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On the Existence of a Codimension 1 Completely Integrable Totally Geodesic Distribution on a Pseudo-Riemannian Heisenberg Group

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Abstract. In this note we prove that the Heisenberg group with a left-invariant pseudo-Riemannian metric admits a completely integrable totally geodesic distribution of codimension 1. This is on the contrary to the Riemannian case, as it was proved by T. Hangan.

Key words: Heisenberg group; pseudo-Riemannian metrics; geodesics; codimension 1 distributions; completely integrable distributions

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1 Introduction

Given an n-dimensional C^∞ manifold M, a distribution D of codimension k of M is an $(n-k)$-dimensional subbundle of the tangent space TM. A distribution D of a pseudo-Riemannian manifold (M, g) is called *totally geodesic* if every geodesic tangent to D at some point remains everywhere tangent to D. Among 2-step nilpotent Lie groups with left-invariant metrics, the Heisenberg group is of particular significance. In [8], the second author proved that there are three nonisometric left-invariant Lorentzian metrics on the 3-dimensional Heisenberg group, one of which is flat. This is a strong contrast to the Riemannian case in which there is only one (up to positive homothety) and it is not flat.

Another major difference to the Riemannian case appears. Indeed, T. Hangan proved that the Heisenberg group endowed with a left-invariant Riemannian metric does not admit any codimension 1 completely integrable totally geodesic distribution (see [5] for the three-dimensional case and [6] for the higher dimensional case). Recently, in [9] the second author and N. Rahmani proved that the 3-dimensional Heisenberg group admits a left-invariant Lorentzian metric for which there exists a codimension 1 completely integrable totally geodesic distribution. It is an interesting problem to investigate whether and to what extent results valid in Riemannian Heisenberg group can be extended to the pseudo-Riemannian case. In fact, in a high-dimensional Heisenberg group there are different types of pseudo-Riemannian metrics while there is still essentially only one (up to positive homothety) Riemannian metric. So, the purpose of this paper is to prove, contrary to the Riemannian case, the existence of a codimension 1 completely integrable totally geodesic distribution on a pseudo-Riemannian Heisenberg group of dimension $2p+1$.

2 Geodesics and totally geodesic distributions on Heisenberg group

Consider $\mathbb{R}^{2p+1}$ with the elements of the form $X = (x_1, \dots, x_p, y_1, \dots, y_p, z)$. Define the product on $\mathbb{R}^{2p+1}$, by

$$X\tilde{X} = \left(x_1 + \tilde{x}_1, \dots, x_p + \tilde{x}_p, y_1 + \tilde{y}_1, \dots, y_p + \bar{y}_p, z + \tilde{z} - \sum_{i=1}^{p} x_i \tilde{y}_i\right),$$

where $X = (x_1, \dots, x_p, y_1, \dots, y_p, z)$, $\tilde{X} = (\tilde{x}_1, \dots, \tilde{x}_p, \tilde{y}_1, \dots, \tilde{y}_p, \tilde{z})$.

Let $H_{2p+1} = \left(\mathbb{R}^{2p+1}, g\right)$ be the $(2p+1)$-dimensional Heisenberg group with the left-invariant pseudo-Riemannian metric g defined by

$$g = -\sum_{i=1}^{p} (dx_i)^2 + \sum_{i=1}^{p} (dy_i)^2 + \left(dz + \sum_{i=1}^{p} x_i dy_i\right)^2. \tag{1}$$

With respect to the basis of coordinate vector fields $\left\{\partial_{x_i} = \frac{\partial}{\partial x_i}, \partial_{y_i} = \frac{\partial}{\partial y_i}, \partial_z = \frac{\partial}{\partial z}\right\}$, $i = 1, 2, \dots, p$ for which (1) holds, the nonvanishing metric components are

$$\begin{aligned} &g_{\partial_{x_i}\partial_{x_i}} = -1, \qquad g_{\partial_{y_i}\partial_{y_i}} = 1 + (x_i)^2, \qquad g_{\partial_{y_i}\partial_z} = x_i \qquad \text{for} \quad 1 \le i \le p, \\ &g_{\partial_z\partial_z} = 1, \qquad g_{\partial_{y_i}\partial_{y_j}} = x_i x_j \qquad \text{for} \quad 1 \le i \ne j \le p. \end{aligned} \tag{2}$$

The geodesics equations are obtained in [2] (for the Riemannian case) and in [1] and [4] (for the pseudo-Riemannian case), in more general context of two-step nilpotent Lie groups with a left-invariant metric. Jang and Parker [7] later gave explicit formulas for geodesics on the 3-dimensional Heisenberg group endowed with a left-invariant Lorentzian metric. A corresponding description was made in [5] in the Riemannian case.

Here we will find the geodesics equations of H_{2p+1} for the left-invariant pseudo-Riemannian metric (1) by integrating their Euler–Lagrange equations. Notice that Hangan obtained in [6] all geodesics of the Riemannian Heisenberg group of dimension $2p+1$.

So, let $\gamma(t)$ be a geodesic curve on H_{2p+1}, determined, with respect to the basis of coordinate vector fields $\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_p}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_p}, \frac{\partial}{\partial z}\right\}$, by its components $(x(t), y(t), z(t))$ where $x(t) = (x_1(t), \dots, x_p(t))$, $y(t) = (y_1(t), \dots, y_p(t))$ and $z(t) \in \mathbb{R}$. Using (2), the Lagrangian associated to the metric (1) is

$$L = \frac{1}{2}\left\{-\sum_{i=1}^{p} \left(x_i'\right)^2 + \sum_{i=1}^{p} \left(y_i'\right)^2 + \left[z' + \sum_{i=1}^{p} x_i y_i'\right]^2\right\},$$

with $x'(t) = (x_1'(t), \dots, x_p'(t))$ and $y'(t) = (y_1'(t), \dots, y_p'(t))$. Hence, the corresponding Euler–Lagrange equations are

$$\begin{aligned} &x'' + \left(z' + \sum_{i=1}^{p} x_i y_i'\right) y' = 0, \qquad y'' + \left(z' + \sum_{i=1}^{p} x_i y_i'\right) x' = 0, \\ &z'' + \sum_{i=1}^{p} \left(x_i' y_i' + x_i y_i''\right) = 0, \end{aligned} \tag{3}$$

and we will integrate the system (3) with the initial conditions

$$\begin{aligned} &x(0) = x_0, && y(0) = y_0, && z(0) = z_0, \\ &x'(0) = x'_0, && y'(0) = y'_0, && z'(0) = z'_0. \end{aligned}$$

The first integration of the last equation of the Euler–Lagrange equations, reduces the system (3) to

$$x'' + \alpha y' = 0, \qquad y'' + \alpha x' = 0, \qquad z' + \sum_{i=1}^{p} x_i y'_i = \alpha, \tag{4}$$

where $\alpha = z'_0 + \sum_{i=1}^{p} x_i(0) y'_i(0)$ is a real constant. For the system (4) we distinguish two cases.

If $\alpha = 0$, the geodesics curves are given by

$$\begin{aligned} &x(t) = x'_0 t + x_0, \qquad y(t) = y'_0 t + y_0, \\ &z(t) = z_0 - \sum_{i=1}^{p} \left(x'_i(0)\frac{t}{2} + x_i(0) \right) y'_i(0) t. \end{aligned} \tag{5}$$

If $\alpha \neq 0$, the geodesics curves are given by

$$\begin{aligned} x(t) &= \frac{1}{\alpha}\big(\sinh(\alpha t) x'_0 - \cosh(\alpha t) y'_0 + \alpha x_0 + y'_0 \big), \\ y(t) &= \frac{1}{\alpha}\big(-\cosh(\alpha t) x'_0 + \sinh(\alpha t) y'_0 + \alpha y_0 + x'_0 \big), \\ z(t) &= \frac{1}{\alpha^2}\sum_{i=1}^{p} \left(\alpha x_i(0) + y'_i(0)\right) \left(x'_i(0)\cosh(\alpha t) - y'_i(0)\sinh(\alpha t)\right) \\ &\quad - \frac{1}{2\alpha^2}\sum_{i=1}^{p} x'_i(0) y'_i(0)\cosh(2\alpha t) \\ &\quad + \frac{1}{4\alpha^2}\sum_{i=1}^{p} \left[(x'_i(0))^2 + (y'_i(0))^2\right]\sinh(2\alpha t) + \left(\alpha - \frac{1}{2\alpha}\sum_{i=1}^{p} \left[(x'_i(0))^2 - (y'_i(0))^2\right] \right) t \\ &\quad + z_0 - \frac{1}{2\alpha^2}\sum_{i=1}^{p} x'_i(0)\left(2\alpha x_i(0) + y'_i(0)\right). \end{aligned} \tag{6}$$

It must be noted that the pseudo-Riemannian metric g on a $(2p+1)$-dimensional Heisenberg group is geodesically complete. This is a general truth for pseudo-Riemannian 2-step nilpotent Lie groups. In fact, in [3] Guediri proved that every left-invariant pseudo-Riemannian metric on a 2-step nilpotent Lie groups is geodesically complete.

We now prove the following result

Theorem 1. *The pseudo-Riemannian metric* (1) *admits a codimension* 1 *completely integrable totally geodesic distribution on the Heisenberg group* H_{2p+1}.

Proof. Let γ be a geodesic curve on H_{2p+1} with $\gamma(0) = (x_0, y_0, z_0)$ and $\gamma'(0) = (x'_0, y'_0, z'_0)$. Consider in H_{2p+1} the 1-form $\theta = \sum_{i=1}^{p} (dx_i - dy_i)$.

If $z'_0 + \sum_{i=1}^{p} x_i(0) y'_i(0) = 0$ then γ is given by (5) and we have

$$x'(t) = x'_0, \qquad y'(t) = y'_0.$$

If $z_0' + \sum_{i=1}^{p} x_i(0)y_i'(0) \neq 0$ then γ is given by (6) and we have

$$x_i'(t) = \left(\cosh(\alpha t)x_i'(0) - \sinh(\alpha t)y_i'(0)\right), \qquad y_i'(t) = \left(-\sinh(\alpha t)x_i'(0) + \cosh(\alpha t)y_i'(0)\right),$$

for all $i \in \{1, 2, \dots, p\}$. From (5) and (6) one gets

$$\theta\left(\gamma'(t)\right) = \sum_{i=1}^{p}\left(x_i'(0) - y_i'(0)\right) \qquad \text{if} \quad z_0' + \sum_{i=1}^{p} x_i(0)y_i'(0) = 0,$$

$$\theta\left(\gamma'(t)\right) = [\cosh(\alpha t) + \sinh(\alpha t)]\left(\sum_{i=1}^{p}\left(x_i'(0) - y_i'(0)\right)\right) \qquad \text{if} \quad z_0' + \sum_{i=1}^{p} x_i(0)y_i'(0) \neq 0,$$

for all $t \in \mathbb{R}$. This means that, in both cases, if $\theta\left(\gamma'(0)\right) = 0$ then $\theta\left(\gamma'(t)\right) = 0$ for all $t \in \mathbb{R}$. In other words, the kernel of the given 1-form θ defines a codimension 1 totally geodesic distribution. Since θ is closed, the distribution $D = \ker\theta$ is involutive and from the Frobenius theorem it follows that it is completely integrable. ■

Notice that since θ is the differential of the function $f = \sum_{i=1}^{p}(x_i - y_i)$, the submanifolds given by $f = \text{const}$ are the maximal integral submanifolds of D.

Remark. Since the dimension of the center of a Heisenberg group is 1, it seems natural to pose the study of the existence of totally geodesic distributions on 2-step nilpotent Lie groups endowed with a pseudo-Riemannian metric with index equal to the dimension of the center of the Lie algebra.

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