Yang–Baxter Maps from the Discrete BKP Equation⋆

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Abstract. We construct rational and piecewise-linear Yang–Baxter maps for a general N-reduction of the discrete BKP equation.

Key words: Yang–Baxter map; discrete BKP equation

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1 Introduction

Let R be a map $R : \mathbb{C}^2 \to \mathbb{C}^2$ and let $R_{ij} : \mathbb{C}^n \to \mathbb{C}^n$ be the map that acts as R on the i-th and j-th factors in the direct product $\mathbb{C} \times \cdots \times \mathbb{C}$ and identically on the others. A map R is called a Yang–Baxter map if it satisfies the Yang–Baxter equation (with parameters) [23, 24],

$$R_{12}(\lambda_1,\lambda_2)R_{13}(\lambda_1,\lambda_3)R_{23}(\lambda_2,\lambda_3) = R_{23}(\lambda_2,\lambda_3)R_{13}(\lambda_1,\lambda_3)R_{12}(\lambda_1,\lambda_2). \tag{1}$$

An important example of a Yang–Baxter map can be derived from the 3D-consistency condition for the discrete potential KdV equation [17]:

$$R(\kappa,\mu) : \ (u,v) \mapsto \left(vp, \frac{u}{p}\right), \qquad p = \frac{1+\kappa uv}{1+\mu uv}, \tag{2}$$

which is the H_{III}^B map in the classification of quadrirational Yang–Baxter maps found in [16] (which is in fact equivalent to the F_{III} map in the Adler–Bobenko–Suris classification [1]). An important property of the map (2) is that it is related to a geometric crystal of $A_1^{(1)}$-type [5]. Indeed, if we set

$$u = e^{U/\epsilon}, \qquad v = e^{V/\epsilon}, \qquad p = e^{P/\epsilon}, \qquad \kappa = e^{-K/\epsilon}, \qquad \mu = e^{-M/\epsilon},$$

and take the ultra-discrete limit (henceforth "ud-limit") $\epsilon \to +0$ [22], we obtain a piecewise-linear Yang–Baxter map,

$$(U,V) \mapsto (V+P, U-P), \qquad P = \max(0, U+V-K) - \max(0, U+V-M), \tag{3}$$

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where we have used the formula

$$\lim_{\epsilon\to+0} \epsilon \log\left(e^{A/\epsilon} + e^{B/\epsilon}\right) = \max(A, B), \qquad A, B \in \mathbb{R}. \tag{4}$$

It can be verified that (3) coincides with the combinatorial R-matrix of $A_1^{(1)}$-type [6], and is in fact equivalent to the evolution rule of a soliton cellular automata (a box-ball system "with a carrier") [20]. Furthermore, if we choose $\kappa = 0$, the map (2) coincides with Hirota's discrete KdV equation [8] and is therefore related to the Takahashi–Satsuma soliton cellular automata [21] as discussed in [11].

In a previous paper [11], we showed how several Yang–Baxter maps can be directly obtained from the discrete KP hierarchy. The aim of the present paper is to investigate the BKP counterpart of this construction and of the map (2) in particular.

2 Hirota–Miwa equation and Harrison-type map

In this section, we briefly describe a non-autonomous version of the method that enabled us to obtain the map (2) from the Hirota–Miwa equation,

$$\begin{aligned}(b-c)\tau(\ell+1,m,n)\tau(\ell,m+1,n+1) + (c-a)\tau(\ell,m+1,n)\tau(\ell+1,m,n+1)\\ + (a-b)\tau(\ell,m,n+1)\tau(\ell+1,m+1,n) = 0,\end{aligned} \tag{5}$$

which was introduced by Hirota [9]. The connection between (5) and the the A-type KP hierarchy was clarified by Miwa [13]. A non-autonomous version of the Hirota–Miwa equation,

$$\begin{aligned}\{b(m) - c(n)\}\,\tau(100)\tau(011) + \{c(n) - a(\ell)\}\,\tau(010)\tau(101)\\ + \{a(\ell) - b(m)\}\,\tau(001)\tau(110) = 0,\end{aligned} \tag{6}$$

was studied in [25, 26]. Here we have used the abbreviations, $\tau(100) = \tau(\ell+1,m,n)$, $\tau(101) = \tau(\ell+1,m,n+1)$, etc. The non-autonomous Hirota–Miwa (naHM) equation (6) arises as the compatibility condition for the linear system (cf. [3, 14] for autonomous case),

$$\begin{aligned}\psi(010) &= \frac{a(\ell)}{b(m)}\psi(100) + \left\{1 - \frac{a(\ell)}{b(m)}\right\} u(000)\psi(000),\\ \psi(001) &= \frac{a(\ell)}{c(n)}\psi(100) + \left\{1 - \frac{a(\ell)}{c(n)}\right\} v(000)\psi(000),\end{aligned} \tag{7}$$

where $u(\ell,m,n)$, $v(\ell,m,n)$ are defined as

$$u(000) = \frac{\tau(000)\tau(110)}{\tau(100)\tau(010)}, \qquad v(000) = \frac{\tau(000)\tau(101)}{\tau(100)\tau(001)}. \tag{8}$$

We now impose the following 2-reduction condition with respect to the variable ℓ;

$$\begin{aligned}&u(\ell+2,m,n) = u(\ell,m,n), \qquad v(\ell+2,m,n) = v(\ell,m,n),\\ &\psi(\ell+2,m,n) = \lambda\psi(\ell,m,n), \qquad a(\ell+2) = a(\ell) \qquad \forall\, \ell \in \mathbb{Z}.\end{aligned} \tag{9}$$

We furthermore require that the evolution in the remaining lattice directions is autonomous:

$$b(m) = b, \qquad c(n) = c \qquad \forall\, m, n \in \mathbb{Z}. \tag{10}$$

Then the linear system (7) is reduced to the 2×2 system

$$\begin{bmatrix}\psi_0(m+1,n)\\ \psi_1(m+1,n)\end{bmatrix} = \mathcal{U}(m,n)\begin{bmatrix}\psi_0(m,n)\\ \psi_1(m,n)\end{bmatrix}, \qquad \begin{bmatrix}\psi_0(m,n+1)\\ \psi_1(m,n+1)\end{bmatrix} = \mathcal{V}(m,n)\begin{bmatrix}\psi_0(m,n)\\ \psi_1(m,n)\end{bmatrix}, \tag{11}$$

where $\psi_i(m,n) = \psi(i,m,n)$ $(i = 0,1)$ and the coefficient matrices $\mathcal{U}(m,n)$, $\mathcal{V}(m,n)$ are given by

$$\begin{aligned}
\mathcal{U}(m,n) &= \begin{bmatrix} \{1 - a(0)/b\}\, u_0(m,n) & a(0)/b \\ \lambda a(1)/b & \{1 - a(1)/b\}\, u_1(m,n) \end{bmatrix}, \\
\mathcal{V}(m,n) &= \begin{bmatrix} \{1 - a(0)/c\}\, v_0(m,n) & a(0)/c \\ \lambda a(1)/c & \{1 - a(1)/c\}\, v_1(m,n) \end{bmatrix}, \\
u_\ell(m,n) &= u(\ell,m,n), \qquad v_\ell(m,n) = v(\ell,m,n).
\end{aligned}$$

We remark that $u_\ell(m,n)$, $v_\ell(m,n)$ $(\ell = 0,1)$ obey the constraints

$$u_0(m,n)u_1(m,n) = v_0(m,n)v_1(m,n) = 1, \tag{12}$$

under the conditions (9), (10).

The discrete Lax equation

$$\mathcal{U}(m,n+1)\mathcal{V}(m,n) = \mathcal{V}(m+1,n)\mathcal{U}(m,n), \tag{13}$$

follows from the compatibility condition for (11) and gives

$$u_\ell(m,n)v_\ell(m+1,n) = u_\ell(m,n+1)v_\ell(m,n), \tag{14}$$

$$\begin{aligned}
&\{a(\ell) - b\}\, u_\ell(m,n+1) - \{a(\ell) - c\}\, v_\ell(m+1,n) \\
&\quad = \{a(\ell+1) - b\}\, u_{\ell+1}(m,n) - \{a(\ell+1) - c\}\, v_{\ell+1}(m,n),
\end{aligned} \tag{15}$$

for $\ell = 0,1$. Using (12) to eliminate the u_1, v_1 variables, one can solve (14) and (15) for $u_0(m,n+1)$ and $v_0(m+1,n)$, to obtain the 2-reduced non-autonomous discrete KP (na-dKP) equation,

$$\begin{aligned}
&u_0(m,n+1) = \frac{\tilde{F}(m,n)}{v_0(m,n)}, \qquad v_0(m+1,n) = \frac{\tilde{F}(m,n)}{u_0(m,n)}, \\
&\tilde{F}(m,n) = \frac{\{a(1) - b\}\, v_0(m,n) - \{a(1) - c\}\, u_0(m,n)}{\{a(0) - b\}\, u_0(m,n) - \{a(0) - c\}\, v_0(m,n)}.
\end{aligned} \tag{16}$$

Alternatively, we can use (12), (14) and (15) to express $u_0(m,n+1)$, $v_0(m,n)$ in terms of $u_0(m,n)$, $v_0(m+1,n)$:

$$\begin{aligned}
&u_0(m,n+1) = v_0(m+1,n)F(m,n), \qquad v_0(m,n) = \frac{u_0(m,n)}{F(m,n)}, \\
&F(m,n) = \frac{\{a(1) - b\} + \{a(0) - c\}\, u_0(m,n)v_0(m+1,n)}{\{a(1) - c\} + \{a(0) - b\}\, u_0(m,n)v_0(m+1,n)}.
\end{aligned} \tag{17}$$

Motivated by (16) and (17), we define the following birational maps $\tilde{R}(b,c)$, $R(b,c)$:

$$\begin{aligned}
&\tilde{R}(b,c):\ (u,v) \mapsto \left(\frac{\tilde{F}}{v}, \frac{\tilde{F}}{u}\right), \qquad \tilde{F} = \frac{\{a(1) - b\}\, v - \{a(1) - c\}\, u}{\{a(0) - b\}\, u - \{a(0) - c\}\, v}, \\
&R(b,c):\ (u,v) \mapsto \left(vF, \frac{u}{F}\right), \qquad F = \frac{\{a(1) - b\} + \{a(0) - c\}\, uv}{\{a(1) - c\} + \{a(0) - b\}\, uv}.
\end{aligned} \tag{18}$$

From now on we shall refer to the map $\tilde{R}(b,c)$ as the "solitonic" map, since $\tilde{R}$ is nothing but the evolution rule for the discrete soliton equation (16). On the other hand, if we rescale as $u \mapsto \{a(1) - b\}\, u$, $v \mapsto \{a(1) - c\}\, v$, the map $R(b,c)$ coincides with the map (2) with $\kappa = \{a(0) - c\}\, \{a(1) - c\}$, $\mu = \{a(0) - b\}\, \{a(1) - b\}$. Hence $R(b,c)$ is a genuine Yang–Baxter map, i.e. it satisfies the Yang–Baxter equation with parameters (1). As discussed in [18], this is a direct consequence of the 3D-consistency of the discrete Lax equation (13). The solitonic map $\tilde{R}(b,c)$ satisfies a similar but slightly different relation,

$$\tilde{R}_{12}(b,c)\tilde{R}_{13}(b,d)R_{23}(c,d) = R_{23}(c,d)\tilde{R}_{13}(b,d)\tilde{R}_{12}(b,c), \tag{19}$$

which is also obtained from the 3D-consistency of the Lax pair.

Remark. Here we use the notation $\tilde{R}$ for "solitonic" maps and R for Yang–Baxter maps, whereas the opposite notation was used in [11].

If we impose the N-reduction conditions,

$$\begin{aligned}
&u(\ell+N,m,n) = u(\ell,m,n), \qquad v(\ell+N,m,n) = v(\ell,m,n),\\
&\psi(\ell+N,m,n) = \lambda\psi(\ell,m,n), \qquad a(\ell+N) = a(\ell) \qquad \forall\,\ell\in\mathbb{Z},\\
&b(m) = b, \qquad c(n) = c \qquad \forall\,m,n\in\mathbb{Z},
\end{aligned} \tag{20}$$

instead of the 2-reduction conditions (9), (10), the resulting Yang–Baxter map are related to a geometric crystal of $A^{(1)}_{N-1}$-type [5]. Its ud-limit coincides with a combinatorial R-matrix of $A^{(1)}_{N-1}$-type and hence with the evolution rule for $A^{(1)}_{N-1}$-soliton cellular automata [6].

3 From the Miwa equation to Yang–Baxter maps

In [13], Miwa introduced a discrete analogue of the bilinear BKP equation,

$$\begin{aligned}
&(a+b)(a+c)(b-c)\tau(100)\tau(011) + (b+c)(b+a)(c-a)\tau(010)\tau(101)\\
&\quad + (c+a)(c+b)(a-b)\tau(001)\tau(110) + (a-b)(b-c)(c-a)\tau(111)\tau(000) = 0.
\end{aligned} \tag{21}$$

It is straightforward to generalize (21) to the non-autonomous case:

$$\begin{aligned}
&\{a(\ell)+b(m)\}\{a(\ell)+c(n)\}\{b(m)-c(n)\}\tau(100)\tau(011)\\
&\quad + \{b(m)+c(n)\}\{b(m)+a(\ell)\}\{c(n)-a(\ell)\}\tau(010)\tau(101)\\
&\quad + \{c(n)+a(\ell)\}\{c(n)+b(m)\}\{a(\ell)-b(m)\}\tau(001)\tau(110)\\
&\quad + \{a(\ell)-b(m)\}\{b(m)-c(n)\}\{c(n)-a(\ell)\}\tau(111)\tau(000) = 0.
\end{aligned} \tag{22}$$

The non-autonomous Miwa (naM) equation (22) arises as the compatibility condition for the linear system (cf. [4, 15] for the autonomous case),

$$\begin{aligned}
&\{a(\ell)+b(m)\}\psi(010) - \{a(\ell)-b(m)\}u(000)\psi(110)\\
&\quad + \{a(\ell)-b(m)\}u(000)\psi(000) - \{b(m)+a(\ell)\}\psi(100) = 0,\\
&\{a(\ell)+c(n)\}\psi(001) - \{a(\ell)-c(n)\}v(000)\psi(101)\\
&\quad + \{a(\ell)-c(n)\}u(000)\psi(000) - \{c(n)+a(\ell)\}\psi(100) = 0,
\end{aligned}$$

where $u(\ell,m,n)$, $v(\ell,m,n)$ are defined as in (8).

We now impose the N-reduction condition (20) to obtain the discrete linear equations

$$\mathcal{U}_1(m,n)\begin{bmatrix}\psi_0(m+1,n)\\ \vdots\\ \psi_{N-1}(m+1,n)\end{bmatrix} = \mathcal{U}_2(m,n)\begin{bmatrix}\psi_0(m,n)\\ \vdots\\ \psi_{N-1}(m,n)\end{bmatrix}, \tag{23}$$

$$\mathcal{V}_1(m,n)\begin{bmatrix}\psi_0(m,n+1)\\ \vdots\\ \psi_{N-1}(m,n+1)\end{bmatrix} = \mathcal{V}_2(m,n)\begin{bmatrix}\psi_0(m,n)\\ \vdots\\ \psi_{N-1}(m,n)\end{bmatrix}, \tag{24}$$

where $\psi_i(m,n) = \psi(i,m,n)$ $(i = 0,1)$ and the coefficient matrices $\mathcal{U}_i(m,n)$, $\mathcal{V}_i(m,n)$ $(i = 1,2)$ are given by

$$\begin{aligned}
&\mathcal{U}_1 = \mathcal{I} + \mathcal{U}^{(0)}\Lambda, \quad \mathcal{U}_2 = \mathcal{U}^{(0)} + \Lambda, \qquad \mathcal{U}^{(0)} = \operatorname{diag}\left[\beta_0 u_0, \ldots, \beta_{N-1} u_{N-1}\right],\\
&\mathcal{V}_1 = \mathcal{I} + \mathcal{V}^{(0)}\Lambda, \qquad \mathcal{V}_2 = \mathcal{V}^{(0)} + \Lambda, \qquad \mathcal{V}^{(0)} = \operatorname{diag}\left[\gamma_0 v_0, \ldots, \gamma_{N-1} v_{N-1}\right],\\
&u_\ell(m,n) = u(\ell,m,n), \qquad v_\ell(m,n) = v(\ell,m,n),\\
&\Lambda = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ \lambda & & & 0 \end{bmatrix}, \qquad \beta_\ell = \frac{b - a(\ell)}{b + a(\ell)}, \qquad \gamma_\ell = \frac{c - a(\ell)}{c + a(\ell)},
\end{aligned} \tag{25}$$

where $\mathcal{I}$ is the identity matrix. We remark that $u_\ell(m,n)$, $v_\ell(m,n)$ $(\ell = 0,1,\ldots,N-1)$ obey the constraints

$$u_0(m,n)\cdots u_{N-1}(m,n) = v_0(m,n)\cdots v_{N-1}(m,n) = 1, \tag{26}$$

under the conditions (20). The compatibility condition of (23) and (24) is

$$\begin{aligned}
&\mathcal{U}_1(m,n+1)^{-1}\mathcal{U}_2(m,n+1)\mathcal{V}_1(m,n)^{-1}\mathcal{V}_2(m,n)\\
&\qquad = \mathcal{V}_1(m+1,n)^{-1}\mathcal{V}_2(m+1,n)\mathcal{U}_1(m,n)^{-1}\mathcal{U}_2(m,n).
\end{aligned} \tag{27}$$

We remark that $\det\mathcal{U}_1 = \det\mathcal{V}_1 = 1 + (-1)^{N-1}\lambda\prod_{j=0}^{N-1}\beta_j$ and hence $\mathcal{U}_1$, $\mathcal{V}_1$ are invertible.

To write down the Yang–Baxter map and the solitonic map associated with the discrete Lax equation (27), we use the abbreviations, $u_j := u_j(0,0)$, $v_j := v_j(0,0)$, $\bar{u}_j := u_j(0,1)$, $\bar{v}_j := v_j(1,0)$, $\boldsymbol{u} = (u_0,\ldots,u_{N-1})$, $\boldsymbol{v} = (v_0,\ldots,v_{N-1})$, $\bar{\boldsymbol{u}} = (\bar{u}_0,\ldots,\bar{u}_{N-1})$, $\bar{\boldsymbol{v}} = (\bar{v}_0,\ldots,\bar{v}_{N-1})$. Using these abbreviations, we have the following discrete evolution equations:

$$\begin{aligned}
&\bar{u}_j = u_j\frac{f_{j+1}(\boldsymbol{u},\boldsymbol{v})}{f_j(\boldsymbol{u},\boldsymbol{v})}, \qquad \bar{v}_j = v_j\frac{f_{j+1}(\boldsymbol{u},\boldsymbol{v})}{f_j(\boldsymbol{u},\boldsymbol{v})}, \qquad j = 0,1,\ldots,N-1,\\
&f_j(\boldsymbol{u},\boldsymbol{v}) = \sum_{k=0}^{N-1}\prod_{l=0}^{k-1}\beta_{j+l}\gamma_{j+l}u_{j+l}v_{j+l}\left(\beta_{j+k}u_{j+k} - \gamma_{j+k}v_{j+k}\right).
\end{aligned} \tag{28}$$

Here and in what follows, the subscripts are considered as elements of $\mathbb{Z}/N\mathbb{Z}$.

One can show that the discrete Lax equation (27) is equivalent to

$$\begin{aligned}
&\bar{u}_j = u_j\frac{g_j(\boldsymbol{u},\bar{\boldsymbol{v}})}{g_{j+1}(\boldsymbol{u},\bar{\boldsymbol{v}})}, \qquad v_j = \bar{v}_j\frac{g_{j+1}(\boldsymbol{u},\bar{\boldsymbol{v}})}{g_j(\boldsymbol{u},\bar{\boldsymbol{v}})}, \qquad j = 0,1,\ldots,N-1,\\
&g_j(\boldsymbol{u},\bar{\boldsymbol{v}}) = \sum_{k=0}^{N-1}\left\{\prod_{l=0}^{k-1}\gamma_{j+l}\bar{v}_{j+l}\cdot\left(\beta_{j+k}\gamma_{j+k}u_{j+k}\bar{v}_{j+k} - 1\right)\cdot\prod_{l=k+1}^{N-1}\beta_{j+l}u_{j+l}\right\}.
\end{aligned} \tag{29}$$

We will give a derivation of the equations (28), (29) in Appendix A.

In analogy with (18) we shall refer to the equations (28) as a solitonic map:

$$\begin{array}{rcccc}
\tilde{R}(\boldsymbol{\beta},\boldsymbol{\gamma}) & : & \mathbb{C}^N\times\mathbb{C}^N & \to & \mathbb{C}^N\times\mathbb{C}^N \\
 & & \Downarrow & & \Downarrow \\
 & & (\boldsymbol{u},\boldsymbol{v}) & \mapsto & (\bar{\boldsymbol{u}},\bar{\boldsymbol{v}})
\end{array}. \tag{30}$$

The corresponding Yang–Baxter map is obtained from (29):

$$\begin{array}{rcccc}
R(\boldsymbol{\beta},\boldsymbol{\gamma}) & : & \mathbb{C}^N\times\mathbb{C}^N & \to & \mathbb{C}^N\times\mathbb{C}^N \\
 & & \Downarrow & & \Downarrow \\
 & & (\boldsymbol{u},\bar{\boldsymbol{v}}) & \mapsto & (\bar{\boldsymbol{u}},\boldsymbol{v})
\end{array}. \tag{31}$$

$$
R\colon \quad \begin{array}{ccc} & (U_0, U_1) & \\ (\bar{V}_0, \bar{V}_1) & & (V_0, V_1) \\ & (\bar{U}_0, \bar{U}_1) & \end{array}
$$

Figure 1. Graphical representation.

The birational map (31) satisfies the Yang–Baxter equation (1), whereas when considered together, the Yang–Baxter map (31) and the BKP solitonic map (30) satisfy (19). We remark again that these "Yang–Baxter" properties are mere consequences of the 3D consistency of the discrete Lax equation (27).

Next we consider the ud-limit of the Yang–Baxter map (29), (31). To do this, we replace the parameters β and γ by

$$
\beta_j \mapsto \sqrt{-1}\beta_j, \qquad \gamma_j \mapsto \sqrt{-1}\gamma_j, \qquad j = 0, 1, \ldots, N-1,
$$

and introduce new independent variables U_j, V_j, $\bar{U}_j$, $\bar{V}_j$ as defined in

$$
e^{U_j/\epsilon} = \beta_j u_j, \qquad e^{V_j/\epsilon} = \gamma_j v_j \qquad e^{\bar{U}_j/\epsilon} = \beta_j \bar{u}_j, \qquad e^{\bar{V}_j/\epsilon} = \gamma_j \bar{v}_j, \qquad j = 0, 1, \ldots, N-1.
$$

It follows from (26) that $\sum_{j=0}^{N-1} U_j = \sum_{j=0}^{N-1} \bar{U}_j$ and $\sum_{j=0}^{N-1} V_j = \sum_{j=0}^{N-1} \bar{V}_j$ are independent of m and n. Then taking the ud-limit of (29) gives

$$
\begin{aligned}
&\bar{U}_j = U_j + G_j(\boldsymbol{U}, \bar{\boldsymbol{V}}) - G_{j+1}(\boldsymbol{U}, \bar{\boldsymbol{V}}), \\
&V_j = \bar{V}_j + G_{j+1}(\boldsymbol{U}, \bar{\boldsymbol{V}}) - G_j(\boldsymbol{U}, \bar{\boldsymbol{V}}), \qquad j = 0, 1, \ldots, N-1, \\
&G_j(\boldsymbol{U}, \bar{\boldsymbol{V}}) = \max_{0 \le k < N} \left(\sum_{l=0}^{k-1} \bar{V}_{j+1} + \max(0, U_{j+k} + \bar{V}_{j+k}) + \sum_{l=k+1}^{N-1} U_{j+l} \right), \\
&\boldsymbol{U} = (U_0, \ldots, U_{N-1}), \qquad \boldsymbol{V} = (V_0, \ldots, V_{N-1}), \\
&\bar{\boldsymbol{U}} = (\bar{U}_0, \ldots, \bar{U}_{N-1}), \qquad \bar{\boldsymbol{V}} = (\bar{V}_0, \ldots, \bar{V}_{N-1}),
\end{aligned} \tag{32}
$$

which defines a piecewise-linear Yang–Baxter map $R^{(\mathrm{ud})}$:

$$
\begin{array}{cccc} R^{(\mathrm{ud})} \; : & \mathbb{R}^N & \to & \mathbb{R}^N \\ & \Uparrow & & \Uparrow \\ & (\boldsymbol{U}, \bar{\boldsymbol{V}}) & \mapsto & (\bar{\boldsymbol{U}}, \boldsymbol{V}) \end{array}.
$$

We remark that if we impose the condition $U_j, \bar{V}_j \ge 0$ ($j = 0, 1, \ldots, N-1$), the Yang–Baxter map $R^{(\mathrm{ud})}$ coincides with the combinatorial R-matrix of type $A_{N-1}^{(1)}$ (see (2.12) in [19]).

However, the piecewise-linear map (32) also admits soliton-type solutions in the general case. To describe these solitons, we use the notation of Fig. 1. Figs. 2 and 3 show soliton-type phenomena in the 3-reduced case, where we have set $U_0 + U_1 + U_2 = \bar{U}_0 + \bar{U}_1 + \bar{U}_2 = 1$ and $\bar{V}_0 + \bar{V}_1 + \bar{V}_2 = V_0 + V_1 + V_2 = 0$. The evolution shown in Fig. 3 is reminiscent of the interactions of solitons with stationary solutions that have been studied by Hirota in case of the ultra-discrete limit of the discrete Sawada–Kotera equation [10].

4 Concluding remarks

In this article, we have constructed rational and piecewise-linear Yang–Baxter maps from the N-reduction of the discrete BKP equation. It should be remarked that the 4D consistency property [12] of the (non-reduced) BKP equation was observed in [1].

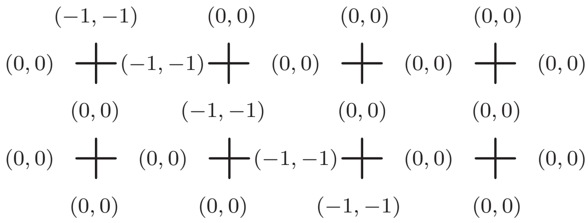

Figure 2. 1-soliton propagation.

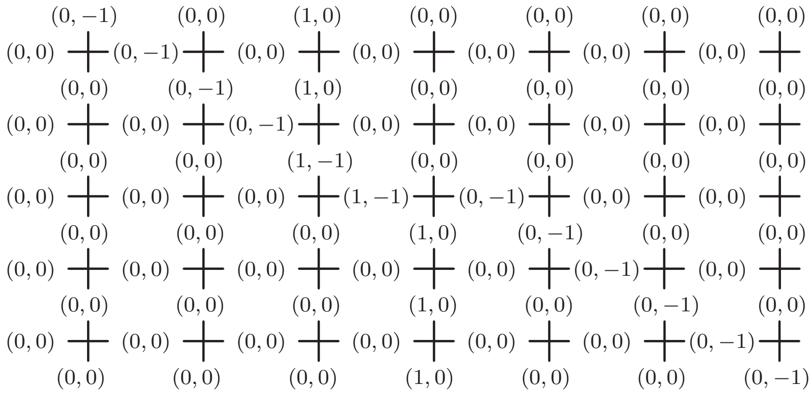

Figure 3. 2-soliton interaction.

From the Lie-algebraic viewpoint, the $(2n+1)$-reduced BKP equation has the symmetry of the affine Lie algebra of $A_{2n}^{(2)}$-type and the $(2n)$-reduced one has that of $D_{n+1}^{(2)}$ [2]. It is to be expected that the reduced ultra-discrete BKP system (32) is somehow related to crystal bases of the above types. However, in the 3-reduced case, our piecewise-linear map (32) seems to be different from the combinatorial R-matrix of $A_2^{(2)}$-type, discussed explicitly in [7], as (32) has a rotational symmetry which the $A_2^{(2)}$-type map in [7] does not possess. The exact relationship between the N-reduced discrete BKP equation and $A_2^{(2)}$-type crystal bases needs further clarification.

A Derivation of the N-reduced BKP equation and related Yang–Baxter map

Here we give a derivation of the N-reduced BKP equation (28) and the related Yang–Baxter map (29).

In the case $N = 2$, the nonlinear equations associated with the discrete Lax equation (27) are the same as in the 2-reduced A-case, i.e., those in Section 2 (as is to be expected for Lie-algebraic reasons). The first non-trivial example is the 3-reduced case and hereafter we assume $N \geq 3$.

If $\beta_j \neq 0$, $\gamma_j \neq 0$ $(j = 0, 1, \dots, N-1)$ in (25), we can set $\beta_j = \gamma_j = 1$ $(j = 0, 1, \dots, N-1)$ by rescaling the variables $\{u_j\}$, $\{v_j\}$. In this appendix, we choose $\beta_j = \gamma_j = 1$.

We first remark that the inverse of $\mathcal{U}_1$ of (25) is of the form,

$$\mathcal{U}_1^{-1} = \left\{1 + (-1)^{N-1}\lambda \prod_{i=0}^{N-1} \beta_i\right\}^{-1} \left\{\mathcal{I} + \sum_{j=1}^{N-1} (-1)^j \left(\prod_{k=0}^{j-1} \mathcal{U}^{(k)}\right) \Lambda^j\right\}$$

where the matrices

$$\mathcal{U}^{(k)} = \operatorname{diag}\left[u_k, u_{k+1}, \dots, u_{k+N-1}\right], \qquad k = 0, 1, \dots, N-1,$$

have the properties,

$$\mathcal{U}^{(k+N)} = \mathcal{U}^{(k)}, \qquad \Lambda\mathcal{U}^{(k)} = \mathcal{U}^{(k+1)}\Lambda, \qquad k = 0, 1, \ldots, N-1. \tag{33}$$

We then introduce a degree on Mat(N), the space of $N \times N$ matrices:

$$\forall\, X \in \mathrm{Mat}(N) \qquad \deg(X) = n \;\Leftrightarrow\; [d, X] = nX, \qquad d = N\lambda\frac{d}{d\lambda} - \mathrm{diag}\,[1, 2, \ldots, N]\,. \tag{34}$$

Substituting (33) into (27) and arranging in order of the degree (34), we obtain a set of equations for u_j, v_j, $\bar{u}_j$, $\bar{v}_j$. To write down these equations, we prepare some additional notation:

$$\boldsymbol{\mathcal{U}} = \left(\mathcal{U}^{(0)}, \mathcal{U}^{(1)}, \ldots, \mathcal{U}^{(N-1)}\right), \qquad \boldsymbol{\mathcal{V}} = \left(\mathcal{V}^{(0)}, \mathcal{V}^{(1)}, \ldots, \mathcal{V}^{(N-1)}\right),$$
$$\bar{\boldsymbol{\mathcal{U}}} = \left(\bar{\mathcal{U}}^{(0)}, \bar{\mathcal{U}}^{(1)}, \ldots, \bar{\mathcal{U}}^{(N-1)}\right), \qquad \bar{\boldsymbol{\mathcal{V}}} = \left(\bar{\mathcal{V}}^{(0)}, \bar{\mathcal{V}}^{(1)}, \ldots, \bar{\mathcal{V}}^{(N-1)}\right),$$

where $\bar{\mathcal{U}}^{(j)} = \mathcal{U}^{(j)}(0,1)$, $\bar{\mathcal{V}}^{(j)} = \mathcal{V}^{(j)}(1,0)$. We denote by $h_k\left(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}\right)\Lambda^k$ the degree k terms in $(\det\bar{\mathcal{U}}_1)(\det\mathcal{V}_1)\bar{\mathcal{U}}_1^{-1}\bar{\mathcal{U}}_2\mathcal{V}_1^{-1}\mathcal{V}_2$, i.e.,

$$(\det\bar{\mathcal{U}}_1)(\det\mathcal{V}_1)\bar{\mathcal{U}}_1^{-1}\bar{\mathcal{U}}_2\mathcal{V}_1^{-1}\mathcal{V}_2 = \sum_{k=0}^{2N} h_k\left(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}\right)\Lambda^k.$$

For example, $h_0\left(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}\right)$, $h_1\left(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}\right)$ are given by

$$h_0\left(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}\right) = \bar{\mathcal{U}}^{(0)}\mathcal{V}^{(0)},$$
$$h_1\left(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}\right) = \bar{\mathcal{U}}^{(0)}\left(\mathcal{I} - \mathcal{V}^{(0)}\mathcal{V}^{(1)}\right) + \left(\mathcal{I} - \bar{\mathcal{U}}^{(0)}\bar{\mathcal{U}}^{(1)}\right)\mathcal{V}^{(1)}.$$

By using this notation, the discrete Lax equation (27) can be rewritten as

$$h_k\left(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}\right) = h_k\left(\bar{\boldsymbol{\mathcal{V}}}, \boldsymbol{\mathcal{U}}\right), \qquad k = 0, 1, \ldots, 2N. \tag{35}$$

The system of polynomial equations (35) is of course overdetermined. However, it has a unique nontrivial solution. For the moment, we consider only two equations,

$$h_0\left(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}\right) = h_0\left(\bar{\boldsymbol{\mathcal{V}}}, \boldsymbol{\mathcal{U}}\right), \qquad h_1\left(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}\right) = h_1\left(\bar{\boldsymbol{\mathcal{V}}}, \boldsymbol{\mathcal{U}}\right). \tag{36}$$

To solve (36), we introduce another set of variables [5, 18]:

$$x_j = \prod_{i=0}^{j} v_k, \qquad \bar{x}_j = \prod_{i=0}^{j} \bar{v}_k, \qquad j = 0, 1, \ldots, N-2.$$

It is straightforward to show that $\vec{x} = {}^t\,[1 : x_0 : \cdots : x_{N-2}] \in \mathbb{P}^{N-1}$ satisfies a linear equation,

$$M\vec{x} = \vec{0}, \tag{37}$$
$$M = \begin{bmatrix} -u_0\bar{v}_0p_1 & q_0 & -p_0 & & & & \\ & -u_1\bar{v}_1p_2 & q_1 & & -p_1 & & \\ & & \ddots & & \ddots & \ddots & \\ & & & -u_{N-3}\bar{v}_{N-3}p_{N-2} & & q_{N-3} & -p_{N-3} \\ -p_{N-2}V & & & & & -u_{N-2}\bar{v}_{N-2}p_{N-1} & q_{N-2} \\ q_{N-1}V & -p_{N-1}V & & & & & -u_{N-1}\bar{v}_{N-1}p_0 \end{bmatrix}$$

where $p_j = u_j\bar{v}_j - 1$, $q_j = p_ju_{j+1} + \bar{v}_jp_{j+1}$ $(j = 0, 1, \ldots, N-1)$, and $V = \prod_{k=0}^{N-1} v_k = \prod_{k=0}^{N-1} \bar{v}_k$.

Theorem 1. *Assume that*

$$u_j\bar{v}_j \neq 1, \qquad \sum_{k=0}^{n}\left\{\prod_{l=0}^{k-1}\bar{v}_{j+l}\cdot(u_{j+k}\bar{v}_{j+k}-1)\cdot\prod_{l=k+1}^{n}u_{j+l}\right\}\neq 0, \qquad j=0,1,\dots,N-1. \tag{38}$$

Under this assumption, the space of solutions for the linear equation (37) *is one-dimensional, and a basis is given by*

$${}^t\left[1 : x_0 : x_1 : \cdots : x_{N-2}\right] = {}^t\left[g_0 : \bar{v}_0 g_1 : \bar{v}_0\bar{v}_1 g_2 : \cdots : (\bar{v}_0\cdots\bar{v}_{N-2})g_{N-1}\right], \tag{39}$$

where $g_j = g_j(\boldsymbol{u},\bar{\boldsymbol{v}})$ $(j=0,1,\dots,N-1)$ *are defined as* (29)*.*

To prove this theorem, we prepare two lemmas.

Lemma 1. *Let* $D_n(m)$ *be the determinant of order* n*:*

$$D_n(m) = \begin{vmatrix} q_m & -p_m & & \\ -u_{m+1}\bar{v}_{m+1}p_{m+2} & q_{m+1} & \ddots & \\ & \ddots & \ddots & -p_{m+n-2} \\ & & -u_{m+n-1}\bar{v}_{m+n-1}p_{m+n} & q_{m+n-1} \end{vmatrix}.$$

Then $D_n(m)$ *can be expressed as*

$$D_n(m) = \prod_{i=1}^{n-1}(u_{m+i}\bar{v}_{m+i}-1)\cdot\sum_{j=0}^{n}\left\{\prod_{k=0}^{j-1}\bar{v}_{m+k}\cdot(u_{m+j}\bar{v}_{m+j}-1)\cdot\prod_{k=j+1}^{n}u_{m+k}\right\}. \tag{40}$$

Proof. It is enough to prove the case $m=0$. Using the recurrence relation $D_{n+2}(0) = q_{n+1}D_{n+1}(0) - u_{n+1}\bar{v}_{n+1}p_n p_{n+2}D_n(0)$, one can prove the Lemma 1 by induction on n. ∎

Lemma 2. $\det M = 0$.

Proof. The desired result follows from (40) and a judicious expansion of M involving the $(N-1)$-th and N-th rows. ∎

Proof of Theorem 1. Denote by M_N^1 the matrix that results from M by deleting the N-th row and the first column. Under the assumption (38), $\det M_N^1 = D_{N-1}(0)\neq 0$. Together with $\det M = 0$ (Lemma 2), it follows that $\dim\operatorname{Ker} M = 1$. The solution (39) can be checked by direct substitution into (37). ∎

Thus we have obtained (29). Next we will show that every relation in (35) holds if the variables $\boldsymbol{u}$, $\boldsymbol{v}$, $\bar{\boldsymbol{u}}$, $\bar{\boldsymbol{v}}$ satisfy the relations (29).

For $\boldsymbol{a} = (a_0,\dots,a_{N-1})$, $\boldsymbol{b} = (b_0,\dots,b_{N-1})$, we define polynomials $g_j^{(m)}(\boldsymbol{a},\boldsymbol{b})$ $(j,m=0,1,\dots,N-1)$ as

$$g_j^{(m)}(\boldsymbol{a},\boldsymbol{b}) = \sum_{k=0}^{m}\left\{\prod_{l=0}^{k-1}a_{j+l}\cdot(a_{j+k}b_{j+k}-1)\cdot\prod_{l=k+1}^{m}b_{j+l}\right\}.$$

Lemma 3. *If the variables* $\boldsymbol{u}$*,* $\boldsymbol{v}$*,* $\bar{\boldsymbol{u}}$*,* $\bar{\boldsymbol{v}}$ *obey the relation* (29)*, then the following relations hold:*

$$g_j^{(m)}(\bar{\boldsymbol{u}},\boldsymbol{v}) = g_j^{(m)}(\bar{\boldsymbol{v}},\boldsymbol{u}), \qquad j,m=0,1,\dots,N-1.$$

Proof. Since the equation is invariant under the rotation of the suffices, i.e. $0 \mapsto 1 \mapsto 2 \mapsto \cdots \mapsto N-1 \mapsto 0$, one can set $j = 0$ without loss of generality. The desired result can be proved by induction on m, using the recurrence relation

$$g_0^{(m+1)}(\boldsymbol{a}, \boldsymbol{b}) = g_0^{(m)}(\boldsymbol{a}, \boldsymbol{b})\, b_m + (a_0 \cdots a_{m-1})(a_m b_m - 1). \qquad \blacksquare$$

A straightforward calculation shows that

$$\begin{aligned}
h_0(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}) &= g_0^{(0)}(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}) + \mathcal{I},\\
h_1(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}) &= -g_0^{(1)}(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}),\\
h_j(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}) &= (-1)^j \left\{ g_0^{(j)}(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}) + g_0^{(j-2)}(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}) \right\}, \qquad 2 \le j \le N-1,\\
h_N(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}) &= (-1)^N \left\{ \bar{\mathcal{U}}^{(0)} \mathcal{V}^{(0)} g_1^{(N-2)}(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}) - g_0^{(N-2)}(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}) - \prod_{i=0}^{N-1} \bar{\mathcal{U}}^{(i)} - \prod_{i=0}^{N-1} \mathcal{V}^{(i)} \right\},\\
h_{2N-k}(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}) &= (-1)^k \prod_{i=0}^{N-k} (\bar{\mathcal{U}}^{(i)} \mathcal{V}^{(i)}) g_{N-k+1}^{(k-2)}(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}) \\
&\quad + (-1)^{k+1} \prod_{i=0}^{N-k-2} (\bar{\mathcal{U}}^{(i)} \mathcal{V}^{(i)}) g_{N-k-1}^{(k)}(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}) \qquad (2 \le k \le N-1),\\
h_{2N-1}(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}) &= \prod_{i=0}^{N-3} (\bar{\mathcal{U}}^{(i)} \mathcal{V}^{(i)}) g_{N-2}^{(1)}(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}),\\
h_{2N}(\bar{\boldsymbol{\mathcal{U}}}, \boldsymbol{\mathcal{V}}) &= \prod_{i=0}^{N-2} (\bar{\mathcal{U}}^{(i)} \mathcal{V}^{(i)}).
\end{aligned} \tag{41}$$

Lemma 3 and relation (41) imply that the overdetermined system (35) is satisfied if the variables $\boldsymbol{u}$, $\boldsymbol{v}$, $\bar{\boldsymbol{u}}$, $\bar{\boldsymbol{v}}$ obey the relation (29). In other words, this means that the discrete Lax equation (27) is equivalent to (29).

Just as the Yang–Baxter map (29) was shown to be equivalent to (36), under the condition (38), the solitonic map (28) can also be shown to be equivalent to (36). We omit however the details of this proof.

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