mathcal{A})$*.*

Proof. Assume that Λ and T provides a unitary representation and that a basis has been chosen in which D and $\tilde{D}$ are diagonal. Since Λ is a unitary matrix at least one of its matrix element has to be non-zero, say $\Lambda_{ij} \neq 0$. Then equation (17) must hold, which is impossible if $\lambda < -2$ or $\lambda > 2$. ■

From the above result it follows that whenever a unitary representation exists, then there exists a $\beta \in [0, \pi/2]$ such that $\lambda = 2\cos 2\beta$. We will now proceed in analogy with the proofs in [3, 2] to which we refer for details.

Let us start by studying the case when $\lambda \neq 0$, and let $\vec{x} = (e^{i\varphi}, e^{i\tilde{\varphi}})$ be a vector such that $\varphi - \tilde{\varphi} = \pm 2\beta$. Then it is easy to calculate that $s(\vec{x}) = (e^{i(\varphi \pm 2\beta)}, e^{i\varphi})$. Thus, the maps s preserves the condition (17) provided that we start with a vector fulfilling the condition.

This implies that if G_Λ has a loop (i.e. a directed cycle) on k vertices, then we must have that $s^k(\vec{x}_i) = \vec{x}_i$ for some $i \in V$. Moreover, it is a trivial fact that every directed graph of a unitary matrix must have a loop. Hence, β must be such that $e^{i2k\beta} = 1$ for some integer $k > 0$. Since the map s is invertible, given any $\vec{x}_i$ in the graph uniquely determines $\vec{x}_j$ for all other vertices in the graph. Hence, we can partition the vertices into subsets $V_1, \ldots, V_k$ such that $\vec{x}_i = \vec{x}_j$ if and only if $i, j \in V_l$ for some $l \in \{1, \ldots, k\}$. It follows that all edges of G_Λ are of the form (i, j) with $i \in V_l$ and $j \in V_{l+1}$ (where we identify $k + 1 \equiv 1$). Thus, we can permute the vertices to bring the matrix Λ to the following form

$$\Lambda = \begin{pmatrix} 0 & \Lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \Lambda_{k-1} \\ \Lambda_k & 0 & \cdots & 0 & 0 \end{pmatrix}$$

with Λ_i being unitary matrices for $i = 1, \ldots, k$. Moreover, there exists a unitary matrix such that $U^\dagger \Lambda U$ is of the above form but each Λ_i is diagonal. This means that the directed graph of $U^\dagger \Lambda U$ is a direct sum of k loops (this also holds for T since $T = \tilde{D}\Lambda^\dagger$), and each of these loops correspond to an irreducible representation. However, to calculate the representation graph, we must go to the basis in which Λ is diagonal. The matrix corresponding to a single loop on n vertices has the n roots of unity as eigenvalues. Therefore, in the basis in which Λ is diagonal, the directed graph of T will be the representation graph. It is easy to see that the matrices D and $\tilde{D}$ will act as shift operators on the eigenvectors of Λ, which implies that $T = \tilde{D}\Lambda^\dagger$ will also act as a shift operator in this basis. Thus, the directed graph of T will be a single loop. We conclude that this representation is precisely the Fuzzy torus representation presented above.

Let us turn to the case when $\lambda = 0$, i.e. $\mu = 2$. In this case, there is no restriction on $\varphi - \tilde{\varphi}$, but instead one notes that $s^4(\vec{x}) = \vec{x}$ for any $\vec{x} \in \mathbb{C}^2$. Thus, the vertices of G_Λ can be split into four disjoint subsets, and we conclude that all irreducible representations are 4-dimensional. However, since $\varphi - \tilde{\varphi}$ does not have to be related to β, the action of T on the eigenbasis of Λ will not simply be a shift. Therefore, the representation graph will be one of the two in Fig. 5.

When $\lambda = 2$ ($\mu = 0$) then $\varphi = \tilde{\varphi}$ and we see that any vector of the form $(e^{i\varphi}, e^{i\varphi})$ is a fixpoint of s. Hence, all irreducible representations are 1-dimensional. This agrees with the result in [11] which states that, when $\mathrm{Spec}(\mathcal{A}) = \{0\}$, all irreducible finite-dimensional representations of $\mathfrak{U}_d(\mathcal{A})$ are 1-dimensional.

Proposition 8. *Assume that $\mu \neq 2$ and let $\mathcal{A}$ be a DMSA with $\mathrm{Spec}(\mathcal{A}) = \{\mu\}$. If ϕ is an n-dimensional irreducible unitary representation of $\mathfrak{U}_4(\mathcal{A})$ then ϕ is equivalent to a representation ϕ' with $\phi'(t_1) = e^{i\theta} g$ and $\phi'(t_2) = e^{i\theta'} h$ for some $\theta, \theta' \in \mathbb{R}$. Moreover, there exists a $\beta \in \mathbb{R}$ such that $\mu = 4\sin^2(\beta)$ and $e^{i2n\beta} = 1$.*

Proposition 9. *Let $\mathcal{A}$ be a DMSA with $\mathrm{Spec}(\mathcal{A}) = \{2\}$ and let ϕ be an irreducible unitary representation of $\mathfrak{U}_4(\mathcal{A})$. Then ϕ is 4-dimensional and the representation graph of ϕ is one of the two in Fig. 5.*

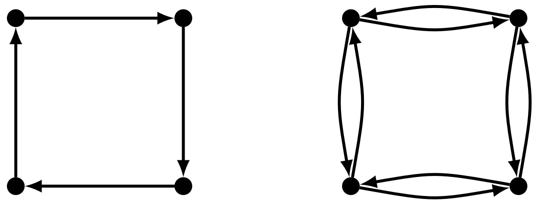

Figure 5. The two different types of representation graphs for irreducible unitary representations when $\mu_1 = \cdots = \mu_4 = 2$.

The graph to the right in Fig. 5 is the Cartesian product of two representation graphs corresponding to 2-dimensional representations defined by (6) and (7). However, one can check that there are 4-dimensional unitary representations that can not be written as a tensor product of two such representations.

3.4 Representations induced by $\mathfrak{sl}_3$

We will now present a DMSA $\mathcal{A}$ constructed from $\mathfrak{sl}_3$, whose representations give rise to normal representations of $\mathfrak{U}_4(\mathcal{A})$. The vertices of the representation graph will be the weight diagram of the $\mathfrak{sl}_3$-representation.

Let α and β be the simple roots of $\mathfrak{sl}_3$. By setting

$$t_1 = e^{i\theta}\big(h_\alpha + e^{i\pi/3} h_\beta\big), \qquad s_1 = e^{-i\theta}\big(h_\alpha + e^{-i\pi/3} h_\beta\big),$$

$$t_2 = e^{i\theta'}\big(e_\alpha + e^{i\varphi_1}e_\beta + e^{i\varphi_2}e_{-\alpha-\beta}\big), \qquad s_2 = e^{-i\theta'}\big(e_{-\alpha} + e^{-i\varphi_1}e_{-\beta} + e^{-i\varphi_2}e_{\alpha+\beta}\big)$$

we obtain a DMSA with $[t_1, s_1] = [t_2, s_2] = 0$ and

$$\big[[t_1, t_2], s_2\big] = \frac{3}{2}l^2 t_1, \qquad \big[[s_1, t_2], s_2\big] = \frac{3}{2}l^2 s_1,$$
$$\big[[t_2, t_1], s_1\big] = \frac{3}{4}l^4 t_2, \qquad \big[[s_2, t_1], s_1\big] = \frac{3}{4}l^4 s_2.$$

In the current convention, a compact real form of $\mathfrak{sl}_3$ is provided by

$$ih_\alpha,\ ih_\beta,\ e_\alpha^+,\ e_\alpha^-,\ e_\beta^+,\ e_\beta^-,\ e_{\alpha+\beta}^+,\ e_{\alpha+\beta}^-$$

as defined in Lemma 2. Hence, any representation is equivalent to one where these elements are represented by *anti-Hermitian* matrices, which implies that $\phi(e_\gamma)^\dagger = \phi(e_{-\gamma})$ and $\phi(h_\gamma)^\dagger = \phi(h_\gamma)$ for $\gamma = \alpha, \beta, \alpha+\beta$. It follows that $\phi(t_1)^\dagger = \phi(s_1)$ and $\phi(t_2)^\dagger = \phi(s_2)$. The weight diagram of a representation is usually presented as vectors with respect to an orthonormal basis of the Cartan subalgebra. In $\mathfrak{sl}_3$ we can construct an orthonormal basis by setting

$$h_1 = \frac{1}{l}h_\alpha, \qquad h_2 = \frac{1}{l\sqrt{3}}(h_\alpha + 2h_\beta),$$

and from this we calculate that

$$t_1 = \frac{l\sqrt{3}}{2}e^{i(\theta-\pi/6)}(h_1 + ih_2).$$

Hence, the eigenvalues of $\phi(t_1)$ will be the weights of the (scaled and rotated) weight diagram of the representation ϕ. As an example, let us study the representations of the kind $\{n, 0\}$, i.e. representations of highest weight nw_1, where w_1, w_2 are the fundamental weights. These representations have dimension $(n+1)(n+2)/2$ and all weights have multiplicity one. Therefore, in a representation ϕ, where the elements of the Cartan subalgebra are diagonal, the representation graph is given by the directed graph of $T = \phi(t_2)$. Since t_2 is a linear combination of e_α, e_β and $e_{-\alpha-\beta}$ we can construct the representation graph by drawing arrows in the direction of these roots in the weight diagram, see Fig. 6.

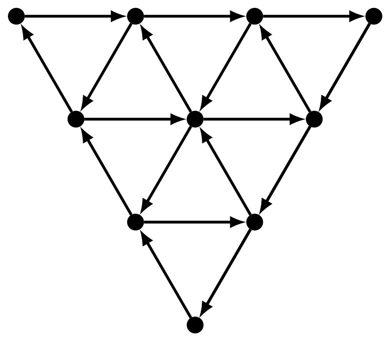

Figure 6. The representation graph corresponding to the $\{3, 0\}$ representation of $\mathfrak{sl}_3$.

3.5 Representations induced from a Clifford algebra

As we have already noted, Clifford algebras satisfy the relations in the enveloping algebra and therefore, any Clifford algebra representation will give a representation of $\mathfrak{U}_d(\mathcal{A})$. As an example, we consider the unique (up to equivalence) irreducible representation of $Cl_{4,0}$ given by $e_i = \sigma_1 \otimes \sigma_i$, for $i = 1, 2, 3$, and $e_4 = \sigma_3 \otimes \mathbb{1}_2$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The *Hermitian* matrices e_1, e_2, e_3, e_4 satisfy the relations $e_ie_j + e_je_i = 2\delta_{ij}\mathbb{1}_4$. In this example, no combination of the form $e_k + ie_l$ will be diagonalizable which, in particular, means that such a combination is never a normal matrix. However, as this is an irreducible representation, any Jordan basis of $\Lambda = e_1 + ie_2$, is a Jordan basis of Λ with respect to $T = e_3 + ie_4$. The Jordan normal form of Λ, as well as the matrix form of T in this basis, are easily computed to be

$$P^{-1}\Lambda P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad P^{-1}TP = \begin{pmatrix} 1+i & 0 & 1 & 0 \\ 0 & -1-i & 0 & -1 \\ -2i & 0 & -1-i & 0 \\ 0 & 2i & 0 & 1+i \end{pmatrix}.$$

Hence, a representation graph $G_\phi = (G, \lambda)$ is given as in Fig. 7 together with $\lambda : \{1, 2, 3, 4\} \to \mathbb{C}$ defined by $\lambda(i) = 0$ for all i. We note that even though the graph is disconnected the representation is irreducible.

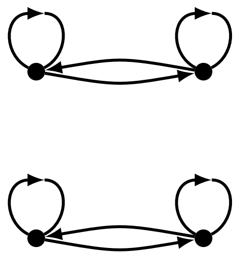

Figure 7. The representation graph of an irreducible representation related to the Clifford algebra $Cl_{4,0}$.

4 Summary

Motivated by several examples, in which double commutator matrix equations arise, we have considered the relations

$$\sum_{j=1}^{d} \big[[x_i, x_j], x_j\big] = \mu_i x_i \tag{18}$$

in a general (Lie) algebraic setting. Some examples can easily be constructed from subsets of semi-simple Lie algebras. Via the Diamond lemma we can show that it is consistent to impose relations (18) in a free associative algebra, and a basis for the corresponding enveloping algebra was computed.

In contrast to the case when $\mu_i = 0$ for $i = 1, \dots, d$ (in which case all irreducible finite-dimensional representations are one dimensional [11]), the representation theory for arbitrary μ_i's has a rich structure. We have considered the case when $d \leq 4$ in detail and introduced the *representation graph*, which encodes the structure of a finite-dimensional representation in terms of a directed graph. The connectivity of the graph provides information on the irreducible components of the representation, and the tensor product can (generically) be described by the Cartesian product of graphs. All unitary representations when $\mathrm{Spec}(\mathcal{A}) = \{\mu\}$ were then classified, and it was shown that essentially all such representations are equivalent to the Fuzzy torus algebra. Several other examples were provided to demonstrate that the representation theory is non-trivial. A particular feature, that in general distinguishes the representation theory from that of Lie algebras, is that the tensor product of two irreducible representations can again be irreducible.

While we think relations (18) are interesting in themselves – as a class of algebras containing Clifford algebras and semi-simple Lie algebras – let us end by noting that we expect matrix sequences corresponding to surfaces of genus $g \geq 2$ to exist, even for $d \leq 4$ and $\mu_1 = \cdots = \mu_4$.

Acknowledgement

We would like to thank the Marie Curie Research Training Network ENIGMA and the Swedish Research Council, as well as the IHES, the Sonderforschungsbereich "Raum-Zeit-Materie" (SFB647) and ETH Zürich, for financial support respectively hospitality – and Martin Bordemann for many discussions and collaboration on related topics.

References

[1] Arnlind J., Graph techniques for matrix equations and eigenvalue dynamics, PhD Thesis, Royal Institute of Technology, 2008.

[2] Arnlind J., Representation theory of C-algebras for a higher-order class of spheres and tori, *J. Math. Phys.* **49** (2008), 053502, 13 pages, arXiv:0711.2943.

[3] Arnlind J., Bordemann M., Hofer L., Hoppe J., Shimada H., Noncommutative Riemann surfaces by embeddings in $\mathbb{R}^3$, *Comm. Math. Phys.* **288** (2009), 403–429, arXiv:0711.2588.

[4] Arnlind J., Hoppe J., Theisen S., Spinning membranes, *Phys. Lett. B* **599** (2004), 118–128.

[5] Berger R., Dubois-Violette M., Inhomogeneous Yang–Mills algebras, *Lett. Math. Phys.* **76** (2006), 65–75, math.QA/0511521.

[6] Bergman G.M., The diamond lemma for ring theory, *Adv. in Math.* **29** (1978), 178–218.

[7] Bianchi L., Sugli spazi a tre dimensioni che ammettono un gruppo continuo di movimenti, *Mem. Soc. Ital. delle Scienze (3)* **11** (1898), 267–352.

[8] Connes A., Dubois-Violette M., Yang–Mills algebra, *Lett. Math. Phys.* **61** (2002), 149–158, math.QA/0206205.

[9] Fairlie D.B., Fletcher P., Zachos C.K., Trigonometric structure constants for new infinite-dimensional algebras, *Phys. Lett. B* **218** (1989), 203–206.

[10] Harary F., Trauth C.A. Jr., Connectedness of products of two directed graphs. *SIAM J. Appl. Math.* **14** (1966), 250–254.

[11] Herscovich E., Solotar A., Representation theory of Yang–Mills algebras, arXiv:0807.3974.

[12] Hoppe J., Quantum theory of a massless relativistic surface and a two-dimensional bound state problem, PhD Thesis, MIT, 1982, available at `http://dspace.mit.edu/handle/1721.1/15717`.

[13] Hoppe J., Diffeomorphism groups, quantization, and SU(∞), *Internat. J. Modern Phys. A* **4** (1989), 5235–5248.

[14] Hoppe J., Some classical solutions of membrane matrix model equations, hep-th/9702169.

[15] Lawson H.B. Jr., Complete minimal surfaces in S^3, *Ann. of Math. (2)* **92** (1970), 335–374.

[16] Nekrasov N., Lectures on open strings, and noncommutative gauge theories, hep-th/0203109.

[17] Sabidussi G., Graph multiplication, *Math. Z.* **72** (1959/1960), 446–457.