Symmetry, Integrability and Geometry: Methods and Applications SIGMA **6** (2010), 027, 18 pages

Level Set Structure of an Integrable Cellular Automaton*

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Received October 23, 2009, in final form March 15, 2010; Published online March 31, 2010
doi:10.3842/SIGMA.2010.027

Abstract. Based on a group theoretical setting a sort of discrete dynamical system is constructed and applied to a combinatorial dynamical system defined on the set of certain Bethe ansatz related objects known as the rigged configurations. This system is then used to study a one-dimensional periodic cellular automaton related to discrete Toda lattice. It is shown for the first time that the level set of this cellular automaton is decomposed into connected components and every such component is a torus.

Key words: periodic box-ball system; rigged configuration; invariant torus

2010 Mathematics Subject Classification: 82B23; 37K15; 68R15; 37B15

1 Introduction

The Liouville's theorem on completely integrable systems is one of the most fundamental results in classical mechanics. By V.I. Arnold's formulation one of the claims in the theorem says that any compact connected level set of completely integrable systems with N degrees of freedom is diffeomorphic to an N-dimensional torus [1]. It is called an invariant torus and the theorem also claims that the phase flow with the Hamiltonian function determines a conditionally periodic motion on it.

In this paper we study the level set structure of a one-dimensional cellular automaton known as the periodic box-ball system (pBBS) [2, 3] and construct its invariant tori. It is one of the (ultra-)discrete dynamical systems associated with integrable non-linear evolution equations. See the "picture" in Subsection 4.1, just below Remark 4, for an example of the time evolution of this system, in which mutually interacting solitons are traveling along it. This system is attracting attentions because of its relations with discrete Toda lattice [4], Bethe ansatz of integrable quantum spin chains [5], tropical geometry [6] and ultradiscrete Riemann theta functions [7].

The purpose of this paper is twofold. The first is to construct a discrete dynamical system through a group theoretical setting which has potentially several applications in (ultra-)discrete integrable systems (Theorem 1). The second is to make the structure of the level set of pBBS perfectly clear as one of its applications (Theorem 2). It is shown for the first time that the level set is decomposed into connected components and every such component is a torus, as if it were a compact level set of completely integrable systems in Hamiltonian mechanics.

Let us begin by describing backgrounds and motivations on the problem more precisely. Consider the completely integrable systems with N degrees of freedom again. Such a system has N independent first integrals. Its level set is determined by fixing the values of the first integrals, and becomes an N-dimensional manifold in the $2N$-dimensional phase space. Inspired by this

*This paper is a contribution to the Proceedings of the Workshop "Geometric Aspects of Discrete and Ultra-Discrete Integrable Systems" (March 30 – April 3, 2009, University of Glasgow, UK). The full collection is available at http://www.emis.de/journals/SIGMA/GADUDIS2009.html picture in Hamiltonian mechanics, A. Kuniba, A. Takenouchi and the author introduced the notion of the level set of pBBS [5], and provided its volume formula. In fact it was exactly equal to the same formula for the enumeration of the off-diagonal solutions to the string center equations in combinatorial Bethe ansatz [8]. It is based on the notion of rigged configurations [9, 10], and is described in the following way. Let m_j be the number of solitons with amplitude j. In what follows m_j are supposed to be positive for $1 \leq j \leq N$ and to be zero for $j > N$ for simplicity. One can think of $\boldsymbol{m} = (m_j)_{1\leq j\leq N}$ as a collection of fixed values of the first integrals. We denote by $\mathcal{J}(\boldsymbol{m})$ the level set specified by $\boldsymbol{m}$ and by $\Omega(\boldsymbol{m})$ its volume. Then we have [5]

$$\Omega(\boldsymbol{m}) = (\det F) \prod_{1\leq j\leq N} \frac{1}{m_j} \binom{p_j + m_j - 1}{m_j - 1}. \tag{1}$$

Here p_j are positive integers and F is an $N \times N$ matrix with integer entries, which are explicitly expressed in terms of $\boldsymbol{m}$ and the system size L.

It can be shown that if there is no solitons of common amplitudes then the level set $\mathcal{J}(\boldsymbol{m})$ becomes a torus $F\mathbb{Z}^N \backslash \mathbb{Z}^N$.[1] This fact is suggested by the volume formula (1) that reduces to $\Omega(\boldsymbol{m}) = \det F$ when $m_j = 1$ for every j. However this simple picture fails when there are multiple solitons of common amplitudes. If so, the right hand side of (1) can not be regarded as the volume of a torus any more. An expression for the level set in this generalized case is given by

$$\mathcal{J}(\boldsymbol{m}) = A\mathbb{Z}^{m_1+\cdots+m_N} \backslash (\mathcal{I}_{m_1} \times \cdots \times \mathcal{I}_{m_N}), \tag{2}$$

where $\mathcal{I}_n = \mathfrak{S}_n \backslash (\mathbb{Z}^n - \Delta_n)$ is the n dimensional lattice without the diagonal points $\Delta_n = \{(z_1, \ldots, z_n) \in \mathbb{Z}^n \mid z_\alpha = z_\beta \text{ for some } 1 \leq \alpha \neq \beta \leq n\}$ and identified under the permutations $\mathfrak{S}_n$. The A is a $\gamma \times \gamma$ symmetric matrix ($\gamma = m_1 + \cdots + m_N$) with integer entries [5].

In this paper we show that the level set (2) has generally many connected components, and every such component is written as $F^{(\boldsymbol{\alpha})}\mathbb{Z}^N \backslash \mathbb{Z}^N$ where $F^{(\boldsymbol{\alpha})}$ is an $N \times N$ matrix determined by the above F and the symmetry $\boldsymbol{\alpha}$ of the system that depends on the initial conditions. It is also shown that the time evolutions of the cellular automaton yield straight motions on the torus, just like the phase flows on the invariant torus generated by first integrals.

The layout of the rest of this paper is as follows. In Section 2 we begin with an abstract group theoretical setting and present a general result, and one of its specializations called a direct product model. Here we establish our first main result, Theorem 1. This model is interpreted as a discrete dynamical system. In Section 3 we construct a specific example of the direct product model associated with the rigged configurations. We give a review on the periodic box-ball system in Section 4, and then construct its invariant tori based on the results in previous sections. Our second main result is Theorem 2. Properties of the cellular automaton that follows from this result are discussed in Section 5. Two elementary lemmas are given in Appendix A, and an algorithm for calculating rigged configurations is presented in Appendix B.

2 Construction of a discrete dynamical system

2.1 Group theoretical setting

Let X be a set, $\mathfrak{S}(X)$ be the group of all bijections from X to itself. Then $\mathfrak{S}(X)$ acts on X from the left by $\sigma \cdot x = \sigma(x)$ ($\forall\, x \in X$, $\forall\, \sigma \in \mathfrak{S}(X)$). From now on we assume that every group action is left action and omit the word "from the left". Given any group G acting on X there

[1] More precisely, it is not a torus but a set of all integer points on the torus $F\mathbb{Z}^N \backslash \mathbb{R}^N$. For the sake of simplicity we call such a set a torus. Besides, in this paper we write all the quotient sets as left quotient ones.

is an equivalence relation associated with its action. Its equivalence classes are called G-orbits on X. If there is only one G-orbit, the action of G is called *transitive*.

The G-orbit containing $x \in X$ is denoted $G \cdot x$, and is called the G-orbit of x. The set of all G-orbits on X is denoted $G\backslash X$. Though one can think of $G \cdot x$ either as an element of $G\backslash X$ or as a subset of X, we shall adopt the latter interpretation and elements of $G\backslash X$ will be written as $[x]_G$. The map $X \ni x \mapsto [x]_G \in G\backslash X$ is called the *canonical map* associated with $G\backslash X$. There is a transitive action of G on $G \cdot x (\subset X)$. The following lemma is elementary.

Lemma 1. *Let G and H be any groups that act on X commutatively. Then G naturally acts on $H\backslash X$. Namely their is a unique action of G on $H\backslash X$ that is commutative with the canonical map $X \ni x \mapsto [x]_H \in H\backslash X$.*

Proof. Given any $p \in H\backslash X$ one can write it as $p = [x]_H$ for some $x \in X$. We define $\Gamma : G \times H\backslash X \to H\backslash X$ to be a map given by the relation $\Gamma(g, p) = [g \cdot x]_H$ for all $g \in G$. It is easy to see that this map is well-defined and yields the desired action of G by $g \cdot p = \Gamma(g, p)$. ∎

Let G_1 and G_2 be subgroups of $\mathfrak{S}(X)$ that act on X commutatively. Then $G_1 \cap G_2$ is also a subgroup of $\mathfrak{S}(X)$ and its action on X is commutative with those of G_1 and G_2. By Lemma 1 there is a natural action of G_1 on $G_2\backslash X$. Let x be any element of X. Then $G_1 \cdot [x]_{G_2}$ is a subset of $G_2\backslash X$ where G_1 acts on transitively. Similarly $G_1 \cdot x$ is a subset of X where G_1 and $G_1 \cap G_2$ act on commutatively. Hence by Lemma 1 there is a natural action of G_1 on $(G_1 \cap G_2)\backslash(G_1 \cdot x)$.

Proposition 1. *Let G_1 and G_2 be subgroups of $\mathfrak{S}(X)$ and suppose their actions on X is commutative. Suppose we have $(G_1 \cdot x) \cap (G_2 \cdot x) \subset (G_1 \cap G_2) \cdot x$ for some $x \in X$. Then there is a bijection between $G_1 \cdot [x]_{G_2}$ and $(G_1 \cap G_2)\backslash(G_1 \cdot x)$ that is commutative with the action of G_1.*

Proof. Let z be any element of $G_1 \cdot [x]_{G_2}$. Then there exists $g \in G_1$ such that $z = g \cdot [x]_{G_2}$. Define $\Xi : G_1 \cdot [x]_{G_2} \to (G_1 \cap G_2)\backslash(G_1 \cdot x)$ by $\Xi(z) = [g \cdot x]_{G_1 \cap G_2}$.

Well-definedness: Suppose there is another $g' \in G_1$ such that $z = g' \cdot [x]_{G_2}$. Since the actions of G_1 and G_2 are commutative we have $[g \cdot x]_{G_2} = z = [g' \cdot x]_{G_2}$. Hence there exists $h \in G_2$ such that $h \cdot (g \cdot x) = g' \cdot x$. This implies $h \cdot x = (g^{-1} \circ g') \cdot x \in (G_1 \cdot x) \cap (G_2 \cdot x) \subset (G_1 \cap G_2) \cdot x$. Hence one can take h as an element of $G_1 \cap G_2$. Thus $[g' \cdot x]_{G_1 \cap G_2} = [g \cdot x]_{G_1 \cap G_2}$.

Commutativity with G_1 action: Let $g' \in G_1$. Then $g' \cdot z = (g' \circ g) \cdot [x]_{G_2}$. Hence $\Xi(g' \cdot z) = [(g' \circ g) \cdot x]_{G_1 \cap G_2} = [g' \cdot (g \cdot x)]_{G_1 \cap G_2} = g' \cdot ([g \cdot x]_{G_1 \cap G_2}) = g' \cdot \Xi(z)$.

Injectivity: Suppose $\Xi(z) = \Xi(z')$ for $z, z' \in G_1 \cdot [x]_{G_2}$. Then there exist $g, g' \in G_1$ such that $z = g \cdot [x]_{G_2}$ and $z' = g' \cdot [x]_{G_2}$ with the property $[g \cdot x]_{G_1 \cap G_2} = [g' \cdot x]_{G_1 \cap G_2}$. Hence there exists $h \in G_1 \cap G_2$ such that $h \cdot (g \cdot x) = g' \cdot x$. Then $g \cdot ([x]_{G_2}) = [g \cdot x]_{G_2} = [h \cdot (g \cdot x)]_{G_2} = [g' \cdot x]_{G_2} = g' \cdot ([x]_{G_2})$. Hence $z = z'$.

Surjectivity: Choose an arbitrary element C of $(G_1 \cap G_2)\backslash(G_1 \cdot x)$. Then there exists $y \in G_1 \cdot x$ such that $C = [y]_{G_1 \cap G_2}$. Since y lies in $G_1 \cdot x$ there exists $g \in G_1$ such that $y = g \cdot x$. Let $z = g \cdot [x]_{G_2} \in G_1 \cdot [x]_{G_2}$. Then $\Xi(z) = [g \cdot x]_{G_1 \cap G_2} = [y]_{G_1 \cap G_2} = C$. ∎

Among the assumptions of Proposition 1, the condition $(G_1 \cdot x) \cap (G_2 \cdot x) \subset (G_1 \cap G_2) \cdot x$ is rather specific. In the next subsection we construct an example of the triplet (G_1, G_2, X) that satisfies this condition for all $x \in X$.

2.2 Direct product model

Let X_1, X_2 be sets and H_1, H_2 groups that act on X_1, X_2 respectively. Choose an arbitrary subgroup H_2' of H_2. We define

$$X_2' = \{y \in X_2 \mid g \cdot y = y \text{ if and only if } g \in H_2'\}. \tag{3}$$

In other words X_2' is the set of all elements of X_2 whose stabilizer associated with the H_2 action is H_2'.

In what follows H_2 is supposed to be abelian, and we denote the multiplication as a sum $g+h \in H_2$ for any $g, h \in H_2$. Then X_2' is invariant under the action of H_2.

Remark 1. Suppose there are several different actions of the group G on the set X. To distinguish them we write $\Gamma : G \times X \to X$ for example to denote a specific action of G on X. For any such Γ there is a group homomorphism $\rho_\Gamma : G \to \mathfrak{S}(X)$ such that the relation $\Gamma(g,x) = \rho_\Gamma(g) \cdot x$ holds for all $g \in G, x \in X$. Such group homomorphisms are called *permutation representations*.

Let $X = X_1 \times X_2'$. We define $\Gamma_1 : H_1 \times X \to X$ to be a diagonal action of H_1 on X in which its action on X_2' is trivial. In other words we have $\Gamma_1(g,x) = (g \cdot x_1, x_2)$ for all $g \in H_1, x = (x_1, x_2) \in X$.

Suppose there exists a map $\Gamma_2 : H_2 \times X \to X$ that has the following property. For all $g \in H_2, x = (x_1, x_2) \in X$ we have $\Gamma_2(g,x) = (\varphi(g, x_1, x_2), g \cdot x_2)$ where $\varphi : H_2 \times X_1 \times X_2' \to X_1$ is a map satisfying the relation $\varphi(g+h, x_1, x_2) = \varphi(h, \varphi(g, x_1, x_2), g \cdot x_2)$ for all $g, h \in H_2$. Then Γ_2 yields an action of H_2 on X.

By definition we have $\Gamma_2(g,x) = (\varphi(g, x_1, x_2), x_2)$ for all $g \in H_2'$. This implies that $\Gamma_2|_{H_2' \times X}$ yields an action of H_2' on X in which its action on X_2' is trivial.

Let $G_1 = \rho_{\Gamma_1}(H_1), G_2 = \rho_{\Gamma_2}(H_2)$, and $G_2' = \rho_{\Gamma_2}(H_2')$ where ρ_{Γ_1} and ρ_{Γ_2} are the permutation representations. Note that G_1 and G_2 depend on the choice of H_2'.

In what follows the action of H_1 on X_1 is supposed to be transitive.

Lemma 2. $G_2' \subset G_1 \cap G_2$.

Proof. By definition we have $G_2' \subset G_2$. The inclusion $G_2' \subset G_1$ holds since the action of G_2' on X_2' is trivial, and the action of G_1 on X_1 is transitive. ∎

Lemma 3. *The following relation holds*

$$G_2' \cdot x = (G_1 \cap G_2) \cdot x = (G_1 \cdot x) \cap (G_2 \cdot x),$$

for all $x \in X = X_1 \times X_2'$.

Proof. By Lemma 2 we have

$$G_2' \cdot x \subset (G_1 \cap G_2) \cdot x \subset (G_1 \cdot x) \cap (G_2 \cdot x),$$

for all $x \in X$, where the latter inclusion is obvious. The opposite inclusions are proved as follows. Take any $x = (x_1, x_2) \in X$. Choose an arbitrary element y of $(G_1 \cdot x) \cap (G_2 \cdot x)$. Since y lies in $G_2 \cdot x$ there exists $g \in H_2$ such that $y = \rho_{\Gamma_2}(g) \cdot x = (\varphi(g, x_1, x_2), g \cdot x_2)$. Then since y lies in $G_1 \cdot x$ this implies $g \cdot x_2 = x_2$, forcing g to be an element of H_2' by (3). Hence y is an element of $G_2' \cdot x$. ∎

In what follows the actions of G_1 and G_2 on X are supposed to be commutative.

By Proposition 1 and Lemma 3 there is a bijection between $G_1 \cdot [x]_{G_2}$ and $G_1 \cap G_2 \backslash (G_1 \cdot x)$ that is commutative with the action of $G_1 = \rho_{\Gamma_1}(H_1)$. Note that Lemma 3 implies $G_2' = G_1 \cap G_2$. Note also that for any $x = (x_1, x_2) \in X$ we have $G_1 \cdot x = X_1 \times \{x_2\} \cong X_1$ as a subset of X, for the action of G_1 on X_1 is transitive and that on X_2 is trivial. Thus we have the following.

Theorem 1. *Let the sets* X_1, X_2', *the groups* G_1, G_2, G_2', *and their actions on the set* $X = X_1 \times X_2'$ *be defined as above, and* x *be any element of* X. *Then there is a bijection between* $G_1 \cdot [x]_{G_2}$ *and* $G_2' \backslash X_1$ *that is commutative with the action of* G_1.

This is our first main result in this paper.

2.3 Interpretation as a dynamical system

In this subsection we present, without mathematical rigor, an interpretation of the direct product model as a dynamical system. It is intended to help readers to have some intuitive physical images on the model.

Consider any completely integrable system with N degrees of freedom in Hamiltonian mechanics. There are N independent first integrals. By fixing their values we obtain the level set, an N-dimensional manifold in the phase space $\mathbb{R}^{2N}$. On the level set there are N mutually commuting phase flows associated with the first integrals. Note that the time evolution of the system is one of the phase flows since the Hamiltonian itself is one of the first integrals.

One can think of the triplet (G_1, G_2, X) in Subsection 2.2 as a discrete analogue of such a completely integrable system. Let $\tilde{X} := X_1 \times X_2$ be a non-compact level set and suppose the groups G_1, G_2 are acting on $\tilde{X}$ commutatively. We regard the group action of G_2 as the symmetry of the system and assume that it yields a compact (but not necessarily connected) level set $G_2\backslash\tilde{X}$. Then each state of the system is represented by a G_2-orbit $[x]_{G_2}$ for some $x \in \tilde{X}$. We regard G_1 as the group generated by the mutually commuting phase flows associated with the first integrals. Then one can think of $G_1 \cdot [x]_{G_2} (\subset G_2\backslash\tilde{X})$ as a connected component of the compact level set $G_2\backslash\tilde{X}$ containing the state $[x]_{G_2}$.

Suppose the group G_2 acts not only on $\tilde{X}$ but also on X_2. Choose an arbitrary subgroup G_2' of G_2 and denote by X_2' the set of all elements of X_2 whose stabilizer associated with the G_2 action is G_2'. Let x be any element of $X = X_1 \times X_2'$. Now the model in Subsection 2.2 can be interpreted as follows. There is a bijection between the connected component of the level set $G_1 \cdot [x]_{G_2}$ and the quotient set $G_2'\backslash X_1$ that is commutative with the phase flows of the system. In particular the time evolution of the system is commutative with this bijection.

Consider the completely integrable system in Hamiltonian mechanics again. Suppose the level set is compact and connected. Then by Arnold–Liouville theorem it must be diffeomorphic to an N-dimensional torus [1]. It is called an *invariant torus*. In the following sections we construct a direct product model with $X_1 = \mathbb{Z}^N$ and G_2' being a sub-lattice of $\mathbb{Z}^N$ which acts on X_1 by left translation. Now the quotient set $G_2'\backslash X_1$ is, roughly speaking, a discrete analogue of invariant torus.

3 A dynamical system on rigged configurations

3.1 Extended rigged configurations

The *rigged configuration* is an ingenious device utilized in combinatorial Bethe ansatz [9, 10]. We construct an example of the direct product model associated with the rigged configurations. The result of this section will be used in Subsection 4.4.

We introduce a pair of sets X_1 and X_2. Choose an arbitrary positive integer N and define $X_1 = \mathbb{Z}^N$. For any pair of positive integers m, p we define

$$\Lambda(m,p) = \{(\lambda_i)_{i\in\mathbb{Z}} \mid \lambda_i \in \mathbb{Z},\ \lambda_1 = 0,\ \lambda_i \leq \lambda_{i+1},\ \lambda_{i+m} = \lambda_i + p \ \text{ for all } i\}. \tag{4}$$

Given any positive integer sequences $(m_j)_{1\leq j\leq N}$, $(p_j)_{1\leq j\leq N}$ we define $X_2 = \prod_{j=1}^N \Lambda(m_j, p_j)$. Each element of X_2 is written, for instance, as $\boldsymbol{\lambda} = (\lambda^{(j)})_{1\leq j\leq N}$ with $\lambda^{(j)} \in \Lambda(m_j, p_j)$ or $\boldsymbol{\lambda} = (\lambda_i^{(j)})_{i\in\mathbb{Z},1\leq j\leq N}$.

Example 1. Let $(m_1, m_2, m_3) = (3, 2, 1)$ and $(p_1, p_2, p_3) = (12, 6, 4)$. Each element of the set $X_2 = \Lambda(3, 12) \times \Lambda(2, 6) \times \Lambda(1, 4)$ is written as $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$. It is labeled by three integers $a = \lambda_2^{(1)}$, $b = \lambda_3^{(1)}$ and $c = \lambda_2^{(2)}$ satisfying the conditions $0 \leq a \leq b \leq 12$ and $0 \leq c \leq 6$. Note that $\lambda^{(3)} = (\lambda_i^{(3)})_{i\in\mathbb{Z}}$ has a unique element $\lambda_i^{(3)} = 4(i - 1)$.

The set $\tilde{X} = X_1 \times X_2$ with the above X_1, X_2 is regarded as the non-compact level set considered in Subsection 2.3. It will be identified with the set of *extended rigged configurations* in Subsection 4.3 under a certain condition imposed on the values of m_j and p_j. For the time being we ignore this condition and call $\tilde{X}$ itself the set of extended rigged configurations.

We introduce a pair of abelian groups H_1 and H_2. Suppose H_1 acts on X_1 by

$$\Gamma : \ H_1 \times X_1 \to X_1, \qquad (g, \boldsymbol{\omega}) \mapsto \Gamma(g, \boldsymbol{\omega}) = \boldsymbol{\omega} + \psi_\Gamma(g),$$

where $\psi_\Gamma : H_1 \to X_1$ is a map common to every $\boldsymbol{\omega} \in X_1$. This map must be linear, i.e. for all $g, h \in H_1$ we have $\psi_\Gamma(g+h) = \psi_\Gamma(g) + \psi_\Gamma(h)$. We assume that this action is transitive, i.e. for any $\boldsymbol{\omega}, \boldsymbol{\omega}' \in X_1$ there exists $g \in H_1$ such that $\Gamma(g, \boldsymbol{\omega}) = \boldsymbol{\omega}'$. An example of such a map will appear in the next section, just above Lemma 6.

Remark 2. Since $X_1 = \mathbb{Z}^N$, any sub-lattice of X_1 can be thought of as an abelian group which acts on X_1 by translation. As a sub-lattice of X_1 we can take X_1 itself, which leads to the above H_1 and its action on X_1.

Suppose H_2 is a free abelian group with generators $s_1, \ldots, s_N$. We define $\Upsilon : H_2 \times X_2 \to X_2$ to be an action of H_2 on X_2 by

$$\Upsilon(g, \boldsymbol{\lambda}) = \big(\lambda^{(j)}_{n_j+i} - \lambda^{(j)}_{n_j+1}\big)_{i \in \mathbb{Z}, 1 \le j \le N}, \tag{5}$$

for each $g = \sum_{j=1}^N n_j s_j \in H_2$ and $\boldsymbol{\lambda} = (\lambda^{(j)}_i)_{i \in \mathbb{Z}, 1 \le j \le N} \in X_2$.

Example 2. Consider the X_2 in Example 1. Let $\overline{\boldsymbol{\lambda}} = \Upsilon(g, \boldsymbol{\lambda})$ for $g = n_1 s_1 + n_2 s_2 + n_3 s_3 \in H_2$. Then $\overline{\boldsymbol{\lambda}} = \big(\overline{\lambda}^{(1)}, \overline{\lambda}^{(2)}, \overline{\lambda}^{(3)}\big)$ is specified by three integers $\overline{a} = \overline{\lambda}^{(1)}_2, \overline{b} = \overline{\lambda}^{(1)}_3$ and $\overline{c} = \overline{\lambda}^{(2)}_2$ as

$$(\overline{a}, \overline{b}) = \begin{cases} (a, b) & n_1 \equiv 0 \pmod 3, \\ (b-a, 12-a) & n_1 \equiv 1 \pmod 3, \\ (12-b, 12-b+a) & n_1 \equiv 2 \pmod 3, \end{cases} \qquad \overline{c} = \begin{cases} c & n_2 \equiv 0 \pmod 2, \\ 6-c & n_2 \equiv 1 \pmod 2. \end{cases}$$

Here a, b, c are the labels for $\boldsymbol{\lambda}$ in Example 1.

3.2 Cyclic group structures

Let $\rho_\Upsilon : H_2 \to \mathfrak{S}(X_2)$ be the permutation representation for the action of Υ in the previous subsection. Then $\rho_\Upsilon(H_2)$ is isomorphic to $\mathcal{C}_{m_1} \times \cdots \times \mathcal{C}_{m_N}$, where $\mathcal{C}_m$ is an order m cyclic group. This in particular determines an action of $\mathcal{C}_m$ on $\Lambda(m, p)$. The following facts are well known in the theory of cyclic groups.

Proposition 2. *Every subgroup of a cyclic group is a cyclic group.*

Proposition 3. *Let m be a positive integer and n be a divisor of m. Then there exists a unique subgroup of $\mathcal{C}_m$ that is isomorphic to $\mathcal{C}_n$.*

Let α be a common divisor of m and p. As a subset of $\Lambda(m, p)$ we define

$$\Lambda^{(\alpha)}(m, p) = \{\lambda \in \Lambda(m/\alpha, p/\alpha) \mid \lambda \notin \Lambda(m/\alpha', p/\alpha') \text{ for every common divisor } \alpha' > \alpha\}. \tag{6}$$

In other words $\Lambda^{(\alpha)}(m, p)$ is the set of all elements of $\Lambda(m, p)$ whose stabilizer associated with the action of $\mathcal{C}_m$ is isomorphic to $\mathcal{C}_\alpha$. Now we have the decomposition $\Lambda(m, p) = \bigsqcup_\alpha \Lambda^{(\alpha)}(m, p)$ where α runs over every common divisor of m and p.

Choose a sequence $\boldsymbol{\alpha} = (\alpha_j)_{1 \le j \le N}$ with each α_j being a common divisor of m_j and p_j. Let $H_2^{(\boldsymbol{\alpha})}$ be a group generated by $(m_1/\alpha_1)s_1, \ldots, (m_N/\alpha_N)s_N$. It is a subgroup of H_2 whose image $\rho_\Upsilon(H_2^{(\boldsymbol{\alpha})})$ by the permutation representation is isomorphic to $\mathcal{C}_{\alpha_1} \times \cdots \times \mathcal{C}_{\alpha_N}$. Let $X_2^{(\boldsymbol{\alpha})} = \prod_{j=1}^N \Lambda^{(\alpha_j)}(m_j, p_j)$. In other words $X_2^{(\boldsymbol{\alpha})}$ is the set of all elements of X_2 whose stabilizer associated with the H_2 action is $H_2^{(\boldsymbol{\alpha})}$.

Example 3. Consider the X_2 in Example 1. We have $\Lambda(3,12) = \Lambda^{(1)}(3,12) \sqcup \Lambda^{(3)}(3,12)$, $\Lambda(2,6) = \Lambda^{(1)}(2,6) \sqcup \Lambda^{(2)}(2,6)$ and $\Lambda(1,4) = \Lambda^{(1)}(1,4)$. If we set $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) = (3,1,1)$, then $(a,b) = (4,8)$ and $c \neq 3$ for any $\boldsymbol{\lambda} \in X_2^{(\boldsymbol{\alpha})}$. It follows from Example 2 that the set of all $g \in H_2$ satisfying $\Upsilon(g, \boldsymbol{\lambda}) = \boldsymbol{\lambda}$ is $H_2^{(\boldsymbol{\alpha})} = \{g = n_1 s_1 + 2n_2 s_2 + n_3 s_3 \,|\, n_1, n_2, n_3 \in \mathbb{Z}\}$.

Since H_2 is abelian, $X_2^{(\boldsymbol{\alpha})} (\subset X_2)$ is invariant under the action of H_2. Let

$$X^{(\boldsymbol{\alpha})} = X_1 \times X_2^{(\boldsymbol{\alpha})}. \tag{7}$$

The $X_2^{(\boldsymbol{\alpha})}$ here corresponds to the X_2', and the $X^{(\boldsymbol{\alpha})}$ to the X in Subsection 2.2. The set $\tilde{X} = X_1 \times X_2$ is decomposed as $\tilde{X} = \bigsqcup_{\boldsymbol{\alpha}} X^{(\boldsymbol{\alpha})}$ where each α_j runs over every common divisor of m_j and p_j.

3.3 Group actions on the set of rigged configurations

We define a pair of commutative actions of H_1 and H_2 on $\tilde{X} = X_1 \times X_2$. As in Subsection 2.2, let $\Gamma_1 : H_1 \times \tilde{X} \to \tilde{X}$ be a diagonal action of H_1 on $\tilde{X}$ in which its action on X_2 is trivial, i.e. for each $f \in H_1$ and $x = (\boldsymbol{\omega}, \boldsymbol{\lambda}) \in \tilde{X}$ we set $\Gamma_1(f, x) = (\boldsymbol{\omega} + \psi_\Gamma(f), \boldsymbol{\lambda})$. Let $\Gamma_2 : H_2 \times \tilde{X} \to \tilde{X}$ be an action of H_2 on $\tilde{X}$ which is defined as follows.

Fix an arbitrary set of N^2 integers $(B_{j,k})_{1 \le j,k \le N} \in \mathbb{Z}^{N^2}$. In the next section we take $B_{j,k}$ to be those given in (12). For each $g \in H_2$ and $x = (\boldsymbol{\omega}, \boldsymbol{\lambda}) \in \tilde{X}$ we set $\Gamma_2(g, x) = (\varphi(g, \boldsymbol{\omega}, \boldsymbol{\lambda}), \Upsilon(g, \boldsymbol{\lambda}))$ where $\Upsilon(g, \boldsymbol{\lambda})$ is given by (5) and $\varphi : H_2 \times X_1 \times X_2 \to X_1$ is a map defined by

$$\varphi(g, \boldsymbol{\omega}, \boldsymbol{\lambda}) = \boldsymbol{\omega} + \left(\lambda^{(j)}_{n_j+1} + \sum_{k=1}^N B_{j,k} n_k \right)_{1 \le j \le N}, \tag{8}$$

for each $g = \sum_{k=1}^N n_k s_k \in H_2$, $\boldsymbol{\omega} \in X_1$ and $\boldsymbol{\lambda} = (\lambda_i^{(j)})_{i \in \mathbb{Z}, 1 \le j \le N} \in X_2$. It is easy to see that the map Γ_2 indeed yields an action of H_2 on $\tilde{X}$, since the relation $\varphi(g + h, \boldsymbol{\omega}, \boldsymbol{\lambda}) = \varphi(h, \varphi(g, \boldsymbol{\omega}, \boldsymbol{\lambda}), \Upsilon(g, \boldsymbol{\lambda}))$ holds for all $g, h \in H_2$. It is also easy to see that H_1 and H_2 act on $\tilde{X}$ commutatively by Γ_1 and Γ_2.

Example 4. Consider the X_2 in Example 1. For $g = n_1 s_1 + n_2 s_2 + n_3 s_3 \in H_2$ and $x = (\boldsymbol{\omega}, \boldsymbol{\lambda}) \in \tilde{X}$ we write $\Gamma_2(g, x) = (\overline{\boldsymbol{\omega}}, \overline{\boldsymbol{\lambda}}) \in \tilde{X}$. Here $\overline{\boldsymbol{\lambda}} = \Upsilon(g, \boldsymbol{\lambda})$ is the one given in Example 2, and $\overline{\boldsymbol{\omega}} = \varphi(g, \boldsymbol{\omega}, \boldsymbol{\lambda})$ is written as

$$\varphi(g, \boldsymbol{\omega}, \boldsymbol{\lambda}) = \boldsymbol{\omega} + \left(\mu_j + \sum_{k=1}^3 B_{j,k} n_k \right)_{1 \le j \le 3}.$$

Here we set

$$\mu_1 = \begin{cases} 12 l_1 & n_1 = 3 l_1, \\ 12 l_1 + a & n_1 = 3 l_1 + 1, \\ 12 l_1 + b & n_1 = 3 l_1 + 2, \end{cases} \qquad \mu_2 = \begin{cases} 6 l_2 & n_2 = 2 l_2, \\ 6 l_2 + c & n_2 = 2 l_2 + 1, \end{cases}$$

for any $l_1, l_2 \in \mathbb{Z}$, and $\mu_3 = 4 n_3$. The a, b, c are the labels for $\boldsymbol{\lambda}$ in Example 1.

By definition the set $X^{(\boldsymbol{\alpha})}(\subset \tilde{X})$ is invariant under the action of H_1. It is also invariant under H_2. Thus by restricting their domains from $\tilde{X}$ to $X^{(\boldsymbol{\alpha})}$, $\varGamma_1$ and $\varGamma_2$ yield their actions on $X^{(\boldsymbol{\alpha})}$. In what follows we denote them by the same symbols $\varGamma_1$ and $\varGamma_2$ for simplicity.

We define $G_1 = \rho_{\varGamma_1}(H_1), G_2 = \rho_{\varGamma_2}(H_2)$, and $G_2^{(\boldsymbol{\alpha})} = \rho_{\varGamma_2}(H_2^{(\boldsymbol{\alpha})})$ to be subgroups of $\mathfrak{S}(X^{(\boldsymbol{\alpha})})$ where $\rho_{\varGamma_1}$ and $\rho_{\varGamma_2}$ are the permutation representations.

Lemma 4. *Consider the action of $G_2^{(\boldsymbol{\alpha})}$ on $X^{(\boldsymbol{\alpha})} = X_1 \times X_2^{(\boldsymbol{\alpha})}$.*

(i) It is trivial on the $X_2^{(\boldsymbol{\alpha})}$-part.

(ii) On the X_1-part, $G_2^{(\boldsymbol{\alpha})}$ acts as a sub-lattice of X_1 by translation.

(iii) The action on the X_1-part is independent of the $X_2^{(\boldsymbol{\alpha})}$-part.

Proof. Since the $G_2^{(\boldsymbol{\alpha})}$ corresponds to the G_2' in Subsection 2.2, we have $G_2^{(\boldsymbol{\alpha})} = G_1 \cap G_2$ by Lemma 3. In particular $G_2^{(\boldsymbol{\alpha})}$ is a subgroup of G_1, hence follow items (i) and (ii). Consider item (iii). The X_1-part is given by the map φ in (8). We show that for each $g = \sum_{k=1}^N n_k(m_k/\alpha_k)s_k \in H_2^{(\boldsymbol{\alpha})}$ and $\boldsymbol{\omega} \in X_1$ the image $\varphi(g, \boldsymbol{\omega}, \boldsymbol{\lambda})$ of the map φ is common to all $\boldsymbol{\lambda} \in X_2^{(\boldsymbol{\alpha})}$. In fact for any $\boldsymbol{\lambda} \in X_2$ we have

$$\varphi(g, \boldsymbol{\omega}, \boldsymbol{\lambda}) = \boldsymbol{\omega} + \left(\lambda^{(j)}_{(m_j/\alpha_j)n_j+1} + \sum_{k=1}^N (m_k/\alpha_k) B_{j,k} n_k \right)_{1 \le j \le N}. \tag{9}$$

Note that if $\boldsymbol{\lambda}$ lies in $X_2^{(\boldsymbol{\alpha})}$ then we have $\lambda^{(j)}_{(m_j/\alpha_j)n_j+1} = (p_j/\alpha_j)n_j$ from (4) and (6). Hence $\varphi(g, \boldsymbol{\omega}, \boldsymbol{\lambda})$ is independent of the choice of $\boldsymbol{\lambda} \in X_2^{(\boldsymbol{\alpha})}$. ■

We define $F^{(\boldsymbol{\alpha})} = (F^{(\boldsymbol{\alpha})}_{j,k})_{1 \le j,k \le N}$ to be an $N \times N$ matrix whose elements are given by

$$F^{(\boldsymbol{\alpha})}_{j,k} = (\delta_{j,k} p_k + B_{j,k} m_k)/\alpha_k, \tag{10}$$

and regard $\boldsymbol{\omega}$, $\boldsymbol{n} = {}^t(n_1, \dots, n_N)$ and $\varphi(g, \boldsymbol{\omega}, \boldsymbol{\lambda})$ as column vectors. Then (9) is written as $\varphi(g, \boldsymbol{\omega}, \boldsymbol{\lambda}) = \boldsymbol{\omega} + F^{(\boldsymbol{\alpha})}\boldsymbol{n}$ when $\boldsymbol{\lambda}$ lies in $X_2^{(\boldsymbol{\alpha})}$. By Lemma 4 we can think of $G_2^{(\boldsymbol{\alpha})}$ as a subgroup of $\mathfrak{S}(X_1)$ rather than that of $\mathfrak{S}(X^{(\boldsymbol{\alpha})})$. In this sense the group $G_2^{(\boldsymbol{\alpha})}$ is isomorphic to the lattice $F^{(\boldsymbol{\alpha})}\mathbb{Z}^N$ which acts on $X_1 = \mathbb{Z}^N$ by translation.

Thus Theorem 1 for the present construction with Lemma 7 (to appear in the appendix) is now stated as follows.

Proposition 4. *Let the set $X^{(\boldsymbol{\alpha})}$, the matrix $F^{(\boldsymbol{\alpha})}$, the groups H_1, H_2 and their actions on $X^{(\boldsymbol{\alpha})}$ be defined as above, and x be any element of $X^{(\boldsymbol{\alpha})}$. Then there is a bijection between $H_1 \cdot [x]_{H_2}$ and $F^{(\boldsymbol{\alpha})}\mathbb{Z}^N \backslash \mathbb{Z}^N$ that is commutative with the action of H_1.*

The set $F^{(\boldsymbol{\alpha})}\mathbb{Z}^N \backslash \mathbb{Z}^N$ becomes compact (and is a torus) if and only if $\det F^{(\boldsymbol{\alpha})} \neq 0$. A way to achieve this condition is given as follows. Let L be an integer satisfying $L \ge 2\sum_{k=1}^N k m_k$ and set

$$p_j = L - 2\sum_{k=1}^N \min(j,k) m_k, \tag{11}$$

$$B_{j,k} = 2\min(j,k). \tag{12}$$

Then we have $L > p_1 > p_2 > \cdots > p_N = L - 2\sum_{k=1}^N k m_k \ge 0$ and

$$\det F^{(\boldsymbol{\alpha})} = L p_1 p_2 \cdots p_{N-1}/(\alpha_1 \cdots \alpha_N) > 0.$$

Note that $p_N = 0$ is allowed here. In this case we have $\Lambda(m_N, p_N) = \Lambda^{(m_N)}(m_N, p_N)$ which has a unique element $(0)_{i \in \mathbb{Z}}$.

Example 5. The p_j and m_j in Example 1 satisfy the relation (11) with $L = 24$. We set $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) = (3, 1, 1)$ as in Example 3. By taking $B_{j,k}$ as in (12) we have

$$\begin{aligned} F^{(\boldsymbol{\alpha})} &= \begin{pmatrix} p_1/\alpha_1 & 0 & 0 \\ 0 & p_2/\alpha_2 & 0 \\ 0 & 0 & p_3/\alpha_3 \end{pmatrix} + \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 6 \end{pmatrix} \begin{pmatrix} m_1/\alpha_1 & 0 & 0 \\ 0 & m_2/\alpha_2 & 0 \\ 0 & 0 & m_3/\alpha_3 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 4 & 2 \\ 2 & 14 & 4 \\ 2 & 8 & 10 \end{pmatrix}, \end{aligned}$$

and $\det F^{(\boldsymbol{\alpha})} = 576$. For instance, let $x = (\boldsymbol{\omega}, \boldsymbol{\lambda})$ be the element of $X^{(\boldsymbol{\alpha})}$ specified by $\boldsymbol{\omega} = (0,0,0) \in X_1 = \mathbb{Z}^3$, and $\boldsymbol{\lambda} \in X_2^{(\boldsymbol{\alpha})}$ that is labeled by $(a, b, c) = (4, 8, 1)$. Then there is a bijection between $H_1 \cdot [x]_{H_2}$ and the three dimensional torus $F^{(\boldsymbol{\alpha})}\mathbb{Z}^3 \backslash \mathbb{Z}^3$ that is commutative with the action of H_1.

Remark 3. Given any positive integer s, choose an increasing sequence of positive integers $j_1, \ldots, j_s$ and define $\mathcal{H}$ to be the set $\mathcal{H} = \{j_1, \ldots, j_s\}$. A generalization of the arguments in this section is given by replacing the positive integer sequence $(m_j)_{1 \le j \le N}$ by $(m_j)_{j \in \mathcal{H}}$, which fits to the case in [5].

All the results in this paper can be extended to this generalized setting: We set $X_1 = \mathbb{Z}^s$, and H_2 to be a free abelian group with generators $\{s_j\}_{j \in \mathcal{H}}$. Given any positive integer sequence $(p_j)_{j \in \mathcal{H}}$ we set $X_2 = \prod_{j \in \mathcal{H}} \Lambda(m_j, p_j)$. Proposition 4 becomes a claim on the bijection between $H_1 \cdot [x]_{H_2}$ and $F^{(\boldsymbol{\alpha})}\mathbb{Z}^s \backslash \mathbb{Z}^s$ where the $F^{(\boldsymbol{\alpha})} = (F^{(\boldsymbol{\alpha})}_{j,k})_{j,k \in \mathcal{H}}$ is now an $s \times s$ matrix. The other definitions and statements can be modified similarly.

4 A one-dimensional integrable cellular automaton

4.1 The periodic box-ball system

The periodic box-ball system (pBBS) is a one-dimensional cellular automaton with periodic boundary conditions. We give a brief review on this system based on [5] from here to Subsection 4.3. Let L be a positive integer and p be a sequence of letters 1 and 2 under the conditions $\#(1) \ge \#(2)$ and $\#(1) + \#(2) = L$. Such sequences are called *paths* of positive weight and of length L. Denote by $\mathcal{P}$ the set of all such paths. We can define a commuting family of time evolutions T_k $(k = 1, 2, \ldots)$ acting on $\mathcal{P}$. It is a collection of update procedures for the cellular automaton. In this paper we write nT_k for an n times repeated application of T_k instead of T_k^n, regarding T_ks as generators of an abelian group (see Subsection 4.3).

The action of T_1 is given by a cyclic shift by one digit to the right. The definition of T_k for the other k's is given by means of the crystal basis of the quantized envelope algebra [11] and is available in Section 2.2 of [5]. Here we review it shortly.

Let B_k be the set of one-row semistandard tableaux of length k with entries 1 and 2. For instance, $B_1 = \{1, 2\}$, $B_2 = \{11, 12, 22\}$ and $B_3 = \{111, 112, 122, 222\}$. The combinatorial R map $R : B_k \times B_1 \to B_1 \times B_k$ is defined as follows. If we depict the relation $R(x, y) = (\tilde{y}, \tilde{x})$ by

$$\begin{array}{ccc} & y & \\ x & + & \tilde{x} \\ & \tilde{y} & \end{array}$$

then the definition of R is given by the following diagrams:

$$\overbrace{1\,\cdots\cdots\,1}^{k} \;\overset{1}{\underset{1}{-\!\!\!-\!\!\!+\!\!\!-\!\!\!-}}\; \overbrace{1\,\cdots\cdots\,1}^{k} \qquad \overbrace{2\,\cdots\cdots\,2}^{k} \;\overset{2}{\underset{2}{-\!\!\!-\!\!\!+\!\!\!-\!\!\!-}}\; \overbrace{2\,\cdots\cdots\,2}^{k}$$

$$\underset{(0<a\le k)}{\overbrace{1\cdots1}^{k-a}\overbrace{2\cdots2}^{a} \;\overset{1}{\underset{2}{-\!\!\!-\!\!\!+\!\!\!-\!\!\!-}}\; \overbrace{1\cdots1}^{k-a+1}\overbrace{2\cdots2}^{a-1}} \qquad \underset{(0\le a<k)}{\overbrace{1\cdots1}^{k-a}\overbrace{2\cdots2}^{a} \;\overset{2}{\underset{1}{-\!\!\!-\!\!\!+\!\!\!-\!\!\!-}}\; \overbrace{1\cdots1}^{k-a-1}\overbrace{2\cdots2}^{a+1}}$$

By repeated use of this R we define the following map

$$\begin{aligned} &B_k \times (B_1 \times \cdots \times B_1) \to (B_1 \times \cdots \times B_1) \times B_k, \\ &v \times (b_1 \times \cdots \times b_L) \mapsto (b'_1 \times \cdots \times b'_L) \times v'. \end{aligned} \tag{13}$$

Example 6. Set $k = 3$, $L = 13$, $v = 122$ and $b_1 \dots b_{13} = 1122211112212$. (We omit the symbol $\times$ here and in what follows.) Then we have $b'_1 \dots b'_{13} = 2211122211121$ and $v' = 122$. It is verified by the following diagram

$$122 \overset{1}{\underset{2}{+}} 112 \overset{1}{\underset{2}{+}} 111 \overset{2}{\underset{1}{+}} 112 \overset{2}{\underset{1}{+}} 122 \overset{2}{\underset{1}{+}} 222 \overset{1}{\underset{2}{+}} 122 \overset{1}{\underset{2}{+}} 112 \overset{1}{\underset{2}{+}} 111 \overset{1}{\underset{1}{+}} 111 \overset{2}{\underset{1}{+}} 112 \overset{2}{\underset{1}{+}} 122 \overset{1}{\underset{2}{+}} 112 \overset{2}{\underset{1}{+}} 122$$

In this example we have $v = v'$. In fact one can always find an element of B_k with this property.

Proposition 5 ([5]). *Given any $b_1 \times \cdots \times b_L \in \mathcal{P} \subset B_1^{\times L}$, let $v_0 \in B_k$ be the one defined in the same way as* (13) *by*

$$\overbrace{1\dots1}^{k} \times (b_1 \times \cdots \times b_L) \mapsto (b''_1 \times \cdots \times b''_L) \times v_0.$$

Then we have $v = v'$ in (13) *when we adopt this v_0 as the v there.*

By this choice of v, we define the time evolution T_k by $T_k(b_1 \dots b_L) = b'_1 \dots b'_L$.

Example 7. We have $T_3(1122211112212) = 2211122211121$ by Example 6.

Remark 4. Given any initial condition, the actions of T_ks for sufficiently large k's are common to them and determine a unique time evolution. The original study of pBBS [2, 3] considers only this time evolution which we denote by T_∞. In the context of the dynamical system in Subsection 2.3, we can regard the T_∞ as the unique time evolution in this dynamical system, and the other T_ks can be thought of as the phase flows associated with the other first integrals.

Here we give an example of the time evolution of this cellular automaton. (The case of $L = 31$, and evolved by $T_4 = T_\infty$.)

$$\begin{array}{ll} t=0: & 2\,2\,2\,2\,1\,1\,1\,2\,1\,1\,2\,2\,1\,1\,1\,1\,1\,2\,2\,2\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1 \\ t=1: & 1\,1\,1\,1\,2\,2\,2\,1\,2\,2\,1\,1\,2\,2\,1\,1\,1\,1\,1\,1\,2\,2\,2\,1\,1\,1\,1\,1\,1\,1\,1 \\ t=2: & 1\,1\,1\,1\,1\,1\,1\,2\,1\,1\,2\,2\,1\,1\,2\,2\,2\,2\,1\,1\,1\,1\,1\,2\,2\,2\,1\,1\,1\,1\,1 \\ t=3: & 1\,1\,1\,1\,1\,1\,1\,1\,2\,1\,1\,1\,2\,2\,1\,1\,1\,1\,2\,2\,2\,2\,1\,1\,1\,1\,2\,2\,2\,1\,1 \\ t=4: & 2\,1\,1\,1\,1\,1\,1\,1\,1\,2\,1\,1\,1\,1\,2\,2\,1\,1\,1\,1\,1\,1\,2\,2\,2\,2\,1\,1\,1\,2\,2 \\ t=5: & 1\,2\,2\,2\,2\,1\,1\,1\,1\,1\,2\,1\,1\,1\,1\,1\,2\,2\,1\,1\,1\,1\,1\,1\,1\,1\,1\,2\,2\,2\,1\,1 \end{array}$$

$t=6:$ 2 1 1 1 1 2 2 2 2 1 1 2 1 1 1 1 1 1 2 2 1 1 1 1 1 1 1 1 1 2 2
$t=7:$ 1 2 2 2 1 1 1 1 1 2 2 1 2 2 2 1 1 1 1 1 2 2 1 1 1 1 1 1 1 1 1
$t=8:$ 1 1 1 1 2 2 2 1 1 1 1 2 1 1 1 2 2 2 2 1 1 1 2 2 1 1 1 1 1 1 1
$t=9:$ 1 1 1 1 1 1 1 2 2 2 1 1 2 1 1 1 1 1 1 2 2 2 1 1 2 2 2 1 1 1 1

If we denote by $p \in \mathcal{P}$ the sequence for $t = 0$ then the sequence for $t = n$ stands for $(nT_4) \cdot p \in \mathcal{P}$.

When sufficiently separated from the other 2's, one can think of a consecutive sequence of 2's of length k as a soliton of amplitude k. By an appropriate definition one can say that the number of solitons conserves for each amplitude. In this example there are four solitons of distinct amplitudes (1, 2, 3 and 4) in every time step. For instance, in the state for $t = 9$ in the above example, the sequence 2 2 2 1 1 2 2 2 should be interpreted as an intermediate stage of the collision of two solitons of amplitudes 4 and 2.

4.2 Soliton content and rigged configurations

Let $\mathcal{P}_+$ be the set of all ballot sequences in $\mathcal{P}$: A path $p = b_1 b_2 \dots b_L$ is called a ballot sequence if and only if the number of 1's in the prefix $b_1 \dots b_k$ is at least as large as the number of 2's in that prefix for every k. For any $p \in \mathcal{P}$ there exist an integer d and a sequence $p_+ \in \mathcal{P}_+$ such that $p = (dT_1) \cdot p_+$ where T_1 is the right cyclic shift operator.

Given any $p_+ \in \mathcal{P}_+$, a bijective map ϕ due to Kerov–Kirillov–Reshetikhin [9, 10] yields the following data $(\mathcal{H}, \boldsymbol{m}, \boldsymbol{J})$. Here $\mathcal{H} = \{j_1, \dots, j_s\}$ is a set of positive integers satisfying the condition $j_1 < \dots < j_s$, and $\boldsymbol{m} = (m_{j_1}, \dots, m_{j_s})$ is an array of positive integers. The $\boldsymbol{J} = \Big((J_i^{(j_1)})_{1 \le i \le m_{j_1}}, \dots, (J_i^{(j_s)})_{1 \le i \le m_{j_s}} \Big)$ is an array of partitions: Each $(J_i^{(j)})_{1 \le i \le m_j}$ is a partition of non-negative integers into at most m_j parts in increasing order, with largest part $\le p_j$ where $p_j = L - 2 \sum_{k \in \mathcal{H}} \min(j,k) m_k$. In symbols it is written as $\phi(p_+) = (\boldsymbol{m}, \boldsymbol{J})$. The collection $(\boldsymbol{m}, \boldsymbol{J})$ is called a rigged configuration with configuration $\boldsymbol{m}$ and riggings $\boldsymbol{J}$, and p_j are called *vacancy numbers*. The set of all rigged configurations with configuration $\boldsymbol{m}$ is denoted $\mathrm{Rig}(\boldsymbol{m})$. An algorithm to obtain the data $(\mathcal{H}, \boldsymbol{m}, \boldsymbol{J})$ is presented in Appendix B.

Suppose there are two possible ways to write p as $p = (dT_1) \cdot p_+ = (d'T_1) \cdot p'_+$. Then it can be shown that one obtains a common configuration $\boldsymbol{m}$ by $\phi(p_+) = (\boldsymbol{m}, \boldsymbol{J})$ and $\phi(p'_+) = (\boldsymbol{m}, \boldsymbol{J}')$ with some $\boldsymbol{J}$, $\boldsymbol{J}'$. Hence for each $p \in \mathcal{P}$ one can label it with a specific $\boldsymbol{m}$, and we say that such a path has *soliton content* $\boldsymbol{m}$. The set $\mathcal{P}$ is decomposed as $\mathcal{P} = \bigsqcup_{\boldsymbol{m}} \mathcal{P}(\boldsymbol{m})$ where $\mathcal{P}(\boldsymbol{m})$ is the set of all paths of soliton content $\boldsymbol{m}$. It can be shown that each $\mathcal{P}(\boldsymbol{m})$ is invariant under the action of T_k for all k.

Example 8. Consider the paths

$$p = p_+ = 121122111212211222121111 \qquad \text{and} \qquad p'_+ = 111212211222121111121122,$$

in $\mathcal{P}_+$ with $L = 24$ which are related by $p = (6T_1) \cdot p'_+$. By the bijective map ϕ we obtain $\phi(p) = (\boldsymbol{m}, \boldsymbol{J})$ and $\phi(p'_+) = (\boldsymbol{m}, \boldsymbol{J}')$ with

$$\begin{aligned}
&\boldsymbol{m} = (m_1, m_2, m_3) = (3, 2, 1), \\
&\boldsymbol{J} = \big((J_1^{(1)}, J_2^{(1)}, J_3^{(1)}), (J_1^{(2)}, J_2^{(2)}), (J_1^{(3)}) \big) = ((0,4,8), (0,1), (0)), \\
&\boldsymbol{J}' = \big((J_1'^{(1)}, J_2'^{(1)}, J_3'^{(1)}), (J_1'^{(2)}, J_2'^{(2)}), (J_1'^{(3)}) \big) = ((2,6,10), (1,6), (0)).
\end{aligned}$$

4.3 Direct scattering transform

By means of the map ϕ one can construct a bijection between the set of all states in the cellular automaton and the set of extended rigged configurations divided by a group action. The map from the former to the latter is called the direct scattering transform (and its inverse is referred to as the inverse scattering transform), named after a similar transform in the theory of integrable non-linear evolution equations [12].

For the sake of simplicity we assume $\mathcal{H} = \{1, \dots, N\}$ in what follows. Then we have $\boldsymbol{m} = (m_j)_{1 \le j \le N}$, $\boldsymbol{J} = ((J_i^{(j)})_{1 \le i \le m_j})_{1 \le j \le N}$ and the vacancy numbers are given by (11). See Remark 3 for the recipe to recover the original setting for general $\mathcal{H}$.

We define

$$\overline{\mathcal{J}}(m,p) = \{(J_i)_{i \in \mathbb{Z}} \mid J_i \in \mathbb{Z},\ J_i \le J_{i+1},\ J_{i+m} = J_i + p \ \text{ for all } i\}, \tag{14}$$

and $\overline{\mathcal{J}}(\boldsymbol{m}) = \prod_{j=1}^N \overline{\mathcal{J}}(m_j, p_j)$. There is a map

$$\begin{array}{llcl} \iota: & \mathrm{Rig}(\boldsymbol{m}) & \longrightarrow & \overline{\mathcal{J}}(\boldsymbol{m}) \\ & ((J_i^{(j)})_{1 \le i \le m_j})_{1 \le j \le N} & \mapsto & (J_i^{(j)})_{i \in \mathbb{Z}, 1 \le j \le N} \end{array} \tag{15}$$

where the infinite sequence $(J_i^{(j)})_{i \in \mathbb{Z}} \in \overline{\mathcal{J}}(m_j, p_j)$ is the one that extends $(J_i^{(j)})_{1 \le i \le m_j}$ quasi-periodically as $J_{i+m_j}^{(j)} = J_i^{(j)} + p_j$ for all $i \in \mathbb{Z}$. In this article we call the elements of $\overline{\mathcal{J}}(\boldsymbol{m})$ extended rigged configurations. The relation between $\overline{\mathcal{J}}(\boldsymbol{m})$ and the set $\tilde{X}$ in Subsection 3.1 will be explained in the next subsection.

Let $\mathcal{A}$ be a free abelian group with generators $\sigma_1, \dots, \sigma_N$. We define its action on the set $\overline{\mathcal{J}}(\boldsymbol{m})$. Given any $g = \sum_{k=1}^N n_k \sigma_k \in \mathcal{A}$ and $\boldsymbol{J} = (J_i^{(j)})_{i \in \mathbb{Z}, 1 \le j \le N} \in \overline{\mathcal{J}}(\boldsymbol{m})$ we set

$$g \cdot \boldsymbol{J} = \left(J_{i+n_j}^{(j)} + 2 \sum_{k=1}^N n_k \min(j,k) \right)_{i \in \mathbb{Z}, 1 \le j \le N}.$$

Recall the family of time evolutions T_k $(k = 1, 2, \dots)$ introduced in Subsection 4.1. Let $\mathcal{T}$ be the free abelian group generated by $T_1, \dots, T_N$ which acts on $\mathcal{P}(\boldsymbol{m})$. We define its actions on the set $\overline{\mathcal{J}}(\boldsymbol{m})$. Given any $h = \sum_{k=1}^N n_k T_k \in \mathcal{T}$ and $\boldsymbol{J} = (J_i^{(j)})_{i \in \mathbb{Z}, 1 \le j \le N} \in \overline{\mathcal{J}}(\boldsymbol{m})$ we set

$$h \cdot \boldsymbol{J} = \left(J_i^{(j)} + \sum_{k=1}^N n_k \min(j,k) \right)_{i \in \mathbb{Z}, 1 \le j \le N}.$$

The direct scattering transform Φ ((3.15) in [5]) is given as follows:

$$\begin{array}{lccccccc} \Phi: & \mathcal{P}(\boldsymbol{m}) & \longrightarrow & \mathbb{Z} \times \mathcal{P}_+(\boldsymbol{m}) & \longrightarrow & \overline{\mathcal{J}}(\boldsymbol{m}) & \longrightarrow & \mathcal{A} \backslash \overline{\mathcal{J}}(\boldsymbol{m}) \\ & p & \longmapsto & (d, p_+) & \longmapsto & \iota(\boldsymbol{J}) + d & \longmapsto & [\iota(\boldsymbol{J}) + d]_{\mathcal{A}} \end{array} \tag{16}$$

Here the second map is given by ϕ together with the map ι (15), where for any $d \in \mathbb{Z}$ and $\boldsymbol{J} = ((J_i^{(j)})_{1 \le i \le m_j})_{1 \le j \le N} \in \mathrm{Rig}(\boldsymbol{m})$ we write $\iota(\boldsymbol{J}) + d = (J_i^{(j)} + d)_{i \in \mathbb{Z}, 1 \le j \le N} \in \overline{\mathcal{J}}(\boldsymbol{m})$. As in Subsection 4.2, suppose there are two possible ways to write p as $p = (dT_1) \cdot p_+ = (d'T_1) \cdot p'_+$. Then one generally has $\iota(\boldsymbol{J}') + d' \neq \iota(\boldsymbol{J}) + d$ in $\overline{\mathcal{J}}(\boldsymbol{m})$. However it can be shown that the relation $[\iota(\boldsymbol{J}') + d']_{\mathcal{A}} = [\iota(\boldsymbol{J}) + d]_{\mathcal{A}}$ holds, permitting the map Φ to be well-defined.

Example 9. Let $\boldsymbol{J}$ and $\boldsymbol{J}'$ be the ones in Example 8. Then we have $(\sigma_1 + \sigma_2) \cdot (\iota(\boldsymbol{J}) + d) = \iota(\boldsymbol{J}') + d'$ with $d = 0$, $d' = 6$. Hence for the p and $\boldsymbol{m}$ in that example, $\Phi(p) = [\iota(\boldsymbol{J}) + d]_{\mathcal{A}} = [\iota(\boldsymbol{J}') + d']_{\mathcal{A}}$.

It is easy to see that the actions of $\mathcal{A}$ and $\mathcal{T}$ on $\overline{\mathcal{J}}(\boldsymbol{m})$ is commutative. Hence by Lemma 1 there is a natural action of $\mathcal{T}$ on $\mathcal{A} \backslash \overline{\mathcal{J}}(\boldsymbol{m})$.

The main result of [5] is as follows.

Proposition 6 ([5, Theorem 3.12]). *The map Φ is a bijection between $\mathcal{P}(\boldsymbol{m})$ and $\mathcal{A}\backslash\overline{\mathcal{J}}(\boldsymbol{m})$ that is commutative with the action of $\mathcal{T}$.*

In particular one has $|\mathcal{P}(\boldsymbol{m})| = |\mathcal{A}\backslash\overline{\mathcal{J}}(\boldsymbol{m})|$. This quantity is written as $\Omega(\boldsymbol{m})$ in (1) where the $N \times N$ matrix F is set to be the $F^{(\boldsymbol{\alpha})}$ in (10) with (11), (12) and $\boldsymbol{\alpha} = (1, \dots, 1)$. The level set itself is written as $\mathcal{J}(\boldsymbol{m}) := \mathcal{A}\backslash\overline{\mathcal{J}}(\boldsymbol{m})$ in (2) where the $\gamma \times \gamma$ matrix A is defined by $A = (A_{j\alpha,k\beta})_{1\le j,k\le N, 1\le\alpha\le m_j, 1\le\beta\le m_k}$ with $A_{j\alpha,k\beta} = \delta_{j,k}\delta_{\alpha,\beta}(p_j + m_j) + 2\min(j,k) - \delta_{j,k}$.

4.4 Construction of the invariant tori

Let α be a common divisor of m and p. As a subset of $\overline{\mathcal{J}}(m,p)$ we define

$$\overline{\mathcal{J}}^{(\alpha)}(m,p) = \{J \in \overline{\mathcal{J}}(m/\alpha, p/\alpha) \mid J \notin \overline{\mathcal{J}}(m/\alpha', p/\alpha') \, \text{for every common divisor} \, \alpha' > \alpha\}.$$

Choose a sequence $\boldsymbol{\alpha} = (\alpha_j)_{1\le j\le N}$ with each α_j being a common divisor of m_j and p_j. We define $\overline{\mathcal{J}}^{(\boldsymbol{\alpha})}(\boldsymbol{m}) = \prod_{j=1}^{N} \overline{\mathcal{J}}^{(\alpha_j)}(m_j, p_j)$. This $\boldsymbol{\alpha}$ represents the symmetry of the system that depends on the initial conditions. As a subset of $\overline{\mathcal{J}}(\boldsymbol{m})$ it is invariant under the actions of $\mathcal{A}$ and $\mathcal{T}$. Let $\mathcal{P}^{(\boldsymbol{\alpha})}(\boldsymbol{m}) = \Phi^{-1}(\overline{\mathcal{J}}^{(\boldsymbol{\alpha})}(\boldsymbol{m}))$. As a corollary of Proposition 6 we have the following.

Corollary 1. *The map Φ is a bijection between $\mathcal{P}^{(\boldsymbol{\alpha})}(\boldsymbol{m})$ and $\mathcal{A}\backslash\overline{\mathcal{J}}^{(\boldsymbol{\alpha})}(\boldsymbol{m})$ that is commutative with the action of $\mathcal{T}$.*

Let $X^{(\boldsymbol{\alpha})} = X_1 \times X_2^{(\boldsymbol{\alpha})}$ be the set introduced in (7). In what follows we identify the group H_2 in Subsection 3.1 with the group $\mathcal{A}$ by $s_i = \sigma_i$, and set p_j and $B_{j,k}$ as in (11) and (12).

Lemma 5. *There is a bijection between $\overline{\mathcal{J}}^{(\boldsymbol{\alpha})}(\boldsymbol{m})$ and $X^{(\boldsymbol{\alpha})}$ that is commutative with the action of $\mathcal{A}$.*

Proof. For each $\boldsymbol{J} = \big(J_i^{(j)}\big)_{i\in\mathbb{Z}, 1\le j\le N} \in \overline{\mathcal{J}}^{(\boldsymbol{\alpha})}(\boldsymbol{m})$ let

$$\boldsymbol{\omega} = \big(J_1^{(j)}\big)_{1\le j\le N}, \qquad \boldsymbol{\lambda} = \big(J_i^{(j)} - J_1^{(j)}\big)_{i\in\mathbb{Z}, 1\le j\le N}.$$

Then the map $\overline{\mathcal{J}}^{(\boldsymbol{\alpha})}(\boldsymbol{m}) \ni \boldsymbol{J} \mapsto (\boldsymbol{\omega}, \boldsymbol{\lambda}) \in X^{(\boldsymbol{\alpha})}$ yields the desired bijection. ∎

By this lemma we can identify the set $\tilde{X} = X_1 \times X_2$ in Subsection 3.1 with the set $\overline{\mathcal{J}}(\boldsymbol{m})$.

Example 10. Let $\boldsymbol{J}$, $\boldsymbol{J}'$, d, d' be the ones in Examples 8 and 9. Then $\iota(\boldsymbol{J}) + d$ and $\iota(\boldsymbol{J}') + d'$ are elements of $\overline{\mathcal{J}}^{(\boldsymbol{\alpha})}(\boldsymbol{m})$ with $\boldsymbol{m}$ in those examples, and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) = (3, 1, 1)$. The above map sends $\iota(\boldsymbol{J}) + d$ to the $(\boldsymbol{\omega}, \boldsymbol{\lambda}) \in X^{(\boldsymbol{\alpha})}$ in Example 5. Write the image of $\iota(\boldsymbol{J}') + d'$ as $(\boldsymbol{\omega}', \boldsymbol{\lambda}') \in X^{(\boldsymbol{\alpha})}$. Then $\boldsymbol{\omega}' = (8, 7, 6) \in X_1 = \mathbb{Z}^3$, and $\boldsymbol{\lambda}'$ is the one labeled by $(a, b, c) = (4, 8, 5)$. By using the formula in Example 4 one can show that $(s_1 + s_2) \cdot (\boldsymbol{\omega}, \boldsymbol{\lambda}) = (\boldsymbol{\omega}', \boldsymbol{\lambda}')$. Compared with Example 9 this shows the commutativity of the action of $\mathcal{A}$ with the bijection in Lemma 5.

By Lemmas 5 and 8 (to appear in the appendix) there is a natural action of $\mathcal{T}$ on $X^{(\boldsymbol{\alpha})}$ that is commutative with the action of $\mathcal{A}$. Also one has a bijection between $\mathcal{A}\backslash\overline{\mathcal{J}}^{(\boldsymbol{\alpha})}(\boldsymbol{m})$ and $\mathcal{A}\backslash X^{(\boldsymbol{\alpha})}$ that is commutative with the action of $\mathcal{T}$. Now by Corollary 1 we have the following.

Proposition 7. *There is a bijection between $\mathcal{P}^{(\boldsymbol{\alpha})}(\boldsymbol{m})$ and $\mathcal{A}\backslash X^{(\boldsymbol{\alpha})}$ that is commutative with the action of $\mathcal{T}$.*

We denote by Ψ the map representing this bijection. Then for any $p \in \mathcal{P}^{(\boldsymbol{\alpha})}(\boldsymbol{m})$ there exists some $x \in X^{(\boldsymbol{\alpha})}$ such that $\Psi(p) = [x]_{\mathcal{A}}$. Now as a corollary of Proposition 7 we have the following.

Proposition 8. *Given $p \in \mathcal{P}^{(\boldsymbol{\alpha})}(\boldsymbol{m})$ and $x \in X^{(\boldsymbol{\alpha})}$ as above, there is a bijection between $\mathcal{T} \cdot p (\subset \mathcal{P}^{(\boldsymbol{\alpha})}(\boldsymbol{m}))$ and $\mathcal{T} \cdot [x]_{\mathcal{A}} (\subset \mathcal{A} \backslash X^{(\boldsymbol{\alpha})})$ that is commutative with the action of $\mathcal{T}$.*

The action of the abelian group $\mathcal{T}$ on the set $X^{(\boldsymbol{\alpha})} = X_1 \times X_2^{(\boldsymbol{\alpha})}$ keeps $X_2^{(\boldsymbol{\alpha})}$ untouched. Thus this action is essentially defined on the set $X_1 = \mathbb{Z}^N$. By Lemma 8 it is uniquely determined from the action of $\mathcal{T}$ on the set $\mathcal{P}(\boldsymbol{m})$ in Subsection 4.3. An explicit description of this action is as follows. For each $h = \sum_{k=1}^N n_k T_k \in \mathcal{T}$ and $\boldsymbol{\omega} = (\omega^{(j)})_{1 \le j \le N} \in X_1$ we set $h \cdot \boldsymbol{\omega} = \boldsymbol{\omega} + \sum_{k=1}^N n_k \boldsymbol{h}_k$ where $\boldsymbol{h}_k = (\min(j,k))_{1 \le j \le N} \in X_1$.

Lemma 6. *The action of the group $\mathcal{T}$ on the set $X_1 = \mathbb{Z}^N$ is transitive.*

Proof. Let $\boldsymbol{\omega}$, $\boldsymbol{\omega'}$ be any two elements of $\mathbb{Z}^N$, thought of as column vectors. We define the $N \times N$ matrix $M = (\boldsymbol{h}_1, \dots, \boldsymbol{h}_N)$. It is easy to see that $\det M = 1$. Hence by $\boldsymbol{n} = M^{-1}(\boldsymbol{\omega'} - \boldsymbol{\omega})$ we obtain a column vector $\boldsymbol{n} = {}^t(n_1, \dots, n_N)$ with integer entries. Then we have $\boldsymbol{\omega'} = (n_1 T_1 + n_2 T_2 + \cdots + n_N T_N) \cdot \boldsymbol{\omega}$. ■

Remark 5. For the generalized case in Remark 3 we define $\mathcal{T}$ to be the free abelian group generated by $T_1, T_{j_1+1}, \dots, T_{j_{s-1}+1}$. Then one can show that the action of $\mathcal{T}$ on the set $X_1 = \mathbb{Z}^s$ is transitive in the same way as above.

By Lemma 6 one can adopt $\mathcal{T}$ as the abelian group H_1 in Subsection 3.1. Given any $p \in \mathcal{P}^{(\boldsymbol{\alpha})}(\boldsymbol{m})$, a path of soliton content $\boldsymbol{m}$ and with the symmetry $\boldsymbol{\alpha}$, one can obtain the matrix $F^{(\boldsymbol{\alpha})}$ by (10), (11) and (12). By Propositions 4 and 8 we finally obtain our second main result.

Theorem 2. *There is a bijection between $\mathcal{T} \cdot p$ and $F^{(\boldsymbol{\alpha})}\mathbb{Z}^N \backslash \mathbb{Z}^N$ that is commutative with the action of $\mathcal{T}$.*

As we have shown at the end of Subsection 3.3 the $F^{(\boldsymbol{\alpha})}\mathbb{Z}^N \backslash \mathbb{Z}^N$ is an N-dimensional discrete torus. By the considerations given just above Lemma 6 one also knows that each time evolution T_k yields a straight motion on the torus with the "velocity" vector $\boldsymbol{h}_k = (\min(j,k))_{1 \le j \le N}$.

Example 11. For $p = 121122111212211222121111$ and

$$F^{(\boldsymbol{\alpha})} = \begin{pmatrix} 6 & 4 & 2 \\ 2 & 14 & 4 \\ 2 & 8 & 10 \end{pmatrix},$$

there is a bijection between $\mathcal{T} \cdot p$ and $F^{(\boldsymbol{\alpha})}\mathbb{Z}^3 \backslash \mathbb{Z}^3$. This is a consequence of Examples 5, 8 and 10.

5 Periods and the level set structure

5.1 Dynamical periods

Given $p \in \mathcal{P}^{(\boldsymbol{\alpha})}(\boldsymbol{m})$, a path of soliton content $\boldsymbol{m}$ and with the symmetry $\boldsymbol{\alpha}$, there is a smallest positive integer n_k such that $(n_k T_k) \cdot p = p$ for each k. It is called the *dynamical period* of p associated with the time evolution T_k. Since the time evolutions are mapped to straight motions on the torus $F^{(\boldsymbol{\alpha})}\mathbb{Z}^N \backslash \mathbb{Z}^N$, it is easy to obtain an explicit formula for n_k.

We fix a bijection $\widehat{\Psi}$ from $\mathcal{T} \cdot p$ to $F^{(\boldsymbol{\alpha})}\mathbb{Z}^N \backslash \mathbb{Z}^N$ whose existence was ensured by Theorem 2. Then since p lies in $\mathcal{T} \cdot p$ there exists $\boldsymbol{\omega} \in \mathbb{Z}^N$ such that $\widehat{\Psi}(p) = [\boldsymbol{\omega}]_{F^{(\boldsymbol{\alpha})}\mathbb{Z}^N}$. By the definition of the action of $\mathcal{T}$ on $F^{(\boldsymbol{\alpha})}\mathbb{Z}^N \backslash \mathbb{Z}^N$ we have

$$(n_k T_k) \cdot \widehat{\Psi}(p) = [(n_k T_k) \cdot \boldsymbol{\omega}]_{F^{(\boldsymbol{\alpha})}\mathbb{Z}^N} = [\boldsymbol{\omega} + n_k \boldsymbol{h}_k]_{F^{(\boldsymbol{\alpha})}\mathbb{Z}^N}.$$

On the other hand since $\widehat{\Psi}$ is commutative with the action of $\mathcal{T}$ we have

$$(n_k T_k) \cdot \widehat{\Psi}(p) = \widehat{\Psi}((n_k T_k) \cdot p) = \widehat{\Psi}(p) = [\boldsymbol{\omega}]_{F^{(\boldsymbol{\alpha})}\mathbb{Z}^N}.$$

Thus $n_k \boldsymbol{h}_k$ must lie in the $F^{(\boldsymbol{\alpha})}\mathbb{Z}^N$-orbit of 0, implying that n_k is defined as the smallest positive integer satisfying $n_k \boldsymbol{h}_k \in F^{(\boldsymbol{\alpha})}\mathbb{Z}^N$. This condition is essentially the same one given in (4.32) of [5] which yields an explicit formula for this quantity ((4.26) of [5]). Note that our formulation uses no Bethe ansatz considerations to obtain this result.

Example 12. For $p = 121122111212211222121111$ one can observe the dynamical periods $n_1 = 24$, $n_2 = 48$ and $n_3 = 72$ by directly applying the time evolutions in Subsection 4.1. By using the above considerations they are also calculated as

$$n_k = \text{L.C.M.}\left(\frac{\det F^{(\boldsymbol{\alpha})}}{\det F^{(\boldsymbol{\alpha})}[1]}, \frac{\det F^{(\boldsymbol{\alpha})}}{\det F^{(\boldsymbol{\alpha})}[2]}, \frac{\det F^{(\boldsymbol{\alpha})}}{\det F^{(\boldsymbol{\alpha})}[3]} \right),$$

where $F^{(\boldsymbol{\alpha})}$ is the one in Example 11 and $F^{(\boldsymbol{\alpha})}[i]$ is the matrix obtained from $F^{(\boldsymbol{\alpha})}$ by replacing its i-th column by $\boldsymbol{h}_k$. Here L.C.M. stands for the least common multiple. For instance, we have

$$n_3 = \text{L.C.M.}\left(24, \frac{72}{5}, \frac{72}{17} \right) = 72.$$

5.2 Decomposition of the level set

Each element $\lambda = (\lambda_i)_{i\in\mathbb{Z}}$ of the set $\Lambda(m,p)$ is specified by a non-decreasing sequence of integers $(0 \le)\lambda_2 \le \lambda_3 \le \cdots \le \lambda_m (\le p)$, or a partition $(\lambda_m, \ldots, \lambda_2)$. Hence the cardinality of $\Lambda(m,p)$ is given by the binomial coefficient $|\Lambda(m,p)| = \binom{p+m-1}{m-1}$. Recall the decomposition of the set $\Lambda(m,p) = \bigsqcup_\alpha \Lambda^{(\alpha)}(m,p)$, where α runs over every common divisor of m and p. Thus we have the relation

$$\binom{p+m-1}{m-1} = \sum_\alpha |\Lambda^{(\alpha)}(m,p)|.$$

The formula (1) for the cardinality of the whole level set with soliton content $\boldsymbol{m}$ is now written as

$$\begin{aligned} \Omega(\boldsymbol{m}) &= (\det F) \prod_{1\le j\le N} \frac{1}{m_j} \binom{p_j + m_j - 1}{m_j - 1} \\ &= \sum_{\alpha_1} \cdots \sum_{\alpha_N} \left(\det F^{(\boldsymbol{\alpha})} \right) \prod_{1\le j\le N} \frac{|\Lambda^{(\alpha_j)}(m_j, p_j)|}{m_j/\alpha_j}. \end{aligned} \tag{17}$$

This expression for the volume formula shows the decomposition of the level set (2) into the invariant tori: For each $\boldsymbol{\alpha} = (\alpha_j)_{1\le j\le N}$ we have the tori $F^{(\boldsymbol{\alpha})}\mathbb{Z}^N \backslash \mathbb{Z}^N$ with the multiplicity $\prod_{1\le j\le N} \frac{|\Lambda^{(\alpha_j)}(m_j,p_j)|}{m_j/\alpha_j}$. See [13, Lemmas 3.1 and 3.2] for a proof of this statement. While the smallest torus is given by $\boldsymbol{\alpha} = (\alpha_j)_{1\le j\le N}$ with every α_j being the greatest common divisor of m_j and p_j, the largest torus is given by $\boldsymbol{\alpha} = (1)_{1\le j\le N}$ when $p_N > 0$, or $\boldsymbol{\alpha} = (1, \ldots, 1, m_N)$ when $p_N = 0$.

One can check that this expression for the multiplicity is indeed an integer. Let s be the generator of the cyclic group $\mathcal{C}_m$ that acts on $\Lambda^{(\alpha)}(m,p)$. For each $\lambda \in \Lambda^{(\alpha)}(m,p)$ the m/α elements $(ks) \cdot \lambda \, (0 \le k \le m/\alpha - 1)$ are all distinct and hence $|\Lambda^{(\alpha)}(m,p)|$ is divisible by m/α.

An expression for the quantity $|\Lambda^{(\alpha)}(m,p)|$ is given as follows. Let $\mu(n)$ be the Möbius function of number theory [14]; that is, $\mu(1) = 1, \mu(n) = 0$ if n is divisible by the square of an integer greater than one, and $\mu(n) = (-1)^r$ if n is the product of r distinct primes. Then we have

$$|\Lambda^{(\alpha)}(m,p)| = \sum_{\beta} \mu(\beta/\alpha) \binom{\frac{p+m}{\beta} - 1}{\frac{m}{\beta} - 1}, \tag{18}$$

where β runs over every common divisor of m and p that is a multiple of α.

Example 13. For $L = 24$ and $\boldsymbol{m} = (m_1, m_2, m_3) = (3,2,1)$ we have $(p_1, p_2, p_3) = (12, 6, 4)$. The former expression of (17) gives

$$\begin{aligned}\Omega(\boldsymbol{m}) &= (\det F) \frac{1}{m_1} \binom{p_1 + m_1 - 1}{m_1 - 1} \frac{1}{m_2} \binom{p_2 + m_2 - 1}{m_2 - 1} \frac{1}{m_3} \binom{p_3 + m_3 - 1}{m_3 - 1} \\ &= 1728 \cdot \frac{1}{3} \binom{14}{2} \cdot \frac{1}{2} \binom{7}{1} = 183456,\end{aligned}$$

where $F = F^{(\boldsymbol{\alpha}_1)}$ is given below. Let $\boldsymbol{\alpha}_1 = (1,1,1)$, $\boldsymbol{\alpha}_2 = (1,2,1)$, $\boldsymbol{\alpha}_3 = (3,1,1)$, $\boldsymbol{\alpha}_4 = (3,2,1)$. Then

$$F^{(\boldsymbol{\alpha}_1)} = \begin{pmatrix} 18 & 4 & 2 \\ 6 & 14 & 4 \\ 6 & 8 & 10 \end{pmatrix}, \qquad F^{(\boldsymbol{\alpha}_2)} = \begin{pmatrix} 18 & 2 & 2 \\ 6 & 7 & 4 \\ 6 & 4 & 10 \end{pmatrix},$$
$$F^{(\boldsymbol{\alpha}_3)} = \begin{pmatrix} 6 & 4 & 2 \\ 2 & 14 & 4 \\ 2 & 8 & 10 \end{pmatrix}, \qquad F^{(\boldsymbol{\alpha}_4)} = \begin{pmatrix} 6 & 2 & 2 \\ 2 & 7 & 4 \\ 2 & 4 & 10 \end{pmatrix}.$$

By using (18) one has $|\Lambda^{(1)}(3,12)| = 90$, $|\Lambda^{(3)}(3,12)| = 1$, $|\Lambda^{(1)}(2,6)| = 6$ and $|\Lambda^{(2)}(2,6)| = 1$. See Example 3 for the decomposition of the set $\Lambda(m,p)$ in the present example. Now the latter expression of (17) gives

$$\Omega(\boldsymbol{m}) = 90 \det F^{(\boldsymbol{\alpha}_1)} + 30 \det F^{(\boldsymbol{\alpha}_2)} + 3 \det F^{(\boldsymbol{\alpha}_3)} + \det F^{(\boldsymbol{\alpha}_4)} = 183456.$$

This enumeration reflects the following decomposition of the level set into invariant tori:

$$\mathcal{J}(\boldsymbol{m}) = 90(F^{(\boldsymbol{\alpha}_1)}\mathbb{Z}^3 \backslash \mathbb{Z}^3) \sqcup 30(F^{(\boldsymbol{\alpha}_2)}\mathbb{Z}^3 \backslash \mathbb{Z}^3) \sqcup 3(F^{(\boldsymbol{\alpha}_3)}\mathbb{Z}^3 \backslash \mathbb{Z}^3) \sqcup (F^{(\boldsymbol{\alpha}_4)}\mathbb{Z}^3 \backslash \mathbb{Z}^3).$$

The $\mathcal{T} \cdot p \cong F^{(\boldsymbol{\alpha})}\mathbb{Z}^3 \backslash \mathbb{Z}^3$ in Example 11 is one of the three $F^{(\boldsymbol{\alpha}_3)}\mathbb{Z}^3 \backslash \mathbb{Z}^3$s in this $\mathcal{J}(\boldsymbol{m})$.

A Two elementary lemmas

The following lemma is used in Subsection 3.3

Lemma 7. *Let G, H be groups and X be a set. Suppose there are commutative actions of G and H on X. Denote by Γ (resp. Γ') the action of G (resp. H) on X, and by ρ_Γ (resp. $\rho_{\Gamma'}$) its permutation representation. Then:*

(i) The actions of $\rho_\Gamma(G)$ and H on X are commutative.

(ii) There is a bijection between $G \backslash X$ and $\rho_\Gamma(G) \backslash X$ that is commutative with the action of H.

(iii) For all $x \in X$ there is a bijection between $H \cdot [x]_G$ and $\rho_{\Gamma'}(H) \cdot [x]_{\rho_\Gamma(G)}$ that is commutative with the action of H.

Proof. (i) For all $g \in G, h \in H$ and $x \in X$ we have $h \cdot (\rho_\Gamma(g) \cdot x) = h \cdot \Gamma(g, x) = \Gamma(g, h \cdot x) = \rho_\Gamma(g) \cdot (h \cdot x)$. (ii) Given any $p \in G\backslash X$ it can be written as $p = [x]_G$ with some $x \in X$. Define $\psi : G\backslash X \to \rho_\Gamma(G)\backslash X$ by the relation $\psi(p) = [x]_{\rho_\Gamma(G)}$. It is easy to see that this map is well-defined and yields the desired bijection. (iii) The bijection is given from that in (ii) by restricting its domain to $H \cdot [x]_G$. ■

The following lemma is used in Subsection 4.4

Lemma 8. *Let G be a group and X_1, X_2 be sets. Suppose there are actions of G on X_1 and X_2, and there is a bijection $\phi : X_1 \to X_2$ that is commutative with the actions of G. Let H be another group, and suppose there is an action of H on X_1 that is commutative with the action of G. Then:*

(i) *There is a unique action of H on X_2 that is commutative with ϕ, and commutative with the action of G on X_2.*

(ii) *There is a bijection between $G\backslash X_1$ and $G\backslash X_2$ that is commutative with the action of H.*

Proof. (i) For all $h \in H$ and $x \in X_2$ the desired action is uniquely determined by $h \cdot x = \phi(h \cdot \phi^{-1}(x))$. (ii) Given any $p \in G\backslash X_1$ it can be written as $p = [x]_G$ with some $x \in X_1$. Define $\bar{\phi} : G\backslash X_1 \to G\backslash X_2$ by the relation $\bar{\phi}(p) = [\phi(x)]_G$. It is easy to see that this map is well-defined and yields the desired bijection. ■

B An algorithm for Kerov–Kirillov–Reshetikhin map

We present an algorithm for the map of Kerov–Kirillov–Reshetikhin based on [15].

Let $p = b_1 \dots b_L$ be a ballot sequence of letters 1 and 2. We define M_1, M_2 to be a pair of subsets of $\{1, \dots, L\}$ that is specified by the following condition: The integer j lies in M_1 (resp. M_2) if and only if $b_j = 1$ and $b_{j+1} = 2$ (resp. $b_j = 2$ and $b_{j+1} = 1$) . Here we interpret $b_{L+1} = 1$.

We define Op_1, Op_2 to be a pair of operators acting on finite sets of distinct integers: Given any such set M, the Op_1 (resp. Op_2) replaces its i-th smallest element, say x, by $x - (2i - 1)$ (resp. $x - 2i$) from $i = 1$ to $i = |M|$.

Let $\overline{\varphi} = \{\}$ be an empty data set. We repeat the following procedure until we have $M_1 = M_2 = \varnothing$.

(i) Let $i = 0$.

(ii) While $M_1 \cap M_2 = \varnothing$, apply Op_1 to M_1 and Op_2 to M_2, and replace i by $i + 1$.

(iii) When $M_1 \cap M_2 \neq \varnothing$ is attained, append $\{M_1 \cap M_2, i\}$ to $\overline{\varphi}$, and replace M_1 by $M_1 \backslash (M_1 \cap M_2)$ and M_2 by $M_2 \setminus (M_1 \cap M_2)$.

In the above procedure the multiplicity of the elements should be respected. For instance $\{1, 2, 2, 3\} \cap \{2, 2, 4, 5\}$ is equal to $\{2, 2\}$, not to $\{2\}$. And $\{1, 2, 2, 3\} \setminus \{2, 3\}$ is equal to $\{1, 2\}$, not to $\{1\}$. At the end we obtain such type of data $\overline{\varphi} = \{\{S_1, i_1\}, \{S_2, i_2\}, \dots, \{S_s, i_s\}\}$ for some s, where S_a are sets of integers and i_a are positive integers.

Now we have $\mathcal{H} = \{j_1, \dots, j_s\}$ with $j_1 = i_1, j_2 = i_1 + i_2, \dots, j_s = i_1 + \cdots + i_s$, and $\boldsymbol{m} = (m_{j_1}, \dots, m_{j_s}) = (|S_1|, \dots, |S_s|)$. For the riggings $\boldsymbol{J}$, each $(J_i^{(j_k)})_{1 \le i \le m_{j_k}}$ is obtained from S_k by ordering its elements in increasing order.

Example 14. Consider the path $p = 11112222111112222111211122211122221222$. Then $M_1 = \{4, 13, 19, 23, 29, 34\}$ and $M_2 = \{8, 16, 20, 26, 33, 38\}$. By applying procedure (ii) once, we obtain $M_1 = \{3, 10, 14, 16, 20, 23\}$ and $M_2 = \{6, 12, 14, 18, 23, 26\}$. Then by procedure (iii) one has $\overline{\varphi} = \{\{\{14, 23\}, 1\}\}$, $M_1 = \{3, 10, 16, 20\}$, and $M_2 = \{6, 12, 18, 26\}$.

In the second turn of the algorithm, we obtain $M_1 = \{1, 4, 6, 6\}$ and $M_2 = \{2, 4, 6, 10\}$ after applying procedure (ii) twice. Thus one has $\overline{\varphi} = \{\{\{14, 23\}, 1\}, \{\{4, 6\}, 2\}\}$, $M_1 = \{1, 6\}$, and $M_2 = \{2, 10\}$.

After two more turns, one has $\overline{\varphi} = \{\{\{14, 23\}, 1\}, \{\{4, 6\}, 2\}, \{\{0\}, 1\}, \{\{0\}, 3\}\}$ and $M_1 = M_2 = \varnothing$. Hence we have $\mathcal{H} = \{1, 3, 4, 7\}$, $\boldsymbol{m} = (m_1, m_3, m_4, m_7) = (2, 2, 1, 1)$ and

$$\boldsymbol{J} = \big(\big(J_1^{(1)}, J_2^{(1)}\big), \big(J_1^{(3)}, J_2^{(3)}\big), \big(J_1^{(4)}\big), \big(J_1^{(7)}\big)\big) = ((14, 23), (4, 6), (0), (0)).$$

Acknowledgements

The author thanks Atsuo Kuniba for valuable discussions.

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