On Classical Dynamics of Affinely-Rigid Bodies Subject to the Kirchhoff–Love Constraints*

Vasyl KOVALCHUK

Institute of Fundamental Technological Research, Polish Academy of Sciences,
5^B Pawińskiego Str., 02-106 Warsaw, Poland
E-mail: *vkoval@ippt.gov.pl*

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Abstract. In this article we consider the affinely-rigid body moving in the three-dimensional physical space and subject to the Kirchhoff–Love constraints, i.e., while it deforms homogeneously in the two-dimensional central plane of the body it simultaneously performs one-dimensional oscillations orthogonal to this central plane. For the polar decomposition we obtain the stationary ellipsoids as special solutions of the general, strongly nonlinear equations of motion. It is also shown that these solutions are conceptually different from those obtained earlier for the two-polar (singular value) decomposition.

Key words: affinely-rigid bodies with degenerate dimension; Kirchhoff–Love constraints; polar decomposition; Green deformation tensor; deformation invariants; stationary ellipsoids as special solutions

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1 Introduction

The special interest in the present work is devoted to the classical description of an affinely-rigid (homogeneously deforming) mechanical system subject to the Kirchhoff–Love constraints. We know that the standard continuum theory as well as some fundamental theories deal with such objects as membranes, plates, discs, etc. So, the main contribution of this work is to present a toy model for the analytical description of the above-mentioned objects.

The structure of this article is as follows: Firstly, we will present the main notions about the concept of the affinely-rigid body, as a generalization of the metrically-rigid one, and of its special case, i.e., the affinely-rigid body with degenerate dimension. Secondly, for convenience of the Reader the main results obtained earlier for the case of two-polar (singular value) decomposition are remembered. Thirdly, an alternative (polar) decomposition is introduced and the equations of motion for our toy model are obtained for the general form of the inertial tensor, i.e., when $J_1 \neq J_2 \neq J_3$. And finally, three main branches of special solutions (stationary ellipsoids) for our strongly nonlinear equations of motion are gathered in the form of Proposition 1. Additionally some remarks about the complementarity of the obtained results to those described in our previous work [6] are presented in the Summary.

So, let us remind some basic facts generally concerning the notion of affinely-rigid bodies [14, 15, 17, 18].

Let $(M, V, \rightarrow)$ be an affine space and $(M, V, \rightarrow, g)$ be the corresponding Euclidean one, where M is a physical space in which the classical system of material points (discrete or continuous) is placed, V is a linear space of translations (free vectors) in M, and $g \in V^* \otimes V^*$ is the metric

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tensor. Also let us introduce an affine $(N, U, \rightarrow)$ and the corresponding Euclidean $(N, U, \rightarrow, \eta)$ spaces, where N is the material space of labels which are assigned to every material point of our body in some way, U is the corresponding linear space of translations in N, and $\eta \in U^* \otimes U^*$ is the metric tensor. Then the position of the a-th material point at the time instant t will be denoted by $x(t, a)$ $(x \in M,\ a \in N)$ and an affine mapping from the material space into the physical one is as follows:

$$x^i(t, a) = r^i(t) + \varphi^i{}_A(t) a^A,$$

where $\varphi(t)$ is a linear part of the affine mapping (φ is non-singular for any time instant t), i.e., $\varphi(t) \in \mathrm{LI}(U, V)$, where $\mathrm{LI}(U, V)$ is a manifold of linear isomorphisms from the linear space U into the linear space V, $r(t)$ is the radius-vector of the centre of mass of our body if in the material space the position of the centre of mass is $a^A = 0$. If the system is continuous, then the label a becomes the Lagrangian radius-vector (material variables) and x becomes the Eulerian radius-vector (physical variables). Thus, at any fixed $t \in \mathbb{R}$ the configuration space Q of our problem is given by the following expression:

$$Q = \mathrm{AfI}(N, M) = Q_{\mathrm{tr}} \times Q_{\mathrm{int}} = M \times \mathrm{LI}(U, V),$$

where "tr" and "int" refer to the translational (spatial translations) and internal (rotations and homogeneous deformations) motions respectively.

The considered system is called an affinely-rigid body [14, 15, 16, 17, 18], i.e., during any admissible motion all affine relations between constituents of the body are invariant (the material straight lines remain straight lines, their parallelism is conserved, and all mutual ratios of segments placed on the same straight lines are constant). The conception of the affinely-rigid body is a generalization of the usual metrically-rigid body, in which during any admissible motion all distances (metric relations) between constituents of the body are constant (see, for example, [1, 4]).

In this article we concentrate mainly on the case of such an affinely-rigid body that is subject to the additional constraints, i.e., it can deform homogeneously in the two-dimensional central plane of the body and simultaneously performs one-dimensional oscillations orthogonal to this central plane. Then the material space N is presented as the Cartesian product $\mathbb{R}^+ \times \mathbb{R}^2$ and the group of material transformations has the form $\mathbb{R}^+ \times \mathrm{GL}\,(2, \mathbb{R})$, where $\mathbb{R}^+$ is the dilatation group in the third dimension and the material transformations in $\mathbb{R}^2$ act as in the case of the usual affinely-rigid body with degenerate dimension [11, 12].

We can identify configurations $\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with the pairs (ϱ, φ), where φ describes the immersion of the central plane in the physical space, i.e., analytically $\varphi^i{}_A$ is the 3×2 matrix. An element (k, B) acts on (ϱ, φ) as follows:

$$(k, B) \in \mathbb{R}^+ \times \mathrm{GL}(2, \mathbb{R}) : \quad (\varrho, \varphi) \mapsto (k\varrho, \varphi B).$$

The conservation of orthogonality of the direction of dilatations to the central plane means that the matrix

$$\Phi = \begin{bmatrix} \Phi^1{}_1 & \Phi^1{}_2 & \Phi^1{}_3 \\ \Phi^2{}_1 & \Phi^2{}_2 & \Phi^2{}_3 \\ \Phi^3{}_1 & \Phi^3{}_2 & \Phi^3{}_3 \end{bmatrix}$$

fulfils the condition that third column has to be proportional to the vector product of first and second ones. If we consider $\Phi^a{}_1$, $\Phi^b{}_2$, $a, b = 1, 2, 3$, as independent and arbitrary, then $\Phi^a{}_3 = \ell\ \varepsilon^a{}_{bc} \Phi^b{}_1 \Phi^c{}_2$, where ε_{abc} is the completely antisymmetrical Levi-Civita (permutation) symbol, ℓ is the parameter which depends both on the variable describing one-dimensional oscillations orthogonal to the central plane of the body and on the ones describing the state of deformation in this central plane, e.g., for the two-polar (singular value) (1) and the polar (4) decompositions we have respectively that

$$\ell_{\text{two-polar}} = \frac{\varrho}{\lambda\mu}, \qquad \ell_{\text{polar}} = \frac{\varrho}{\xi\zeta - \alpha^2},$$

where the meaning of variables λ, μ, α, ξ, ζ, ϱ is clear from the expressions (2) and (5) below.

The above-described orthogonality is well known in the theory of plates and shells as the Kirchhoff–Love condition [8].

2 Two-polar decomposition

In [6] we discussed the language of the two-polar (singular value) decomposition:

$$\Phi\left(\overline{k}; \lambda, \mu, \varrho; \theta\right) = R\left(\overline{k}\right) D\left(\lambda, \mu, \varrho\right) U\left(\theta\right)^{-1}, \qquad \lambda, \mu, \varrho > 0, \tag{1}$$

where $R, U \in \mathrm{SO}(3, \mathbb{R})$ are proper orthogonal matrices (whereas $\overline{k}$ is a rotation vector, i.e., a non-normalized vector codirectional with the rotation axis whose magnitude is equal to the rotation angle) and D is diagonal, i.e.,

$$D(\lambda, \mu, \varrho) = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \varrho \end{bmatrix}, \qquad U(\theta)^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

Then the co-moving angular velocities for R- and U-tops [17, 18] are as follows:

$$\omega = R^{-1}\dot{R} = R^T\dot{R} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}, \qquad \omega^T = -\omega,$$

and

$$\vartheta = U^{-1}\dot{U} = U^T\dot{U} = \dot{\theta}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \vartheta^T = -\vartheta.$$

For $\dot{\Phi}$ and $\dot{\Phi}^T$ we have the following expressions:

$$\dot{\Phi} = R\big(\dot{D} + \omega D - D\vartheta\big)U^{-1}, \qquad \dot{\Phi}^T = U\big(\dot{D} + \vartheta D - D\omega\big)R^T.$$

The kinetic energy is assumed to have the usual form (we have only to substitute the constraints):

$$T = \tfrac{1}{2}\mathrm{Tr}\big(J\dot{\Phi}^T\dot{\Phi}\big) = \tfrac{1}{2}\mathrm{Tr}\big(U^{-1}JU\big[\dot{D} + \vartheta D - D\omega\big]\big[\dot{D} + \omega D - D\vartheta\big]\big),$$

where $J \in U \otimes U$ is the twice contravariant, symmetric, non-singular, positively-definite tensor describing the inertial properties of our affinely-rigid body. If we take J in the diagonal form $J = \mathrm{Diag}\left(J_1, J_2, J_3\right)$, then the above kinetic energy can be rewritten as follows:

$$T = \frac{J_1\cos^2\theta + J_2\sin^2\theta}{2}\left(\frac{d\lambda}{dt}\right)^2 + \frac{J_1\sin^2\theta + J_2\cos^2\theta}{2}\left(\frac{d\mu}{dt}\right)^2$$
$$+ \frac{J_3}{2}\left(\frac{d\varrho}{dt}\right)^2 + \frac{\left(J_1\sin^2\theta + J_2\cos^2\theta\right)\mu^2 + J_3\varrho^2}{2}\,\omega_1^2$$

$$
\begin{aligned}
&+\frac{\left(J_1\cos^2\theta+J_2\sin^2\theta\right)\lambda^2+J_3\varrho^2}{2}\,\omega_2^2+\left(J_1+J_2\right)\lambda\mu\omega_3\frac{d\theta}{dt}\\
&+\left(J_1-J_2\right)\sin 2\theta\left[\left(\mu\frac{d\mu}{dt}-\lambda\frac{d\lambda}{dt}\right)\frac{d\theta}{dt}+\left(\lambda\frac{d\mu}{dt}-\mu\frac{d\lambda}{dt}\right)\omega_3+\lambda\mu\omega_1\omega_2\right]\\
&+\frac{\left(J_1\cos^2\theta+J_2\sin^2\theta\right)\lambda^2+\left(J_1\sin^2\theta+J_2\cos^2\theta\right)\mu^2}{2}\,\omega_3^2\\
&+\frac{\left(J_1\sin^2\theta+J_2\cos^2\theta\right)\lambda^2+\left(J_1\cos^2\theta+J_2\sin^2\theta\right)\mu^2}{2}\left(\frac{d\theta}{dt}\right)^2. \qquad (3)
\end{aligned}
$$

The above expressions significantly simplify when we consider the isotropic case in the central plane of the body, i.e., when we have $J_1 = J_2 = J$. Then

$$
\begin{aligned}
T=&\frac{J}{2}\left[\left(\frac{d\lambda}{dt}\right)^2+\left(\frac{d\mu}{dt}\right)^2\right]+\frac{J_3}{2}\left(\frac{d\varrho}{dt}\right)^2+\frac{J\mu^2+J_3\varrho^2}{2}\omega_1^2\\
&+\frac{J\lambda^2+J_3\varrho^2}{2}\omega_2^2+2J\lambda\mu\omega_3\frac{d\theta}{dt}+\frac{J}{2}\left(\lambda^2+\mu^2\right)\left[\omega_3^2+\left(\frac{d\theta}{dt}\right)^2\right].
\end{aligned}
$$

We also remind here that the corresponding expression for the kinetic energy in the canonical variables has the following form:

$$
\begin{aligned}
\mathcal{T}=&\frac{s_1^2}{2\left(J\mu^2+J_3\varrho^2\right)}+\frac{s_2^2}{2\left(J\lambda^2+J_3\varrho^2\right)}\\
&+\frac{\left(\lambda^2+\mu^2\right)\left(s_3^2+p_\theta^2\right)-4\lambda\mu p_\theta s_3}{2J\left(\lambda^2-\mu^2\right)^2}+\frac{p_\lambda^2+p_\mu^2}{2J}+\frac{p_\varrho^2}{2J_3}.
\end{aligned}
$$

Then introducing some modelled potentials in [6] we obtained the Hamiltonian (total energy) and calculated the corresponding equations of motion for the isotropic case with the help of the Poisson brackets. In the present article we concentrate mainly on the alternative decomposition, i.e., the polar one. The main advantages of this decomposition are the more physically intuitive division on three main terms in the kinetic energy expression (see the formulas (6)–(9) below) and the possibility to obtain the equations of motion in the quite simple form (see the expressions (11)–(17) below) even for the general case, when the inertial tensor is not isotropic in the central plane ($J_1 \neq J_2$).

3 Polar decomposition

Instead of (1) we can also use the language of the polar decomposition, i.e.,

$$
\Phi\left(\overline{\kappa};\alpha,\xi,\zeta,\varrho\right)=L\left(\overline{\kappa}\right)S\left(\alpha,\xi,\zeta,\varrho\right), \qquad (4)
$$

where $L \in \mathrm{SO}(3,\mathbb{R})$ is a proper orthogonal matrix and $S \in \mathrm{Sym}(3,\mathbb{R})$ is symmetrical. The connection between the polar and two-polar decompositions is given by the following expressions:

$$
\begin{aligned}
&L=RU^{-1},\\
&\nu=L^{-1}\dot{L}=-\nu^T=\begin{bmatrix}0&\nu_3&-\nu_2\\-\nu_3&0&\nu_1\\\nu_2&-\nu_1&0\end{bmatrix}=U\left(\omega-\vartheta\right)U^{-1}\\
&=\begin{bmatrix}0&\omega_3+\dot\theta&-\omega_1\sin\theta-\omega_2\cos\theta\\-\omega_3-\dot\theta&0&\omega_1\cos\theta-\omega_2\sin\theta\\\omega_1\sin\theta+\omega_2\cos\theta&\omega_2\sin\theta-\omega_1\cos\theta&0\end{bmatrix},
\end{aligned}
$$

$$S = \begin{bmatrix} \xi & \alpha & 0 \\ \alpha & \zeta & 0 \\ 0 & 0 & \varrho \end{bmatrix} = UDU^{-1} = \begin{bmatrix} \lambda\cos^2\theta + \mu\sin^2\theta & (\lambda-\mu)\sin\theta\cos\theta & 0 \\ (\lambda-\mu)\sin\theta\cos\theta & \lambda\sin^2\theta + \mu\cos^2\theta & 0 \\ 0 & 0 & \varrho \end{bmatrix}, \tag{5}$$

and then the Green deformation tensor, which is not sensitive with respect to the left orthogonal mappings, is as follows:

$$\begin{aligned} G = \Phi^T\Phi = S^2 &= \begin{bmatrix} \xi^2+\alpha^2 & (\xi+\zeta)\,\alpha & 0 \\ (\xi+\zeta)\,\alpha & \zeta^2+\alpha^2 & 0 \\ 0 & 0 & \varrho^2 \end{bmatrix} = UD^2U^{-1} \\ &= \begin{bmatrix} \lambda^2\cos^2\theta + \mu^2\sin^2\theta & \left(\lambda^2-\mu^2\right)\sin\theta\cos\theta & 0 \\ \left(\lambda^2-\mu^2\right)\sin\theta\cos\theta & \lambda^2\sin^2\theta + \mu^2\cos^2\theta & 0 \\ 0 & 0 & \varrho^2 \end{bmatrix}, \end{aligned}$$

where for the positive definiteness the parameters have to fulfil the conditions

$$\begin{aligned} &\xi = \lambda\cos^2\theta + \mu\sin^2\theta > 0, \qquad && \zeta = \lambda\sin^2\theta + \mu\cos^2\theta > 0, \\ &\xi\zeta - \alpha^2 = \lambda\mu > 0, && \varrho > 0. \end{aligned}$$

For the polar decomposition we can as well introduce the concept of deformation invariants $\mathcal{K}^a$, $a = 1, 2, 3$, which may be chosen, e.g., as the eigenvalues of the symmetric matrix G:

$$\det\left[G - \mathcal{K}\mathbb{I}_3\right] = 0,$$

where $\mathbb{I}_3$ is the 3×3 identity matrix, and the solutions are as follows:

$$\mathcal{K}_{1,2} = \tfrac{1}{2}\left(\xi^2 + \zeta^2 + 2\alpha^2 \pm (\xi+\zeta)\sqrt{(\xi-\zeta)^2 + 4\alpha^2}\right), \qquad \mathcal{K}_3 = \varrho^2.$$

The above deformation invariants are not sensitive with respect to both the spatial and material rigid rotations (isometries).

Let us consider the Lagrangian $L = T - V(\Phi)$ and then the Hamiltonian $H = \mathcal{T} + V(\Phi)$, where the kinetic energy (3) can be rewritten for the polar decomposition as follows:

$$T = T_{\text{rot}} + T_{\text{rot-def}} + T_{\text{def}}, \tag{6}$$

where

$$\begin{aligned} T_{\text{rot}} &= \frac{J_1\alpha^2 + J_2\zeta^2 + J_3\varrho^2}{2}\nu_1^2 + \frac{J_1\xi^2 + J_2\alpha^2 + J_3\varrho^2}{2}\nu_2^2 \\ &\quad + \frac{J_1\xi^2 + J_2\zeta^2 + (J_1+J_2)\,\alpha^2}{2}\nu_3^2 - (J_1\xi + J_2\zeta)\,\alpha\nu_1\nu_2 \end{aligned} \tag{7}$$

describes the coupling between the angular velocity ν of the L-top and deformation matrix S,

$$T_{\text{rot-def}} = \left(J_1\alpha\frac{d\xi}{dt} - J_2\alpha\frac{d\zeta}{dt} - (J_1\xi - J_2\zeta)\,\frac{d\alpha}{dt}\right)\nu_3 \tag{8}$$

describes the connection between the angular and deformation velocities, and finally

$$T_{\text{def}} = \frac{J_1+J_2}{2}\left(\frac{d\alpha}{dt}\right)^2 + \frac{J_1}{2}\left(\frac{d\xi}{dt}\right)^2 + \frac{J_2}{2}\left(\frac{d\zeta}{dt}\right)^2 + \frac{J_3}{2}\left(\frac{d\varrho}{dt}\right)^2 \tag{9}$$

describes the kinetic energy of the deformation oscillations, whereas the potential term $V(\Phi)$ depends on Φ only through the Green deformation tensor $G = S^2$, i.e., the potential term adapted to the polar decomposition is a function only of α, ξ, ζ, and ϱ.

Performing the Legendre transformation we obtain that

$$\begin{aligned}
\pi_1 &= \frac{\partial T}{\partial \nu_1} = \left(J_1\alpha^2 + J_2\zeta^2 + J_3\varrho^2\right)\nu_1 - \left(J_1\xi + J_2\zeta\right)\alpha\nu_2,\\
\pi_2 &= \frac{\partial T}{\partial \nu_2} = \left(J_1\xi^2 + J_2\alpha^2 + J_3\varrho^2\right)\nu_2 - \left(J_1\xi + J_2\zeta\right)\alpha\nu_1,\\
\pi_3 &= \frac{\partial T}{\partial \nu_3} = \left(J_1\xi^2 + J_2\zeta^2 + \left(J_1 + J_2\right)\alpha^2\right)\nu_3 + J_1\alpha\dot{\xi} - J_2\alpha\dot{\zeta} - \left(J_1\xi - J_2\zeta\right)\dot{\alpha},\\
p_\alpha &= \frac{\partial T}{\partial \dot{\alpha}} = \left(J_1 + J_2\right)\dot{\alpha} - \left(J_1\xi - J_2\zeta\right)\nu_3,\\
p_\xi &= \frac{\partial T}{\partial \dot{\xi}} = J_1\left(\dot{\xi} + \alpha\nu_3\right),\\
p_\zeta &= \frac{\partial T}{\partial \dot{\zeta}} = J_2\left(\dot{\zeta} - \alpha\nu_3\right),\\
p_\varrho &= \frac{\partial T}{\partial \dot{\varrho}} = J_3\dot{\varrho},
\end{aligned}$$

where π_i are canonical "spin" variables conjugate to angular velocities ν_i.

Therefore after inverting the above dependencies, i.e.,

$$\begin{aligned}
\nu_1 &= \frac{\left(J_1\xi^2 + J_2\alpha^2 + J_3\varrho^2\right)\pi_1 + \left(J_1\xi + J_2\zeta\right)\alpha\pi_2}{J_1J_2\left(\alpha^2 - \xi\zeta\right)^2 + \left[J_1\xi^2 + J_2\zeta^2 + \left(J_1 + J_2\right)\alpha^2\right]J_3\varrho^2 + J_3^2\varrho^4},\\
\nu_2 &= \frac{\left(J_1\xi + J_2\zeta\right)\alpha\pi_1 + \left(J_1\alpha^2 + J_2\zeta^2 + J_3\varrho^2\right)\pi_2}{J_1J_2\left(\alpha^2 - \xi\zeta\right)^2 + \left[J_1\xi^2 + J_2\zeta^2 + \left(J_1 + J_2\right)\alpha^2\right]J_3\varrho^2 + J_3^2\varrho^4},\\
\nu_3 &= \frac{\left(J_1 + J_2\right)\left[\pi_3 + \alpha\left(p_\zeta - p_\xi\right)\right] + \left(J_1\xi - J_2\zeta\right)p_\alpha}{J_1J_2\left(\xi + \zeta\right)^2},\\
\frac{d\alpha}{dt} &= \frac{\left(J_1\xi - J_2\zeta\right)\left[\pi_3 + \alpha\left(p_\zeta - p_\xi\right)\right] + \left(J_1\xi^2 + J_2\zeta^2\right)p_\alpha}{J_1J_2\left(\xi + \zeta\right)^2},\\
\frac{d\xi}{dt} &= \frac{p_\xi}{J_1} - \alpha\frac{\left(J_1 + J_2\right)\left[\pi_3 + \alpha\left(p_\zeta - p_\xi\right)\right] + \left(J_1\xi - J_2\zeta\right)p_\alpha}{J_1J_2\left(\xi + \zeta\right)^2},\\
\frac{d\zeta}{dt} &= \frac{p_\zeta}{J_2} + \alpha\frac{\left(J_1 + J_2\right)\left[\pi_3 + \alpha\left(p_\zeta - p_\xi\right)\right] + \left(J_1\xi - J_2\zeta\right)p_\alpha}{J_1J_2\left(\xi + \zeta\right)^2},\\
\frac{d\varrho}{dt} &= \frac{p_\varrho}{J_3},
\end{aligned}$$

we obtain the kinetic energy in the canonical variables as follows:

$$\begin{aligned}
\mathcal{T} &= \frac{\left(J_1\xi^2 + J_2\alpha^2 + J_3\varrho^2\right)\pi_1^2 + \left(J_1\alpha^2 + J_2\zeta^2 + J_3\varrho^2\right)\pi_2^2}{2\left(J_1J_2\left(\alpha^2 - \xi\zeta\right)^2 + \left[J_1\xi^2 + J_2\zeta^2 + \left(J_1 + J_2\right)\alpha^2\right]J_3\varrho^2 + J_3^2\varrho^4\right)}\\
&+ \frac{\left(J_1\xi + J_2\zeta\right)\alpha\pi_1\pi_2}{J_1J_2\left(\alpha^2 - \xi\zeta\right)^2 + \left[J_1\xi^2 + J_2\zeta^2 + \left(J_1 + J_2\right)\alpha^2\right]J_3\varrho^2 + J_3^2\varrho^4}\\
&+ \frac{J_1 + J_2}{2J_1J_2\left(\xi + \zeta\right)^2}\left[\pi_3 + \alpha\left(p_\zeta - p_\xi\right)\right]^2 + \frac{J_1\xi^2 + J_2\zeta^2}{2J_1J_2\left(\xi + \zeta\right)^2}p_\alpha^2\\
&+ \frac{J_1\xi - J_2\zeta}{J_1J_2\left(\xi + \zeta\right)^2}\left[\pi_3 + \alpha\left(p_\zeta - p_\xi\right)\right]p_\alpha + \frac{p_\xi^2}{2J_1} + \frac{p_\zeta^2}{2J_2} + \frac{p_\varrho^2}{2J_3}.
\end{aligned} \tag{10}$$

From the above kinetic energy expressions (6)–(9) one can see that the generalized velocities $\dot{\alpha}$, $\dot{\xi}$, $\dot{\zeta}$ corresponding to α, ξ, ζ and other variables describing the motion in the central plane of the body are separated from the generalized velocity $\dot{\varrho}$ describing the one-dimensional oscillations orthogonal to this central plane. The same can be said also about the above expression in the canonical variables (10), i.e., the momentum p_ϱ conjugated to ϱ is orthogonal (in the sense of metrics encoded in the kinetic energy expression) to the other canonical momenta. Hence, the most simple are those dynamical models in which also the isotropic potential will have the separated form:

$$V(\alpha, \xi, \zeta, \varrho) = V_{\text{plane}}(\alpha, \xi, \zeta) + V_\varrho(\varrho),$$

where as the potential V_ϱ we can take, e.g., the following potential which describes the nonlinear oscillations and is in accordance with the main demands of the elasticity theory, i.e.,

$$V_\varrho(\varrho) = \frac{a}{\varrho} + \frac{b}{2}\varrho^2, \qquad a, b > 0,$$

where the first term prevents from the unlimited compressing of the body, whereas the second one restricts the motion for large values of ϱ, i.e., prevents from the non-physical unlimited stretching of the body.

So, the Hamiltonian (total energy) can be written as follows:

$$H = \mathcal{T} + V_{\text{plane}}(\alpha, \xi, \zeta) + V_\varrho(\varrho),$$

where $\mathcal{T}$ is taken in the form of (10). Then the equations of motion can be calculated with the help of the following Poisson brackets:

$$\frac{d\pi_i}{dt} = \{\pi_i, H\}, \qquad \frac{dp_\alpha}{dt} = \{p_\alpha, H\}, \qquad \frac{dp_\xi}{dt} = \{p_\xi, H\},$$
$$\frac{dp_\zeta}{dt} = \{p_\zeta, H\}, \qquad \frac{dp_\varrho}{dt} = \{p_\varrho, H\}.$$

The only non-zero basic Poisson brackets are

$$\{\alpha, p_\alpha\} = \{\xi, p_\xi\} = \{\zeta, p_\zeta\} = \{\varrho, p_\varrho\} = 1, \qquad \{\pi_i, \pi_j\} = -\varepsilon_{ij}{}^k \pi_k,$$

where the former expressions follow directly from the definition of the Poisson bracket and the latter ones are based on the structure constants of the special orthogonal group $\mathrm{SO}(3, \mathbb{R})$.

First of all, let us rewrite the kinetic energy (10) in a more symbolic way, i.e.,

$$\mathcal{T} = \frac{\Omega(\pi_1, \pi_2)}{2\Xi} + \frac{\Upsilon(\pi_3 + \alpha(p_\zeta - p_\xi), p_\alpha)}{2J_1J_2(\xi + \zeta)^2} + \frac{p_\xi^2}{2J_1} + \frac{p_\zeta^2}{2J_2} + \frac{p_\varrho^2}{2J_3},$$

where

$$\Xi = J_1J_2\left(\alpha^2 - \xi\zeta\right)^2 + \left[J_1\xi^2 + J_2\zeta^2 + (J_1 + J_2)\alpha^2\right]J_3\varrho^2 + J_3^2\varrho^4,$$

and two expressions built of the canonical momenta are as follows:

$$\begin{aligned} \Omega(\pi_1, \pi_2) = {} & \left(J_1\xi^2 + J_2\alpha^2 + J_3\rho^2\right)\pi_1^2 + 2(J_1\xi + J_2\zeta)\alpha\pi_1\pi_2 \\ & + \left(J_1\alpha^2 + J_2\zeta^2 + J_3\rho^2\right)\pi_2^2, \\ \Upsilon(\pi_3 + \alpha(p_\zeta - p_\xi), p_\alpha) = {} & (J_1 + J_2)\left[\pi_3 + \alpha(p_\zeta - p_\xi)\right]^2 + \left(J_1\xi^2 + J_2\zeta^2\right)p_\alpha^2 \\ & + 2(J_1\xi - J_2\zeta)\left[\pi_3 + \alpha(p_\zeta - p_\xi)\right]p_\alpha. \end{aligned}$$

Then we obtain the following equations of motion:

$$\frac{d\pi_1}{dt} = -\frac{\left[(J_1\xi + J_2\zeta)\alpha\pi_1 + \left(J_1\alpha^2 + J_2\zeta^2 + J_3\varrho^2\right)\pi_2\right]\pi_3}{\Xi}$$

$$+\frac{\pi_2\left[\left(J_1+J_2\right)\left[\pi_3+\alpha\left(p_\zeta-p_\xi\right)\right]+\left(J_1\xi-J_2\zeta\right)p_\alpha\right]}{J_1J_2\left(\xi+\zeta\right)^2}, \tag{11}$$

$$\frac{d\pi_2}{dt}=\frac{\left[\left(J_1\xi^2+J_2\alpha^2+J_3\varrho^2\right)\pi_1+\left(J_1\xi+J_2\zeta\right)\alpha\pi_2\right]\pi_3}{\Xi}$$
$$-\frac{\pi_1\left[\left(J_1+J_2\right)\left[\pi_3+\alpha\left(p_\zeta-p_\xi\right)\right]+\left(J_1\xi-J_2\zeta\right)p_\alpha\right]}{J_1J_2\left(\xi+\zeta\right)^2}, \tag{12}$$

$$\frac{d\pi_3}{dt}=\frac{\left(J_1\xi+J_2\zeta\right)\alpha\left(\pi_1^2-\pi_2^2\right)+\left[J_1\left(\alpha^2-\xi^2\right)+J_2\left(\zeta^2-\alpha^2\right)\right]\pi_1\pi_2}{\Xi}, \tag{13}$$

$$\frac{d\alpha}{dt}=-\frac{\partial V_{\text{plane}}}{\partial\alpha}-\frac{\left(J_2\pi_1^2+J_1\pi_2^2\right)\alpha+\left(J_1\xi+J_2\zeta\right)\pi_1\pi_2}{\Xi}$$
$$+\frac{2J_1J_2\alpha\left(\alpha^2-\xi\zeta\right)+\left(J_1+J_2\right)\alpha J_3\varrho^2}{\Xi^2}\Omega\left(\pi_1,\pi_2\right)$$
$$-\frac{\left(J_1+J_2\right)\left[\pi_3+\alpha\left(p_\zeta-p_\xi\right)\right]+\left(J_1\xi-J_2\zeta\right)p_\alpha}{J_1J_2\left(\xi+\zeta\right)^2}\left(p_\zeta-p_\xi\right), \tag{14}$$

$$\frac{d\xi}{dt}=-\frac{\partial V_{\text{plane}}}{\partial\xi}-\frac{J_1\xi\pi_1^2+J_1\alpha\pi_1\pi_2}{\Xi}+\frac{J_1J_2\zeta\left(\xi\zeta-\alpha^2\right)+J_1\xi J_3\varrho^2}{\Xi^2}\Omega\left(\pi_1,\pi_2\right)$$
$$-\frac{J_1\xi p_\alpha^2+J_1\left[\pi_3+\alpha\left(p_\zeta-p_\xi\right)\right]p_\alpha}{J_1J_2\left(\xi+\zeta\right)^2}+\frac{\Upsilon\left(\pi_3+\alpha\left(p_\zeta-p_\xi\right),p_\alpha\right)}{J_1J_2\left(\xi+\zeta\right)^3}, \tag{15}$$

$$\frac{d\zeta}{dt}=-\frac{\partial V_{\text{plane}}}{\partial\zeta}-\frac{J_2\zeta\pi_2^2+J_2\alpha\pi_1\pi_2}{\Xi}+\frac{J_1\xi J_2\left(\xi\zeta-\alpha^2\right)+J_2\zeta J_3\varrho^2}{\Xi^2}\Omega\left(\pi_1,\pi_2\right)$$
$$-\frac{J_2\zeta p_\alpha^2-J_2\left[\pi_3+\alpha\left(p_\zeta-p_\xi\right)\right]p_\alpha}{J_1J_2\left(\xi+\zeta\right)^2}+\frac{\Upsilon\left(\pi_3+\alpha\left(p_\zeta-p_\xi\right),p_\alpha\right)}{J_1J_2\left(\xi+\zeta\right)^3}, \tag{16}$$

$$\frac{d\varrho}{dt}=-\frac{dV_\varrho}{d\varrho}-\frac{J_3\rho}{\Xi}\left(\pi_1^2+\pi_2^2\right)+\frac{J_3\varrho}{\Xi^2}\left[J_1\xi^2+J_2\zeta^2+\left(J_1+J_2\right)\alpha^2+2J_3\rho^2\right]\Omega\left(\pi_1,\pi_2\right). \tag{17}$$

The structure of the above expressions implies that even in the simplest case of the completely separated potential the dynamical coupling between the parameter describing one-dimensional oscillations orthogonal to the central plane of the body and the variables living in this central plane is present.

4 Stationary ellipsoids as special solutions

Our equations of motion (11)–(17) are strongly nonlinear and in a general case there is hardly a hope to solve them analytically. Nevertheless, there exists a way for imaging some features of the phase portrait of such a dynamical system, i.e., we have to find some special solutions, namely, the stationary ellipsoids [14, 15], which are analogous to the ellipsoidal figures of equilibrium well known in astro- [2] and geophysics, e.g., in the theory of the Earth's shape [3].

In the case of the two-polar (singular value) decomposition (1) we obtained the above-mentioned special solutions just putting the deformation invariants λ, μ, ϱ and the angular velocities ω, ϑ equal to some constant values [6]. But now, in the case of the polar decomposition (4), we see that the Green deformation tensor G, therefore the deformation matrix S, and the angular velocity ν of the L-top have to be constant [15], i.e.,

$$\frac{dG}{dt}=\frac{d}{dt}\left(\Phi^T\Phi\right)=\frac{d}{dt}\left(S^2\right)=0,\qquad \frac{d\nu}{dt}=\frac{d}{dt}\left(L^{-1}\dot{L}\right)=0.$$

This means that the L-top performs the stationary rotation, i.e., if at the initial time $t=0$ we have that the configuration of the body is L_0, then at the time instant t the configuration will be as follows:

$$L_0 \circ e^{\nu t},$$

where $\circ$ is the function composition symbol. We see that the whole affinely-rigid body, which at the initial time $t = 0$ has the internal configuration $\Phi_0 = L_0 \circ S$, at the time instant t will be in the following configuration:

$$\Phi(t) = L_0 \circ e^{\nu t} \circ S = e^{\widehat{\nu} t} \circ L_0 \circ S = e^{\widehat{\nu} t} \circ \Phi_0, \tag{18}$$

where $\widehat{\nu} = L_0 \circ \nu \circ L_0^{-1}$.

Proposition 1. *While the affinely-rigid body rotates in the stationary way around the axis fixed in the physical and material spaces, the deformation and the angular velocity of rotation are not independent and related by some algebraic expressions.*

Proof. The trajectories of type (18) are the orbits of the Euler (spatial) action of the one-parameter orthogonal group $\left\{e^{\widehat{\nu} t} : t \in \mathbb{R}\right\} \subset \mathrm{SO}(V, g)$. Nevertheless, during the motion the configurations of the body are deformed. At the same time the Green deformation tensor does not perform any oscillations, it is constant. This means that an equilibrium is set between the centrifugal forces coming from the rotation of the body and the elastic forces coming from the fact that $S \neq \mathrm{Id}_U$. This kind of equilibrium is possible only if between the constant values of ν, S are set some algebraic relations that guarantee the balance of the above-described forces. These algebraic relations between ν, S obviously come from the equations of motion (11)–(17). Hence, we propose to divide them into the following three main branches:

(i) $\nu_1 \neq 0$, $\nu_2 = \nu_3 = 0$, then $\pi_1, \pi_2 \neq 0$ and $\pi_3 = p_\alpha = p_\xi = p_\zeta = p_\varrho = 0$;

(ii) $\nu_2 \neq 0$, $\nu_1 = \nu_3 = 0$, then $\pi_1, \pi_2 \neq 0$ and $\pi_3 = p_\alpha = p_\xi = p_\zeta = p_\varrho = 0$.

For the first two cases the relations take the same form, i.e.,

$$\frac{\partial V_{\text{plane}}}{\partial \alpha} = -\frac{\left(J_2\pi_1^2 + J_1\pi_2^2\right)\alpha + \left(J_1\xi + J_2\zeta\right)\pi_1\pi_2}{\Xi} + \frac{2J_1J_2\alpha\left(\alpha^2 - \xi\zeta\right) + \left(J_1 + J_2\right)\alpha J_3\varrho^2}{\Xi^2}\Omega\left(\pi_1, \pi_2\right), \tag{19}$$

$$\frac{\partial V_{\text{plane}}}{\partial \xi} = -\frac{J_1\xi\pi_1^2 + J_1\alpha\pi_1\pi_2}{\Xi} + \frac{J_1J_2\zeta\left(\xi\zeta - \alpha^2\right) + J_1\xi J_3\varrho^2}{\Xi^2}\Omega\left(\pi_1, \pi_2\right), \tag{20}$$

$$\frac{\partial V_{\text{plane}}}{\partial \zeta} = -\frac{J_2\zeta\pi_2^2 + J_2\alpha\pi_1\pi_2}{\Xi} + \frac{J_1\xi J_2\left(\xi\zeta - \alpha^2\right) + J_2\zeta J_3\varrho^2}{\Xi^2}\Omega\left(\pi_1, \pi_2\right), \tag{21}$$

$$\frac{dV_\varrho}{d\varrho} = -\frac{J_3\rho}{\Xi}\left(\pi_1^2 + \pi_2^2\right) + \frac{J_3\varrho}{\Xi^2}\left[J_1\xi^2 + J_2\zeta^2 + \left(J_1 + J_2\right)\alpha^2 + 2J_3\rho^2\right]\Omega\left(\pi_1, \pi_2\right), \tag{22}$$

with the compatibility condition

$$\left(J_1\xi + J_2\zeta\right)\alpha\left(\pi_1^2 - \pi_2^2\right) + \left[J_1\left(\alpha^2 - \xi^2\right) + J_2\left(\zeta^2 - \alpha^2\right)\right]\pi_1\pi_2 = 0.$$

We see that, while our parameters ν_1 or ν_2 take completely arbitrary constant values, the above equations (19)–(22) describe their interrelation with the elements of the symmetrical matrix S, i.e., with α, ξ, ζ, ϱ.

(iii) $\nu_3 \neq 0$, $\nu_1 = \nu_2 = 0$, then $\pi_3, p_\alpha, p_\xi, p_\zeta \neq 0$ and $\pi_1 = \pi_2 = p_\varrho = 0$, whereas

$$p_\alpha = \frac{\left(J_2\zeta - J_1\xi\right)\pi_3}{J_1\xi^2 + J_2\zeta^2 + \left(J_1 + J_2\right)\alpha^2},$$

$$p_\xi = \frac{J_1\alpha\pi_3}{J_1\xi^2 + J_2\zeta^2 + \left(J_1 + J_2\right)\alpha^2},$$

$$p_\zeta = -\frac{J_2\alpha\pi_3}{J_1\xi^2 + J_2\zeta^2 + (J_1+J_2)\,\alpha^2}.$$

So, for the third case we obtain the following relations:

$$\frac{\partial V_{\text{plane}}}{\partial\alpha} = \frac{(J_1+J_2)\left[\pi_3 + \alpha\left(p_\zeta - p_\xi\right)\right] + \left(J_1\xi - J_2\zeta\right)p_\alpha}{J_1J_2\left(\xi+\zeta\right)^2}\left(p_\xi - p_\zeta\right), \tag{23}$$

$$\frac{\partial V_{\text{plane}}}{\partial\xi} = -\frac{J_1\xi p_\alpha^2 + J_1\left[\pi_3 + \alpha\left(p_\zeta - p_\xi\right)\right]p_\alpha}{J_1J_2\left(\xi+\zeta\right)^2} + \frac{\Upsilon\left(\pi_3 + \alpha\left(p_\zeta - p_\xi\right), p_\alpha\right)}{J_1J_2\left(\xi+\zeta\right)^3}, \tag{24}$$

$$\frac{\partial V_{\text{plane}}}{\partial\zeta} = -\frac{J_2\zeta p_\alpha^2 - J_2\left[\pi_3 + \alpha\left(p_\zeta - p_\xi\right)\right]p_\alpha}{J_1J_2\left(\xi+\zeta\right)^2} + \frac{\Upsilon\left(\pi_3 + \alpha\left(p_\zeta - p_\xi\right), p_\alpha\right)}{J_1J_2\left(\xi+\zeta\right)^3}, \tag{25}$$

$$\frac{dV_\varrho}{d\varrho} = 0. \tag{26}$$

This time our parameter ν_3 has a completely arbitrary constant value and the above equations (23)–(26) describe the way in which α, ξ, ζ, ϱ are related to it. ■

Remark 1. It should be mentioned that the name "stationary ellipsoids" is not the most adequate for the description of the above-obtained stationary solutions. Of course, for the non-restricted affinely-rigid body we can visualize this kind of special solutions as follows:

- At the beginning the body stays in the equilibrium configuration.
- Then we switch on some mechanical device which deforms our body in the homogeneous way, i.e., this deformation is the superposition of three material stretchings with the coefficients D_1, D_2, D_3, where D_i, $i = 1, 2, 3$, are the diagonal elements of the deformation matrix D in the two-polar decomposition $\Phi = RDU^{-1}$.
- After this we start to rotate our mechanical device with the constant angular velocity ϑ around one of the main axes of the Green deformation tensor $G = \Phi^T\Phi = UD^2U^{-1}$ so that the state of material deformation follows this movement of the device with the same angular velocity (note that our body itself does not rotate!).
- And finally, the whole system consisting of the already rotating mechanical device and the body starts also to rotate with the constant angular velocity ω around the corresponding main axis of the Cauchy deformation tensor $C = \Phi^{-1T}\Phi^{-1} = RD^2R^{-1}$ (this time both the state of deformation and our body rotate!).

We see that the whole system reminds the gimbals equipped with additional mechanical device deforming the body.

If our parameters D_1, D_2, D_3, ϑ, ω are chosen in such a way that they fulfill the algebraic relations obtained from the equations of motion, then even when we switch off the mechanical device which generates the state of deformation in the material of our body, nothing will change, i.e., the stretchings will continue to rotate with the same constant angular velocity ϑ around the same axis in the material and the body will be rotating with the same angular velocity ω around the same axis in the space. The deformation invariants D_1, D_2, D_3 also will be constant during the above-described two types of rotation.

But in our case of the affinely-rigid body subject to the Kirchhoff–Love constraints we have the homogeneous deformation only in the central plane of the body, whereas in the perpendicular direction the body performs some nonlinear oscillations. Hence, in this situation the more appropriate name for our special solutions is "elliptical", but we have kept the generic name "ellipsoidal" for the matter of convenience.

5 Summary

It is interesting to note that the special solutions obtained for the polar decomposition case are conceptually different from those obtained for the two-polar one [6] because here the Green deformation tensor $G = S^2$ has a constant value (i.e., $\dot{G} = 2S\dot{S} = 0$) contrary to the situation described in [6] when the Green deformation tensor $G = \Phi^T\Phi = UD^2U^{-1}$, as well as the Cauchy one $C = \Phi^{-1T}\Phi^{-1} = RD^2R^{-1}$, depended on time explicitly through the time dependence of U and R respectively, i.e.,

$$\frac{dG}{dt} = U\left(\vartheta D^2 - D^2\vartheta\right)U^{-1} \neq 0, \qquad \frac{dC}{dt} = R\left(\omega D^2 - D^2\omega\right)R^{-1} \neq 0,$$

and performed the stationary rotations around their principal axes, whereas the deformation invariants λ, μ, ϱ had the constant values.

So, if we additionally keep in mind that in [6] we obtained the stationary solutions only for the isotropic model $J_1 = J_2 = J$ and here the general situation $J_1 \neq J_2$ is allowed, then we can compare the four (one in [6] and three here) studied cases according to the following scheme:

- The only degrees of freedom we can manipulate are the rotational degrees of R- and U-tops, because the deformation matrix D is constant for this type of stationary solutions.
- To achieve the constant behaviour of the Green deformation tensor $G = S^2 = \left(UDU^{-1}\right)^2 = UD^2U^{-1}$ we have to suppose that the U-top is fixed and does not rotate at all. If U is constant, then the principal axes of the R- and $L\left(= RU^{-1}\right)$-tops (for the two-polar and polar decompositions respectively) rotate in the same manner, i.e., at any moment ones can be obtained from others with the help of applying some constant orthogonal transformation. This situation corresponds to the above-mentioned three cases (i)–(iii) describing the stationary rotations of the L-top around its three principal axes.
- If U-top is not fixed, then the Green deformation is not constant and we have to consider three branches of the stationary motion for R- and U-tops when they rotate not independently but in the correlated manner, i.e., either both around their first principal axes or both around the second ones or both around the third ones [15]. Nevertheless, for our affinely-rigid body subject to the Kirchhoff–Love constraints only the third case is possible and exactly this situation was studied in the previous paper [6].

Hence, we see that in the above-described sense the results obtained in this article are essentially different from and simultaneously complementary to those obtained in [6].

Let us mention that the affine models of degrees of freedom for structured bodies have been studied by many authors. The thorough analysis of some stationary motions for affine bodies and their stability was presented in [5, 7, 9, 10, 13]. However, in this article we have discussed other problems.

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