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Integral calculus on $E_q(2)$*

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Abstract. The complexes of integral forms on the quantum Euclidean group $E_q(2)$ and the quantum plane are defined and their isomorphisms with the corresponding de Rham complexes are established.

Key words: integral forms; hom-connection; quantum Euclidean group

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1 Introduction

In [5, Chapter 4] the Berezin integral on a supermanifold is explained in terms of the complex of *integral forms*. The boundary operator on this complex is derived from a unique *right connection*, a notion originating from the analysis of supersymmetric $\mathcal{D}$-modules [6]. A right connection is a version of a covariant derivative, whose covariance properties are different from (one might say: dual to) that of the usual connection. The existence of a right connection can be proven by a direct construction in terms of local coordinates.

The notions of a right connection and a complex of integral forms were extended to differential operators over commutative algebras in [9] and, further, over a commutative algebra in the braided category of Yetter–Drinfeld modules in [8].

Studies of right connections for non-commutative algebras were initiated in [1]. Right connections for a non-commutative algebra are defined as maps between modules of (right) linear homomorphisms from (homogeneous parts of) a differential graded algebra to a module over the zero degree part of this algebra, and hence termed *hom-connections* (as opposed to usual connections which operate between the tensor product of a module and a differential graded algebra). The local arguments which lead to the existence of right connections on supermanifolds are not available in the realm of noncommutative geometry. In their stead an effective method of constructing differential graded algebras with a unique hom-connection was presented in [2]. This method is based on the use of twisted multi-derivations.

The aim of this note is to calculate explicitly two examples of complexes of integral forms based on twisted multi-derivations, and to show that these complexes are isomorphic to the corresponding de Rham complexes, which implies (a version of) the Poincaré duality. The examples in question are the quantum Euclidean group $E_q(2)$ obtained by the contraction of the quantum group $SU_q(2)$ in [11], and its homogeneous space – the quantum plane.

The paper is organised as follows. In Section 2 we briefly recall from [1] and [2] definitions of hom-connections and integral forms and the method of constructing hom-connections in differential calculi determined by q-skew derivations. In Section 3 we recall from [11] the definition of the quantum group $E_q(2)$, describe a three-dimensional differential calculus on it

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and then explicitly construct a flat hom-connection and the complex of integral forms on $E_q(2)$. The explicit isomorphism of the complexes of differential and integral forms is then presented. In Section 4 we derive a hom-connection and the complex of integral forms on the quantum plane understood as the (quantum) homogeneous space of $E_q(2)$. The paper is concluded with comments and outlook.

We hope that the material contained in Section 2 of this note will provide the reader with a concise introduction to non-commutative integral forms. We also hope that explicitly calculated examples (in Sections 3 and 4), which append examples presented in [2], will indicate the methods and techniques involved in construction of integral forms on quantum spaces.

2 Noncommutative integral forms

An algebra means an associative algebra with identity over a field $\Bbbk$. By a *differential graded algebra over an algebra* A we mean a non-negatively graded algebra

$$\Omega(A) := \bigoplus_{k=0}^{\infty} \Omega^k(A),$$

such that $\Omega^0(A) = A$, together with a degree one operation $d : \Omega^k(A) \to \Omega^{k+1}(A)$ that is nilpotent, i.e. $d \circ d = 0$, and satisfies the graded Leibniz rule, i.e., for all $\omega \in \Omega^k(A)$ and $\omega' \in \Omega(A)$,

$$d(\omega\omega') = d\omega\omega' + (-1)^k \omega d\omega'.$$

Each of the $\Omega^k(A)$ is an A-bimodule, and we denote the right dual of $\Omega^k(A)$ (i.e. the collection of all right A-linear maps $\Omega^k(A) \to A$) by $\mathfrak{I}^k(A)$, that is

$$\mathfrak{I}^k(A) := \mathrm{Hom}_A(\Omega^k(A), A).$$

Each of the $\mathfrak{I}^k(A)$ is an A-bimodule with the actions defined by

$$a \cdot \phi \cdot b(\omega) := a\phi(b\omega), \qquad \text{for all } \ a, b \in A,\ \phi \in \mathfrak{I}^k(A),\ \omega \in \Omega^k(A). \tag{1}$$

To ease the notation we write $a\phi b$ for $a \cdot \phi \cdot b$. The right action (1) is a special case of more general operation that, for any $\omega \in \Omega^l(A)$, sends $\phi \in \mathfrak{I}^{k+l}(A)$ to $\phi\omega \in \mathfrak{I}^k(A)$. The latter is defined by

$$\phi\omega(\omega') := \phi(\omega\omega'), \qquad \text{for all } \ \omega' \in \Omega^k(A).$$

A *hom-connection* on a $\Bbbk$-algebra A (with respect to the differential graded algebra $\Omega(A)$ over A) is a $\Bbbk$-linear map $\nabla : \mathfrak{I}^1(A) \to A$ such that,

$$\nabla(\phi a) = \nabla(\phi)a + \phi(da), \qquad \text{for all } \ a \in A,\ \phi \in \mathfrak{I}^1(A).$$

Any hom-connection can be extended to a family of maps $\nabla_k : \mathfrak{I}^{k+1}(A) \to \mathfrak{I}^k(A)$ by the formula

$$\nabla_k(\phi)(\omega) := \nabla(\phi\omega) + (-1)^{k+1}\phi(d\omega), \qquad \text{for all } \ \phi \in \mathfrak{I}^{k+1}(A),\ \omega \in \Omega^k(A).$$

The ∇_k satisfy the following graded Leibniz rule

$$\nabla_l(\phi\omega) = \nabla_{k+l}(\phi)\omega + (-1)^{k+l}\phi d\omega, \qquad \text{for all } \ \phi \in \mathfrak{I}^{k+l+1}(A),\ \omega \in \Omega^k(A). \tag{2}$$

The map $F := \nabla \circ \nabla_1$ is a right A-module homomorphism which is called the *curvature* of ∇. A hom-connection ∇ is said to be *flat* provided $F = 0$. To a flat hom-connection ∇ one associates a chain complex $\left(\bigoplus_{k=0}\mathfrak{I}^k(A),\nabla\right)$. This complex is termed a *complex of integral forms* on A, and the canonical map

$$\Lambda: \ A \longrightarrow \mathrm{coker}(\nabla) = A/\mathrm{Im}\nabla$$

is called a ∇-*integral on* A (or simply an integral on A).

A general construction of hom-connections based on twisted multi-derivations was presented in [2]. This construction is applicable to all left-covariant differential calculi on quantum groups. Presently we outline this construction in the special case of *q-skew derivations* and the reader is referred to [2, Section 3] for more details and for the general case.

Let A be a $\Bbbk$-algebra. Following [3], a linear map $\partial : A \to A$ is called a *q-skew derivation* provided there exists an algebra automorphism $\sigma : A \to A$ and a scalar q such that $\sigma^{-1}\circ\partial\circ\sigma = q\partial$ and, for all $a, b \in A$,

$$\partial(ab) = \partial(a)\sigma(b) + a\partial(b), \tag{3}$$

i.e. ∂ satisfies the σ-*twisted Leibniz rule*. Starting with q_i-skew derivations ∂_i, $i = 1, 2, \ldots, n$, (with corresponding automorphisms σ_i) one constructs a first order differential calculus on A as follows. $\Omega^1(A)$ is a free left A-module $\bigoplus_{i=1}^n A\omega_i$ with basis $\omega_1, \ldots, \omega_n$ and right A-action given by

$$\omega_i a = \sigma_i(a)\omega_i, \qquad i = 1, 2, \ldots, n. \tag{4}$$

The exterior differential $d : A \to \Omega^1(A)$ is defined by the formula

$$da = \sum_i \partial_i(a)\omega_i = \sum_{i,j} \omega_i\sigma_i^{-1}\left(\partial_j\left(a\right)\right). \tag{5}$$

Often one reserves the term *differential calculus* to the pair $(\Omega^1(A), d)$ that is *dense* in the sense that every element of $\Omega^1(A)$ is of the form $\sum_i a_i db_i$, for some $a_i, b_i \in A$. The calculus described above is dense if and only if there exist two finite subsets $\{a_{i\,t}\}$, $\{b_{i\,t}\}$ of elements of A such that,

$$\sum_t a_{i\,t}\partial_k(b_{i\,t}) = \delta_{ik}, \qquad \text{for all} \ \ i, k = 1, \ldots, n.$$

The examples of calculi discussed in Sections 3 and 4 are dense in this sense.

$(\Omega^1(A), d)$ can be extended to a full differential graded algebra $(\Omega(A), d)$ in the standard way, that is by using the graded Leibniz rule and the A-bimodule structure of $\Omega^1(A)$.

The calculus $(\Omega^1(A), d)$ determined by the formulae (4), (5) admits a hom-connection $\nabla : \mathfrak{I}^1(A) \to A$, given by, for all $f \in \mathfrak{I}^1(A)$,

$$\nabla(f) = \sum_i q_i\partial_i(f(\omega_i)). \tag{6}$$

This is a unique hom-connection on $\Omega(A)$ with the property that $\nabla(\xi_i) = 0$, where $\xi_i \in \mathfrak{I}^1(A)$ are such that $\xi_i(\omega_j) = \delta_{ij}$, $i, j = 1, 2, \ldots, n$.

3 Integral geometry of $E_q(2)$

In this and the following section the algebras are over the field of complex numbers. The (polynomial part of the) quantum Euclidean group $E_q(2)$, obtained by the contraction of $SU_q(2)$ in [11], is a $*$-Hopf algebra generated by v, n subject to relations

$$nn^* = n^*n, \qquad vv^* = v^*v = 1, \qquad vn = qnv, \qquad vn^* = qn^*v,$$

where $q \in \mathbb{R}$. The elements v, v^* are grouplike, while

$$\Delta(n) = v \otimes n + n \otimes v^*, \qquad \varepsilon(n) = 0, \qquad S(n) = -q^{-1}n, \qquad S(n^*) = -qn^*.$$

By setting $z = vn$, the algebra $E_q(2)$ can be equivalently defined as generated by a unitary element v and elements z, z^* subject to relations

$$vz = qzv, \qquad vz^* = qz^*v, \qquad zz^* = q^2 z^* z. \tag{7}$$

The algebra $E_q(2)$ has a $\mathbb{Z}$-grading, defined by

$$|v| = |n^*| = 1, \qquad |v^*| = |n| = -1.$$

In particular, the elements z, z^* generate the zero-degree subalgebra of $E_q(2)$ which we denote by $\mathbb{C}_q$.

A three-dimensional left-covariant calculus $\Omega^1(E_q(2))$ on $E_q(2)$ can be obtained by contraction of the 3D calculus on $SU_q(2)$ introduced in [10]. $\Omega^1(E_q(2))$ is generated by left-invariant forms ω_0, $\omega_\pm$ subject to relations

$$\begin{aligned} &\omega_0 v = q^{-2} v\omega_0, \qquad \omega_0 n = q^2 n\omega_0, \qquad \omega_0 n^* = q^{-2} n^*\omega_0, \qquad \omega_0 v^* = q^2 v^*\omega_0,\\ &\omega_\pm v = q^{-1} v\omega_\pm, \qquad \omega_\pm n = qn\omega_\pm, \qquad \omega_\pm n^* = q^{-1} n^*\omega_\pm, \qquad \omega_\pm v^* = qv^*\omega_\pm. \end{aligned} \tag{8}$$

This is a $*$-calculus with $\omega_0^* = -\omega_0$, $\omega_\pm^* = q^{\mp 1}\omega_\mp$. The action of the exterior differential d on the generators is

$$\begin{aligned} &dv = v\omega_0, \qquad && dn = -q^2 n\omega_0 + v\omega_-,\\ &dn^* = n^*\omega_0 + q^2 v^*\omega_+, \qquad && dv^* = -q^2 v^*\omega_0. \end{aligned} \tag{9}$$

Consequently,

$$\omega_0 = v^* dv, \qquad \omega_- = v^* dn - q^{-1} n dv^*, \qquad \omega_+ = q^{-2} v dn^* - q^{-1} n^* dv. \tag{10}$$

In view of commutation rules (8), the module structure of $\Omega^1(E_q(2))$ is of the type described by (4) with automorphisms σ_0 and $\sigma_+ = \sigma_-$ given by, for all homogeneous $a \in A$ (with the $\mathbb{Z}$-degree $|a|$),

$$\sigma_0(a) = q^{-2|a|} a, \qquad \sigma_\pm(a) = q^{-|a|} a. \tag{11}$$

The formulae (9) or (10) indicate that the $\mathbb{Z}$-grading of $E_q(2)$ can be extended to a $\mathbb{Z}$-grading of $\Omega^1(E_q(2))$ such that the differential d is the degree preserving map, by setting

$$|\omega_0| = 0, \qquad |\omega_\pm| = \pm 2.$$

Equations (9) determine (or can be understood as determined by as in (5)) maps $\partial_i : E_q(2) \to E_q(2)$ that satisfy the σ_i-twisted derivation properties (3). Explicitly, in terms of actions on generators of $E_q(2)$ the maps ∂_i are

$$\begin{aligned} &\partial_0(v) = v, \qquad &&\partial_0(n) = -q^2 v, \qquad &&\partial_0(n^*) = n^*, \qquad &&\partial_0(v^*) = -q^2 v^*,\\ &\partial_+(v) = 0, &&\partial_+(n) = 0, &&\partial_+(n^*) = q^2 v^*, &&\partial_+(v^*) = 0,\\ &\partial_-(v) = 0, &&\partial_-(n) = v, &&\partial_-(n^*) = 0, &&\partial_-(v^*) = 0. \end{aligned}$$

Since $da = \partial_-(a)\omega_- + \partial_0(a)\omega_0 + \partial_+(a)\omega_+$, the derivations have $\mathbb{Z}$-degrees

$$|\partial_0| = 0, \qquad |\partial_\pm| = \mp 2, \tag{12}$$

i.e., for all homogeneous $a \in E_q(2)$, $|\partial_0(a)| = |a|$, $|\partial_\pm(a)| = |a| \mp 2$. Combining equations (11) with (12) one easily finds that the maps ∂_0, ∂_+, ∂_- are q_i-skew derivations with constants 1, q^{-2} and q^2, respectively. Therefore, there is a hom-connection on $E_q(2)$ (with respect to $\Omega^1(E_q(2))$) given by the formula (6), that is, for all $f \in \mathfrak{I}^1(E_q(2))$,

$$\nabla(f) = \partial_0\left(f\left(\omega_0\right)\right) + q^{-2}\partial_+\left(f\left(\omega_+\right)\right) + q^2\partial_-\left(f\left(\omega_-\right)\right). \tag{13}$$

The calculus $\Omega^1(E_q(2))$ can be extended to the full differential graded algebra. The relations in the bimodule $\Omega^2(E_q(2))$ (deformed exterior product) are

$$\omega_i^2 = 0, \qquad \omega_+\omega_- = -q^2\omega_-\omega_+, \qquad \omega_0\omega_- = -q^4\omega_-\omega_0, \qquad \omega_+\omega_0 = -q^4\omega_0\omega_+, \tag{14}$$

and the exterior derivative is

$$d\omega_0 = 0, \qquad d\omega_+ = q^2(q^2+1)\,\omega_0\omega_+, \qquad d\omega_- = q^2(q^2+1)\omega_-\omega_0. \tag{15}$$

Thus $\Omega^2(E_q(2))$ is a free module generated by three closed forms $\omega_-\omega_0$, $\omega_0\omega_+$ and $\omega_-\omega_+$. In degree 3, $\Omega^3(E_q(2))$ is generated by the form $\omega_-\omega_0\omega_+$.

It might be worth noticing at this point that the differential graded algebra $\Omega^\bullet(E_q(2))$ can be equivalently (and conveniently) described in terms of generators v, z, z^*. An easy calculation reveals that

$$\omega_+ = q^{-3}v^2dz^*, \qquad \omega_- = v^{*2}dv, \tag{16}$$

so that

$$dz^* = q^3v^{*2}\omega_+, \qquad dz = v^2\omega_-. \tag{17}$$

Therefore, the $*$-calculus $\Omega^1(E_q(2))$ is freely generated by dv, dz and dz^* subject to relations

$$\begin{aligned} &vdv = q^2dvv, \qquad v^*dv = q^{-2}dvv^*, \qquad zdv = q^{-1}dvz, \qquad z^*dv = q^{-1}dvz^*,\\ &vdz = qdzv, \qquad vdz^* = qdz^*v, \qquad zdz = q^{-2}dzz, \qquad z^*dz = q^{-2}dzz^*. \end{aligned} \tag{18}$$

The wedge product calculated from relations (14) comes out as

$$\begin{aligned} &(dz)^2 = (dv)^2 = dvdv^* = 0, \qquad dvdz = -qdzdv,\\ &dvdz^* = -qdz^*dv, \qquad dzdz^* = -q^2dz^*dz. \end{aligned} \tag{19}$$

Note that relations (19) are simply obtained by differentiating rules (18) (and observing that v^* is the inverse of v).

Every element of $\Omega^3(E_q(2))$ can be written as a linear combination of

$$v^kz^{*l-1}z^{m-1}dzdvdz^*, \qquad k \in \mathbb{Z},\ l, m \in \mathbb{N}.$$

On the other hand, relations (18) and the Leibniz rule imply that

$$d\big(v^kz^{*l-1}z^m\big) = d\big(v^kz^{*l-1}\big)z^m + [m]_{q^2}v^kz^{*l-1}z^{m-1}dz,$$

where

$$[m]_x = 1 + x + \cdots + x^{m-1}$$

is a notation for the x-integer. Therefore,

$$v^kz^{*l-1}z^{m-1}dzdvdz^* = d\left(\frac{1}{[m]_{q^2}}v^kz^{*l-1}z^mdvdz^*\right), \tag{20}$$

so the third de Rham cohomology group of $E_q(2)$ is trivial, $H^3(E_q(2)) = 0$.

Our next aim is to show that ∇ defined by equation (13) is a flat hom-connection and thus defines a complex of integral forms on $E_q(2)$. To this end we need first to extend ∇ to $\nabla_1 : \mathfrak{I}^2(E_q(2)) \to \mathfrak{I}^1(E_q(2))$. Define $\phi_0, \phi_\pm \in \mathfrak{I}^2(E_q(2))$ by

$$\phi_0(\omega_-\omega_+) = 1, \qquad \phi_+(\omega_-\omega_0) = 1, \qquad \phi_-(\omega_0\omega_+) = 1,$$

and zero on other generators. By inspection of relations (14) one concludes that any $f \in \mathfrak{I}^2(E_q(2))$ can be written as $f = \phi_0 a_0 + \phi_+ a_+ + \phi_- a_-$, for suitably defined $a_i \in E_q(2)$. Since the curvature of a hom-connection is a right $E_q(2)$-linear map, it suffices to compute it on the ϕ_i. Let $\xi_0, \xi_\pm \in \mathfrak{I}^1(E_q(2))$ be the right dual basis to ω_0, $\omega_\pm$, i.e. such that $\xi_0(\omega_0) = \xi_\pm(\omega_\pm) = 1$ and zero on other generators of $\Omega^1(E_q(2))$. Using commutation rules (14) one easily computes that

$$\begin{aligned}
&\phi_0\omega_0 = 0, &&\phi_0\omega_+ = -q^2\xi_-, &&\phi_0\omega_- = \xi_+,\\
&\phi_+\omega_0 = -q^4\xi_-, &&\phi_+\omega_+ = 0, &&\phi_+\omega_- = \xi_0,\\
&\phi_-\omega_0 = \xi_+, &&\phi_-\omega_+ = -q^4\xi_0, &&\phi_-\omega_- = 0.
\end{aligned} \tag{21}$$

By definition, for all one-forms ω and all $\phi \in \mathfrak{I}^2(E_q(2))$,

$$\nabla_1(\phi)(\omega) = \nabla(\phi\omega) + \phi(d\omega). \tag{22}$$

Combining (21) with (22) and (15), and remembering that ∇ given by (13) is the unique hom-connection such that $\nabla(\xi_i) = 0$, $i = 0, +, -$, we obtain

$$\nabla_1(\phi_0) = 0, \qquad \nabla_1(\phi_+) = q^2(q^2+1)\xi_-, \qquad \nabla_1(\phi_-) = q^2(q^2+1)\xi_+.$$

Since $\nabla(\xi_i) = 0$, $\nabla \circ \nabla_1(\phi_i) = 0$, for all $i = 0, +, -$, and the hom-connection (13) is flat.

The module $\mathfrak{I}^3(E_q(2))$ is generated by $\phi \in \mathfrak{I}^3(E_q(2))$ defined by $\phi(\omega_-\omega_0\omega_+) = 1$. Using commutation rules (14), one finds

$$\phi\omega_-\omega_0 = \xi_+, \qquad \phi\omega_0\omega_+ = q^6\xi_-, \qquad \phi\omega_-\omega_+ = -q^4\xi_0. \tag{23}$$

Since, for any 2-form ω, $\nabla_2(\phi)(\omega) = \nabla(\phi\omega) - \phi(d\omega)$, the forms generating $\Omega^2(E_q(2))$ are closed and $\nabla(\xi_i) = 0$, $i = 0, +, -$, we conclude that

$$\nabla_2(\phi) = 0.$$

This completes the description of the complex

$$0 \longrightarrow \mathfrak{I}^3(E_q(2)) \xrightarrow{\nabla_2} \mathfrak{I}^2(E_q(2)) \xrightarrow{\nabla_1} \mathfrak{I}^1(E_q(2)) \xrightarrow{\nabla} E_q(2) \xrightarrow{\Lambda} \mathrm{coker}\nabla \longrightarrow 0 \tag{24}$$

of integral forms on $E_q(2)$. Here Λ is the associated integral on $E_q(2)$. We will presently show that $E_q(2)$ enjoys the strong Poincaré duality in the sense that the complex (24) is isomorphic to the de Rham complex. In view of the triviality of $H^2(E_q(2))$ this will allow us to deduce that $\mathrm{coker}\nabla = 0$, hence the integral Λ is zero. The isomorphism of the de Rham and integral complexes means the following commutative diagram

$$\begin{tikzcd}
E_q(2) \arrow[r, "d"] \arrow[d, "\Theta^*"'] & \Omega^1(E_q(2)) \arrow[r, "d"] \arrow[d, "\Phi"'] & \Omega^2(E_q(2)) \arrow[r, "d"] \arrow[d, "\Psi"'] & \Omega^3(E_q(2)) \\
\mathfrak{I}^3(E_q(2)) \arrow[r, "\nabla_2"] & \mathfrak{I}^2(E_q(2)) \arrow[r, "\nabla_1"] & \mathfrak{I}^1(E_q(2)) \arrow[r, "\nabla"] & E_q(2) \arrow[u, "\Theta"']
\end{tikzcd} \tag{25}$$

in which all columns are (right A-module) isomorphisms. These are defined as follows. $\Theta^*(a) = \phi a$, $\Theta(a) = \omega_-\omega_0\omega_+ a$, and

$$\Phi(\omega_- a + \omega_0 b + \omega_+ c) = \phi_- a - q^4\phi_0 b + q^6\phi_+ c,$$
$$\Psi(\omega_-\omega_0 a + \omega_-\omega_+ b + \omega_0\omega_+ c) = \xi_+ a - q^4\xi_0 b + q^6\xi_- c,$$

for all $a, b, c \in E_q(2)$ (compare the form of Ψ with relations (23)). The commutativity of the diagram (25) can be checked by a straightforward albeit lengthy calculation. For example, the commutativity of the leftmost square can be verified as follows. Take a homogeneous $a \in E_q(2)$ with the $\mathbb{Z}$-degree $|a|$. In view of the form of commutation rules (8) or the automorphisms $\sigma_\pm$, σ_0 in (11), and the grading of derivations $\partial_\pm$, ∂_0, one easily computes

$$\begin{aligned}\nabla_2 \circ \Theta^*(a)(\omega_-\omega_0) &= \nabla_2(\phi)(\omega_-\omega_0)a + \phi da(\omega_-\omega_0)\\ &= \phi(\partial_+(a)\omega_+\omega_-\omega_0) = q^{4|a|-8}\phi(\omega_+\omega_-\omega_0)\partial_+(a) = q^{4|a|-2}\partial_+(a),\end{aligned}$$

where the first equality follows by the Leibniz rule for a hom-connection (2), and the last two equalities by the right linearity and the definition of ϕ combined with the commutation rules (14). On the other hand

$$\begin{aligned}\Phi(da)(\omega_-\omega_0) &= \Phi(\partial_-(a)\omega_- + \partial_0(a)\omega_0 + \partial_+(a)\omega_+)(\omega_-\omega_0)\\ &= q^{|a|+4}\phi_+(\partial_+(a)\omega_-\omega_0) = q^{4|a|-2}\partial_+(a),\end{aligned}$$

where the equations (11), the definitions of Φ and of $\phi_\pm$, ϕ_0 were used. In the same way one computes

$$\nabla_2 \circ \Theta^*(a)(\omega_-\omega_+) = -q^{4|a|+4}\partial_0(a) = \Phi(da)(\omega_-\omega_+)$$

and

$$\nabla_2 \circ \Theta^*(a)(\omega_0\omega_+) = q^{4|a|+8}\partial_-(a) = \Phi(da)(\omega_0\omega_+).$$

Therefore,

$$\nabla_2 \circ \Theta^* = \Phi \circ d,$$

as required. The commutativity of the other squares in diagram (25) is proven similarly.

The commutativity of (25) implies that coker∇ is isomorphic to the third de Rham cohomology group $H^3(E_q(2))$, and thus allows one to compute the form of the former (and hence the integral on $E_q(2)$). Recall that $H^3(E_q(2)) = 0$, so also coker$\nabla = 0$, i.e. ∇ is surjective, and the corresponding integral Λ is zero. More specifically, take $a \in E_q(2)$. If $\omega \in \Omega^2(E_q(2))$ is such that $d\omega = \Theta(a)$ (and ω exists by the triviality of $H^3(E_q(2))$), then

$$\nabla(\Psi(\omega)) = a.$$

For an element $a = v^k z^{*l} z^m$ of a basis for $E_q(2)$,

$$\Theta\big(v^k z^{*l} z^m\big) = q^{-4k-8+m+l} v^{k-1} z^{*l} z^m dz dv dz^*,$$

where the definition of Θ and relations (16), (18) and (7) were used. In view of (20) followed by (18) and (17) the corresponding ω comes out as

$$\omega = \frac{q^{-4k-8+m+l}}{[m+1]_{q^2}} v^{k-1} z^{*l} z^{m+1} dv dz^* = \frac{q^{-k-6+2m+2l}}{[m+1]_{q^2}} \omega_0\omega_+ v^{k-2} z^{*l} z^{m+1},$$

therefore,

$$v^k z^{*l} z^m = \nabla\left(\frac{q^{-k+2m+2l}}{[m+1]_{q^2}}\xi_- v^{k-2} z^{*l} z^{m+1}\right),$$

which explicitly proves that ∇ is onto.

Note that although the isomorphisms Φ, Ψ, Θ and Θ^* are right $E_q(2)$-linear, they are not $E_q(2)$-bimodule maps, when the $\mathfrak{I}^k(E_q(2))$ are viewed as $E_q(2)$-bimodules by (1). However, being right $E_q(2)$-module isomorphisms, they can be forced to produce new left $E_q(2)$-actions on the $\mathfrak{I}^k(E_q(2))$ and also to make the complex of integral forms a differential graded algebra, isomorphic to $\Omega(E_q(2))$. The basic integral forms ϕ, $\phi_\pm$, ϕ_0, $\xi_\pm$, ξ_0 themselves form a seven-dimensional *skeletal algebra of integral forms* on $E_q(2)$. The multiplication table can be easily worked out from the definitions of isomorphisms Φ, Ψ, Θ and Θ^*, and from the multiplication rules in $\Omega(E_q(2))$:

	ϕ	ϕ_-	ϕ_0	ϕ_+	ξ_-	ξ_0	ξ_+
ϕ	ϕ	ϕ_-	ϕ_0	ϕ_+	ξ_-	ξ_0	ξ_+
ϕ_-	ϕ_-	0	$-q^{-4}\xi_+$	$-q^{-2}\xi_0$	$q^{-6}\phi$	0	0
ϕ_0	ϕ_0	ξ_+	0	$-q^{-4}\xi_-$	0	$-q^{-4}\phi$	0
ϕ_+	ϕ_+	ξ_0	ξ_-	0	0	0	ϕ
ξ_-	ξ_-	ϕ	0	0	0	0	0
ξ_0	ξ_0	0	$-q^{-4}\phi$	0	0	0	0
ξ_+	ξ_+	0	0	$q^{-6}\phi$	0	0	0

4 Integral geometry of $\mathbb{C}_q$

The subalgebra $\mathbb{C}_q$ of $E_q(2)$ generated by z, z^* is a quantum homogeneous space of $E_q(2)$. It is also a base algebra for the quantum principal bundle with the quantum total space $E_q(2)$ and the structure group $U(1)$ identified with the Hopf algebra of Laurent polynomials in variable u, $\mathbb{C}[u,u^{-1}]$ (u is a grouplike element). In other words, $E_q(2)$ is a principal comodule algebra. The coaction of $\mathbb{C}[u,u^{-1}]$ on $E_q(2)$ is given by the $\mathbb{Z}$-grading, i.e. for any $a \in E_q(2)$ of $\mathbb{Z}$-degree $|a|$,

$$a \longmapsto a \otimes u^{|a|}. \tag{26}$$

$\mathbb{C}[u,u^{-1}]$ has a natural $*$-Hopf algebra structure given by $u^* = u^{-1}$. With respect to this, the coaction (26) is a $*$-algebra map.

The calculus described by equations (18) and (19) restricts to the calculus on $\mathbb{C}_q$. By [2, Theorem 4.3], the hom-connection (13) restricts to the hom-connection on the quantum plane $\mathbb{C}_q$,

$$\nabla: \ \mathfrak{I}^1(\mathbb{C}_q) \to \mathbb{C}_q, \qquad \nabla f = q^2\partial_-(\hat{f}(\omega_-)) + q^{-1}\partial_+(\hat{f}(\omega_+)), \tag{27}$$

where

$$\hat{f}(\omega_-) = f\big(\omega_- v^2\big)v^{*2}, \qquad \hat{f}(\omega_+) = f\big(\omega_+ v^{*2}\big)v^2.$$

The skew derivations ∂_+ and ∂_- are not defined on $\mathbb{C}_q$ but on $E_q(2)$, hence (27) is not the optimal description of the hom-connection ∇ on $\mathbb{C}_q$ as it seems to depend on the embedding of $\mathbb{C}_q$ into $E_q(2)$. The calculus $\Omega^1(\mathbb{C}_q)$ is freely generated by the holomorphic form dz and the antiholomorphic form dz^*. Using the right $\mathbb{C}_q$-linearity of $f \in \mathfrak{I}^1(\mathbb{C}_q)$ and commutation rules (18) one easily finds that

$$\hat{f}(\omega_-) = q^{-2}f(dz)v^{*2}, \qquad \hat{f}(\omega_+) = q^{-1}f(dz^*)v^2. \tag{28}$$

Next, introduce (twisted) derivatives ∂, $\bar\partial$ associated to dz, dz^* by the formula

$$d(a) = \partial(a)dz + \bar\partial(a)dz^*, \qquad \text{for all } \ a \in \mathbb{C}_q.$$

Computing $d(av^{*2})$ and $d(av^2)$ with the help of equations (17) and commutation rules in $\Omega^1(E_q(2))$, one easily finds that, for all $a \in \mathbb{C}_q$,

$$d(av^{*2}) = q^2\partial(a)\omega_- + \bar\partial(a)dz^*v^{*2}, \qquad d(av^2) = \partial(a)dzv^2 + q\bar\partial(a)\omega_+.$$

This implies that

$$\partial_-(av^{*2}) = q^2\partial(a), \qquad \partial_+(av^2) = q\bar\partial(a)\omega_+, \qquad \text{for all } \ a \in \mathbb{C}_q. \tag{29}$$

Putting equations (28), (29) and (27) together we obtain the following formula for a hom-connection on $\mathbb{C}_q$ given in terms of (anti)holomorphic forms,

$$\nabla(f) = q^2\partial\left(f\left(dz\right)\right) + q^{-2}\bar\partial\left(f\left(dz^*\right)\right), \qquad \text{for all } \ f \in \mathfrak{I}^1(\mathbb{C}_q). \tag{30}$$

Working entirely in terms of dz and dz^*, the module of integral forms $\mathfrak{I}^1(\mathbb{C}_q)$ is generated by ξ, $\bar\xi$ defined by

$$\xi(dz) = \bar\xi(dz^*) = 1, \qquad \xi(dz^*) = \bar\xi(dz) = 0.$$

The formula (30) immediately implies that $\nabla(\xi) = \nabla(\bar\xi) = 0$.

The degree-two integral forms $\mathfrak{I}^2(\mathbb{C}_q)$ are generated by ψ determined by $\psi(dzdz^*) = 1$, as one easily finds that $\Omega^2(\mathbb{C}_q)$ is generated by $dzdz^*$; see relations (19). The latter imply that

$$\psi dz = \bar\xi, \qquad \psi dz^* = -q^{-2}\xi,$$

and thus arguments analogous to those in the case of $E_q(2)$ affirm flatness of ∇.

Similarly to the quantum Euclidean group, the quantum plane $\mathbb{C}_q$ enjoys the strong Poincaré duality, i.e. there is a commutative diagram

$$\begin{tikzcd}
\mathbb{C}_q \arrow[r, "d"] \arrow[d, "\Theta^*"'] & \Omega^1(\mathbb{C}_q) \arrow[r, "d"] \arrow[d, "\Phi"'] & \Omega^2(\mathbb{C}_q) \\
\mathfrak{I}^2(\mathbb{C}_q) \arrow[r, "\nabla_1"] & \mathfrak{I}^1(\mathbb{C}_q) \arrow[r, "\nabla"] & \mathbb{C}_q \arrow[u, "\Theta"']
\end{tikzcd},$$

in which vertical maps are right $\mathbb{C}_q$-module isomorphisms. These are given by, for all $a, b \in \mathbb{C}_q$,

$$\Theta(a) = dzdz^*a, \qquad \Phi(dza + dz^*a) = -\bar\xi a + q^{-2}\xi b, \qquad \Theta^*(a) = \psi a.$$

The second de Rham cohomology group is trivial, hence, consequently, the integral on $\mathbb{C}_q$ is zero.

5 Comments

In this note we presented two explicit examples of complexes of integral forms. In relation to such specific examples one can pose two more general questions. First, the quantum group $E_q(2)$ and its differential calculus are obtained by contraction of $SU_q(2)$ and a calculus on it. It might be interesting to study in general the effect of contraction on integral geometry or to develop the contraction procedure for integral forms. Second, together with examples in [2], the examples presented here provide an indication that the (strong) Poincaré duality in noncommutative geometry could be understood as existence of an isomorphism between differential and integral complexes. One could even venture a suggestion that a (compact without boundary) noncommutative differentiable or smooth manifold should be understood as a differential graded algebra $\Omega(A)$ over A which is isomorphic (as a complex) to the complex of integral forms $(\mathfrak{I}^\bullet(A), \nabla)$ (with respect to $\Omega(A)$). It would be rather interesting to investigate, what classes of differential graded algebras can be characterised by this property, and how this viewpoint on the Poincaré duality compares with the (algebraic, homological) noncommutative Poincaré duality of Van den Bergh [7]; see also [4].

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