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Epsilon Systems on Geometric Crystals of type A_n⋆

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Abstract. We introduce an epsilon system on a geometric crystal of type A_n, which is a certain set of rational functions with some nice properties. We shall show that it is equipped with a product structure and that it is invariant under the action of tropical R maps.

Key words: geometric crystal; epsilon system; tropical R map

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1 Introduction

In the theory of crystal bases, the piecewise-linear functions ε_i and φ_i play many crucial roles, e.g., description of highest weight vectors, tensor product of crystals, extremal vectors, etc. There exist counterparts for geometric crystals, denoted also $\{\varepsilon_i\}$, which are rational functions with several nice properties, indeed, they are needed to describe the product structure of geometric crystals (see Section 2) and in $\mathfrak{sl}_2$-case, the universal tropical R map is presented by using them [12].

In [1], higher objects $\varepsilon_{i,j}$ and $\varepsilon_{j,i}$ are introduced in order to prove the existence of product structure of geometric crystals induced from unipotent crystals, which satisfy the relation $\varepsilon_i\varepsilon_j = \varepsilon_{i,j}+\varepsilon_{j,i}$ if the vertices i and j are simply laced. It motivates us to define further higher objects, "epsilon system".

The aim of the article is to define an "epsilon system" for type A_n and reveal its basic properties, e.g., product structures and invariance under the action of tropical R maps. An epsilon system is a certain set of rational functions on a geometric crystal, which satisfy some relations with each other and have simple forms of the action by e_i^c's.

We found its prototype on the geometric crystal of the opposite Borel subgroup $B^-\subset SL_{n+1}(\mathbb{C})$. In that case, indeed, the epsilon system is realized as a set of matrix elements and minor determinants of unipotent part of a group element in B^- (see Section 6). Therefore, we know that geometric crystals induced from unipotent crystals are equipped with an epsilon system naturally.

We shall introduce two remarkable properties of epsilon system: One is a product structure of epsilon systems. That is, for two geometric crystals with epsilon systems, say $\mathbb{X}$ and $\mathbb{Y}$, there exists canonically an epsilon system on the product of geometric crystals $\mathbb{X}\times\mathbb{Y}$ (see Section 5.3). The other is an invariance by tropical R maps: Let $\mathcal{R}:\mathbb{X}\times\mathbb{Y}\to\mathbb{Y}\times\mathbb{X}$ be a tropical R map (see Section 4) and $\varepsilon_J^{\mathbb{X}\times\mathbb{Y}}$ (resp. $\varepsilon_J^{\mathbb{Y}\times\mathbb{X}}$) an arbitrary element in the epsilon system on $\mathbb{X}\times\mathbb{Y}$ (resp. $\mathbb{Y}\times\mathbb{X}$) obtained from the ones on $\mathbb{X}$ and $\mathbb{Y}$. Then, we have the invariant property:

$$\varepsilon_J^{\mathbb{Y}\times\mathbb{X}}(\mathcal{R}(x,y)) = \varepsilon_J^{\mathbb{X}\times\mathbb{Y}}(x,y).$$

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In the last section, we shall give an application of these results, which shows the uniqueness of tropical R map on some geometric crystals of type $A_n^{(1)}$.

Since the $\mathfrak{sl}_2$-universal tropical R map is presented by using the rational functions $\{\varepsilon_i\}$ as mentioned above, we expect that epsilon systems would be a key to find universal tropical R maps of higher ranks. For further aim, we would like to extend this notion to other simple Lie algebras, e.g., B_n, C_n, and D_n. These problems will be discussed elsewhere.

2 Geometric crystals and unipotent crystals

The notations and definitions here follow [2, 5, 6, 7, 8, 9, 10, 11].

2.1 Geometric crystals

Fix a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$ with a finite index set I. Let $(\mathfrak{t}, \{\alpha_i\}_{i\in I}, \{h_i\}_{i\in I})$ be the associated root data satisfying $\alpha_j(h_i) = a_{ij}$. Let $\mathfrak{g} = \mathfrak{g}(A) = \langle \mathfrak{t}, e_i, f_i (i \in I)\rangle$ be the Kac–Moody Lie algebra associated with A [5]. Let $P \subset \mathfrak{t}^*$ (resp. $Q := \oplus_i \mathbb{Z}\alpha_i$, $Q^\vee := \oplus_i \mathbb{Z}h_i$) be a weight (resp. root, coroot) lattice such that $\mathbb{C} \otimes P = \mathfrak{t}^*$ and $P \subset \{\lambda \,|\, \lambda(Q^\vee) \subset \mathbb{Z}\}$, whose element is called a weight.

Define the simple reflections $s_i \in \mathrm{Aut}(\mathfrak{t})$ $(i \in I)$ by $s_i(h) := h - \alpha_i(h)h_i$, which generate the Weyl group W. Let G be the Kac–Moody group associated with $(\mathfrak{g}, P)$ [6, 7]. Let $U_\alpha := \exp \mathfrak{g}_\alpha$ $(\alpha \in \Delta^{\mathrm{re}})$ be the one-parameter subgroup of G. The group G (resp. $U^\pm$) is generated by $\{U_\alpha | \alpha \in \Delta^{\mathrm{re}}\}$ (resp. $\{U_\alpha | \alpha \in \Delta^{\mathrm{re}} \cap (\oplus_i \pm \mathbb{Z}\alpha_i)\}$). Here $U^\pm$ is a unipotent subgroup of G. For any $i \in I$, there exists a unique group homomorphism $\phi_i \colon SL_2(\mathbb{C}) \to G$ such that

$$\phi_i\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) = \exp(te_i), \qquad \phi_i\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) = \exp(tf_i), \qquad t \in \mathbb{C}.$$

Set $\alpha_i^\vee(c) := \phi_i\left(\left(\begin{smallmatrix} c & 0 \\ 0 & c^{-1} \end{smallmatrix}\right)\right)$, $x_i(t) := \exp(te_i)$, $y_i(t) := \exp(tf_i)$, $G_i := \phi_i(SL_2(\mathbb{C}))$, $T_i := \alpha_i^\vee(\mathbb{C}^\times)$ and $N_i := N_{G_i}(T_i)$. Let T be the subgroup of G with P as its weight lattice which is called a *maximal torus* in G, and let $B^\pm(\supset T)$ be the Borel subgroup of G. We have the isomorphism $\phi : W \xrightarrow{\sim} N/T$ defined by $\phi(s_i) = N_iT/T$. An element $\overline{s}_i := x_i(-1)y_i(1)x_i(-1)$ is in $N_G(T)$, which is a representative of $s_i \in W = N_G(T)/T$.

Definition 2.1. Let X be an ind-variety over $\mathbb{C}$, γ_i and ε_i $(i \in I)$ rational functions on X, and $e_i : \mathbb{C}^\times \times X \to X$ a rational $\mathbb{C}^\times$-action. A quadruple $(X, \{e_i\}_{i\in I}, \{\gamma_i,\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ is a G (or $\mathfrak{g}$)-*geometric crystal* if

changed

(i) $(\{1\} \times X) \cap \mathrm{dom}(e_i)$ is open dense in $\{1\} \times X$ for any $i \in I$, where $\mathrm{dom}(e_i)$ is the domain of definition of $e_i \colon \mathbb{C}^\times \times X \to X$.

(ii) The rational functions $\{\gamma_i\}_{i\in I}$ satisfy $\gamma_j(e_i^c(x)) = c^{a_{ij}}\gamma_j(x)$ for any $i, j \in I$.

(iii) e_i and e_j satisfy the following relations:

$$\begin{aligned}
&e_i^{c_1}e_j^{c_2} = e_j^{c_2}e_i^{c_1} && \text{if } a_{ij} = a_{ji} = 0,\\
&e_i^{c_1}e_j^{c_1c_2}e_i^{c_2} = e_j^{c_2}e_i^{c_1c_2}e_j^{c_1} && \text{if } a_{ij} = a_{ji} = -1,\\
&e_i^{c_1}e_j^{c_1^2c_2}e_i^{c_1c_2}e_j^{c_2} = e_j^{c_2}e_i^{c_1c_2}e_j^{c_1^2c_2}e_i^{c_1} && \text{if } a_{ij} = -2,\ a_{ji} = -1,\\
&e_i^{c_1}e_j^{c_1^3c_2}e_i^{c_1^2c_2}e_j^{c_1^3c_2^2}e_i^{c_1c_2}e_j^{c_2} = e_j^{c_2}e_i^{c_1c_2}e_j^{c_1^3c_2^2}e_i^{c_1^2c_2}e_j^{c_1^3c_2}e_i^{c_1} && \text{if } a_{ij} = -3,\ a_{ji} = -1.
\end{aligned}$$

(iv) The rational functions $\{\varepsilon_i\}_{i\in I}$ satisfy $\varepsilon_i(e_i^c(x)) = c^{-1}\varepsilon_i(x)$ and $\varepsilon_i(e_j^c(x)) = \varepsilon_i(x)$ if $a_{i,j} = a_{j,i} = 0$.

The relations in (iii) is called *Verma relations.* If $\chi = (X, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ satisfies the conditions (i), (ii) and (iv), we call χ a *pre-geometric crystal.*

Remark. The last condition (iv) is slightly modified from [2, 9, 10, 11, 12] since all ε_i appearing in these references satisfy the new condition '$\varepsilon_i(e_j^c(x)) = \varepsilon_i(x)$ if $a_{i,j} = a_{j,i} = 0$' and we need this condition to define "epsilon systems" later.

2.2 Unipotent crystals

In the sequel, we denote the unipotent subgroup U^+ by U. We define unipotent crystals (see [1, 9]) associated to Kac–Moody groups.

Definition 2.2. Let X be an ind-variety over $\mathbb{C}$ and $\alpha : U \times X \to X$ be a rational U-action such that α is defined on $\{e\} \times X$. Then, the pair $\mathbf{X} = (X, \alpha)$ is called a *U-variety.* For U-varieties $\mathbf{X} = (X, \alpha_X)$ and $\mathbf{Y} = (Y, \alpha_Y)$, a rational map $f : X \to Y$ is called a *U-morphism* if it commutes with the action of U.

Now, we define a U-variety structure on $B^- = U^-T$. As in [8], the Borel subgroup B^- is an ind-subgroup of G and hence an ind-variety over $\mathbb{C}$. The multiplication map in G induces the open embedding; $B^- \times U \hookrightarrow G$, which is a birational map. Let us denote the inverse birational map by $g : G \longrightarrow B^- \times U$ and let rational maps $\pi^- : G \to B^-$ and $\pi : G \to U$ be $\pi^- := \mathrm{proj}_{B^-} \circ g$ and $\pi := \mathrm{proj}_U \circ g$. Now we define the rational U-action α_{B^-} on B^- by

$$\alpha_{B^-} := \pi^- \circ m : \; U \times B^- \longrightarrow B^-,$$

where m is the multiplication map in G. Then we get U-variety $\mathbf{B}^- = (B^-, \alpha_{B^-})$.

Definition 2.3.

(i) Let $\mathbf{X} = (X, \alpha)$ be a U-variety and $f : X \to B^-$ a U-morphism. The pair $(\mathbf{X}, f)$ is called a *unipotent G-crystal* or, for short, *unipotent crystal.*

(ii) Let $(\mathbf{X}, f_X)$ and $(\mathbf{Y}, f_Y)$ be unipotent crystals. A U-morphism $g : \mathbf{X} \to \mathbf{Y}$ is called a *morphism of unipotent crystals* if $f_X = f_Y \circ g$. In particular, if g is a birational map of ind-varieties, it is called an *isomorphism of unipotent crystals.*

We define a product of unipotent crystals following [1]. For unipotent crystals $(\mathbf{X}, f_X)$, $(\mathbf{Y}, f_Y)$, define a rational map $\alpha_{X\times Y} : U \times X \times Y \to X \times Y$ by

$$\alpha_{X\times Y}(u, x, y) := (\alpha_X(u, x), \alpha_Y(\pi(u \cdot f_X(x)), y)).$$

Theorem 2.4 ([1]).

(i) The rational map $\alpha_{X\times Y}$ defined above is a rational U-action on $X \times Y$.

(ii) Let $\mathbf{m} : B^- \times B^- \to B^-$ be a multiplication map and $f = f_{X\times Y} : X \times Y \to B^-$ be the rational map defined by

$$f_{X\times Y} := \mathbf{m} \circ (f_X \times f_Y).$$

Then $f_{X\times Y}$ is a U-morphism and $(\mathbf{X} \times \mathbf{Y}, f_{X\times Y})$ is a unipotent crystal, which we call a product of unipotent crystals $(\mathbf{X}, f_X)$ and $(\mathbf{Y}, f_Y)$.

(iii) Product of unipotent crystals is associative.

2.3 From unipotent crystals to geometric crystals

For $i \in I$, set $U_i^{\pm} := U^{\pm} \cap \bar{s}_i U^{\mp} \bar{s}_i^{-1}$ and $U_{\pm}^i := U^{\pm} \cap \bar{s}_i U^{\pm} \bar{s}_i^{-1}$. Indeed, $U_i^{\pm} = U_{\pm\alpha_i}$. Set

$$Y_{\pm\alpha_i} := \langle x_{\pm\alpha_i}(t) U_\alpha x_{\pm\alpha_i}(-t) | t \in \mathbb{C},\ \alpha \in \Delta_{\pm}^{\mathrm{re}} \setminus \{\pm\alpha_i\} \rangle,$$

where $x_{\alpha_i}(t) := x_i(t)$ and $x_{-\alpha_i}(t) := y_i(t)$. We have the unique decomposition;

$$U^- = U_i^- \cdot Y_{\pm\alpha_i} = U_{-\alpha_i} \cdot U_-^i.$$

By using this decomposition, we get the canonical projection $\xi_i : U^- \to U_{-\alpha_i}$. Now, we define the function on U^- by

$$\chi_i := y_i^{-1} \circ \xi_i : \ U^- \longrightarrow U_{-\alpha_i} \xrightarrow{\sim} \mathbb{C},$$

and extend this to the function on B^- by $\chi_i(u \cdot t) := \chi_i(u)$ for $u \in U^-$ and $t \in T$. For a unipotent G-crystal $(\mathbf{X}, f_X)$, we define a function $\varepsilon_i := \varepsilon_i^X : X \to \mathbb{C}$ by

$$\varepsilon_i := \chi_i \circ f_X,$$

and a rational function $\gamma_i : X \to \mathbb{C}$ by

$$\gamma_i := \alpha_i \circ \mathrm{proj}_T \circ f_X : \ X \to B^- \to T \to \mathbb{C},$$

where proj_T is the canonical projection.

Remark. Note that the function ε_i is denoted by φ_i in [1, 9].

Suppose that the function ε_i is not identically zero on X. We define a morphism e_i: $\mathbb{C}^\times \times X \to X$ by

$$e_i^c(x) := x_i\left(\frac{c-1}{\varepsilon_i(x)}\right)(x).$$

Theorem 2.5 ([1]). *For a unipotent G-crystal $(\mathbf{X}, f_X)$, suppose that the function ε_i is not identically zero for any $i \in I$. Then the rational functions $\gamma_i, \varepsilon_i : X \to \mathbb{C}$ and $e_i : \mathbb{C}^\times \times X \to X$ as above define a geometric G-crystal $(\mathbf{X}, \{e_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$, which is called the induced geometric G-crystals by unipotent G-crystal $(\mathbf{X}, f_X)$.*

Proposition 2.6. *For unipotent G-crystals $(\mathbf{X}, f_X)$ and $(\mathbf{Y}, f_Y)$, set the product $(\mathbf{Z}, f_Z) := (\mathbf{X}, f_X) \times (\mathbf{Y}, f_Y)$, where $Z = X \times Y$. Let $(Z, \{e_i^Z\}_{i\in I}, \{\gamma_i^Z\}_{i\in I}, \{\varepsilon_i^Z\}_{i\in I})$ be the induced geometric G-crystal from $(\mathbf{Z}, f_Z)$. Then we obtain:*

(i) For each $i \in I$, $(x, y) \in Z$,

$$\gamma_i^Z(x, y) = \gamma_i^X(x)\gamma_i^Y(y), \qquad \varepsilon_i^Z(x, y) = \varepsilon_i^X(x) + \frac{\varepsilon_i^Y(y)}{\gamma_i^X(x)}.$$

(ii) For any $i \in I$, the action $e_i^Z : \mathbb{C}^\times \times Z \to Z$ is given by: $(e_i^Z)^c(x, y) = ((e_i^X)^{c_1}(x), (e_i^Y)^{c_2}(y))$, where

$$c_1 = \frac{c\gamma_i^X(x)\varepsilon_i^X(x) + \varepsilon_i^Y(y)}{\gamma_i^X(x)\varepsilon_i^X(x) + \varepsilon_i^Y(y)}, \qquad c_2 = \frac{c(\gamma_i^X(x)\varepsilon_i^X(x) + \varepsilon_i^Y(y))}{c\gamma_i^X(x)\varepsilon_i^X(x) + \varepsilon_i^Y(y)}.$$

Here note that $c_1 c_2 = c$. The formula c_1 and c_2 in [1] seem to be different from ours.

3 Prehomogeneous geometric crystal

Definition 3.1. Let $\chi = (X, \{e_i^c\}, \{\gamma_i\}, \{\varepsilon_i\})$ be a geometric crystal. We say that χ is *prehomogeneous* if there exists a Zariski open dense subset $\Omega \subset X$ which is an orbit by the actions of the e_i^c's.

Lemma 3.2 ([3]). *Let $\chi_j = (X_j, \{e_i^c\}, \{\gamma_i\}, \{\varepsilon_i\})$ $(j = 1, 2)$ be prehomogeneous geometric crystals. Let $\Omega_1 \subset X_1$ be an open dense orbit in X_1. For isomorphisms of geometric crystals $\phi, \phi' \colon \chi_1 \to \chi_2$, suppose that there exists $p_1 \in \Omega_1$ such that $\phi(p_1) = \phi'(p_1) \in X_2$. Then, we have $\phi = \phi'$ as rational maps.*

Theorem 3.3 ([3]). *Let $\chi = (X, \{e_i^c\}, \{\gamma_i\}, \{\varepsilon_i\})$ be a finite-dimensional positive geometric crystal with the positive structure $\theta \colon (\mathbb{C}^\times)^{\dim(X)} \to X$ and $B := \mathcal{UD}_\theta(\chi)$ the crystal obtained as the ultra-discretization of χ. If B is a connected crystal, then χ is prehomogeneous.*

In [2, 3], we showed that ultra-discretization of the affine geometric crystal $\mathcal{V}(\mathfrak{g})_l$ $(l > 0)$ is a limit of perfect crystal $B_\infty(\mathfrak{g}^L)$, where $\mathfrak{g}^L$ is the Langlands dual of $\mathfrak{g}$. Since for any $k \in \mathbb{Z}_{>0}$ a tensor product $B_\infty(\mathfrak{g}^L)^{\otimes k}$ is connected by the perfectness of $B_\infty(\mathfrak{g}^L)$ and we have the isomorphism of crystals

$$\mathcal{UD}(\mathcal{V}(\mathfrak{g})_{L_1} \times \cdots \times \mathcal{V}(\mathfrak{g})_{L_k}) \cong B_\infty\big(\mathfrak{g}^L\big)^{\otimes k}, \qquad L_1, \ldots, L_k > 0,$$

by Theorem 3.3 we obtain the following:

Corollary 3.4 ([3]). *$\mathcal{V}(\mathfrak{g})_{L_1} \times \cdots \times \mathcal{V}(\mathfrak{g})_{L_k}$ is prehomogeneous.*

4 Tropical R maps

Definition 4.1. Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of geometric crystals with the product structures, where Λ is an index set. A birational map $\mathcal{R}_{\lambda\mu} : X_\lambda \times X_\mu \longrightarrow X_\mu \times X_\lambda$ is called a *tropical R map* if they satisfy:

$$\big(e_i^{X_\mu \times X_\lambda}\big)^c \circ \mathcal{R}_{\lambda\mu} = \mathcal{R}_{\lambda\mu} \circ \big(e_i^{X_\lambda \times X_\mu}\big)^c, \tag{4.1}$$
$$\varepsilon_i^{X_\lambda \times X_\mu} = \varepsilon_i^{X_\mu \times X_\lambda} \circ \mathcal{R}_{\lambda\mu}, \tag{4.2}$$
$$\gamma_i^{X_\lambda \times X_\mu} = \gamma_i^{X_\mu \times X_\lambda} \circ \mathcal{R}_{\lambda\mu}, \tag{4.3}$$
$$\mathcal{R}_{\lambda\mu}\mathcal{R}_{\mu\nu}\mathcal{R}_{\lambda\mu} = \mathcal{R}_{\mu\nu}\mathcal{R}_{\lambda\mu}\mathcal{R}_{\mu\nu} \tag{4.4}$$

for any $i \in I$ and $\lambda, \mu, \nu \in \Lambda$.

Tropical R maps for certain affine geometric crystals of type $A_n^{(1)}$, $B_n^{(1)}$, $D_n^{(1)}$, $D_{n+1}^{(2)}$, $A_{2n-1}^{(2)}$ and $A_{2n}^{(2)}$ are described explicitly [3, 4].

The following is immediate from Lemma 3.2 and Corollary 3.4.

Theorem 4.2 ([3]). *Let $\mathcal{R}, \mathcal{R}' : \mathcal{V}_L \times \mathcal{V}_M \to \mathcal{V}_M \times \mathcal{V}_L$ be tropical R maps. Suppose that there exists $p \in \mathcal{V}_L \times \mathcal{V}_M$ such that $\mathcal{R}(p) = \mathcal{R}'(p)$. Then we have $\mathcal{R} = \mathcal{R}'$ as birational maps.*

Let us introduce an example of a tropical R map of type $A_n^{(1)}$.

Example 4.3. Set $\mathcal{B}_L := \{l = (l_1, \ldots, l_{n+1}) \,|\, l_1 l_2 \cdots l_{n+1} = L\}$, which is equipped with an $A_n^{(1)}$-geometric crystal structure by:

$$e_i^c(l) = (\ldots, c\, l_i, c^{-1}\, l_{i+1}, \ldots), \qquad \gamma_i(l) = l_i/l_{i+1}, \qquad \varepsilon_i(l) = l_{i+1}, \qquad i = 0, 1, \ldots, n.$$

The tropical R map on $\{\mathcal{B}_L\}_{L\in\mathbb{R}_{>0}}$ is given by [4]:

$$\mathcal{R}:\ \mathcal{B}_L\times\mathcal{B}_M\to\mathcal{B}_M\times\mathcal{B}_L,\qquad \mathcal{R}(l,m)=(l',m'),\tag{4.5}$$
$$l'_i:=m_i\frac{P_i(l,m)}{P_{i-1}(l,m)},\qquad m'_i:=l_i\frac{P_{i-1}(l,m)}{P_i(l,m)},\qquad \text{where}\quad P_i(l,m):=\sum_{k=1}^{n+1}\prod_{j=k}^{n+1}l_{i+j}\prod_{j=1}^{k}m_{i+j}.$$

Remark. In the case $\mathfrak{g}=A_n^{(1)}$, we have $\mathcal{B}_L\cong\mathcal{V}_L$.

5 Epsilon systems

5.1 Definition of epsilon systems

Definition 5.1. For an interval $J:=\{s,s+1,\dots,t-1,t\}\subset I=\{1,2,\dots,n\}$ a set of intervals $P=\{I_1,\dots,I_k\}$ is called a *partition* of J if disjoint intervals $I_1,\dots,I_k$ satisfy $I_1\sqcup\cdots\sqcup I_k=J$ and $\max(I_j)<\min(I_{j+1})$ $(j=1,\dots,k-1)$.

For a partition $P=\{I_1,\dots,I_k\}$ of some interval J, set $l(P):=k$ and called the *length* of P. Let $\mathcal{J}$ be the set of all intervals in I. For an interval $J\in\mathcal{J}$, define

$$\mathcal{P}_J:=\{P\,|\,P \text{ is a partition of } J\}.$$

For a partition $P=\{I_1,\dots,I_k\}$ and symbols ε_{I_j} $(j=1,\dots,k)$, define

$$\varepsilon_P:=\varepsilon_{I_1}\cdot\varepsilon_{I_2}\cdots\varepsilon_{I_k}.$$

Definition 5.2. For an A_n-geometric crystal $\mathbb{X}=(X,\{e_i^c\},\{\gamma_i\},\{\varepsilon_i\})$, the set of rational functions on X, say $E=\{\varepsilon_J,\varepsilon_J^*|J\in\mathcal{J}$ is an interval $\}$, is called an *epsilon system* of $\mathbb{X}$ if they satisfy the following:

$$\varepsilon_J(e_i^cx)=\begin{cases}c^{-1}\varepsilon_J(x) & \text{if } i=s,\\ \varepsilon_J(x) & \text{if } i\neq s-1,s,t+1,\end{cases}$$
$$\varepsilon_J^*(e_i^cx)=\begin{cases}c^{-1}\varepsilon_J^*(x) & \text{if } i=t,\\ \varepsilon_J^*(x) & \text{if } i\neq s-1,t,t+1,\end{cases}\tag{5.1}$$
$$\varepsilon_J^*=\sum_{P\in\mathcal{P}_J}(-1)^{|J|-l(P)}\varepsilon_P,\qquad \text{for any interval } J=\{s,s+1,\dots,t\}\subset I,\tag{5.2}$$

and we set $\varepsilon_J:=\varepsilon_i$ for $J=\{i\}$, which is originally equipped with $\mathbb{X}$. Note that $\varepsilon_i^*=\varepsilon_i$. We call a geometric crystal with an epsilon system an *ε-geometric crystal.*

The actions of e_{s-1}^c and e_{t+1}^c will be described explicitly below, which are derived from (5.1) and (5.2).

Proposition 5.3. *The above definition is well-defined, that is, for any $J\in\mathcal{J}$ we have*

$$\varepsilon_J(e_i^{c_1}e_j^{c_2}x)=\varepsilon_J(e_j^{c_2}e_i^{c_1}x),\qquad \varepsilon_J^*(e_i^{c_1}e_j^{c_2}x)=\varepsilon_J^*(e_j^{c_2}e_i^{c_1}x)\qquad \textit{if}\quad a_{ij}=a_{ji}=0,\tag{5.3}$$
$$\varepsilon_J(e_i^{c_1}e_j^{c_1c_2}e_i^{c_2}x)=\varepsilon_J(e_j^{c_2}e_i^{c_1c_2}e_j^{c_1}x),$$
$$\varepsilon_J^*(e_i^{c_1}e_j^{c_1c_2}e_i^{c_2}x)=\varepsilon_J^*(e_j^{c_2}e_i^{c_1c_2}e_j^{c_1}x),\qquad \textit{if}\quad a_{ij}=a_{ji}=-1.\tag{5.4}$$

More precisely, we claim that if we calculate the both sides of the above equations by using (5.1) *and* (5.2)*, they coincide with each other.*

The proof will be given in the next subsection.

Example 5.4.

(i) $I = \{1, 2\}$: $E = \{\varepsilon_1, \varepsilon_2, \varepsilon_{12}, \varepsilon_{21} = \varepsilon^*_{12}\}$ with the relation $\varepsilon_{21} = \varepsilon_1\varepsilon_2 - \varepsilon_{12}$.

(ii) $I = \{1, 2, 3\}$: $E = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_{12}, \varepsilon^*_{12}, \varepsilon_{23}, \varepsilon^*_{23}, \varepsilon_{123}, \varepsilon^*_{123}\}$ with the relations for rank 2 and $\varepsilon^*_{123} = \varepsilon_{123} - \varepsilon_1\varepsilon_{23} - \varepsilon_{12}\varepsilon_3 + \varepsilon_1\varepsilon_2\varepsilon_3$.

The following lemma will be needed in the sequel. Set $[l, m] := \{l, l+1, \dots, m-1, m\}$ $(l \le m)$ and if $l > m$, $[l, m] := \varnothing$. We also put $\varepsilon_\varnothing(x) = 1$. Let $\{\varepsilon_J\}_{J\in\mathcal{J}}$ be an epsilon system on an ε-geometric crystal $\mathbb{X}$.

Lemma 5.5. *One of the following relations* (5.5) *and* (5.6) *is equivalent to* (5.2)*:*

$$\sum_{j=s-1}^{t} (-1)^j \varepsilon_{[s,j]}\varepsilon^*_{[j+1,t]} = 0, \tag{5.5}$$

$$\sum_{j=s-1}^{t} (-1)^j \varepsilon^*_{[s,j]}\varepsilon_{[j+1,t]} = 0, \tag{5.6}$$

for any $s, t \in I$ such that $s \le t$.

Proof. The proof is easily done by using the induction on $t - s$. ∎

The following describes the explicit action of e^c_i on epsilon systems which is not given in Definition 5.2.

Proposition 5.6. *We have the following formula for $s \le t$:*

$$\varepsilon_{[s,t]}(e^c_{t+1}(x)) = \varepsilon_{[s,t]}(x) + \frac{(c-1)\varepsilon_{[s,t+1]}(x)}{\varepsilon_{t+1}(x)}, \tag{5.7}$$

$$\varepsilon_{[s,t]}(e^c_{s-1}(x)) = c\varepsilon_{[s,t]}(x) + \frac{(1-c)\varepsilon_{[s-1,t]}(x)}{\varepsilon_{s-1}(x)}, \tag{5.8}$$

$$\varepsilon^*_{[s,t]}(e^c_{t+1}(x)) = c\varepsilon^*_{[s,t]}(x) + \frac{(1-c)\varepsilon^*_{[s,t+1]}(x)}{\varepsilon_{t+1}(x)}, \tag{5.9}$$

$$\varepsilon^*_{[s,t]}(e^c_{s-1}(x)) = \varepsilon^*_{[s,t]}(x) + \frac{(c-1)\varepsilon^*_{[s-1,t]}(x)}{\varepsilon_{s-1}(x)}. \tag{5.10}$$

Proof. Let us only show (5.7) since the others are shown similarly. It follows from (5.5) for $[s, t+1]$ that

$$\varepsilon_{[s,t]}(x) = \frac{1}{\varepsilon_{t+1}(x)}\left(\varepsilon_{[s,t+1]}(x) + \sum_{i=s-1}^{t-1}(-1)^{t-i-1}\varepsilon_{[s,i]}(x)\varepsilon^*_{[i+1,t+1]}(x)\right).$$

Then applying e^c_t to x, we get

$$\begin{aligned}
\varepsilon_{[s,t]}(e^c_{t+1}x) &= \frac{1}{c^{-1}\varepsilon_{t+1}(x)}\left(\varepsilon_{[s,t+1]}(x) + c^{-1}\sum_{i=s-1}^{t-1}(-1)^{t-i-1}\varepsilon_{[s,i]}(x)\varepsilon^*_{[i+1,t+1]}(x)\right)\\
&= \frac{1}{\varepsilon_{t+1}(x)}(c\varepsilon_{[s,t+1]}(x) + \varepsilon_{[s,t]}(x)\varepsilon_{t+1}(x) - \varepsilon_{[s,t+1]}(x))\\
&= \varepsilon_{[s,t]}(x) + \frac{(c-1)\varepsilon_{[s,t+1]}(x)}{\varepsilon_{t+1}(x)}.
\end{aligned}$$
∎

By this result, we know that all explicit forms of the action by e^c_i on epsilon systems.

5.2 The proof of Proposition 5.3

Let us prove Proposition 5.3. It is trivial to show (5.3). As for (5.4), the crucial cases are: for $J=[s,t]$

$$\varepsilon_J(e_{s-1}^{c_1}e_s^{c_1c_2}e_{s-1}^{c_2}x)=\varepsilon_J(e_s^{c_2}e_{s-1}^{c_1c_2}e_s^{c_1}x), \tag{5.11}$$
$$\varepsilon_J(e_t^{c_1}e_{t+1}^{c_1c_2}e_t^{c_2}x)=\varepsilon_J(e_{t+1}^{c_2}e_t^{c_1c_2}e_{t+1}^{c_1}x), \tag{5.12}$$
$$\varepsilon_J^*(e_{s-1}^{c_1}e_s^{c_1c_2}e_{s-1}^{c_2}x)=\varepsilon_J^*(e_s^{c_2}e_{s-1}^{c_1c_2}e_s^{c_1}x), \tag{5.13}$$
$$\varepsilon_J^*(e_t^{c_1}e_{t+1}^{c_1c_2}e_t^{c_2}x)=\varepsilon_J^*(e_{t+1}^{c_2}e_t^{c_1c_2}e_{t+1}^{c_1}x). \tag{5.14}$$

Using the results in Proposition 5.6, let us show (5.11)

$$\begin{aligned}
\varepsilon_{[s,t]}(e_{s-1}^{c_1}e_s^{c_1c_2}e_{s-1}^{c_2}x)&=c_1\varepsilon_{[s,t]}(e_s^{c_1c_2}e_{s-1}^{c_2}x)+\frac{(1-c_1)\varepsilon_{[s-1,t]}(e_s^{c_1c_2}e_{s-1}^{c_2}x)}{\varepsilon_{s-1}(e_s^{c_1c_2}e_{s-1}^{c_2}x)}\\
&=c_2^{-1}\left(c_2\varepsilon_{[s,t]}(x)+\frac{(1-c_2)\varepsilon_{[s-1,t]}(x)}{\varepsilon_{s-1}(x)}\right)+\frac{(1-c_1)\varepsilon_{[s-1,t]}(x)(\varepsilon_{[s-1,s]}(x)+c_2\varepsilon^*_{[s-1,s]}(x))}{c_2\varepsilon_{s-1}(x)(c_1\varepsilon_{[s-1,s]}(x)+\varepsilon^*_{[s-1,s]}(x))}\\
&=\varepsilon_{[s,t]}(x)+\frac{\varepsilon_{[s-1,t]}(x)}{c_2\varepsilon_{s-1}(x)}\left(1-c_2+\frac{(1-c_1)(\varepsilon_{[s-1,s]}(x)+c_2\varepsilon^*_{[s-1,s]}(x))}{c_1\varepsilon_{[s-1,s]}(x)+\varepsilon^*_{[s-1,s]}(x)}\right)\\
&=\varepsilon_{[s,t]}(x)+\frac{\varepsilon_{[s-1,t]}(x)}{c_2\varepsilon_{s-1}(x)}\frac{(1-c_1c_2)(\varepsilon_{[s-1,s]}(x)+\varepsilon^*_{[s-1,s]}(x))}{c_1\varepsilon_{[s-1,s]}(x)+\varepsilon^*_{[s-1,s]}(x))}\\
&=\varepsilon_{[s,t]}(x)+\frac{(1-c_1c_2)\varepsilon_{[s-1,t]}(x)\varepsilon_s(x)}{c_2(c_1\varepsilon_{[s-1,s]}(x)+\varepsilon^*_{[s-1,s]}(x))},
\end{aligned}$$

where the last equality is derived from $\varepsilon_{[s-1,s]}(x)+\varepsilon^*_{[s-1,s]}(x)=\varepsilon_{s-1}(x)\varepsilon_s(x)$. We also have

$$\begin{aligned}
\varepsilon_{[s,t]}(e_s^{c_2}e_{s-1}^{c_1c_2}e_s^{c_1}x)&=c_2^{-1}\varepsilon_{[s,t]}(e_{s-1}^{c_1c_2}e_s^{c_1}x)=c_2^{-1}\left(c_1c_2\varepsilon_{[s,t]}(e_s^{c_1}x)+\frac{(1-c_1c_2)\varepsilon_{[s-1,t]}(e_s^{c_1}x)}{\varepsilon_{s-1}(e_s^{c_1}x)}\right)\\
&=\varepsilon_{[s,t]}(x)+\frac{(1-c_1c_2)\varepsilon_{[s-1,t]}(x)\varepsilon_s(x)}{c_2(c_1\varepsilon_{[s-1,s]}(x)+\varepsilon^*_{[s-1,s]}(x))}.
\end{aligned}$$

Thus, we obtained (5.11). The others are also shown by direct calculations:

$$\begin{aligned}
\varepsilon_{[s,t]}(e_t^{c_1}e_{t+1}^{c_1c_2}e_t^{c_2}x)&=\varepsilon_{[s,t]}(x)+\frac{(c_1c_2-1)\varepsilon_{[s,t+1]}(x)\varepsilon_t(x)}{\varepsilon_{[t,t+1]}(x)+c_2\varepsilon^*_{[t,t+1]}(x)}=\varepsilon_{[s,t]}(e_{t+1}^{c_2}e_t^{c_1c_2}e_{t+1}^{c_1}x),\\
\varepsilon^*_{[s,t]}(e_{s-1}^{c_1}e_s^{c_1c_2}e_{s-1}^{c_2}x)&=\varepsilon^*_{[s,t]}(x)+\frac{(c_1c_2-1)\varepsilon^*_{[s-1,t]}(x)\varepsilon_s(x)}{c_1\varepsilon_{[s-1,s]}(x)+\varepsilon^*_{[s-1,s]}(x)}=\varepsilon^*_{[s,t]}(e_s^{c_2}e_{s-1}^{c_1c_2}e_s^{c_1}x),\\
\varepsilon^*_{[s,t]}(e_t^{c_1}e_{t+1}^{c_1c_2}e_t^{c_2}x)&=\varepsilon^*_{[s,t]}(x)+\frac{(1-c_1c_2)\varepsilon^*_{[s,t+1]}(x)\varepsilon_t(x)}{c_1(\varepsilon_{[t,t+1]}(x)+c_2\varepsilon^*_{[t,t+1]}(x))}=\varepsilon^*_{[s,t]}(e_{t+1}^{c_2}e_t^{c_1c_2}e_{t+1}^{c_1}x). \qquad \blacksquare
\end{aligned}$$

Let $\mathfrak{g}$ be a Kac–Moody Lie algebra associated with the index set I and $\mathfrak{g}_J$ be a subalgebra associated with a subset $J\subset I$. Let $\mathbf{X}=(X,\{\gamma_i\},\{\varepsilon_i\},\{e_i\})_{i\in I}$ be a $\mathfrak{g}$-geometric crystal. Then it has naturally a $\mathfrak{g}_J$-geometric crystal structure and denote it by $\mathbf{X}_J$.

Definition 5.7. In the above setting, if $\mathfrak{g}_J$ is isomorphic to the Lie algebra of type A_n for some n and the geometric crystal $\mathbf{X}_J$ has an epsilon system $E_{\mathbf{X}_J}$ of type A_n, then we call it a *local epsilon system* of type A_n associated with the index set J.

Remark. An ε-geometric crystal has naturally a local epsilon system associated with each sub-interval of I.

Example 5.8. In Example 5.4(ii), $\{\varepsilon_2,\varepsilon_3,\varepsilon_{23},\varepsilon^*_{23}\}\subset E$ is a local epsilon system of type A_2 associated with the interval $\{2,3\}$.

5.3 Product structures on epsilon systems

Theorem 5.9. *Let $\mathbb{X}$ and $\mathbb{Y}$ be ε-geometric crystals. Suppose that the product $\mathbb{X}\times\mathbb{Y}$ has a geometric crystal structure. Then $\mathbb{X}\times\mathbb{Y}$ turns out to be an ε-geometric crystal as follows: for ε-systems $E_{\mathbb{X}} := \{\varepsilon_J^X, \varepsilon_J^{X*}\}_{J\in\mathcal{J}}$ of $\mathbb{X}$ and $E_{\mathbb{Y}} := \{\varepsilon_J^Y, \varepsilon_J^{Y*}\}_{J\in\mathcal{J}}$ of $\mathbb{Y}$, set*

$$\varepsilon_{[s,t]}(x,y) := \sum_{k=s-1}^{t} \frac{\varepsilon^Y_{[s,k]}(y)\varepsilon^X_{[k+1,t]}(x)}{\prod_{j=s}^{k}\gamma_j^X(x)}, \tag{5.15}$$

$$\varepsilon^*_{[s,t]}(x,y) := \sum_{k=s-1}^{t} \frac{\varepsilon^{*X}_{[s,k]}(x)\varepsilon^{*Y}_{[k+1,t]}(y)}{\prod_{j=k+1}^{t}\gamma_j^X(x)}. \tag{5.16}$$

Then $E_{\mathbb{X}\times\mathbb{Y}} := \{\varepsilon_{[s,t]}(x,y), \varepsilon^*_{[s,t]}(x,y)\}_{[s,t]\in\mathcal{J}}$ defines an epsilon system of $\mathbb{X}\times\mathbb{Y}$.

Example 5.10. We have

$$\varepsilon_{123}(x,y) = \varepsilon_{123}(x) + \frac{\varepsilon_1(y)\varepsilon_{23}(x)}{\gamma_1(x)} + \frac{\varepsilon_{12}(y)\varepsilon_3(x)}{\gamma_1(x)\gamma_2(x)} + \frac{\varepsilon_{123}(y)}{\gamma_1(x)\gamma_2(x)\gamma_3(x)}.$$

Proof. First, we shall show (5.1). For $c\in\mathbb{C}^\times$ and $(x,y)\in X\times Y$, set $e_i^c(x,y) = (e_i^{c_1}x, e_i^{c_2}y)$ where

$$c_1 := \frac{c\varphi_i(x)+\varepsilon_i(y)}{\varphi_i(x)+\varepsilon_i(y)}, \qquad c_2 := \frac{c}{c_1}, \qquad \varphi_i(x) := \varepsilon_i(x)\gamma_i(x).$$

Let us see $\varepsilon_{[s,t]}(e_s^c(x,y)) = c^{-1}\varepsilon_{[s,t]}(x,y)$. Each summand $\frac{\varepsilon^Y_{[s,k]}(y)\varepsilon^X_{[k+1,t]}(x)}{\prod_{j=s}^{k}\gamma_j^X(x)}$ in (5.15) is changed by the action of e_s^c as follows (we omit the superscripts $\mathbb{X}$ and $\mathbb{Y}$):

$$\varepsilon_{[s,t]}(e_s^{c_1}x) = c_1^{-1}\varepsilon_{[s,t]}(x), \qquad k=s-1, \tag{5.17}$$

$$\frac{\varepsilon_s(e_s^{c_2}y)\varepsilon_{[s+1,t]}(e_s^{c_1}x)}{\gamma_s(e_s^{c_1}x)} = \frac{\varepsilon_s(y)}{c_1^2c_2\gamma_s(x)}\left(c_1\varepsilon_{[s+1,t]}(x) + \frac{(1-c_1)\varepsilon_{[s,t]}(x)}{\varepsilon_{s(x)}}\right), \qquad k=s, \tag{5.18}$$

$$\frac{\varepsilon_{[s,k]}(e_s^{c_2}y)\varepsilon_{[k+1,t]}(e_s^{c_1}x)}{\gamma_s(e_s^{c_1}x)\cdots\gamma_k(e_s^{c_1}x)} = \frac{\varepsilon_{[s,k]}(y)\varepsilon_{[k+1,t]}(x)}{c\gamma_s(x)\cdots\gamma_k(x)}, \qquad k>s,$$

where the second formula is obtained by (5.8). Now, taking the summation of (5.17) and (5.18), we have

$$\begin{aligned}
&\varepsilon_{[s,t]}(x)\left(c_1^{-1} + \frac{(1-c_1)\varepsilon_s(y)}{cc_1\varphi_s(x)}\right) + \frac{\varepsilon_s(y)\varepsilon_{[s+1,t]}(x)}{c\gamma_s(x)} \\
&\quad = c_1^{-1}\varepsilon_{[s,t]}(x)\frac{c\varphi_s(x)+\varepsilon_s(y)}{c(\varphi_s(x)+\varepsilon_s(y))} + \frac{\varepsilon_s(y)\varepsilon_{[s+1,t]}(x)}{c\gamma_s(x)} = c^{-1}\left(\varepsilon_{[s,t]}(x) + \frac{\varepsilon_s(y)\varepsilon_{[s+1,t]}(x)}{\gamma_s(x)}\right).
\end{aligned}$$

Thus we have $\varepsilon_{[s,t]}(e_s^c(x,y)) = c^{-1}\varepsilon_{[s,t]}(x)$. The others are obtained by the similar argument.

Next, let us show (5.2). Set the right hand-side of (5.15) (resp. (5.16)) $X_{[s,t]}(x,y)$ (resp. $X^*_{[s,t]}(x,y)$). By Lemma 5.5, it suffices to show that the relation (5.5) holds for $X_{[s,t]}(x,y)$ and $X^*_{[s,t]}(x,y)$

$$\sum_{j=s-1}^{t}(-1)^j X_{[s,j]}(x,y)X^*_{[j+1,t]}(x,y)$$

$$= \sum_{j=s-1}^{t}(-1)^j\left(\sum_{k=s-1}^{j}\frac{\varepsilon_{[s,k]}(y)\varepsilon_{[k+1,j]}(x)}{\prod\limits_{p=s}^{k}\gamma_p(x)}\right)\left(\sum_{m=j}^{t}\frac{\varepsilon^*_{[j+1,m]}(x)\varepsilon^*_{[m+1,t]}(y)}{\prod\limits_{q=m+1}^{t}\gamma_q(x)}\right)$$

$$= \sum_{s-1\le k\le m\le t}\frac{\varepsilon_{[s,k]}(y)}{\prod\limits_{p=s}^{k}\gamma_p(x)}\frac{\varepsilon^*_{[m+1,t]}(y)}{\prod\limits_{q=m+1}^{t}\gamma_q(x)}\left(\sum_{j=k}^{m}(-1)^j\varepsilon_{[k+1,j]}(x)\varepsilon^*_{[j+1,m]}(x)\right)$$

$$= \frac{1}{\prod\limits_{p=s}^{t}\gamma_p(x)}\sum_{s-1\le k\le t}(-1)^k\varepsilon_{[s,k]}(y)\varepsilon^*_{[k+1,t]}(y) = 0$$

since $\sum\limits_{j=k}^{m}(-1)^j\varepsilon_{[k+1,j]}(x)\varepsilon^*_{[j+1,m]}(x) = 0$ for $k < m$ and $\sum\limits_{s-1\le k\le t}(-1)^k\varepsilon_{[s,k]}(y)\varepsilon^*_{[k+1,t]}(y) = 0$ by (5.5). Hence, we showed that $\{X_{[s,t]}(x,y),\, X^*_{[s,t]}(x,y) \,|\, s,t\in I\ (s\le t)\}$ satisfy (5.5), which is equivalent to (5.2) by Lemma 5.5. ■

Example 5.11. $D_5^{(1)}$-case [3, 4]. Set the index set $I = \{0,1,2,3,4,5\}$. The geometric crystal $\mathcal{B}_L$ is defined as follows:

$$\mathcal{B}_L := \{l = (l_1,\dots,l_5,\bar{l}_4,\dots,\bar{l}_1)\in(\mathbb{C}^\times)^9 | l_1l_2\cdots\bar{l}_2\bar{l}_1 = L\},$$

$$\varepsilon_0(l) = l_1\left(\frac{l_2}{\bar{l}_2}+1\right),\qquad \varepsilon_i(l) = \bar{l}_i\left(\frac{l_{i+1}}{\bar{l}_{i+1}}+1\right)\quad(i=1,2,3),\qquad \varepsilon_4(l) = l_5\bar{l}_4,\qquad \varepsilon_5(l) = \bar{l}_4,$$

$$\gamma_0(l) = \frac{\bar{l}_1\bar{l}_2}{l_1l_2},\qquad \gamma_i(l) = \frac{l_i\bar{l}_{i+1}}{\bar{l}_il_{i+1}}\quad(i=1,2,3),\qquad \gamma_4(l) = \frac{l_4}{l_5\bar{l}_4},\qquad \gamma_5(l) = \frac{l_4l_5}{\bar{l}_4},$$

$$e_0^c(l) = (\xi_1^{-1}l_1, c^{-1}\xi_1l_2,\dots,\xi_1\bar{l}_2, c\xi_1^{-1}\bar{l}_1),$$

$$e_i^c(l) = (\dots, c\xi_i^{-1}l_i, c^{-1}\xi_il_{i+1},\dots,\xi_i\bar{l}_{i+1},\xi_i^{-1}\bar{l}_i,\dots)\quad(i=1,2,3),$$

$$e_4^c(l) = (\dots, c\cdot l_4, c^{-1}l_5,\dots),\qquad e_5^c(l) = (\dots, c\cdot l_5, c^{-1}\bar{l}_4,\dots).$$

where $\xi_i = \frac{l_{i+1}+c\bar{l}_{i+1}}{l_{i+1}+\bar{l}_{i+1}}$. There are several local epsilon systems of type A_4 in the $D_5^{(1)}$-geometric crystal $\mathcal{B}_L$ associated with e.g., $\{0,2,3,4\},\{0,2,3,5\}\subset I$. Then, we have

$$\varepsilon_{02}(l) = l_1l_2\left(\frac{l_3}{\bar{l}_3}+1\right),\qquad \varepsilon^*_{02}(l) = l_1\bar{l}_2\left(\frac{l_3}{\bar{l}_3}+1\right),\qquad \varepsilon_{i\,i+1}(l) = \bar{l}_il_{i+1}\left(\frac{l_{i+2}}{\bar{l}_{i+2}}+1\right),$$

$$\varepsilon^*_{i\,i+1}(l) = \bar{l}_i\bar{l}_{i+1}\left(\frac{l_{i+2}}{\bar{l}_{i+2}}+1\right)\quad(i=1,2),\qquad \varepsilon_{34}(l) = \bar{l}_3l_4l_5,$$

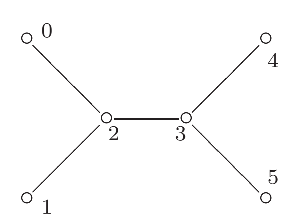
$$\varepsilon^*_{34}(l) = \bar{l}_3\bar{l}_4l_5, \varepsilon_{35}(l) = \bar{l}_3l_4,\qquad \varepsilon^*_{35}(l) = \bar{l}_3\bar{l}_4,$$

$$\varepsilon_{023}(l) = l_1l_2l_3\left(\frac{l_4}{\bar{l}_4}+1\right),\qquad \varepsilon^*_{023}(l) = l_1\bar{l}_2\bar{l}_3\left(\frac{l_4}{\bar{l}_4}+1\right),$$

$$\varepsilon_{234}(l) = \bar{l}_2l_3l_4l_5,\qquad \varepsilon^*_{234}(l) = \bar{l}_2\bar{l}_3\bar{l}_4l_5,$$

$$\varepsilon_{0234}(l) = l_1l_2l_3l_4l_5,\qquad \varepsilon^*_{0234}(l) = l_1\bar{l}_2\bar{l}_3\bar{l}_4l_5,$$

$$\varepsilon_{0235}(l) = l_1l_2l_3l_4,\qquad \varepsilon^*_{0235}(l) = l_1\bar{l}_2\bar{l}_3\bar{l}_4,\quad \dots\quad \text{etc.}$$



6 Epsilon systems on the Borel subgroup B^-

We shall show that there exists a canonical epsilon system on the (opposite) Borel subgroup B^-. Indeed, it is a prototype of general epsilon systems and derives an epsilon system to any A_n-geometric crystals induced from unipotent crystals.

In this section we shall identify B^- with the set of lower triangular matrices in $SL_{n+1}(\mathbb{C})$. Let $T := \{\mathrm{diag}(t_1, \dots, t_{n+1}) \in SL_{n+1}(\mathbb{C})\}$ be the maximal torus in B^-. For $x \in B^-$ there exist a unique unipotent matrix $x_- \in U^-$ and a unique diagonal matrix $x_0 \in T$ such that $x = x_- x_0$. Then the geometric crystal structure on B^- is described as follows: For $x \in B^-$, write $x_0 = \mathrm{diag}(t_1, \dots, t_{n+1})$ and (i, j)-entry of x_- as

$$(x_-)_{i,j} = \begin{cases} u_j & \text{if } i = j+1, \\ u_{j,i-1} & \text{if } i > j+1, \\ 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

For example, for $n = 2$-case,

$$x_- = \begin{pmatrix} 1 & 0 & 0 \\ u_1 & 1 & 0 \\ u_{12} & u_2 & 1 \end{pmatrix}, \qquad x_0 = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix}, \qquad x = \begin{pmatrix} t_1 & 0 & 0 \\ t_1 u_1 & t_2 & 0 \\ t_1 u_{12} & t_2 u_2 & t_3 \end{pmatrix}.$$

Now, the rational functions γ_i and ε_i are given by $\gamma_i(x) = t_i / t_{i+1}$ and $\varepsilon_i(x) = u_i$. The action e_i^c ($c \in \mathbb{C}^\times$) is given by

$$e_i^c(x) = x_i\left(\frac{c-1}{\varepsilon_i(x)}\right)(x) = x_i\left(\frac{c-1}{\varepsilon_i(x)}\right) \cdot x \cdot x_i\left(\frac{c^{-1}-1}{\varphi_i(x)}\right),$$

where $x_i(z) = \mathrm{Id}_{n+1} + z E_{i,i+1} \in B \subset SL_{n+1}(\mathbb{C})$.

Proposition 6.1. *In the above setting, the epsilon system on $B^- \subset SL_{n+1}(\mathbb{C})$ are given by:*

$$\varepsilon_{[s,t]}(x) = u_{s,t}, \qquad \varepsilon^*_{[s,t]}(x) = \det(M_{s,t}), \qquad s < t,$$

where $M_{s,t}$ is the minor of x_- as:

$$M_{s,t} := \begin{pmatrix} u_s & 1 & 0 & \cdots & 0 \\ u_{s,s+1} & u_{s+1} & 1 & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots \\ u_{s,t-1} & u_{s+1,t-1} & \cdots & \cdots & 1 \\ u_{s,t} & u_{s+1,t} & \cdots & u_{t-1,t} & u_t \end{pmatrix}.$$

Proof. By direct calculations, for $x \in B^-$ we have

$$(e_i^c(x))_- = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & & & \\ u_1 & 1 & 0 & \cdots & \cdots & & & \\ u_{12} & u_2 & 1 & 0 & \cdots & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ u_{1,i-1} + \frac{(c-1)u_{1,i}}{u_i} & \cdots & \cdots & u_{i-1} + \frac{(c-1)u_{i-1,i}}{u_i} & 1 & \cdots & \cdots & \cdots \\ u_{1,i} & u_{2,i} & \cdots & u_{i-1,i} & \frac{u_i}{c} & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \frac{u_{i,i+1}}{c} & c\left(u_{i+1} + \frac{(c^{-1}-1)u_{i,i+1}}{u_i}\right) & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ u_{1,n} & u_{2,n} & \cdots & \cdots & \frac{u_{i,n}}{c} & c\left(u_{i+1,n} + \frac{(c^{-1}-1)u_{i,n}}{u_i}\right) & u_n & 1 \end{pmatrix}.$$

The formula (5.1) in Definition 5.2 is shown by using this.

We can easily see

$$\det(M_{s,t}) = u_s \det(M_{s+1,t}) - u_{s,s+1} \det(M_{s+2,t}) + \cdots + (-1)^{t-s} u_{s,t}.$$

Thus, by the induction on $t - s$, we obtain (5.2). ∎

Next, we shall see an epsilon system on a geometric crystal induced from a unipotent crystal. Let (X, f) be a unipotent $SL_{n+1}(\mathbb{C})$ crystal, where $f : X \to B^-$ is a U-morphism. We assume that any rational function ε_i is not identically zero. By Theorem 2.5, we get the geometric crystal $\mathbb{X} = (X, \{e_i^c\}, \{\gamma_i\}, \{\varepsilon_i\})$.

Theorem 6.2. *The geometric crystal $\mathbb{X}$ as above is an ε-geometric crystal. Indeed, by setting*

$$\varepsilon^{\mathbb{X}}_{[s,t]}(x) := \varepsilon^{B^-}_{[s,t]}(f(x)), \tag{6.1}$$

$$\varepsilon^{*\,\mathbb{X}}_{[s,t]}(x) = \sum_{P\in\mathcal{P}_{[s,t]}} (-1)^{(t-s+1)-l(P)}\varepsilon^{\mathbb{X}}_{P}(x), \tag{6.2}$$

the set $E_{\mathbb{X}} := \{\varepsilon^{\mathbb{X}}_{[s,t]}(x), \varepsilon^{*\,\mathbb{X}}_{[s,t]}(x) | 1 \le s \le t \le n\}$ defines an epsilon system on $\mathbb{X}$.

Proof. The relation (5.2) is immediate from (6.2). Since $f(e_i^c(x)) = e_i^c(f(x))$ and $\varepsilon^{B^-}_{[s,t]}$ satisfies (5.1), we can easily obtain (5.1) for $\varepsilon^{\mathbb{X}}_{[s,t]}$. ■

7 Epsilon systems and tropical R maps

Let $\mathbb{X} = (X, \{e_i^c\}, \{\gamma_i\}, E_{\mathbb{X}})$ and $\mathbb{Y} = (Y, \{e_i^c\}, \{\gamma_i\}, E_{\mathbb{Y}})$ be ε-geometric crystals, where $E_{\mathbb{X}}$ (resp. $E_{\mathbb{Y}}$) is the epsilon system of $\mathbb{X}$ (resp. $\mathbb{Y}$). Suppose that there exists a tropical R-map

$$\mathcal{R} :\ X \times Y \to Y \times X.$$

Let $E_{\mathbb{X}\times\mathbb{Y}} = \{\varepsilon^{\mathbb{X}\times\mathbb{Y}}_J, \varepsilon^{*\,\mathbb{X}\times\mathbb{Y}}_J\}_{J\in\mathcal{J}}$ (resp. $E_{\mathbb{Y}\times\mathbb{X}} = \{\varepsilon^{\mathbb{Y}\times\mathbb{X}}_J, \varepsilon^{*\,\mathbb{Y}\times\mathbb{X}}_J\}_{J\in\mathcal{J}}$) be the epsilon system on $\mathbb{X}\times\mathbb{Y}$ (resp. $\mathbb{Y}\times\mathbb{X}$) obtained from $E_{\mathbb{X}}$ and $E_{\mathbb{Y}}$ by Theorem 5.9.

Theorem 7.1. *The epsilon systems $E_{\mathbb{X}\times\mathbb{Y}}$ and $E_{\mathbb{Y}\times\mathbb{X}}$ are invariant by the action of $\mathcal{R}$ in the following sense:*

$$\varepsilon^{\mathbb{Y}\times\mathbb{X}}_J(\mathcal{R}(x,y)) = \varepsilon^{\mathbb{X}\times\mathbb{Y}}_J(x,y), \qquad \varepsilon^{*\,\mathbb{Y}\times\mathbb{X}}_J(\mathcal{R}(x,y)) = \varepsilon^{*\,\mathbb{X}\times\mathbb{Y}}_J(x,y), \tag{7.1}$$

for any $J \in \mathcal{J}$.

Proof. Let us show the theorem by the induction on the length of intervals J, denoted $|J|$. Assume that we have (7.1) for $|J| < n$. The case $|J| = 1$ is obtained from (4.2) in Definition 4.1: the definition of tropical R maps. Now, we shall show

$$\varepsilon^{\mathbb{Y}\times\mathbb{X}}_I(\mathcal{R}(x,y)) = \varepsilon^{\mathbb{X}\times\mathbb{Y}}_I(x,y), \qquad \varepsilon^{*\,\mathbb{Y}\times\mathbb{X}}_I(\mathcal{R}(x,y)) = \varepsilon^{*\,\mathbb{X}\times\mathbb{Y}}_I(x,y).$$

By Theorem 5.9, we have

$$\varepsilon^{*\,\mathbb{X}\times\mathbb{Y}}_I(x,y) = \sum_{P\in\mathcal{P}_I} (-1)^{n-l(P)}\varepsilon^{\mathbb{X}\times\mathbb{Y}}_P(x,y). \tag{7.2}$$

In the above summation, if a partition $P \in \mathcal{P}_I$ is different from $I = \{1, 2, \dots, n\}$, then $\varepsilon^{\mathbb{X}\times\mathbb{Y}}_P(x,y)$ is invariant by the induction hypothesis. Thus, we have

$$\varepsilon^{*\,\mathbb{Y}\times\mathbb{X}}_I(\mathcal{R}(x,y)) = (-1)^{n-1}\varepsilon^{\mathbb{Y}\times\mathbb{X}}_I(\mathcal{R}(x,y)) + \sum_{P\in\mathcal{P}_I,\ P\ne I} (-1)^{n-l(P)}\varepsilon^{\mathbb{X}\times\mathbb{Y}}_P(x,y). \tag{7.3}$$

It follows from (7.2) and (7.3) that

$$\varepsilon^{*\,\mathbb{Y}\times\mathbb{X}}_I(\mathcal{R}(x,y)) - \varepsilon^{*\,\mathbb{X}\times\mathbb{Y}}_I(x,y) = (-1)^{n-1}(\varepsilon^{\mathbb{Y}\times\mathbb{X}}_I(\mathcal{R}(x,y)) - \varepsilon^{\mathbb{X}\times\mathbb{Y}}_I(x,y)). \tag{7.4}$$

Here, using (5.1) we have

$$\varepsilon_I^{*\,\mathbb{Y}\times\mathbb{X}}(\mathcal{R}e_1^c(x,y)) = \varepsilon_I^{*\,\mathbb{Y}\times\mathbb{X}}(e_1^c\mathcal{R}(x,y)) = \varepsilon_I^{*\,\mathbb{Y}\times\mathbb{X}}(\mathcal{R}(x,y)),$$
$$\varepsilon_I^{*\,\mathbb{X}\times\mathbb{Y}}(e_1^c(x,y)) = \varepsilon_I^{*\,\mathbb{X}\times\mathbb{Y}}(x,y),$$
$$\varepsilon_I^{\mathbb{Y}\times\mathbb{X}}(\mathcal{R}e_1^c(x,y)) = \varepsilon_I^{\mathbb{Y}\times\mathbb{X}}(e_1^c\mathcal{R}(x,y)) = c^{-1}\varepsilon_I^{\mathbb{Y}\times\mathbb{X}}(\mathcal{R}(x,y)),$$
$$\varepsilon_I^{\mathbb{X}\times\mathbb{Y}}(e_1^c(x,y)) = c^{-1}\varepsilon_I^{\mathbb{X}\times\mathbb{Y}}(x,y).$$

Applying these to (7.4), for any $c \in \mathbb{C}^\times$ one has

$$\varepsilon_I^{*\,\mathbb{Y}\times\mathbb{X}}(\mathcal{R}(x,y)) - \varepsilon_I^{*\,\mathbb{X}\times\mathbb{Y}}(x,y) = (-1)^{n-1}c^{-1}(\varepsilon_I^{\mathbb{Y}\times\mathbb{X}}(\mathcal{R}(x,y)) - \varepsilon_I^{\mathbb{X}\times\mathbb{Y}}(x,y)).$$

Hence, we obtain

$$\varepsilon_I^{*\,\mathbb{Y}\times\mathbb{X}}(\mathcal{R}(x,y)) = \varepsilon_I^{*\,\mathbb{X}\times\mathbb{Y}}(x,y), \qquad \varepsilon_I^{\mathbb{Y}\times\mathbb{X}}(\mathcal{R}(x,y)) = \varepsilon_I^{\mathbb{X}\times\mathbb{Y}}(x,y),$$

which completes the proof. ■

Observing this proof, it is easy to get the following:

Corollary 7.2. *Let $\mathbb{X}$ (resp. $\mathbb{Y}$) be an ε-geometric crystal with the epsilon system $E_{\mathbb{X}} = \{\varepsilon_J^{\mathbb{X}}, \varepsilon_J^{*\,\mathbb{X}}\}_{J\in\mathcal{J}}$ (resp. $E_{\mathbb{Y}} = \{\varepsilon_J^{\mathbb{Y}}, \varepsilon_J^{*\,\mathbb{Y}}\}_{J\in\mathcal{J}}$) and $F : \mathbb{X} \to \mathbb{Y}$ a homomorphism of geometric crystals, that is, F is a rational map commuting with the action of any e_i and preserving the functions ε_i and γ_i. Then we obtain for any $J \in \mathcal{J}$*

$$\varepsilon_J^{\mathbb{Y}}(F(x)) = \varepsilon_J^{\mathbb{X}}(x), \qquad \varepsilon_J^{*\,\mathbb{Y}}(F(x)) = \varepsilon_J^{*\,\mathbb{X}}(x).$$

7.1 Application-uniqueness of $A_n^{(1)}$-tropical R map

Let $\{\mathcal{B}_L\}_{L>0}$ be the family of $A_n^{(1)}$-geometric crystals as in Example 4.3. If we forget the index 0, $\mathcal{B}_L$ can be seen as an A_n-geometric crystal and is equipped with following local epsilon system of type A_n:

$$\varepsilon_i(l) = l_{i+1} = \varepsilon_i^*(l), \qquad \varepsilon^*_{[s,t]}(l) = 0, \qquad \varepsilon_{[s,t]}(l) = l_{s+1}l_{s+2}\cdots l_{t+1},$$

for $l = (l_1, \ldots, l_{n+1}) \in \mathcal{B}_L$.

Proposition 7.3. *Let $\mathcal{R} : \mathcal{B}_L \times \mathcal{B}_M \to \mathcal{B}_M \times \mathcal{B}_L$ be the tropical R map in Example 4.3. Then $\mathcal{R}$ is the unique tropical R maps from $\mathcal{B}_L \times \mathcal{B}_M$ to $\mathcal{B}_M \times \mathcal{B}_L$.*

Proof. For $\mathcal{B}_L \times \mathcal{B}_M$, by Theorem 5.9 we have

$$\varepsilon_i(l,m) = l_{i+1} + \frac{l_{i+1}m_{i+1}}{l_i}, \qquad i = 1, \ldots, n,$$
$$\varepsilon^*_{i,i+1}(l,m) = \varepsilon^*_{i,i+1}(l) + \frac{\varepsilon_{i+1}(m)\varepsilon_i(l)}{\gamma_{i+1}(l)} + \frac{\varepsilon^*_{i,i+1}(m)}{\gamma_i(l)\gamma_{i+1}(l)} = l_{i+2}m_{i+2}, \qquad i = 1, \ldots, n-1.$$

Set $\tilde{l} := L^{\frac{1}{n+1}}$, $\tilde{m} := M^{\frac{1}{n+1}}$, $l_0 := (\tilde{l}, \ldots, \tilde{l}) \in \mathcal{B}_L$ and $m_0 := (\tilde{m}, \ldots, \tilde{m}) \in \mathcal{B}_M$. Then it is immediate from the explicit form of $\mathcal{R}$ as in (4.5) that

$$\mathcal{R}(l_0, m_0) = (m_0, l_0).$$

Let $\mathcal{R}'$ be an arbitrary tropical R-map from $\mathcal{B}_L \times \mathcal{B}_M$ to $\mathcal{B}_M \times \mathcal{B}_L$. Set $(l', m') := \mathcal{R}'(l_0, m_0) \in \mathcal{B}_M \times \mathcal{B}_L$. By Theorem 7.1, for any interval J we have

$$\varepsilon_J(l', m') = \varepsilon_J(\mathcal{R}'(l_0, m_0)) = \varepsilon_J(l_0, m_0), \qquad \varepsilon^*_J(l', m') = \varepsilon^*_J(\mathcal{R}'(l_0, m_0)) = \varepsilon^*_J(l_0, m_0),$$

and $\gamma_i(l',m') = \gamma_i(l_0,m_0)$. These equations can be solved uniquely. Indeed, solving the system of equations $\varepsilon_i(l',m') = \varepsilon_i(l_0,m_0)$, $\gamma_i(l',m') = 1(= \gamma_i(l_0,m_0))$ $(i = 1,\dots,n)$ and $\varepsilon^*_{i,i+1}(l',m') = \varepsilon^*_{i,i+1}(l_0,m_0)$ $(i = 1,\dots,n-1)$, that is, for $l' \in \mathcal{B}_M$ and $m' \in \mathcal{B}_L$,

$$
\begin{aligned}
&l'_{i+1} + \frac{l'_{i+1}m'_{i+1}}{l'_i} = \tilde{l} + \tilde{m}, \qquad l'_i m'_i = l'_{i+1}m'_{i+1}, \qquad i = 1,\dots,n,\\
&l'_{i+2}m'_{i+2} = \tilde{l}\tilde{m}, \qquad i = 1,\dots,n-1,
\end{aligned}
$$

we obtain the unique solution $l'_i = \tilde{m}$ and $m'_i = \tilde{l}$ for any $i = 1,\dots,n+1$, which implies

$$(l',m') = (m_0,l_0),$$

and then $\mathcal{R}'(l_0,m_0) = \mathcal{R}(l_0,m_0)$. According to the remark in Section 4, we have $\mathcal{B}_L \cong \mathcal{V}_L$ and then by Corollary 3.4, $\mathcal{B}_L \times \mathcal{B}_M$ is prehomogeneous. Therefore, by Lemma 3.2 or Theorem 4.2 we have $\mathcal{R} = \mathcal{R}'$. ■

Remark. We expect that this method is applicable to the tropical R maps of other types [3]. But we do not have explicit answers.

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References

[1] Berenstein A., Kazhdan D., Geometric and unipotent crystals, GAFA 2000 (Tel Aviv, 1999), *Geom. Funct. Anal.* (2000), Special Volume, Part I, 188–236, math.QA/9912105.

[2] Kashiwara M., Nakashima T., Okado M., Affine geometric crystals and limit of perfect crystals, *Trans. Amer. Math. Soc.* **360** (2008), 3645–3686, math.QA/0512657.

[3] Kashiwara M., Nakashima T., Okado M., Tropical R maps and affine geometric crystals, *Represent. Theory*, to appear, arXiv:0808.2411.

[4] Kuniba A., Okado M., Takagi T., Yamada Y., Geometric crystals and tropical R for $D_n^{(1)}$, *Int. Math. Res. Not.* **2003** (2003), no. 48, 2565–2620, math.QA/0208239.

[5] Kac V.G., Infinite-dimensional Lie algebras, 3rd ed., Cambridge University Press, Cambridge, 1990.

[6] Kac V.G., Peterson D.H., Defining relations of certain infinite-dimensional groups, in Arithmetic and Geometry, Editors M. Artin and J. Tate, *Progress in Mathematics*, Vol. 36, Birkhäuser Boston, Boston, MA, 1983, 141–166.

[7] Peterson D.H., Kac V.G., Infinite flag varieties and conjugacy theorems, *Proc. Nat. Acad. Sci. USA* **80** (1983), 1778–1782.

[8] Kumar S., Kac–Moody groups, their flag varieties and representation theory, *Progress in Mathematics*, Vol. 204, Birkhäuser Boston, Boston, MA, 2002.

[9] Nakashima T., Geometric crystals on Schubert varieties, *J. Geom. Phys.* **53** (2005), 197–225, math.QA/0303087.

[10] Nakashima T., Geometric crystals on unipotent groups and generalized Young tableaux, *J. Algebra* **293** (2005), 65–88, math.QA/0403234.

[11] Nakashima T., Affine geometric crystal of type $G_2^{(1)}$, in Lie Algebras, Vertex Operator Algebras and Their Applications, *Contemp. Math.*, Vol. 442, Amer. Math. Soc., Providence, RI, 2007, 179–192, math.QA/0612858.

[12] Nakashima T., Universal tropical R map of $\mathfrak{sl}_2$ and prehomogeneous geometric crystals, in New Trends in Combinatorial Representation Theory, *RIMS Kôkyûroku Bessatsu*, B11, Res. Inst. Math. Sci. (RIMS), Kyoto, 2009, 101–116.