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Baker–Akhiezer Modules on Rational Varieties

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Abstract. The free Baker–Akhiezer modules on rational varieties obtained from $\mathbb{C}P^1 \times \mathbb{C}P^{n-1}$ by identification of two hypersurfaces are constructed. The corollary of this construction is the existence of embedding of meromorphic function ring with some fixed pole into the ring of matrix differential operators in n variables.

Key words: commuting differential operators; Baker–Akhiezer modules

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1 Introduction

The Baker–Akhiezer modules (BA-modules) over the ring of differential operators were introduced by Nakayashiki (see [1, 2]). These modules are constructed on the basis of spectral data which include an algebraic variety X and some additional objects. In the one-dimensional case the module elements are the usual Baker–Akhiezer functions.

The BA-module M consists of functions $\psi(x,P)$ which depend on $x \in \mathbb{C}^n$, where $n = \dim_{\mathbb{C}} X$ and $P \in X$. If x is fixed, then the function ψ is the section of a bundle over X, and ψ has an essential singularity on divisor $Y \subset X$. The elements $\psi \in M$ have the following properties:

- $\partial_{x_j}\psi \in M$ and $f(x)\psi \in M$, where $f(x)$ is an analytical function in a neighbourhood of a fixed point x_0;
- if λ is a meromorphic function with a pole on Y, then $\lambda\psi \in M$.

These properties mean that M is the module over the ring of differential operators $\mathcal{D}_n = \mathcal{O}[\partial_{x_1}, \dots, \partial_{x_n}]$, where $\mathcal{O}$ is the ring of analytical functions in a neighbourhood of x_0, and over the ring A_Y of meromorphic functions on X with the pole on Y.

The finitely generated free BA-modules over $\mathcal{D}_n$ are of the main interest as in this case the construction allows to build commutative rings of differential operators. Let us choose the basis $\psi_1(x,P), \dots, \psi_N(x,P)$ in M. Let $\Psi(x,P)$ denote vector-function $(\psi_1(x,P), \dots, \psi_N(x,P))^\top$. Then for $\lambda \in A_Y$ there is only one differential operator $D(\lambda)$ with $N \times N$-matrix coefficients such that

$$D(\lambda)\Psi(x,P) = \lambda(P)\Psi(x,P).$$

The operators $D(\lambda)$ and $D(\mu)$ obviously commutate with each other for different λ and $\mu \in A_Y$. Thus, the considered construction makes it possible to obtain solutions of nonlinear differential equations which are equivalent to the condition of commutation of differential operators.

The following examples of the free BA-modules over $\mathcal{D}_n$ are known. In [1] and [2] it is shown that the BA-modules on Abelian varieties are free under some restrictions on spectral data.

In [2] it is also shown that the restriction of the BA-module from three-dimensional Abelian variety to the shifted theta-divisor remains free (over the ring of differential operators of two variables).

In [3] it is shown[1] that the restriction of BA-module from Abelian variety to the complete intersection of the shifted theta-divisors remains free. On these grounds we found the solutions of multidimensional analog of Kadomtsev–Petviashvili hierarchy

$$[\partial_{t_k} - A_k, \partial_{t_m} - A_m] = 0, \tag{1}$$

where A_k and A_m are matrix differential operators of n variables.

In [4] and [5] the sufficient conditions for the spectral data, which correspond to the free BA-modules were found (see Theorem 4.1 in [4] and Theorem 3.3 in [5]). It is not clear how to find the algebraic varieties, satisfying these conditions. In [4] and [5] there are two examples satisfying these conditions. In [5] sufficient conditions for spectral data corresponding to the solutions of the equation (1) were ascertained; in the paper a corresponding example was also demonstrated.

Note that in all the examples above the construction is either implicit (see [4] and [5]), or the solutions are expressed in terms of theta-functions (see [1, 2], and [3]).

Those who wish to read more widely in the theory of commuting operators of several variables and BA-modules can turn to [6].

The aim of this paper is to demonstrate the construction of Nakayashiki for the rational varieties. For the rational spectral variety, the BA-module elements and coefficients of commutating differential operators are expressed in terms of elementary functions.

Our initial idea was to obtain BA-modules on rational varieties from BA-modules of Nakayashiki by degenerating of Abelian varieties in the same way as soliton solutions of KdV are obtained from finite-gap solutions by degenerating of smooth spectral curves to spheres with double points. We considered many candidates of rational varieties and, as a result, we found varieties Γ and Ω (see below) appropriate for our goals.

In the next section we describe the spectral data used in this paper and formulate our main results. In Sections 3 and 4 we show that the BA-modules on Γ and Ω are free. In Section 5 we present explicit examples of commutating operators. In the Appendix we show that on Γ and Ω there are structures of algebraic varieties (it not follows directly from the definition of Γ and Ω).

2 Main results

Let us fix $a_1, a_2, b_1, b_2 \in \mathbb{C}$ such that $(a_i, b_i) \neq (0, 0)$ and $(a_1 : b_1) \neq (a_2 : b_2)$. Let us also fix nondegenerate linear map $\mathcal{P} : \mathbb{C}^n \to \mathbb{C}^n$. This map induce the map $\mathbb{C}P^{n-1} \to \mathbb{C}P^{n-1}$, which we denote by the same symbol $\mathcal{P}$. Let Γ denote the variety constructed from $\mathbb{C}P^1 \times \mathbb{C}P^{n-1}$ by identification of two hypersurfaces

$$p_1 \times \mathbb{C}P^{n-1} \sim p_2 \times \mathbb{C}P^{n-1}$$

with the use of $\mathcal{P}$, where $p_i = (a_i : b_i)$. Namely, let us identify

$$(a_1 : b_1, t) \sim (a_2 : b_2, \mathcal{P}(t)), \qquad t = (t_1 : \cdots : t_n) \in \mathbb{C}P^{n-1}.$$

Let $f(P)$ be the following function on $\mathbb{C}^{n+2}$

$$f(z_1, z_2, t_1, \ldots, t_n) = \sum_{i=1}^{n} (\alpha_i z_1 t_i + \beta_i z_2 t_i), \qquad \alpha_i, \beta_i \in \mathbb{C}. \tag{2}$$

[1]In the proof of freeness of the BA-modules in [3] there is a gap, an additional proposition is required. The complete proof will appear in the work of K. Cho, A. Mironov, and A. Nakayashiki "Baker–Akhiezer module on the intersection of shifted theta divisors" (submitted to Publ. RIMS).

such that the following identity takes place

$$f(a_1, b_1, t) - Af(a_2, b_2, \mathcal{P}(t)) = 0 \tag{3}$$

for fixed $A \in \mathbb{C}^*$ and every $t = (t_1, \dots, t_n)$. Identity (3) gives the restriction on the choice of α_i, β_i. According to (3), the equation

$$f(z_1 : z_2, t_1 : \dots : t_n) = 0$$

correctly defines a hypersurface in Γ.

We denote eigenvalues and eigenvectors of $\mathcal{P}$ by λ_j and $\mathbf{w}_j$ respectively. Henceforth, we assume that

$$\lambda_j \neq \lambda_k \qquad \text{at} \quad j \neq k, \tag{4}$$

and $f(P)$ is chosen such that

$$f(a_1, b_1, \mathbf{w}_j) \neq 0, \qquad j = 1, \dots, n. \tag{5}$$

We introduce n functions on $\mathbb{C}^{n+2}$

$$f_i(z_1, z_2, t_1, \dots, t_n) = \sum_{k=1}^{n} (\alpha_{ik} z_1 t_k + \beta_{ik} z_2 t_k)$$

such that f_i satisfies the identity:

$$\frac{f_i(a_1, b_1, t)}{f(a_1, b_1, t)} - \frac{f_i(a_2, b_2, \mathcal{P}(t))}{f(a_2, b_2, \mathcal{P}(t))} - c_i = 0, \qquad c_i \in \mathbb{C} \tag{6}$$

for every $t = (t_1, \dots, t_n) \in \mathbb{C}^n$. By (3), this identity is equivalent to

$$f_i(a_1, b_1, t) - Af_i(a_2, b_2, \mathcal{P}(t)) - c_i f(a_1, b_1, t) = 0. \tag{7}$$

The dimension of the space of these functions is equal to $(n+1)$. We choose $f_1, \dots, f_n$ such that $f_1, \dots, f_n$ and f are linearly independent. Moreover, we choose parameters (α, β), (α_i, β_i) in a general position, that means that the parameters belong to some open everywhere dense domain (more precisely, such that equation (18) has no multiple solutions).

Let us fix $\Lambda \in \mathbb{C}$. Let

$$M_\Gamma(k) = \left\{ \psi(x, P) = \frac{h(x_1, \dots, x_n, P)}{f^k(P)} \exp\left(\sum_{j=1}^{n} \frac{f_j(P)}{f(P)} x_j \right) \right\},$$

where

$$\psi(x, a_1 : b_1, t) - \Lambda \psi(x, a_2 : b_2, \mathcal{P}(t)) = 0 \tag{8}$$

for $t = (t_1 : \dots : t_n) \in \mathbb{C}P^{n-1}$. Here $P = (z_1 : z_2, t) \in \mathbb{C}P^1 \times \mathbb{C}P^{n-1}$ and $h(x, P)$ has the form

$$h(x, P) = \sum_{0 \le j \le k,\, |\alpha| = k} h_{j\alpha}(x) z_1^j z_2^{k-j} t^\alpha, \tag{9}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $t^\alpha = t_1^{\alpha_1} \cdot \dots \cdot t_n^{\alpha_n}$.

According to (8), if $\psi \in M_\Gamma(k)$, then $\partial_{x_j} \psi \in M_\Gamma(k)$. Consequently, we have n mapping

$$\partial_{x_j} : \ M_\Gamma(k) \to M_\Gamma(k+1), \qquad j = 1, \dots, n.$$

Thereby, on the set

$$M_\Gamma = \bigcup_{k=1}^{\infty} M_\Gamma(k)$$

the structure of the BA-module over the ring of differential operators $\mathcal{D}_n = \mathcal{O}\left[\partial_{x_1}, \dots, \partial_{x_n}\right]$ is defined.

Theorem 1. *M_Γ is a free $\mathcal{D}_n$-module of rank n generated by n functions from $M_\Gamma(1)$.*

Corollary 1. *There is a ring embedding*

$$D: \ A_f \to \mathrm{Mat}(n, \mathcal{D}_n)$$

of the rings of meromorphic functions on Γ with the poles on hypersurface $f = 0$ into the ring of differential operators in variables $x_1, \dots, x_n$ with the matrix coefficients of size $n \times n$.

For $n = 2$ it is possible to consider another way of identification of two curves in $\mathbb{C}P^1 \times \mathbb{C}P^1$. Let Ω denote the variety which comes out from $\mathbb{C}P^1 \times \mathbb{C}P^1$ by the identification of two lines

$$p_1 \times \mathbb{C}P^1 \sim \mathbb{C}P^1 \times p_2.$$

Videlicet, we identify the following points:

$$(p_1, t) \sim (\mathcal{P}(t), p_2),$$

where $p_i, t \in \mathbb{C}P^1$. We assume that $\mathcal{P}(p_2) \neq p_1$. Therefore, in an appropriate coordinate system, the variety Ω has the form

$$\Omega = \mathbb{C}P^1 \times \mathbb{C}P^1 / \{(1:0, t_1:t_2) \sim (t_1:t_2, 0:1)\}. \tag{10}$$

Indeed, on $\mathbb{C}P^1 \times \mathbb{C}P^1$ we make the following change of coordinates:

$$(z', w') = (z, \mathcal{P}(w)),$$

where (z, w), (z', w') are the old and the new coordinates on $\mathbb{C}P^1 \times \mathbb{C}P^1$ respectively. Then Ω is obtained by the identification of the points:

$$(p_1, \mathcal{P}(t)) \sim (\mathcal{P}(t), \mathcal{P}(p_2)).$$

Now on each of components $\mathbb{C}P^1 \times \mathbb{C}P^1$ we do the same change of coordinates such that the points p_1 and $\mathcal{P}(p_2)$ in the new system have coordinates $(1:0)$ and $(0:1)$ respectively. In the new coordinates the variety Ω has the form (10).

Let g denote the following function

$$g(z_1, z_2, w_1, w_2) = \alpha z_1 w_1 + \beta z_1 w_2 + \gamma z_2 w_1 + \delta z_2 w_2, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{C}$$

such that for $(t_1, t_2) \in \mathbb{C}^2$ the following identity is fulfilled

$$g(1, 0, t_1, t_2) - B g(t_1, t_2, 0, 1) = 0, \tag{11}$$

where $B \in \mathbb{C}^*$ is fixed. We assume that

$$g(0, 1, 0, 1) \neq 0.$$

We introduce two more functions

$$g_i(z_1, z_2, w_1, w_2) = \alpha_i z_1 w_1 + \beta_i z_1 w_2 + \gamma_i z_2 w_1 + \delta_i z_2 w_2, \qquad \alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{C}, \quad i = 1, 2$$

such that for $(t_1, t_2) \in \mathbb{C}^2$ the identity is fulfilled

$$\frac{g_i(1, 0, t_1, t_2)}{g(1, 0, t_1, t_2)} - \frac{g_i(t_1, t_2, 0, 1)}{g(t_1, t_2, 0, 1)} - c_i = 0, \qquad c_i \in \mathbb{C}. \tag{12}$$

The dimension of the space of such functions is equal to 3. According to (11), the identity (12) is equivalent to

$$g_i(1, 0, t_1, t_2) - Bg_i(t_1, t_2, 0, 1) - c_i g(1, 0, t_1, t_2) = 0.$$

Let us choose g_1 and g_2 in such a way that g_1, g_2 and g are linearly independent and the under radical expression in (23) does not vanish (this always can be achieved by the infinitesimal changes of c_1 and c_2). Let

$$G_1(P) = \frac{g_1(P)}{g(P)}, \qquad G_2(P) = \frac{g_2(P)}{g(P)}.$$

Let us fix $\Lambda \in \mathbb{C}$. By $M_\Omega(k)$ we denote the set of functions of the form

$$M_\Omega(k) = \left\{ \varphi = \frac{\tilde{h}(x, y, P)}{g^k(P)} \exp\left(xG_1(P) + yG_2(P)\right) \right\}$$

for which the identity

$$\varphi(x, y, 1 : 0, t_1 : t_2) - \Lambda\varphi(x, y, t_1 : t_2, 0 : 1) = 0,$$

is fulfilled, where $\tilde{h}$ is the function of the form (9).

Let

$$M_\Omega = \bigcup_{k=1}^{\infty} M_\Omega(k),$$

M_Ω is the module over $\mathcal{D} = \mathcal{O}[\partial_x, \partial_y]$.

Theorem 2. *M_Ω is a free $\mathcal{D}$-module of the rank 2 generated by two functions from $M_\Omega(1)$.*

Let A_g denote the ring of the meromorphic functions on Ω with the pole on the curve defined by the equation $g(P) = 0$.

Corollary 2. *There is a ring embedding*

$$D: \; A_g \to \mathrm{Mat}(2, \mathcal{D})$$

of A_g into the ring of 2×2-matrix differential operators in variables x and y.

Remark 1. To be more precise, the freeness of M_Γ and M_Ω is a corollary of the fact that corresponding graded gr$\mathcal{D}$-modules are free, where the graduation is induced by the degree of the operators and the order of poles respectively. Below we virtually prove the freeness of graded modules.

3 Proof of Theorem 1

3.1 Combinatorial structure of M_Γ

We find the rank of the module $M_\Gamma(k)$ over $\mathcal{O}$. The dimension of the space of functions $\{h(x,P)\}$ (see (9)) is equal $C^1_{k+1}C^{n-1}_{k+n-1}$ (for fixed x). The condition (8) with the help of (3) and (6) can be written in the equivalent form

$$h(x,a_1,b_1,t)-h(x,a_2,b_2,\mathcal{P}(t))\Lambda A^k e^{-cx}=0,$$

where $cx=\sum_{j=1}^n c_jx_j$. This equality means that the coefficients of the homogeneous polynomial in $t_1,\dots,t_n$ of the degree k, situated in the left part, are equal to 0. It gives C^{n-1}_{k+n-1} restrictions on the choice of coefficients of $h(x,P)$. Thereby,

$$\mathrm{rank}_{\mathcal{O}}M_\Gamma(k)=(k+1)C^{n-1}_{k+n-1}-C^{n-1}_{k+n-1}=kC^{n-1}_{k+n-1}. \tag{13}$$

Let $\mathcal{D}^{k-1}_n$ be the differential operators of order $k-1$ at most. We have

$$\mathrm{rank}_{\mathcal{O}}\mathcal{D}^{k-1}_n=C^n_{k+n-1}=\frac{(k+n-1)!}{(k-1)!n!}=\frac{k(k+1)\cdots(k+n-1)}{n!}=\frac{k}{n}C^{n-1}_{k+n-1}. \tag{14}$$

Comparing (13) and (14), we can expect that M_Γ is a free module of the rank n generated by n functions from $M_\Gamma(1)$.

3.2 Module N

Let us choose n functions $\psi_1,\dots,\psi_n\in M_\Gamma(1)$ independent over $\mathcal{O}$

$$\psi_k(x,P)=\frac{h_k(x,P)}{f(P)}\exp\left(\sum_{j=1}^n\frac{f_j(P)}{f(P)}x_j\right),$$

where

$$h_k(x,P)=\sum_{i=1}^n\left(h^1_{ki}(x)z_1t_i+h^2_{ki}(x)z_2t_i\right).$$

Consider the module N over $\mathcal{D}_n$ generated by the functions $\psi_1,\dots,\psi_n$

$$N=\left\{\sum_{i=1}^n d_i\psi_i\,|\,d_i\in\mathcal{D}_n\right\}.$$

We show that the module N is free (Lemma 1) and as a consequence from the combinatorial calculation we get that the modules M_Γ and N coincide (Lemma 3).

Lemma 1. *N is a free $\mathcal{D}_n$-module of rank n.*

Proof. Suppose that the assertion is not true, i.e. there are differential operators $d_1,\dots,d_n\in\mathcal{D}_n$ such that

$$d_1\psi_1+\dots+d_n\psi_n=0, \tag{15}$$

where

$$d_j=\sum_{\alpha:|\alpha|\le K}a_{j\alpha}(x)\partial^\alpha_x,$$

K is maximal order of operators d_j.

Let us divide (15) by $\exp\big(\sum \frac{f_j}{f} x_j\big)$, multiply by f^{K+1} and restrict the received equality on the hyperspace $f(P) = 0$. We receive the following equality (for the compactness of the record the arguments are skipped)

$$\begin{aligned} &h_1\big(a_{1(K,0,\dots,0)} f_1^K + a_{1(K-1,1,0,\dots,0)} f_1^{K-1} f_2 + \cdots + a_{1(0,\dots,0,K)} f_n^K\big) + \cdots \\ &\qquad + h_n\big(a_{n(K,0,\dots,0)} f_1^K + a_{n(K-1,1,0,\dots,0)} f_1^{K-1} f_2 + \cdots + a_{n(0,\dots,0,K)} f_n^K\big) = 0. \end{aligned} \tag{16}$$

The hypersurfaces

$$f(P) = 0, \qquad f_j(P) = 0, \qquad j = 1, \dots, n, \quad j \neq k. \tag{17}$$

($f_k(P) = 0$ is left out) have n points of intersections. Indeed, let us consider (17) as linear equations in $t_1, \dots, t_n$. These equations have nonzero solutions if the determinant Δ_j, composed from coefficients (coefficients are linear forms $\alpha_i z_1 + \beta_i z_2$ or $\alpha_{si} z_1 + \beta_{si} z_2$), equals 0

$$\Delta_j = 0. \tag{18}$$

So, (18) is a homogeneous equation in z_1, z_2 of degree n, and by our assumption has no multiple solutions. By P_i^k, $i = 1, \dots, n$ we denote the intersection points of hypersurfaces (17).

Let us substitute P_i^n in (16), divide all equalities by f_n^K ($f_n(P_i^n) \neq 0$, see Lemma 4). We obtain a linear system of equations on coefficients $a_{k\,(0,\dots,0,K)}$ of operators d_k

$$\begin{aligned} &h_1(x, P_1^n) a_{1\,(0,\dots,0,K)} + \cdots + h_n(x, P_1^n) a_{n\,(0,\dots,0,K)} = 0, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &h_1(x, P_n^n) a_{1\,(0,\dots,0,K)} + \cdots + h_n(x, P_n^n) a_{n\,(0,\dots,0,K)} = 0. \end{aligned} \tag{19}$$

We need

Lemma 2. *The inequality holds*

$$\det \begin{pmatrix} h_1(x, P_1^k) & \dots & h_n(x, P_1^k) \\ \vdots & \ddots & \vdots \\ h_1(x, P_n^k) & \dots & h_n(x, P_n^k) \end{pmatrix} \neq 0.$$

The proof of this lemma is given in Subsection 3.3.

By Lemma 2, the solution of the system (19) is

$$a_{1(0,\dots,0,K)} = 0, \quad \dots, \quad a_{n(0,\dots,0,K)} = 0.$$

Similarly, for $k = 1, \dots, n-1$ from $\det\big(h_j(x, P_i^k)\big) \neq 0$ it follows that the coefficients $a_{j(0\dots K\dots 0)}$ must vanish (K is on the k-th place).

To show that the coefficients $a_{j\,(0,\dots 0,m,l)}$, where $m + l = K$, vanish we should restrict (16) on $f(P) = 0$, $f_1(P) = 0$, $\dots$, $f_{n-2}(P) = 0$

$$\begin{aligned} &h_1\big(a_{1(0,\dots,0,K-1,1)} f_{n-1}^{K-1} f_n + \cdots + a_{1(0,\dots,0,1,K-1)} f_{n-1} f_n^{K-1}\big) + \cdots \\ &\qquad + h_n\big(a_{n(0,\dots,0,K-1,1)} f_{n-1}^{K-1} f_n + \cdots + a_{n(0,\dots,0,1,K-1)} f_{n-1} f_n^{K-1}\big) = 0. \end{aligned}$$

Since the product $f_{n-1}(P) f_n(P)$ is not identically zero on this set, then it can be used for dividing. Substituting the points $P = P_i^{n-1}$ and $P = P_i^n$ we obtain sets of equations of the form (19) on coefficients $a_{j\,(0,\dots,1,K-1)}$ and $a_{j\,(0,\dots,K-1,1)}$. Using Lemma 2, we conclude that the matrices $\big(h_j(x, P_i^k)\big)_{j,i=1}^n$ of the corresponding systems are nondegenerated and, consequently, $a_{i(0,\dots,1,K-1)} = a_{i(0,\dots,K-1,1)} = 0$, $i = 1, \dots, n$.

Similarly, one can show that all the leading coefficients of the operators d_k are zero. Thus, we have come to a contradiction with the fact that K is the leading order of operators $d_1, \dots, d_n$. ■

To complete the proof of Theorem 1 we need

Lemma 3. *The modules M_Γ and N coincide.*

Proof. By $N(k)$ we denote the following subset

$$N(k) = \left\{ \sum_{i=1}^{n} d_i \psi_i \mid d_i \in \mathcal{D}_n,\ \operatorname{ord} d_i \leq k-1 \right\}.$$

Since $\mathcal{D}_n$-module N is free,

$$\operatorname{rank}_{\mathcal{O}} N(k) = n \operatorname{rank}_{\mathcal{O}} \{ d\psi_1 \mid m\ d \in \mathcal{D}_n,\ \operatorname{ord} d \leq k-1 \} = n \operatorname{rank}_{\mathcal{O}} \mathcal{D}_n = k C_{k+n-1}^{n-1}.$$

Consequently,

$$\operatorname{rank}_{\mathcal{O}} N(k) = \operatorname{rank}_{\mathcal{O}} M_\Gamma(k).$$

Since there is obvious inclusion $N(k) \subseteq M_\Gamma(k)$, we obtain

$$M_\Gamma = N. \qquad \blacksquare$$

Theorem 1 is proven.

3.3 Proof of subsidiary statements

Choosing on $\mathbb{C}P^1 \times \mathbb{C}P^{n-1}$ a suitable coordinate system, we assume that Γ is obtained by the identification of hypersurfaces $(1:0) \times \mathbb{C}P^{n-1}$ and $(0:1) \times \mathbb{C}P^{n-1}$, i.e. $a_2 = b_1 = 0$. Also, we can assume that $a_1 = b_2 = 1$.

Lemma 4. *Under condition* (5) *hypersurfaces $f = 0,\ f_1 = 0,\ \dots,\ f_n = 0$ do not have common points in $\mathbb{C}P^1 \times \mathbb{C}P^{n-1}$.*

Proof. Let us recall that f and $f_1, \dots, f_n$ are the basis of the form (2), satisfying the identity (6). We find such number k that f_k satisfies (6) for $c_k \neq 0$. Without the loss of generality, we assume $c_n \neq 0$. Then the systems

$$\begin{cases} f = 0, \\ f_1 = 0, \\ \cdots\cdots\cdots \\ f_{n-1} = 0, \\ f_n = 0, \end{cases} \qquad \text{and} \qquad \begin{cases} f = 0, \\ f_1 - \dfrac{c_1}{c_n} f_n = 0, \\ \cdots\cdots\cdots\cdots\cdots \\ f_{n-1} - \dfrac{c_{n-1}}{c_n} f_n = 0, \\ \dfrac{1}{c_n} f_n = 0, \end{cases}$$

are equivalent. It is easy to verify that $f_k - \frac{c_k}{c_n} f_n$ satisfy (3). Let $\tilde{f}_k = f_k - \frac{c_k}{c_n} f_n$ for $1 \leq k \leq n-1$, and $\tilde{f}_n = \frac{1}{c_n} f_n$.

By the definition of f and condition (3), we have

$$\begin{aligned} f(z_1, z_2, t) &= f(z_1, 0, t) + f(0, z_2, t) = z_1 f(1, 0, t) + z_2 f(0, 1, t) \\ &= z_1 A f(0, 1, \mathcal{P}(t)) + z_2 f(0, 1, t). \end{aligned}$$

Since $f(z,t)$ is linear in last n arguments, then the equality $f = 0$ can be rewritten in the following way:

$$f(0, 1, z_1 A \mathcal{P}(t) + z_2 t) = 0.$$

Likewise, we transform the equalities $\tilde{f}_1 = 0, \dots, \tilde{f}_{n-1} = 0$. We obtain the following system of linear equations

$$f(0,1,v) = 0, \qquad \tilde{f}_j(0,1,v) = 0, \qquad j = 1,\dots,n-1,$$

$v = (z_1 A\mathcal{P}(t) + z_2 t) \in \mathbb{C}^n$. By linear independence (over $\mathbb{C}$) of the functions $f, \widetilde{f}_1, \dots, \widetilde{f}_{n-1}$ this system has the unique solution $v = 0$. Thus, the equations $f = 0$, $f_1 = 0, \dots, f_{n-1} = 0$ are equivalent to

$$z_1 A\mathcal{P}(t) + z_2 t = 0.$$

The solutions of this system have the form:

1) $z_1 = z_2 = 0$,
2) $t = 0$,
3) $z_2/(Az_1) = -\lambda_j, \qquad t = \mathbf{w}_j$.

The solutions of the form 1) and 2) do not specify any point in $\mathbb{C}P^1 \times \mathbb{C}P^{n-1}$. Taking into account (4), case 3) gives n different solutions. Now we add to this system the last equality $\tilde{f}_n = 0$. According to (7),

$$\tilde{f}_n(z,t) = z_1 A\tilde{f}_n(0,1,\mathcal{P}(t)) + z_2\tilde{f}_n(0,1,t) + f(1,0,t) = \tilde{f}_n(0,1,z_1 A\mathcal{P}(t) + z_2\, t) + f(1,0,t).$$

Taking into account (5) and 3), we have

$$\tilde{f}_n(z,t) = \tilde{f}_n(0,1,0,\dots,0) + f(1,0,\mathbf{w}_j) = f(1,0,\mathbf{w}_j) \neq 0.$$

This reasoning ends the proving of Lemma 4. ∎

Proof of Lemma 2. We note that since any function of the form (2) is a linear combination of the functions f and $f_1,\dots,f_n$, then from the definition of the module M_Γ it follows that a different choice of the forms f_j corresponds to a nondegenerated linear change of variables $x_1,\dots,x_n$. This implies that if the statement of Theorem 1 is true for any fixed set $f_1,\dots,f_n$ then it is true for any other one. Therefore, we work with easy-to-use basis.

Let us choose the basis such that the points of the intersection $P_i^k = (z_{i1}^k, z_{i2}^k, t_i^k)$ of the hypersurfaces $f(P) = 0$ and $f_j(P) = 0$ ($j = 1,\dots,n$; $j \neq k$) satisfy the following two conditions:

A) first two coordinates z_{i1}^k and z_{i2}^k of the points P_i^k do not vanish;

B) the set of the vectors $(t_1^k,\dots,t_n^k) \in \mathbb{C}^n$ is linearly independent.

It always can be achieved. Indeed, we choose n functions $\tilde{f}_j$ of the form (2), satisfying (3), and such that any $n-1$ from them and the function f are linear independent. In the course of proving of Lemma 4 it was shown that the points of intersection of the hypersurfaces $f = 0$, $\tilde{f}_j = 0$, $j = 1,\dots,n$; $j \neq k$, satisfy the conditions A) and B). We take one more function $\tilde{f}_{n+1}$ of the same form, satisfying (6) at $\tilde{c}_{n+1} = 1$. Since the coordinates of the points of intersections continuously depend on the coefficients of the functions, then by f_j we take $f_j = \tilde{f}_j + c_j\tilde{f}_{n+1}$, where c_j are sufficiently small. From the proof of Lemma 4 we obtain that all points P_j^k are different.

Let us write matrix $(h_j(x, P_i^k))$ in a more convenient form. For this, we note that the condition (8) can be written as the condition on h_k

$$h_k(x, a_1, b_1, t) - h_k(x, a_2, b_2, \mathcal{P}(t))\Lambda A e^{-cx} = 0.$$

Then the following equalities are true:

$$h_j(x,z_1,z_2,t) = h_j(x,z_1,0,t) + h_j(x,0,z_2,t) = z_1\Lambda A e^{-cx} h_j(x,0,1,\mathcal{P}(t)) + z_2 h_j(x,0,1,t)$$
$$= h_j(x,0,1,z_2 t + z_1 \Lambda A e^{-cx}\mathcal{P}(t)). \tag{20}$$

Let $\mathbf{v}_i^k = \left(z_{i2}^k t_i^k + z_{i1}^k e^{-cx}\mathcal{P}(t_i^k)\right)$. Thus, the nondegeneracy condition of the matrix $(h_j(x,P_i^k))$ is written in the following way

$$\det\begin{pmatrix} h_1(x,0,1,\mathbf{v}_1^k) & h_2(x,0,1,\mathbf{v}_1^k) & \dots & h_n(x,0,1,\mathbf{v}_1^k) \\ \vdots & \vdots & \ddots & \vdots \\ h_1(x,0,1,\mathbf{v}_n^k) & h_2(x,0,1,\mathbf{v}_n^k) & \dots & h_n(x,0,1,\mathbf{v}_n^k) \end{pmatrix} \neq 0. \tag{21}$$

Since $h_j(x,P)$ are independent over $\mathcal{O}$, from (20) we obtain that the functions $h_j(x,0,1,\cdot)$, as the functions of the last n arguments, are also independent over $\mathcal{O}$. Then inequality (21) is equivalent to the linear independence of the vectors $\mathbf{v}_i^k$, since h_i are the linear forms of $\mathbf{v}_i^k$.

Let us show that the vectors $\mathbf{v}_i^k$ are linear independent. Suppose that it is wrong, i.e. there are coefficients γ_i (generally speaking dependent on x) such, that $\sum \gamma_i \mathbf{v}_i^k = 0$ or in more detail:

$$\sum_{i=1}^{n}\left(\gamma_i z_{i2} t_i + \gamma_i z_{i1}\Lambda A e^{-cx} s_i\right) = 0,$$

where $s_i^k = \mathcal{P}(t_i^k)$. The last equality can be written in the matrix form

$$\left(T + \Lambda A e^{-cx} S\right)\gamma = 0, \tag{22}$$

where T and S are matrices, composed from vectors $z_{i2}t_i^k$ and $z_{i1}s_i^k$ respectively and independent from x, $\gamma = (\gamma_1,\dots,\gamma_n)^\top$. Since the matrices T and S are nondegenerated, then there are not more then n values μ_j such, that $\det(T+\mu_j S) = 0$ and μ_j also do not depend on x. Consequently, for $\Lambda A e^{-cx} \neq \mu_j$ system (22) has only one solution $\gamma = 0$, i.e., $\mathbf{v}_i^k$ are linear independent for almost every x and, consequently, the determinant $|h_j(x,P_i^k)|$ does not vanish identically in x. Lemma 2 is proven. ■

4 Proof of Theorem 2

Let us choose in $M_\Omega(1)$ two independent over $\mathcal{O}$ functions φ_1 and φ_2

$$\varphi_i = \frac{\tilde{h}_i(x,y,P)}{g(P)}\exp\left(xG_1(P) + yG_2(P)\right),$$

where

$$\tilde{h}_i(x,y,P) = k_i(x,y)z_1z_2 + l_i(x,y)z_1w_2 + m_i(x,y)w_1z_2 + n_i(x,y)w_1w_2, \qquad i = 1,2.$$

The functions $\tilde{h}_i$ satisfy the identity

$$\tilde{h}_i(x,y,1:0,t_1:t_2) - \tilde{h}_i(x,y,t_1:t_2,0:1)\Lambda B\exp(-xc_1 - yc_2) = 0.$$

By P_i and Q_i we denote the points of intersection of the curves, defined by the equations $g_i(P) = 0$ and $g(P) = 0$, $i = 1,2$. By the infinitesimal variations c_1 and c_2, one can obtain that these points are pairwise different.

By means of direct check, we can ascertain that the determinant

$$\det\begin{pmatrix} \tilde{h}_1(x,y,P_i) & \tilde{h}_2(x,y,P_i) \\ \tilde{h}_1(x,y,Q_i) & \tilde{h}_2(x,y,Q_i) \end{pmatrix}$$

up to multiplication by nonvanishing function from $\mathcal{O}$, is equal to

$$\frac{\sqrt{(\gamma\delta_i-\gamma_i\delta)^2+2\gamma\delta c_i(\gamma\delta_i-\gamma_i\delta)+\delta^2(\gamma-2B\delta)^2c_i^2}(\Lambda e^{-c_1x-c_2y}-1)^2}{\gamma\delta_i-\gamma_i\delta}. \tag{23}$$

It is obvious that the inequality

$$\det\begin{pmatrix}\tilde h_1(x,y,P_i) & \tilde h_2(x,y,P_i)\\ \tilde h_1(x,y,Q_i) & \tilde h_2(x,y,Q_i)\end{pmatrix}\neq 0 \tag{24}$$

is fulfilled for almost all x and y.

Further the proof of Theorem 2 verbatim repeats the proof of Theorem 1.

5 Examples

In this section we demonstrate the examples of the commuting differential operators and their common eigenvector-functions defining the basis in the free BA-modules.

5.1 Commuting operators corresponding to the variety Γ

Let us consider the case $n=2$. As a spectral variety we take

$$\Gamma=\mathbb{C}P^1\times\mathbb{C}P^1/\{(1:0,t_1:t_2)\sim(0:1,t_2:t_1)\},$$

i.e. in terms of Section 2 $p_1=(1:0)$, $p_2=(0:1)$, $\mathcal{P}(t_1,t_2)=(t_2,t_1)$.

We introduce three functions

$$f(P)=-z_1t_1-z_2t_2,\qquad f_1(P)=z_1t_2+z_2(t_1-it_2),\qquad f_2(P)=-z_1t_2-z_2(t_1+it_2).$$

Via the direct check, we can ascertain that $f(P)$ satisfies the condition (3) for $A=1$, and f_1, f_2 satisfy the condition (6) for $c_1=c_2=-i$.

Let us choose in $\mathcal{D}$-module M_Γ the basis

$$\psi_1(x,y,P)=\frac{z_1t_1+e^{-i(x+y)}z_2t_2}{f(P)}\exp\left(\frac{f_1}{f}x+\frac{f_2}{f}y\right),$$
$$\psi_2(x,y,P)=\frac{z_1t_2+e^{-i(x+y)}z_2t_1}{f(P)}\exp\left(\frac{f_1}{f}x+\frac{f_2}{f}y\right).$$

We consider the following meromorphic functions on Γ with the poles on the curve $f(P)=0$

$$\lambda_1=\frac{2(z_1t_2+z_2t_1)}{f(P)},\qquad \lambda_2=\frac{iz_1z_2(-t_1^2+t_2^2)}{f(P)^2},\qquad \lambda_3=\frac{z_1^2t_2^2+3z_1z_2t_1t_2+z_2^2t_1^2}{f(P)^2}.$$

Pairwise commuting operators, corresponding to these functions have the forms

$$D(\lambda_1)=\begin{pmatrix}\partial_x-\partial_y & 0\\ 0 & \partial_x-\partial_y\end{pmatrix},$$
$$D(\lambda_2)=\begin{pmatrix}\frac14(\partial_y^2-\partial_x^2)+\frac12\cot\left(\frac{x+y}{2}\right)(\partial_x-\partial_y) & \cot\left(\frac{x+y}{2}\right)-\frac12(\partial_x+\partial_y)\\ -\frac12(\partial_x+\partial_y) & \frac14(\partial_y^2-\partial_x^2)\end{pmatrix}.$$

Operator, corresponding to the function λ_3 has the form

$$[D(\lambda_3)]_{11} = \frac{1}{2}\partial_x^2 + \frac{1}{2}\partial_y^2 - \frac{1}{2}\cot\left(\frac{x+y}{2}\right)(\partial_x + \partial_y),$$

$$[D(\lambda_3)]_{12} = 0, \qquad [D(\lambda_3)]_{21} = \frac{1}{4\sin^2\left(\frac{x+y}{2}\right)}(\partial_x - \partial_y),$$

$$[D(\lambda_3)]_{22} = \frac{1}{2\sin^2\left(\frac{x+y}{2}\right)} - \frac{1}{2}\cot\left(\frac{x+y}{2}\right)(\partial_x + \partial_y) + \frac{1}{2}\partial_x^2 + \frac{1}{2}\partial_y^2.$$

5.2 Commuting operators corresponding to the variety Ω

Let us consider three functions

$$g(P) = z_1w_1 + z_1w_2 + z_2w_2,$$
$$g_1(P) = z_1w_1 + 2z_2w_1 - z_2w_2,$$
$$g_2(P) = -z_1w_1 + 2z_2w_1 + z_2w_2.$$

By the direct check we can ascertain that $g(P)$ satisfies the identity (11) for $B = 1$, $g_1(P)$ and $g_2(P)$ satisfy to the identity (12) for $c_1 = 1$ and $c_2 = -1$ respectively.

The curves $g(P) = 0$ and $g_1(P) = 0$ are intersected in the points

$$P_1 = \left(-2 - \sqrt{2} : 1, -\frac{1}{\sqrt{2}} : 1\right), \qquad Q_1 = \left(-2 + \sqrt{2} : 1, \frac{1}{\sqrt{2}} : 1\right),$$

and the curves $g(P) = 0$ and $g_2(P) = 0$ are intersected in the points

$$P_2 = \left(-\sqrt{2} : 1, -1 + \frac{1}{\sqrt{2}} : 1\right), \qquad Q_2 = \left(\sqrt{2} : 1, -1 - \frac{1}{\sqrt{2}} : 1\right).$$

Chose the basis ψ_1, ψ_2 in $\mathcal{D}$-module M_Ω

$$\psi_1 = \frac{z_2w_1}{g(P)}\exp\left(xG_1(P) + yG_2(P)\right),$$

$$\psi_2 = \frac{z_1w_1e^{y-x} + z_1w_2 + z_2w_2e^{x-y}}{g(P)}\exp\left(xG_1(P) + yG_2(P)\right).$$

Then

$$\frac{\tilde{h}_1(P_1, x, y)}{\tilde{h}_2(P_1, x, y)} = -\frac{e^{x+y}}{\sqrt{2}(e^y - e^x)(-e^x + (1+\sqrt{2})e^y)},$$

$$\frac{\tilde{h}_1(Q_1, x, y)}{\tilde{h}_2(Q_1, x, y)} = -\frac{e^{x+y}}{\sqrt{2}(e^y - e^x)(e^x + (-1+\sqrt{2})e^y)},$$

thus inequality (24) is fulfilled.

Four simplest meromorphic functions on Ω with the poles on the curve $g(P) = 0$ have the form

$$\lambda_1 = \frac{z_2w_1}{g(P)}, \qquad \lambda_2 = \frac{z_1z_2w_1^2}{g(P)^2}, \qquad \lambda_3 = \frac{z_1w_1z_2w_2}{g(P)^2}, \qquad \lambda_4 = \frac{z_1z_2w_2^2 + z_1^2w_1w_2}{g(P)^2}.$$

Pairwise commutating operators, corresponding to these functions have the form

$$D(\lambda_1) = \begin{pmatrix} \frac{1}{4}(\partial_x + \partial_y) & 0 \\ 0 & \frac{1}{4}(\partial_x + \partial_y) \end{pmatrix},$$

$$
\begin{aligned}
[D(\lambda_2)]_{11} &= \frac{e^x}{8(e^x-e^y)}\left(\partial_x^2-\partial_y^2\right) - \frac{e^{x+y}}{4(e^x-e^y)^2}(\partial_x+\partial_y),\\
[D(\lambda_2)]_{12} &= \frac{e^{x+y}}{16(e^x-e^y)^2}(\partial_x+\partial_y)^2,\\
[D(\lambda_2)]_{21} &= \frac{1}{8}(e^{y-x}-e^{x-y}-2)\partial_x^2 + \frac{1}{8}(e^{x-y}-e^{y-x}-2)\partial_y^2 + \frac{1}{2}\partial_x\partial_y\\
&\quad + \frac{e^x+e^{2x-y}+5e^y-e^{2y-x}}{4(e^x-e^y)}\partial_x + \frac{3e^x-e^{2x-y}+3e^y+e^{2y-x}}{4(e^y-e^x)}\partial_y - \frac{e^y(2e^x+e^y)}{(e^x-e^y)^2},\\
[D(\lambda_2)]_{22} &= \frac{e^x}{8(e^y-e^x)}\partial_x^2 - \frac{1}{4}\partial_x\partial_y + \frac{e^x-2e^y}{8(e^y-e^x)}\partial_y^2 + \frac{e^y(2e^x+e^y)}{8(e^x-e^y)^2}(\partial_x+\partial_y).
\end{aligned}
$$

The operator, corresponding to the function λ_3 has the form

$$
\begin{aligned}
[D(\lambda_3)]_{11} &= \frac{(e^x+e^y)}{8(e^y-e^x)}\left(\partial_x^2-\partial_y^2\right) + \frac{(e^{2x}+e^{2y})}{4(e^y-e^x)^2}(\partial_x+\partial_y),\\
[D(\lambda_3)]_{12} &= \frac{e^{x+y}}{8(e^y-e^x)^2}(\partial_x-\partial_y)^2,\\
[D(\lambda_3)]_{21} &= \frac{1}{4}(2+e^{x-y}-e^{y-x})\partial_x^2 - \partial_x\partial_y + \frac{1}{4}(2-e^{x-y}+e^{y-x})\partial_y^2\\
&\quad + \frac{2e^x+e^{2x-y}+4e^y-e^{2y-x}}{2(e^y-e^x)}\partial_x + \frac{4e^x-e^{2x-y}+2e^y+e^{2y-x}}{2(e^x-e^y)}\partial_y\\
&\quad + \frac{e^{2x}+e^{2y}+4e^{x+y}}{(e^x-e^y)^2},\\
[D(\lambda_3)]_{22} &= \frac{3e^x-e^y}{8(e^x-e^y)}\partial_x^2 + \frac{1}{2}\partial_x\partial_y + \frac{e^x-3e^y}{8(e^x-e^y)}\partial_y^2 - \frac{3e^{x+y}}{2(e^y-e^x)^2}(\partial_x+\partial_y).
\end{aligned}
$$

Operator corresponding to the function λ_4 has the form

$$
\begin{aligned}
[D(\lambda_4)]_{11} &= \frac{e^x+3e^y}{4(e^x-e^y)}\partial_x^2 + \frac{1}{2}\partial_x\partial_y - \frac{3e^x+e^y}{4(e^x-e^y)}\partial_y^2\\
&\quad - \frac{e^{2x}+3e^{2y}}{2(e^x-e^y)^2}\partial_x - \frac{3e^{2x}+e^{2y}}{2(e^x-e^y)^2}\partial_y - \frac{2e^{x+y}}{(e^y-e^x)^2},\\
[D(\lambda_4)]_{12} &= \frac{e^{x+y}}{2(e^y-e^x)^2}((\partial_x+\partial_y)^2+\partial_x+\partial_y),\\
[D(\lambda_4)]_{21} &= (e^{y-x}-e^{x-y}-2)\partial_x^2 + 4\partial_x\partial_y + (e^{x-y}-e^{y-x}-2)\partial_y^2\\
&\quad + \frac{5e^x+e^{2x-y}+9e^y-3e^{2y-x}}{e^x-e^y}\partial_x + \frac{9e^x-3e^{2x-y}+5e^y+e^{2y-x}}{e^x-e^y}\partial_y\\
&\quad + \frac{2e^{-x-y}(e^{4x}+e^{4y}-6e^{2(x+y)}-4e^{3x+y}-4e^{x+3y})}{(e^x-e^y)^2},\\
[D(\lambda_4)]_{22} &= \frac{7e^x-3e^y}{4(e^y-e^x)}\partial_x^2 - \frac{3}{2}\partial_x\partial_y + \frac{3e^x-7e^y}{4(e^y-e^x)}\partial_y^2\\
&\quad - \frac{e^{2x}+3e^{2y}-16e^{x+y}}{2(e^x-e^y)^2}\partial_x - \frac{3e^{2x}+e^{2y}-16e^{x+y}}{2(e^x-e^y)^2}\partial_y.
\end{aligned}
$$

A Structures of algebraic varieties on Γ and Ω

We show that on Γ and Ω structures of algebraic varieties can be introduced. For this in the first case we construct a smooth morphism from $\mathbb{C}P^1 \times \mathbb{C}P^{n-1}$ to $\mathbb{C}P^2 \times \mathbb{C}P^{2n-1}$, and in the second case from $\mathbb{C}P^1 \times \mathbb{C}P^1$ to $\mathbb{C}P^{11}$. The morphisms are injective everywhere except gluing hypersurfaces. The images of the morphisms are algebraic varieties which define required structures on Γ and Ω.

A.1 Variety Γ

By choosing the convenient coordinate system on $\mathbb{C}P^1$ we can assume that

$$\Gamma = \mathbb{C}P^1 \times \mathbb{C}P^{n-1}/\{(1:0,t) \sim (0:1,\mathcal{P}(t))\}.$$

We consider the mapping

$$\varphi_1 : \ \mathbb{C}P^1 \times \mathbb{C}P^{n-1} \to \mathbb{C}P^2 \times \mathbb{C}P^{2n-1},$$

defined by the formula:

$$\varphi_1(z,t) = (u,v),$$

where $u = (u_1 : u_2 : u_3)$,

$$u_1 = z_1^2 z_2, \qquad u_2 = z_1 z_2^2, \qquad u_3 = z_1^3 + z_2^3, \qquad v = \left(z_1^2 t + z_2^2 \mathcal{P}^{-1}(t) : z_1 z_2 t\right).$$

Here $v = (\xi_1 : \cdots : \xi_n : \eta_1 : \cdots : \eta_n)$, $\xi_j = z_1 2 t_j + z_2 2 r_j$ and $\eta_j = z_1 z_2 t_j$, where r_j is j-th coordinate $\mathcal{P}^{-1}(t)$, $j = 1, \ldots, n$.

Lemma 5. *The mapping φ_1 is correctly defined on Γ and is the embedding of Γ. The image of φ_1 is defined by the equations:*

$$u_1^3 + u_2^3 = u_1 u_2 u_3, \tag{25}$$
$$u_1^2 \eta + u_2^2 \mathcal{P}^{-1}(\eta) = u_1 u_2 \xi, \tag{26}$$
$$u_3 \eta + u_1 \mathcal{P}(\eta) + u_2 \mathcal{P}^{-1}(\eta) = u_1 \xi + u_2 \mathcal{P}(\xi). \tag{27}$$

Proof. Let us show that the image of the point $(z,t) \in \mathbb{C}P^1 \times \mathbb{C}P^{n-1}$ satisfies the equations (25)–(27). The equalities (25) and (26) obviously follow from the definition of φ_1. The equality (27) for $z_1 \neq 0$ and $z_2 \neq 0$ is the corollary of (25) and (26). Indeed, from (26) we obtain

$$u_1^2 \mathcal{P}(\eta) + u_2^2 \eta = u_1 u_2 \mathcal{P}(\xi).$$

Let us multiply the obtained equality by u_2, equality (26) by u_1 and take a sum. Dividing the result by $u_1 u_2 \neq 0$, we obtain (27). If z_1 or z_2 are equal to zero, then $\eta = 0$, and, consequently, the left and right parts of (27) vanish.

We show that for any point $(u,v) \in \mathbb{C}P^2 \times \mathbb{C}P^{2n-1}$, satisfying (25)–(27), the inverse image can be found. Note that from (25) it follows that u_1 and u_2 can vanish only simultaneously.

If $u_1 = u_2 = 0$, then from (27) we obtain $\eta = 0$ and the inverse image can be either point $(1:0,\xi)$, or point $(0:1,\mathcal{P}(\xi))$, which are identified in Γ.

Now we consider the case $u_1 \neq 0$ and $u_2 \neq 0$. From (26) it follows that $\xi \neq 0$. In this case as the inverse image of the point we can take point $(u_1 : u_2, \eta)$. By easy calculations it can be checked that $\varphi_1(u_1 : u_2, \eta) = (u_1 : u_2 : u_3, \xi : \eta)$. If any other point $B = (a : b, s_1 : \cdots : s_n)$ is the inverse image of $(u_1 : u_2 : u_3, \xi : \eta)$, then $a : b = u_1 : u_2$ and, consequently, $a \neq 0$, $b \neq 0$, $\eta_j = a b s_j$, i.e. $B = (u_1 : u_2, \eta)$, which shows injectivity of the mapping φ_1 on Γ.

By direct calculations one can show that the differential of the mapping φ_1 is nondegenerated. ■

A.2 Variety Ω

Let us consider the mapping

$$\varphi_2 : \ \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^{11},$$

defined by the formula:

$$\varphi_2(z_1 : z_2, w_1 : w_2) = (u_1 : \cdots : u_{12}),$$

where

$$u_1 = z_1^3\big(w_1^3 + w_2^3\big) + (z_2 w_2)^3, \qquad u_2 = z_1^3 w_1^2 w_2 + z_1^2 z_2 w_2^3, \qquad u_3 = z_1^3 w_1 w_2^2 + z_1 z_2^2 w_2^3,$$
$$u_4 = z_1^2 z_2 w_1^3, \qquad u_5 = z_1^2 z_2 w_1^2 w_2, \qquad u_6 = z_1^2 z_2 w_1 w_2^2, \qquad u_7 = z_1 z_2^2 w_1^3, \qquad u_8 = z_1 z_2^2 w_1^2 w_2,$$
$$u_9 = z_1 z_2^2 w_1 w_2^2, \qquad u_{10} = z_2^3 w_1^3, \qquad u_{11} = z_2^3 w_1^2 w_2, \qquad u_{12} = z_2^3 w_1 w_2^2.$$

We can easily ascertain that u_j do not vanish simultaneously (for example, u_1, u_2, u_3 and u_{10} equal zero if and only if $z_1 = z_2 = 0$ or $w_1 = w_2 = 0$).

Lemma 6. *The mapping φ_2 is correctly defined on Ω and is the embedding of Ω.*

Proof. We show that the mapping φ_2 identifies only points $(1 : 0, t_1 : t_2)$ and $(t_1 : t_2, 0 : 1)$ on $\mathbb{C}P^1 \times \mathbb{C}P^1$.

If $u_{10} \neq 0$, then from the definition of φ_2 it follows that the inverse image has the form $(z_1 : z_2, w_1 : w_2) = (u_7 : u_{10}, u_{10} : u_{11})$.

If $u_{10} = 0$, then $u_k = 0$ for $4 \leq k \leq 12$, u_2 and u_3 vanish simultaneously. Two cases are possible:

a) $u_2 \neq 0$ and $u_3 \neq 0$, then inverse image is one of the two points $(1 : 0, u_2 : u_3)$ or $(u_2 : u_3, 0 : 1)$, which are identified in Ω;

b) $u_2 = u_3 = 0$, then $u_1 \neq 0$ and inverse image is one of the three points $(1 : 0, 1 : 0)$, $(1 : 0, 0 : 1)$ or $(0 : 1, 0 : 1)$, which are also identified in Ω.

By direct calculations one can show that the differential of the mapping φ_2 is nondegenerated. ■

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