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Bethe Ansatz, Inverse Scattering Transform and Tropical Riemann Theta Function in a Periodic Soliton Cellular Automaton for $A_n^{(1)\star}$

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Abstract. We study an integrable vertex model with a periodic boundary condition associated with $U_q\big(A_n^{(1)}\big)$ at the crystallizing point $q = 0$. It is an $(n + 1)$-state cellular automaton describing the factorized scattering of solitons. The dynamics originates in the commuting family of fusion transfer matrices and generalizes the ultradiscrete Toda/KP flow corresponding to the periodic box-ball system. Combining Bethe ansatz and crystal theory in quantum group, we develop an inverse scattering/spectral formalism and solve the initial value problem based on several conjectures. The action-angle variables are constructed representing the amplitudes and phases of solitons. By the direct and inverse scattering maps, separation of variables into solitons is achieved and nonlinear dynamics is transformed into a straight motion on a tropical analogue of the Jacobi variety. We decompose the level set into connected components under the commuting family of time evolutions and identify each of them with the set of integer points on a torus. The weight multiplicity formula derived from the $q = 0$ Bethe equation acquires an elegant interpretation as the volume of the phase space expressed by the size and multiplicity of these tori. The dynamical period is determined as an explicit arithmetical function of the n-tuple of Young diagrams specifying the level set. The inverse map, i.e., tropical Jacobi inversion is expressed in terms of a tropical Riemann theta function associated with the Bethe ansatz data. As an application, time average of some local variable is calculated.

Key words: soliton cellular automaton; crystal basis; combinatorial Bethe ansatz; inverse scattering/spectral method; tropical Riemann theta function

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Contents

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1 Introduction

1.1 Background

In [25], a class of soliton cellular automata (SCA) on a periodic one dimensional lattice was introduced associated with the non-exceptional quantum affine algebras $U_q(\mathfrak{g}_n)$ at $q = 0$. In this paper we focus on the type $\mathfrak{g}_n = A^{(1)}_n$. The case $n = 1$ was introduced earlier as the periodic box-ball system [41]. It is a periodic version of Takahashi–Satsuma's soliton cellular automaton [38] originally defined on the infinite lattice. In the pioneering work [40], a closed formula for the dynamical period of the periodic box-ball system was found for the states with a trivial internal symmetry.

After some while, the first complete solution of the initial value problem and its explicit formula by a tropical analogue of Riemann theta function were obtained in [24, 20, 21]. This was done by developing the inverse scattering/spectral method[1] [6] in a tropical (ultradiscrete) setting primarily based on the *quantum integrability* of the system. It provides a linearization scheme of the nonlinear dynamics into straight motions. The outcome is practical as well as conceptual. For instance practically, a closed formula of the exact dynamical period for

[1] Although it is more customary to use 'spectral' in periodic systems, we shall simply call it inverse scattering hereafter for short.

arbitrary states [24, Theorem 4.9] follows straightforwardly without a cumbersome combinatorial argument, see (3.40). Perhaps what is more important conceptually is, it manifested a decent tropical analogue of the theory of quasi-periodic soliton solutions [3, 4] and its quantum aspects in the realm of SCA. In fact, several essential tools in classical integrable systems have found their quantum counterparts at a combinatorial level. Here is a table of rough correspondence.

classical	quantum
solitons	Bethe strings
discrete KP/Toda flow	fusion transfer matrix
action-angle variable	extended rigged configuration mod ($q = 0$ Bethe eq.)
Abel–Jacobi map, Jacobi inversion	modified Kerov–Kirillov–Reshetikhin bijection
Riemann theta function	charge of extended rigged configuration

Here the Kerov–Kirillov–Reshetikhin (KKR) bijection [15, 16] is the celebrated algorithm on rigged configurations in the combinatorial Bethe ansatz explained in Appendix C. As for the last line, see (1.1).

In a more recent development [36], a complete decomposition of the phase space of the periodic box-ball system has been accomplished into connected components under the time evolutions, and each of them is identified with the set of integer points on a certain torus $\mathbb{Z}^g/F\mathbb{Z}^g$ explicitly.

1.2 A quick exposition

The aim of this paper is to extend all these results [24, 20, 21, 36] to a higher rank case based on several conjectures. The results will be demonstrated with a number of examples. We call the system the *periodic* $A^{(1)}_n$ *soliton cellular automaton* (*SCA*). It is a dynamical system on the L numbers $\{1, 2, \ldots, n+1\}^L$ which we call *paths*. There is the n-tuple of commuting series of time evolutions $T^{(1)}_l, \ldots, T^{(n)}_l$ with $l \geq 1$. The system is periodic in that the cyclic shift (2.3) is contained in the member as $T^{(1)}_1$. The casual exposition in the following example may be helpful to grasp the idea quickly.

Example 1.1. Take the path $p = 321113211222111223331111$ of length $L = 24$ with $n = 2$, which will further be treated in Examples 2.3, 3.2, 3.4 3.6 and 4.3.

	$\left(T^{(1)}_3\right)^t(p)$	$\left(T^{(2)}_4\right)^t(p)$
$t = 0:$	321113211222111223331111	321113211222111223331111
$t = 1:$	113211132111222112213331	132111321333111221112221
$t = 2:$	331132111321111221122213	112211132111222331113321
$t = 3:$	213311332113211112211122	213311113211322112211132
$t = 4:$	122233111332132111122111	221122111321133213311112
$t = 5:$	111122332111321332111221	331132111122111321122213
$t = 6:$	211111221332113211332112	112213221133111132132211
$t = 7:$	122211112211332132111331	113311222111221113213321
$t = 8:$	311122211122111321332113	211122333111321111221132
$t = 9:$	133111122211221113211332	221132111222132111331113

By regarding 1 as an empty background, these patterns show the repeated collisions of solitons like 22, 32, 33, 222, 322, 332 and 333. At each time step there are always 5 solitons with amplitudes 3, 3, 2, 2, 2. The dynamics is highly nonlinear. Ultimately in our approach, it will be transformed into the straight motion (Hint: right hand side of (3.15) and Remark 3.1):

$$
\left(T_l^{(1)}\right)^t \left(T_m^{(2)}\right)^s (p) \quad \overset{\Phi}{\longmapsto} \quad
\begin{array}{l}
\square\square\square \; 1+\min(l,3)t \\
\square\square\square \; 0+\min(l,3)t \\
\square\square \; 7+\min(l,2)t \\
\square\square \; 4+\min(l,2)t \\
\square\square \; 2+\min(l,2)t
\end{array}
\qquad
\begin{array}{l}
\square\square\square\square \; 11+\min(m,4)s \\
\square \; 1+\min(m,1)s
\end{array}
$$

Here Φ is the modified Kerov–Kirillov–Reshetikhin (KKR) bijection mentioned in the table. One notices that $T_3^{(1)} = T_\infty^{(1)}$ and $T_4^{(2)} = T_\infty^{(4)}$ in this example. (Of course $\min(m,1) = 1$.) Note also that the first Young diagram (33222) gives the list of amplitudes of solitons. The two Young diagrams are conserved quantities (action variable). A part of a Young diagram having equal width is called a *block*. There are 4 blocks containing 2, 3, 1, 1 rows in this example. The 7 dimensional vector consisting of the listed numbers is the angle variable flowing linearly. It should be understood as an element of (Hint: (4.17))

$$\mathbb{Z}^7/B\mathbb{Z}^7/(\mathfrak{S}_2 \times \mathfrak{S}_3 \times \mathfrak{S}_1 \times \mathfrak{S}_1),$$

where $\mathfrak{S}_m$ is the symmetric group of degree m. We have exhibited the trivial $\mathfrak{S}_1$'s as another hint. The matrix B is given by (4.11). The description is simplified further if one realizes that the relative (i.e., differences of) components within each block remain unchanged throughout. Thus picking the bottom component only from each block, we obtain

$$
\left(T_l^{(1)}\right)^t \left(T_m^{(2)}\right)^s (p) \overset{\Phi_\chi}{\longmapsto}
\begin{pmatrix} 0+\min(l,3)t \\ 2+\min(l,2)t \\ 11+\min(m,4)s \\ 1+\min(m,1)s \end{pmatrix}
\quad \mathrm{mod}\ F\mathbb{Z}^4, \qquad
F = \begin{pmatrix} 16 & 12 & -3 & -1 \\ 8 & 19 & -2 & -1 \\ -6 & -6 & 10 & 2 \\ -2 & -3 & 2 & 3 \end{pmatrix}.
$$

The 4 dimensional vector here is the reduced angle variable (Section 3.3). After all, the dynamics is just a straight motion in $\mathbb{Z}^4/F\mathbb{Z}^4$. The abelian group of the time evolutions acts transitively on this torus. Thus the size of the connected component of p is $\det F = 4656$. The dynamical period $\mathcal{N}_l^{(r)}$ of $T_l^{(r)}$, i.e., the smallest positive integer satisfying $(T_l^{(r)})^{\mathcal{N}_l^{(r)}}(p) = p$ is just the smallest positive integer satisfying

$$
\mathcal{N}_l^{(1)} \begin{pmatrix} \min(l,3) \\ \min(l,2) \\ 0 \\ 0 \end{pmatrix} \in F\mathbb{Z}^4, \qquad
\mathcal{N}_m^{(2)} \begin{pmatrix} 0 \\ 0 \\ \min(m,4) \\ \min(m,1) \end{pmatrix} \in F\mathbb{Z}^4,
$$

where the coefficient vectors are velocities. Solving the elementary exercise, one finds

$\mathcal{N}_1^{(1)}$	$\mathcal{N}_2^{(1)}$	$\mathcal{N}_\infty^{(1)}$	$\mathcal{N}_1^{(2)}$	$\mathcal{N}_2^{(2)}$	$\mathcal{N}_3^{(2)}$	$\mathcal{N}_\infty^{(2)}$
24	12	194	1164	776	582	2328

The level set characterized by the above pair of Young diagrams ((33222), (41)) consists of 139680 paths. It is decomposed into 36 connected components as (see Example 3.6)

$$24\left(\mathbb{Z}^4/F\mathbb{Z}^4\right) \sqcup 12\left(\mathbb{Z}^4/F'\mathbb{Z}^4\right), \qquad 139680 = 24(\det F) + 12(\det F'),$$

where $F' = F \cdot \mathrm{diag}(\frac{1}{2}, 1, 1, 1)$. Here the two kinds of tori arise reflecting that the two kinds of internal symmetries are allowed for the pair ((33222), (41)) under consideration.

1.3 Related works

A few remarks are in order on related works.

1. The solution of initial value problem of the $n = 1$ periodic box-ball system [24, 20, 21] has been reproduced partially by the procedure called 10-elimination [29]. By now the precise relation between the two approaches has been shown [17]. It is not known whether the 10-elimination admits a decent generalization. The KKR bijection on the other hand is a canonical algorithm allowing generalizations not only to type $A^{(1)}_n$ [18] but also beyond [32]. Conjecturally the approach based on the KKR type bijection will work universally for the periodic SCA [25] associated with Kirillov–Reshetikhin crystals as exemplified for $A^{(1)}_1$ higher spin case [22] and $A^{(1)}_n$ [25, 26, this paper]. In fact, the essential results like torus, dynamical period and phase space volume formula are all presented in a universal form by the data from Bethe ansatz and the root system[2]. From a practical point of view, it should also be recognized that the KKR map $\phi^{\pm 1}$ is a delightfully elementary algorithm described in less than half a page in our setting in Appendix C. It is a useful exercise to program it to follow examples in this paper.

2. When there is no duplication of the amplitudes of solitons, the tropical analogue of the Jacobian $\mathcal{J}(\mu)$ obtained in [24] has an interpretation from the tropical geometry point of view [9, 30]. In this paper we are naturally led to a higher rank version of $\mathcal{J}(\mu)$ and the relevant tropical analogue of the Riemann theta function. We will call them 'tropical ...' rather casually without identifying an underlying tropical geometric objects hoping not to cause a too much embarrassment.

3. In [28], the dynamical system on $B^{1,l_1} \otimes \cdots \otimes B^{1,l_L}$ equipped with the unique time evolution $T^{(1)}_\infty$ of the form (2.16) was studied under the name of the generalized periodic box-ball system. See Sections 2.1 and 2.2 for the notations $B^{r,l}$ and $T^{(r)}_l$. Approaches by ultradiscretization of the periodic Toda lattice also capture the $T^{(1)}_\infty$ but conceivably $\{T^{(1)}_l \mid l \geq 1\}$ at most. Our periodic $A^{(1)}_n$ SCA is the generalization of the generalized periodic box-ball system with $\forall\, l_i = 1$ that is furnished with the commuting family of time evolutions $\{T^{(r)}_l \,|\, 1 \leq r \leq n, l \geq 1\}$. As we will see in the rest of the paper, it is crucial to consider this wider variety of dynamics and their whole joint spectrum $\{E^{(r)}_l\}$. They turn out to be the necessary and sufficient conserved quantities to formulate the inverse scattering method. Such a usage of the full family $\{T^{(r)}_l\}$ was firstly proposed in the periodic setting in [25, 26]. In particular in the latter reference, the most general periodic $A^{(1)}_n$ SCA on $B^{r_1,l_1} \otimes \cdots \otimes B^{r_1,l_L}$ endowed with the dynamics $\{T^{(r)}_l\}$ was investigated, and the dynamical period and a phase space volume formula were conjectured using some heuristic connection with combinatorial Bethe ansatz. This paper concerns the basic case $\forall\, r_k = \forall\, l_k = 1$ only but goes deeper to explore the linearization scheme under which the earlier conjectures [25, 26] get refined and become simple corollaries. We expect that essential features in the inverse scattering formalism are not too much influenced by the choice of $\{r_k, l_k\}$.

1.4 Contents of paper

Let us digest the contents of the paper along Sections 2–4.

In Section 2, we formulate the periodic $A^{(1)}_n$ SCA. It is a solvable vertex model [1] associated with the quantum affine algebra $U_q(A^{(1)}_n)$ at $q = 0$ in the sense that notions concerning $U_q(A^{(1)}_n)$ are replaced by those from the crystal theory [14, 12, 13], a theory of quantum group at $q = 0$. The correspondence is shown between the first and the second columns of the following table, where we hope the third column may be more friendly to the readers not necessarily familiar with the crystal theory.

[2] It is a Bethe ansatz folklore that "the solution exists before the model is constructed" according to V.V. Bazhanov.

$U_q(A_n^{(1)})$ vertex model	SCA ($q=0$)	combinatorial description
Kirillov–Reshetikhin module $W_l^{(r)}$	crystal $B^{r,l}$	shape $r \times l$ semistandard tableaux
quantum R	combinatorial R	rule (2.1) on Schensted insertions
fusion transfer matrix	time evolution $T_l^{(r)}$	diagram (2.2) with $v = v' \in B^{r,l}$
conserved quantity	energy $E_l^{(r)}$	n-tuple of Young diagrams μ

The definition of the Kirillov–Reshetikhin module was firstly given in [19, Definition 1.1], although we do not use it in this paper. As mentioned previously, there is the n-tuple of commuting series of time evolutions $T_l^{(1)}, \dots, T_l^{(n)}$ with $l \geq 1$. The first series $T_l^{(1)}$ is the ultradiscrete Toda/KP flow [7, 39], and especially its top $T_\infty^{(1)}$ admits a simple description by a ball-moving algorithm like the periodic box-ball system. See Theorem 2.1 and comments following it. However, as again said previously, what is more essential in our approach is to make use of the entire family $\{T_l^{(r)}\}$ and the associated conserved quantities called energy $\{E_l^{(r)}\}$. It is the totality of the joint spectrum $\{E_l^{(r)}\}$ that makes it possible to characterize the level set $\mathcal{P}(\mu)$ (2.11) by the n-tuple of Young diagrams and the whole development thereafter.

As an illustration, $T_2^{(2)}(241123431) = 423141213$ is calculated as

$$\begin{array}{ccccccccccccccccccc}
 & 2 & & 4 & & 1 & & 1 & & 2 & & 3 & & 4 & & 3 & & 1 & \\
\begin{smallmatrix}1&2\\3&4\end{smallmatrix} & + & \begin{smallmatrix}1&2\\2&3\end{smallmatrix} & + & \begin{smallmatrix}1&3\\2&4\end{smallmatrix} & + & \begin{smallmatrix}1&1\\2&4\end{smallmatrix} & + & \begin{smallmatrix}1&1\\2&4\end{smallmatrix} & + & \begin{smallmatrix}1&1\\2&2\end{smallmatrix} & + & \begin{smallmatrix}1&2\\2&3\end{smallmatrix} & + & \begin{smallmatrix}1&3\\2&4\end{smallmatrix} & + & \begin{smallmatrix}2&3\\3&4\end{smallmatrix} & + & \begin{smallmatrix}1&2\\3&4\end{smallmatrix} \\
 & 4 & & 2 & & 3 & & 1 & & 4 & & 1 & & 2 & & 1 & & 3 &
\end{array}$$

by using the 'hidden variable' called carrier on horizontal edges belonging to $B^{2,2}$. This is a conventional diagram representing the row transfer matrix of a vertex model [1] whose auxiliary (horizontal) space has the 'fusion type' $B^{2,2}$. The general case $T_l^{(r)}$ is similarly defined by using $B^{r,l}$. The peculiarity as a vertex model is that there is no thermal fluctuation due to the crystallizing choice $q = 0$ resulting in a deterministic dynamics. Note that the carrier has been chosen specially so that the leftmost and the rightmost ones coincide to match the periodic boundary condition. This non-local postulate makes the well-definedness of the dynamics highly nontrivial and is in fact a source of the most intriguing features of our SCA. It is an important problem to characterize the situation in which all the time evolutions act stably. With regard to this, we propose a neat sufficient condition in (2.14) under which we will mainly work. See Conjecture 2.1. Our most general claim is an elaborate one in Conjecture 3.4. In Section 2.3, we explain the rigged configurations and Kerov–Kirillov–Reshetikhin bijection which are essential tools from the combinatorial Bethe ansatz [15, 16] at $q = 1$. Based on Theorem 2.2, we identify the solitons and strings in (2.19).

In Section 3, we present the inverse scattering formalism, the solution algorithm of the initial value problem together with applications to the volume formula of the phase space and the dynamical period. The content is based on a couple of conjectures whose status is summarized in Section 3.8. Our strategy is a synthesis of the combinatorial Bethe ansätze at $q = 1$ [15, 16] and $q = 0$ [19], or in other words, a modification of the KKR theory to match the periodic boundary condition. The action-angle variables are constructed from the rigged configurations invented at $q = 1$ by a quasi-periodic extension (3.1) followed by an identification compatible with the $q = 0$ Bethe equation (3.42). More concretely, the action variable is the n-tuple of Young diagrams $\mu = (\mu^{(1)}, \dots, \mu^{(n)})$ preserved under time evolutions. The angle variables live in a tropical analogue of the Jacobi-variety $\mathcal{J}(\mu)$ and undergo a straight motion with the velocity corresponding to a given time evolution. Roughly speaking, the action and angle variables represent the amplitudes and the phases of solitons, respectively as demonstrated in Example 1.1. The modified KKR bijection Φ, Φ^{-1} yield the direct and inverse scattering maps. Schematically these aspects are summarized in the commutative diagram (Conjecture 3.2):

$$\begin{tikzcd} \mathcal{P}(\mu) \arrow[r, "\Phi"] \arrow[d, "\mathcal{T}"'] & \mathcal{J}(\mu) \arrow[d, "\mathcal{T}"] \\ \mathcal{P}(\mu) \arrow[r, "\Phi"] & \mathcal{J}(\mu) \end{tikzcd}$$

where $\mathcal{T}$ stands for the commuting family of time evolutions $\{T^{(r)}_l\}$. Since its action on $\mathcal{J}(\mu)$ is linear, this diagram achieves the solution of the initial value problem conceptually. Practical calculations can be found in Examples 3.2 and 3.5.

In Section 3.3 we decompose the level set $\mathcal{P}(\mu)$ further into connected components, i.e., $\mathcal{T}$-orbits, and identify each of them with $\mathbb{Z}^g/F_{\boldsymbol{\gamma}}\mathbb{Z}^g$, the set of integer points on a torus with g and $F_{\boldsymbol{\gamma}}$ explicitly specified in (2.9) and (3.25). Here $\boldsymbol{\gamma}$ denotes the order of symmetry in the angle variables. As a result we obtain (Theorem 3.1)

$$|\mathcal{P}(\mu)| = \sum_{\boldsymbol{\gamma}} \underbrace{\det F_{\boldsymbol{\gamma}}}_{\text{size of a } \mathcal{T}\text{-orbit}} \underbrace{\prod_{(ai)\in \overline{H}} \frac{|\Lambda_{\gamma^{(a)}_i}(m^{(a)}_i, p^{(a)}_i)|}{m^{(a)}_i/\gamma^{(a)}_i}}_{\text{number of } \mathcal{T}\text{-orbits}} = (\det F) \prod_{(ai)\in \overline{H}} \frac{1}{m^{(a)}_i} \binom{p^{(a)}_i + m^{(a)}_i - 1}{m^{(a)}_i - 1}.$$

The first equality follows from the decomposition into tori and the second one is a slight calculation. The last expression was known as the number of Bethe roots at $q = 0$ [19]. Thus this identity offers the Bethe ansatz formula a most elegant interpretation by the structure of the phase space of the periodic SCA.

Once the linearization scheme is formulated, it is straightforward to determine the dynamical period, the smallest positive integer $\mathcal{N}$ satisfying $T^{\mathcal{N}}(p) = p$ for any time evolution $T \in \mathcal{T}$ and path $p \in \mathcal{P}(\mu)$. The result is given in Theorem 3.2 and Remark 3.4. We emphasize that (3.39) is a closed formula that gives the *exact* (not a multiple of) dynamical period even when there are more than one solitons with equal amplitudes and their order of symmetry $\boldsymbol{\gamma}$ is nontrivial. See Example 3.7 in this paper for $n = 2$ and also [24, Example 4.10] for $n = 1$. In Section 3.6, we explain the precise relation of our linearization scheme to the Bethe ansatz at $q = 0$ [19]. The angle variables are actually in one to one correspondence with what we call the Bethe root at $q = 0$. These results are natural generalizations of the $n = 1$ case proved in [24, 36]. In Section 3.7, we treat the general case (3.46) relaxing the condition (2.14). It turns out that the linearization scheme remains the same provided one discards some time evolutions and restricts the dynamics to a subgroup $\mathcal{T}'$ of $\mathcal{T}$. This uncovers a new feature at $n > 1$.

In Section 4, we derive an explicit formula for the path $p \in \mathcal{P}(\mu)$ that corresponds to a given action-angle variable (Theorem 4.3). This is a tropical analogue of the Jacobi inversion problem, and the result is indeed expressed by a tropical analogue of the Riemann theta function (4.14):

$$\Theta(\mathbf{z}) = -\min_{\mathbf{n}\in\mathbb{Z}^G} \left\{\tfrac{1}{2}\,{}^t\mathbf{n}B\mathbf{n} + {}^t\mathbf{z}\mathbf{n}\right\}$$

having the quasi-periodicity $\Theta(\mathbf{z}+\mathbf{v}) = \Theta(\mathbf{z}) + {}^t\mathbf{v}B^{-1}\big(\mathbf{z} + \frac{\mathbf{v}}{2}\big)$ for $\mathbf{v} \in B\mathbb{Z}^G$. Here the $G \times G$ period matrix B is specified by (4.6) from the n-tuple of Young diagrams μ as in (2.5)–(2.9). Theorems 4.2 and 4.3 are derived from the explicit formula of the KKR map by the tropical tau function τ_{trop} for $A^{(1)}_n$ [23]. It is a piecewise linear function on a rigged configuration related to its charge. The key to the derivation is the identity

$$\Theta = \lim_{\mathrm{RC}\to\infty} (\tau_{\mathrm{trop}}(\mathrm{RC}) - \text{divergent part}) \tag{1.1}$$

first discovered in [20] for $n = 1$. Here the limit sends the rigged configuration RC into the infinitely large rigged configuration obtained by the quasi-periodic extension. See (4.13) for the precise form. It is known [23] that $\tau_{\rm trop}$ is indeed the tropical analogue (ultradiscrete limit) of a tau function in the KP hierarchy [11]. Thus our result can be viewed as a *fermionization* of the Bethe ansatz and quasi-periodic solitons at a combinatorial level. As applications, joint eigenvectors of $\{T^{(r)}_l\}$ are constructed that possess every aspect as the Bethe eigenvectors at $q = 0$ (Section 4.2), the dynamical period is linked with the $q = 0$ Bethe eigenvalue (Section 4.3), and miscellaneous calculations of some time average are presented (Section 4.4).

The main text is followed by 4 appendices. Appendix A recalls the row and column insertions following [5] which is necessary to understand the rule (2.1) that governs the local dynamics. Appendix B contains a proof of Theorem 2.1 based on crystal theory. Appendix C is a quickest and self-contained exposition of the algorithm for the KKR bijection. Appendix D is a proof of Theorem 2.2 using a result by Sakamoto [33] on energy of paths.

2 Periodic $A^{(1)}_n$ soliton cellular automaton

2.1 Crystal $B^{r,l}$ and combinatorial R

Let n be a positive integer. For any pair of integers r, l satisfying $1 \le r \le n$ and $l \ge 1$, we denote by $B^{r,l}$ the set of all $r \times l$ rectangular semistandard tableaux with entries chosen from the set $\{1, 2, \dots, n+1\}$. For example when $n = 2$, one has $B^{1,1} = \{1, 2, 3\}$, $B^{1,2} = \{11, 12, 13, 22, 23, 33\}$ and $B^{2,2} = \left\{ {11 \atop 22}, {11 \atop 23}, {11 \atop 33}, {12 \atop 23}, {12 \atop 33}, {22 \atop 33} \right\}$. The set $B^{r,l}$ is equipped with the structure of crystal [13, 34] for the Kirillov–Reshetikhin module of the quantum affine algebra $U_q\big(A^{(1)}_n\big)$.

There is a canonical bijection, the isomorphism of crystals, between $B^{r,l} \otimes B^{1,1}$ and $B^{1,1} \otimes B^{r,l}$ called *combinatorial R*. It is denoted either by $R(b \otimes c) = c' \otimes b'$ or $R(c' \otimes b') = b \otimes c$, or simply $b \otimes c \simeq c' \otimes b'$, where $b \otimes c \in B^{r,l} \otimes B^{1,1}$ and $c' \otimes b' \in B^{1,1} \otimes B^{r,l}$. R is uniquely determined by the condition that the product tableaux $c \cdot b$ and $b' \cdot c'$ coincide [34]. Here, the product $c \cdot b$ for example signifies the column insertion of c into b, which is also obtained by the row insertion of b into c [5]. Since c and c' are single numbers in our case, it is simplest to demand the equality

$$(c \to b) = (b' \leftarrow c') \tag{2.1}$$

to find the image $b \otimes c \simeq c' \otimes b'$. See Appendix A for the definitions of the row and column insertions. The insertion procedure also determines an integer $H(b \otimes c) = H(c' \otimes b')$ called *local energy*. We specify it as $H = 0$ or 1 according as the shape of the common product tableau is $((l + 1), l^{r-1})$ or $(l^r, 1)$, respectively[3]. Note that R and H refer to the pair (r, l) although we suppress the dependence on it in the notation.

Example 2.1. Let $r = 2$, $l = 3$.

$$R\left(\begin{array}{l}112\\223\end{array} \otimes 1\right) = 2 \otimes \begin{array}{l}111\\223\end{array}, \qquad \text{product tableau} = \begin{array}{l}1112\\223\end{array}, \qquad H = 0,$$

$$R\left(\begin{array}{l}112\\223\end{array} \otimes 3\right) = 1 \otimes \begin{array}{l}122\\233\end{array}, \qquad \text{product tableau} = \begin{array}{l}112\\223\\3\end{array}, \qquad H = 1.$$

We depict the relation $R(b \otimes c) = c' \otimes b'$ (2.1) and $H(b \otimes c) = e$ as

$$\begin{array}{ccc} & c & \\ b & \overset{e}{-\!\!\!-\!\!\!\!+\!\!\!\!-\!\!\!-} & b' \\ & c' & \end{array}$$

[3]Up to a constant shift, this is the original definition of local energy [12] times (-1).

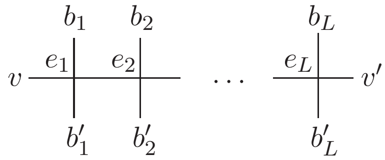
We will often suppress e. The horizontal and vertical lines here carry an element from $B^{r,l}$ and $B^{1,1}$, respectively. We remark that R is trivial, namely $R(b \otimes c) = b \otimes c$, for $(r,l) = (1,1)$ by the definition.

2.2 Definition of dynamics

Fix a positive integer L and set $B = (B^{1,1})^{\otimes L}$. An element of B is called a *path*. A path $b_1 \otimes \cdots \otimes b_L$ will often be written simply as a word $b_1 b_2 \ldots b_L$. Our periodic $A^{(1)}_n$ soliton cellular automaton (SCA) is a dynamical system on a subset of B. To define the time evolution $T^{(r)}_l$ associated with $B^{r,l}$, we consider a bijective map $B^{r,l} \otimes B \to B \otimes B^{r,l}$ and the local energy obtained by repeated use of R. Schematically, the map

$$\begin{array}{ccc} B^{r,l} \otimes B & \rightarrow & B \otimes B^{r,l} \\ v \otimes (b_1 \otimes \cdots \otimes b_L) & \mapsto & (b'_1 \otimes \cdots \otimes b'_L) \otimes v' \end{array}$$

and the local energy $e_1, \ldots, e_L$ are defined by the composition of the previous diagram as follows.



(2.2)

The element v or v' is called a *carrier*. Set $p' = b'_1 \otimes \cdots \otimes b'_L$. Given a path $p = b_1 \otimes \cdots \otimes b_L \in B$, we regard $v' = v'(v;p), e_k = e_k(v;p)$ and $p' = p'(v;p)$ as the functions of v containing p as a 'parameter'. Naively, we wish to define the time evolution $T^{(r)}_l$ of the path p as $T^{(r)}_l(p) = p'$ by using a carrier v that satisfies the periodic boundary condition $v = v'(v;p)$. This idea indeed works without a difficulty for $n = 1$ [24][4]. In addition, $T^{(1)}_1$ is always well defined for general n, yielding the cyclic shift:

$$T^{(1)}_1(b_1 \otimes \cdots \otimes b_{L-1} \otimes b_L) = b_L \otimes b_1 \otimes \cdots \otimes b_{L-1}. \tag{2.3}$$

This is due to the triviality of R on $B^{1,1} \otimes B^{1,1}$ mentioned above. In fact the unique carrier is specified as $v = v' = b_L$. Apart from this however, one encounters the three problems (i)–(iii) to overcome in general for $n \geq 2$. For some $p \in B$, one may suffer from

- (i) Non-existence. There may be no carrier v satisfying $v = v'(v;p)$.
- (ii) Non-uniqueness. There may be more than one carriers, say $v_1, v_2, \ldots, v_m$ such that $v_j = v'(v_j;p)$ but $p'(v_i;p) \neq p'(v_j;p)$ or $e_k(v_i;p) \neq e_k(v_j;p)$ for some i, j, k.

To cope with these problems, we introduce the notion of evolvability of a path according to [26]. A path $p \in B$ is said $T^{(r)}_l$-*evolvable* if there exist $v \in B^{r,l}$ such that $v = v'(v;p)$, and moreover $p' = p'(v;p), e_k = e_k(v;p)$ are unique for possibly non-unique choice of v.[5] In this case we define

$$T^{(r)}_l(p) = p', \qquad E^{(r)}_l(p) = e_1 + \cdots + e_L, \tag{2.4}$$

indicating that the time evolution operator $T^{(r)}_l$ is acting on p. The quantity $E^{(r)}_l(p) \in \mathbb{Z}_{\geq 0}$ is called an *energy* of the path p.

By the standard argument using the Yang–Baxter equation of the combinatorial R, one can show

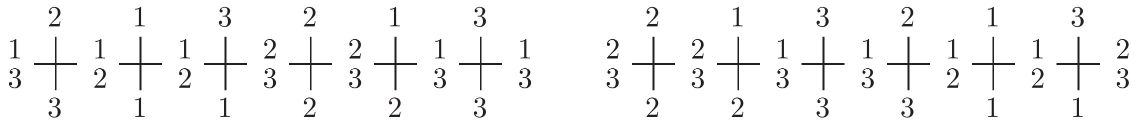
[4] Even for $n = 1$, it becomes nontrivial for higher spin case [22].
[5] We expect that the uniqueness of the local energy $e_k(v;p)$ follows from the uniqueness of $p'(v;p)$ alone.

Proposition 2.1 ([26]). *The commutativity* $T_i^{(a)}T_j^{(b)}(p) = T_j^{(b)}T_i^{(a)}(p)$ *is valid if all the time evolutions* $T_l^{(r)}$ *here are acting on* $T_l^{(r)}$*-evolvable paths. Moreover,* $E_i^{(a)}\big(T_j^{(b)}(p)\big) = E_i^{(a)}(p)$ *and* $E_j^{(b)}\big(T_i^{(a)}(p)\big) = E_j^{(b)}(p)$ *hold.*

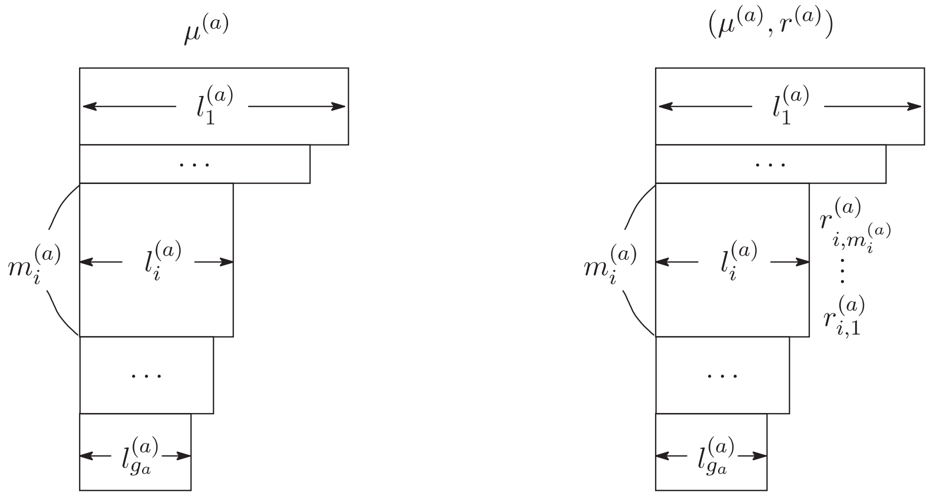
If p is $T_l^{(r)}$-evolvable for all r, l, then it is simply called *evolvable*. The third problem in defining the dynamics is

(*iii*) Even if p is evolvable, $T_l^{(r)}(p)$ is not necessarily so in general.

For instance, $p = 112233 \in \big(B^{1,1}\big)^{\otimes 6}$ is evolvable but $T_1^{(2)}(p) = 213213$ is not $T_1^{(2)}$-evolvable. In fact, the non-uniqueness problem (*ii*) takes place as follows:



To discuss the issue (*iii*), we need to prepare several definitions describing the spectrum $\big\{E_l^{(r)}\big\}$. Let $\mu = (\mu^{(1)}, \dots, \mu^{(n)})$ be an n-tuple of Young diagrams. From $\mu^{(a)}$, we specify the data $m_i^{(a)}$, $l_i^{(a)}$, g_a as in the left diagram in (2.5).



(2.5)

Here, $m_i^{(a)} \times l_i^{(a)}$ rectangle part is called the (a,i) block. Furthermore we introduce[6]

$$\begin{aligned} &H = \big\{(a,i,\alpha) \mid 1 \le a \le n,\ 1 \le i \le g_a,\ 1 \le \alpha \le m_i^{(a)}\big\},\\ &\overline{H} = \{(a,i) \mid 1 \le a \le n,\ 1 \le i \le g_a\}, \end{aligned} \tag{2.6}$$

$$p_i^{(a)} = L\delta_{a1} - \sum_{(bj\beta)\in H} C_{ab}\min\big(l_i^{(a)}, l_j^{(b)}\big) = L\delta_{a1} - \sum_{(bj)\in\overline{H}} C_{ab}\min\big(l_i^{(a)}, l_j^{(b)}\big)m_j^{(b)}, \tag{2.7}$$

$$F = (F_{ai,bj})_{(ai),(bj)\in\overline{H}}, \qquad F_{ai,bj} = \delta_{ab}\delta_{ij}p_i^{(a)} + C_{ab}\min\big(l_i^{(a)}, l_j^{(b)}\big)m_j^{(b)}, \tag{2.8}$$

$$G = |H| = \sum_{a=1}^{n}\sum_{i=1}^{g_a} m_i^{(a)}, \qquad g = |\overline{H}| = \sum_{a=1}^{n} g_a, \tag{2.9}$$

where $(C_{ab})_{1\le a,b\le n}$ is the Cartan matrix of A_n, i.e., $C_{ab} = 2\delta_{ab} - \delta_{a,b+1} - \delta_{a,b-1}$. The quantity $p_i^{(a)}$ is called a *vacancy number*.

[6]The meaning of i in $m_i^{(a)}$ and $p_i^{(a)}$ here has been altered(!) from the literatures [19, 20, 21, 22, 23, 24, 25, 26]. When the space is tight, we will often write $(ai\alpha)$ for $(a,i,\alpha) \in H$ and (ai) for $(a,i) \in \overline{H}$, etc.

Given $\mu = (\mu^{(1)}, \dots, \mu^{(n)})$, we define the sets of paths $B \supset \mathcal{P} \supset \mathcal{P}(\mu) \supset \mathcal{P}_+(\mu)$ by

$$\mathcal{P} = \{p \in B \mid \#(1) \geq \#(2) \geq \cdots \geq \#(n+1)\}, \tag{2.10}$$

$$\mathcal{P}(\mu) = \left\{ p \in \mathcal{P} \mid p : \text{evolvable},\ E_l^{(a)}(p) = \sum_{i=1}^{g_a} \min\left(l, l_i^{(a)}\right) m_i^{(a)} \right\}, \tag{2.11}$$

$$\mathcal{P}_+(\mu) = \{p \in \mathcal{P}(\mu) \mid p : \text{highest}\}, \tag{2.12}$$

where $\#(a)$ is the number of $a \in B^{1,1}$ in $p = b_1 \otimes \cdots \otimes b_L$. In view of the Weyl group symmetry [26, Theorem 2.2], and the obvious property that $T_l^{(r)}$ is weight preserving, we restrict ourselves to $\mathcal{P}$ which is the set of paths with nonnegative weight. A path $b_1 \otimes \cdots \otimes b_L$ is *highest* if the prefix $b_1 \otimes \cdots \otimes b_j$ satisfies the condition $\#(1) \geq \#(2) \geq \cdots \geq \#(n+1)$ for all $1 \leq j \leq L$. We call $\mathcal{P}(\mu)$ the *level set* associated with μ. The n-tuple μ of Young diagrams or equivalently the data $(m_i^{(a)}, l_i^{(a)})_{(ai)\in\overline{H}}$ will be referred to as the *soliton content* of $\mathcal{P}(\mu)$ or the paths contained in it. This terminology comes from the fact that when L is large and $\#(1) \gg \#(2), \dots, \#(n+1)$, it is known that the paths in $\mathcal{P}(\mu)$ consist of $m_i^{(1)}$ solitons with amplitude $l_i^{(1)}$ separated from each other or in the course of collisions. Here a soliton with amplitude l is a part of a path of the form $j_1 \otimes j_2 \otimes \cdots \otimes j_l \in (B^{1,1})^{\otimes l}$ with $j_1 \geq \cdots \geq j_l \geq 2$ by regarding $1 \in B^{1,1}$ as an empty background. Note that a soliton is endowed not only with the amplitude l but also the internal degrees of freedom $\{j_i\}$. The data $m_i^{(a)}$, $l_i^{(a)}$ with $a > 1$ are relevant to the internal degrees of freedom of solitons.

Note that the relation in (2.11) can be inverted as ($E_0^{(a)} := 0$)

$$-E_{l-1}^{(a)} + 2E_l^{(a)} - E_{l+1}^{(a)} = \begin{cases} m_i^{(a)} & \text{if } l = l_i^{(a)} \text{ for some } 1 \leq i \leq g_a, \\ 0 & \text{otherwise.} \end{cases} \tag{2.13}$$

Therefore the correspondence between the soliton content $\mu = \left(\mu^{(1)}, \dots, \mu^{(n)}\right)$ and the energy $\{E_l^{(a)}\}$ is one to one. Either $\{E_l^{(a)}\}$ or μ are conserved quantities in the following sense; if $p \in \mathcal{P}(\mu)$ and $T_l^{(r)}(p)$ is evolvable, then Proposition 2.1 tells that $T_l^{(r)}(p) \in \mathcal{P}(\mu)$. Comparing (2.7) and (2.11), one can express the vacancy number also as

$$p_i^{(a)} = L\delta_{a1} - \sum_{b=1}^{n} C_{ab} E_{l_i^{(a)}}^{(b)}.$$

Now we are ready to propose a sufficient condition under which the annoying feature in problem (*iii*) is absent:

$$\mu = \left(\mu^{(1)}, \dots, \mu^{(n)}\right) \text{ satisfies } p_i^{(a)} \geq 1 \text{ for all } (a, i) \in \overline{H}. \tag{2.14}$$

This condition was first introduced in [22].

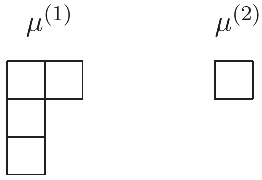
Conjecture 2.1. *Under the condition* (2.14), $T_l^{(r)}(\mathcal{P}(\mu)) = \mathcal{P}(\mu)$ *holds for any* r, l.

For a most general claim, see Conjecture 3.4.

The equality $T_l^{(r)}(\mathcal{P}(\mu)) = \mathcal{P}(\mu)$ implies that one can apply any time evolution in any order for arbitrary times on any path $p \in \mathcal{P}(\mu)$. In particular the inverse $\left(T_l^{(r)}\right)^{-1}$ exists. We denote by $\Sigma(p)$ the set of all paths generated from p in such a manner. By the definition it is a subset of the level set $\mathcal{P}(\mu)$. Call $\Sigma(p)$ the *connected component* of the level set containing the path p.

One can give another characterization of $\Sigma(p)$ in terms of group actions. Let $\mathcal{T}$ be the abelian group generated by all $T_l^{(r)}$'s. Then $\mathcal{T}$ acts on the level set $\mathcal{P}(\mu)$. This group action is not transitive in general, i.e., $\mathcal{P}(\mu)$ is generically decomposed into several $\mathcal{T}$-orbits. In what follows, connected components and $\mathcal{T}$-orbits mean the same thing.

Example 2.2. Let $n = 2, L = 8$ and $\mu = ((211),(1))$. The vacancy numbers are given as $p_1^{(1)} = 1$, $p_2^{(1)} = 3$, $p_1^{(2)} = 1$, hence the condition (2.14) is satisfied.



Then we have $\mathcal{P}(\mu) = \Sigma(p_X) \sqcup \Sigma(p_Y)$ with $p_X = 11221123$ and $p_Y = 11221213$. The connected components are written as $\Sigma(p_X) = \bigsqcup_{i=1}^{9} X_i$ and $\Sigma(p_Y) = \bigsqcup_{i=1}^{9} Y_i$ where X_i and Y_i are given in the following table.

i	X_i	Y_i
1	[11221123]	[11221213]
2	[11122123]	[11121223]
3	[21112123]	[21121123]
4	[21121213]	[21211213]
5	[21212113]	[12212113]
6	[12122113]	[11212213]
7	[12112213]	[11211223]
8	[12111223]	[21211123]
9	[12211123]	[12211213]

Here the symbol [ ] denotes the set of all cyclic shifts of the entry. For instance, $[123] = \{123, 231, 312\}$. For any $i \in \mathbb{Z}/9\mathbb{Z}$ we have $T_1^{(1)}(X_i) = X_i$, $T_{\geq 2}^{(1)}(X_i) = X_{i+1}$, $T_{\geq 1}^{(2)}(X_i) = X_{i+3}$ and the same relations for Y_i. Hence the claim of Conjecture 2.1 is valid for this $\mathcal{P}(\mu)$.

The time evolution $T_\infty^{(1)}$ has an especially simple description, which we shall now explain. Let B_1 be the set of paths in which $1 \in B^{1,1}$ is contained most. Namely,

$$B_1 = \{p \in B \mid \sharp(1) \geq \sharp(a),\ 2 \leq a \leq n+1\}. \tag{2.15}$$

Thus we see $\mathcal{P} \subset B_1 \subset B$. We define the weight preserving map $K_a : B_1 \to B_1$ for $a = 2, 3, \ldots, n+1$. Given a path $p \in B_1$ regarded as a word $p = b_1 b_2 \ldots b_L \in \{1, \ldots, n+1\}^L$, the image $K_a(p)$ is determined by the following procedure.

(i) Ignore all the numbers except 1 and a.

(ii) Connect every adjacent pair $a1$ (not $1a$) by an arc, where 1 is on the right of a *cyclically.*

(iii) Repeat (ii) ignoring the already connected pairs until all a's are connected to some 1.

(iv) Exchange a and 1 within each connected pair.

Theorem 2.1. *Suppose $p \in B_1$ and p is $T_\infty^{(1)}$-evolvable. Then, $T_\infty^{(1)}$ is factorized as follows:*

$$T_\infty^{(1)}(p) = K_2 K_3 \cdots K_{n+1}(p). \tag{2.16}$$

A proof is available in Appendix B, where each K_a is obtained as a gauge transformed simple reflection for type $A_n^{(1)}$ affine Weyl group. The dynamics of the form (2.16) has an interpretation as the ultradiscrete Toda or KP flow [7, 39]. See also [28, Theorem VI.2]. As an additional remark, a factorized time evolution similar to (2.16) or equivalently (B.4) has been formulated for a periodic SCA associated to any non-exceptional affine Lie algebras [25]. In the infinite (non-periodic) lattice case, it was first invented by Takahashi [37] for type $A_n^{(1)}$. The origin of such a factorization is the factorization of the combinatorial R itself in a certain asymptotic domain, which has been proved uniformly for all non-exceptional types in [8].

Example 2.3. In the left case of Example 1.1, the time evolution $T^{(1)}_3$ is actually equal to $T^{(1)}_\infty$. The time evolution from $t = 2$ to $t = 3$ is reproduced by $T^{(1)}_3 = T^{(1)}_\infty = K_2K_3$ as follows:

$$
\begin{array}{ll}
t = 2: & 3311321113211111221122213 \\
& \quad \downarrow K_3 \\
& 1133123311231111221122211 \\
& \quad \downarrow K_2 \\
t = 3: & 2133113321132111122111 22
\end{array}
$$

The procedure (*i*)–(*iv*) is delightfully simple but inevitably non-local as the result of expelling the carrier out from the description. It is an interesting question whether the other typical time evolutions $T^{(2)}_\infty, \dots, T^{(n)}_\infty$ admit a similar description without a carrier.

2.3 Rigged configuration

An n-tuple of Young diagrams $\mu = (\mu^{(1)}, \dots, \mu^{(n)})$ satisfying $p^{(a)}_i \geq 0$ for all $(a, i) \in \overline{H}$ is called a *configuration*. Thus, those μ satisfying (2.14) form a subset of configurations. Consider the configuration μ attached with the integer arrays called *rigging* $\mathbf{r} = (r^{(a)})_{1\leq a\leq n} = (r^{(a)}_{i,\alpha})_{(ai\alpha)\in H}$ as in the right diagram in (2.5). The combined data $(\mu, \mathbf{r}) = ((\mu^{(1)}, r^{(1)}), \dots, (\mu^{(n)}, r^{(n)}))$ is called a *rigged configuration* if the condition

$$0 \leq r^{(a)}_{i,1} \leq r^{(a)}_{i,2} \leq \cdots \leq r^{(a)}_{i,m^{(a)}_i} \leq p^{(a)}_i \quad \text{for all} \quad (a, i) \in \overline{H} \tag{2.17}$$

is satisfied[7]. We let $\mathrm{RC}(\mu)$ denote the set of rigged configurations whose configuration is μ. It is well known [15, 16] that the highest paths in B are in one to one correspondence with the rigged configurations by the Kerov–Kirillov–Reshetikhin (KKR) map

$$\phi : \{p \in B \mid p : \text{highest}\} \;\to\; \sqcup_\mu \mathrm{RC}(\mu), \tag{2.18}$$

where the union is taken over all the configurations. The both ϕ and ϕ^{-1} can be described by an explicit algorithm as described in Appendix C. Our main claim in this subsection is the following, which is a refinement of (2.18) with respect to μ and an adaptation to the periodic setting.

Theorem 2.2. *The restriction of the KKR map ϕ to $\mathcal{P}_+(\mu)$ separates the image according to μ:*

$$\phi : \; \mathcal{P}_+(\mu) \to \mathrm{RC}(\mu).$$

The proof is available in Appendix D. We expect that this restricted injection is still a bijection but we do not need this fact in this paper.

Let us explain some background and significance of Theorem 2.2. Rigged configurations were invented [15] as combinatorial substitutes of solutions to the Bethe equation under *string hypothesis* [2]. The KKR map ϕ^{-1} is the combinatorial analogue of producing the Bethe vector from the solutions to the Bethe equation. The relevant integrable system is sl_{n+1} Heisenberg chain, or more generally the rational vertex model associated with $U_q\big(A^{(1)}_n\big)$ at $q = 1$. In this context, the configuration μ (2.5) is the *string content* specifying that there are $m^{(a)}_i$ strings

[7]We do not include L in the definition of rigged configuration understanding that it is fixed. It is actually necessary to determine the vacancy number $p^{(a)}_i$ (2.7) appearing as the upper bound of the rigging (2.17).

with color a and length $l_i^{(a)}$. Theorem 2.2 identifies the two meanings of μ. Namely, the soliton content for $\mathcal{P}(\mu)$ measured by energy is equal to the string content for $\mathrm{RC}(\mu)$ determined by the KKR bijection. Symbolically, we have the identity

$$\text{soliton} = \text{string}, \tag{2.19}$$

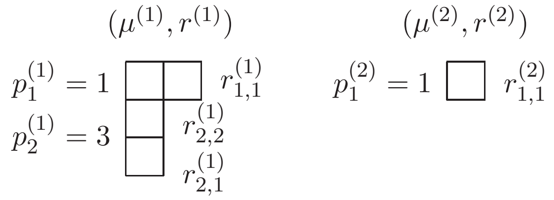
which lies at the heart of the whole combinatorial Bethe ansatz approaches to the soliton cellular automata [20, 21, 22, 23, 24, 25, 26]. It connects the energy and the configuration by (2.13), and in a broader sense, crystal theory and Bethe ansatz. Further arguments will be given around (4.23). For $n = 1$, Theorem 2.2 has been obtained in [24, Proposition 3.4]. In the sequel, we shall call the n-tuple of Young diagrams $\mu = (\mu^{(1)}, \dots, \mu^{(n)})$ either as soliton content, string content or configuration when $p_i^{(a)} \geq 0$ for all $(a, i) \in \overline{H}$.

A consequence of Theorem 2.2 is that for any path $b_1 \otimes \cdots \otimes b_L \in \mathcal{P}(\mu)$ with soliton content μ, the number $\#(a)$ of $a \in B^{1,1}$ in $b_1, \dots, b_L$ is given by

$$\#(a) = \left|\mu^{(a-1)}\right| - \left|\mu^{(a)}\right| \qquad (1 \leq a \leq n+1), \tag{2.20}$$

where we set $|\mu^{(0)}| = L$ and $|\mu^{(n+1)}| = 0$. This is due to a known property of the KKR map ϕ and the fact that μ is a conserved quantity and Conjecture 3.5.

Example 2.4. Let $n = 2, L = 8$ and consider the same configuration $\mu = ((211), (1))$ as Example 2.2. The riggings are to obey the conditions $0 \leq r_{1,1}^{(1)}, r_{1,1}^{(2)} \leq 1$ and $0 \leq r_{2,1}^{(1)} \leq r_{2,2}^{(1)} \leq 3$.



Hence there are $2 \cdot 2 \cdot 10 = 40$ rigged configurations in $\mathrm{RC}(\mu)$. The elements of $\mathcal{P}_+(\mu)$ and the riggings for their images under the KKR map ϕ are given in the following table.

i	highest paths in X_i	riggings	highest paths in Y_i	riggings
1	11221123, 11231122, 12311221	0331, 1110, 0000	11221213, 12131122	0321, 1100
2	11122123, 11221231, 12311122	1331, 0221, 1000	11121223, 11212231	1321, 0211
3	11121232, 11212321	1221, 0111	11211232, 12112321, 11232112	1211, 0101, 0310
4	11212132, 12121321, 12132112	1111, 0001, 0300	12112132, 11213212, 12132121	1101, 1310, 0200
5	12121132, 12113212	1001, 1300	12113122	1200
6	$\varnothing$	$\varnothing$	11212213	0311
7	12112213, 11221312	0301, 0330	11211223, 12112231, 11223112	1311, 0201, 0320
8	12111223, 11122312, 11223121	1301, 1330, 0220	12111232, 11123212, 11232121	1201, 1320, 0210
9	11123122, 11231221	1220, 0110	12131221, 11213122	0100, 1210

Here the riggings denote $r_{1,1}^{(1)} r_{2,2}^{(1)} r_{2,1}^{(1)} r_{1,1}^{(2)}$.

3 Inverse scattering method

3.1 Action-angle variables

By action-angle variables for the periodic $A_n^{(1)}$ SCA, we mean the variables or combinatorial objects that are conserved (action) or growing linearly (angle) under the commuting family of time evolutions $\{T_l^{(r)}\}$. They are scattering data in the context of inverse scattering method [6, 3, 4]. In our approach, the action-angle variables are constructed by a suitable extension of rigged configurations, which exploits a connection to the combinatorial Bethe ansätze both at $q = 1$ [15]

and $q = 0$ [19]. These features have been fully worked out in [24] for $A_1^{(1)}$. More recently it has been shown further that the set of angle variables can be decomposed into connected components and every such component is a torus [36]. Here we present a conjectural generalization of these results to $A_n^{(1)}$ case. It provides a conceptual explanation of the dynamical period and the state counting formula proposed in [25, 26].

Consider the level set $\mathcal{P}(\mu)$ (2.11) with $\mu = (\mu^{(1)}, \dots, \mu^{(n)})$ satisfying the condition (2.14). The action variable for any path $p \in \mathcal{P}(\mu)$ is defined to be μ itself. By Proposition 2.1, it is a conserved quantity under any time evolution.

Recall that each block (a, i) of a rigged configuration is assigned with the rigging $r^{(a)}_{i,\alpha}$ with $\alpha = 1, \dots, m^{(a)}_i$. We extend $r^{(a)}_{i,\alpha}$ to $\alpha \in \mathbb{Z}$ uniquely so that the quasi-periodicity

$$r^{(a)}_{i,\alpha+m^{(a)}_i} = r^{(a)}_{i,\alpha} + p^{(a)}_i \qquad (\alpha \in \mathbb{Z}) \tag{3.1}$$

is fulfilled. Such a sequence $\mathbf{r} = (r^{(a)}_{i,\alpha})_{\alpha\in\mathbb{Z}}$ will be called the *quasi-periodic extension* of $(r^{(a)}_{i,\alpha})_{1\le i\le m^{(a)}_i}$.

Set

$$\tilde{\mathcal{J}}(\mu) = \prod_{(ai)\in\overline{H}} \tilde{\Lambda}\big(m^{(a)}_i, p^{(a)}_i\big),$$

$$\tilde{\Lambda}(m, p) = \{(\lambda_\alpha)_{\alpha\in\mathbb{Z}} \mid \lambda_\alpha \in \mathbb{Z},\ \lambda_\alpha \le \lambda_{\alpha+1},\ \lambda_{\alpha+m} = \lambda_\alpha + p \ \text{ for all } \alpha\} \quad (m \ge 1, p \ge 0), \tag{3.2}$$

where $\prod$ denotes a direct product of sets. Using Theorem 2.2, we define an injection

$$\begin{array}{rcccl} \iota : \mathcal{P}_+(\mu) & \xrightarrow{\phi} & \mathrm{RC}(\mu) & \longrightarrow & \tilde{\mathcal{J}}(\mu) \\ p_+ & \longrightarrow & (\mu, \mathbf{r}) & \longmapsto & \big(r^{(a)}_{i,\alpha}\big)_{(ai)\in\overline{H},\alpha\in\mathbb{Z}}, \end{array}$$

where the bottom right is the quasi-periodic extension of the original rigging $\mathbf{r} = \big(r^{(a)}_{i,\alpha}\big)_{(ai\alpha)\in H}$ in $(\mu, \mathbf{r})$ as explained above. Elements of $\tilde{\mathcal{J}}(\mu)$ will be called *extended rigged configurations*. One may still view an extended rigged configuration as the right diagram in (2.5). The only difference from the original rigged configuration is that the riggings $r^{(a)}_{i,\alpha}$ outside the range $1 \le \alpha \le m^{(a)}_i$ have also been fixed by the quasi-periodicity (3.1). In what follows, either the rigging $\big(r^{(a)}_{i,\alpha}\big)_{(ai\alpha)\in H}$ or its quasi-periodic extension $\big(r^{(a)}_{i,\alpha}\big)_{(ai)\in\overline{H},\alpha\in\mathbb{Z}} \in \tilde{\mathcal{J}}(\mu)$ will be denoted by the same symbol $\mathbf{r}$.

Let $\mathcal{T}$ be the abelian group generated by all the time evolutions $T^{(a)}_l$ with $1 \le a \le n$ and $l \ge 1$. Let further $\mathcal{A}$ be a free abelian group generated by the symbols $s^{(a)}_i$ with $(a, i) \in \overline{H}$. We consider their commutative actions on $\tilde{\mathcal{J}}(\mu)$ as follows:

$$T^{(a)}_l : \ \big(r^{(b)}_{j,\beta}\big)_{(bj)\in\overline{H},\beta\in\mathbb{Z}} \ \mapsto \ \big(r^{(b)}_{j,\beta} + \delta_{ab}\min\big(l, l^{(b)}_j\big)\big)_{(bj)\in\overline{H},\beta\in\mathbb{Z}}, \tag{3.3}$$

$$s^{(a)}_i : \ \big(r^{(b)}_{j,\beta}\big)_{(bj)\in\overline{H},\beta\in\mathbb{Z}} \ \mapsto \ \big(r^{(b)}_{j,\beta+\delta_{ab}\delta_{ij}} + C_{ab}\min\big(l^{(a)}_i, l^{(b)}_j\big)\big)_{(bj)\in\overline{H},\beta\in\mathbb{Z}}. \tag{3.4}$$

The generator $s^{(a)}_i$ is the $A^{(1)}_n$ version of the slide introduced in [24] for $n = 1$. Now we define

$$\mathcal{J}(\mu) = \tilde{\mathcal{J}}(\mu)/\mathcal{A}, \tag{3.5}$$

which is the set of all $\mathcal{A}$-orbits, whose elements are written as $\mathcal{A} \cdot \mathbf{r}$ with $\mathbf{r} \in \tilde{\mathcal{J}}(\mu)$. Elements of $\mathcal{J}(\mu)$ will be called *angle variables*. See also (4.17). Since $\mathcal{T}$ and $\mathcal{A}$ act on $\tilde{\mathcal{J}}(\mu)$ commutatively, there is a natural action of $\mathcal{T}$ on $\mathcal{J}(\mu)$. For $t \in \mathcal{T}$ and $y = \mathcal{A} \cdot \mathbf{r} \in \mathcal{J}(\mu)$, the action is given by $t \cdot y = \mathcal{A} \cdot (t \cdot \mathbf{r})$. In short, the time evolution of extended rigged configurations (3.3) naturally induces the time evolution of angle variables.

Remark 3.1. From (3.4) and (2.7), one can easily check

$$\left(\prod_{(ai)\in\overline{H}} s_i^{(a)\,m_i^{(a)}}\right)(\mathbf{r}) = \left(T_1^{(1)}\right)^L(\mathbf{r}) = \left(L\delta_{b1} + r_{j,\beta}^{(b)}\right)_{(bj)\in\overline{H},\beta\in\mathbb{Z}}$$

for $\mathbf{r} = \left(r_{j,\beta}^{(b)}\right)_{(bj)\in\overline{H},\beta\in\mathbb{Z}} \in \tilde{\mathcal{J}}(\mu)$. Therefore $\mathbf{r}$ and $\left(T_1^{(1)}\right)^L(\mathbf{r})$ define the same angle variable.

3.2 Linearization of time evolution

Given the level set $\mathcal{P}(\mu)$ with μ satisfying the condition (2.14), we are going to introduce the bijection Φ to the set of angle variables $\mathcal{J}(\mu)$. Recall that $\Sigma(p)$ denotes the connected component of $\mathcal{P}(\mu)$ that involves the path p.

Conjecture 3.1. *Under the condition* (2.14), $\Sigma(p)\cap\mathcal{P}_+(\mu)\neq\varnothing$ *holds for any* $p\in\mathcal{P}(\mu)$.

This implies that any path p can be expressed in the form $p = t\cdot p_+$ by some highest path $p_+\in\mathcal{P}_+(\mu)$ and time evolution $t=\prod (T_l^{(r)})^{d_l^{(r)}}\in\mathcal{T}$.[8] Although such an expression is not unique in general, one can go from $\mathcal{P}(\mu)$ to $\mathcal{J}(\mu)$ along the following scheme:

$$\begin{array}{rccccccc} \Phi: & \mathcal{P}(\mu) & \longrightarrow & \mathcal{T}\times\mathcal{P}_+(\mu) & \xrightarrow{\mathrm{id}\times\iota} & \mathcal{T}\times\tilde{\mathcal{J}}(\mu) & \longrightarrow & \mathcal{J}(\mu) \\ & p & \longmapsto & (t,p_+) & \longmapsto & \left(t,\mathbf{r}\right) & \longmapsto & \mathcal{A}\cdot\left(t\cdot\mathbf{r}\right). \end{array} \tag{3.6}$$

Note that the abelian group $\mathcal{T}$ of time evolutions acts on paths by (2.4) and on extended rigged configurations by (3.3). The composition (3.6) serves as a definition of a map Φ only if the non-uniqueness of the decomposition $p\mapsto(t,p_+)$ is 'canceled' by regarding $t\cdot\mathbf{r}$ mod $\mathcal{A}$.

Our main conjecture in this subsection is the following.

Conjecture 3.2. *Suppose the condition* (2.14) *is satisfied. Then the map* Φ (3.6) *is well-defined, bijective and commutative with the action of* $\mathcal{T}$. *Namely, the following diagram is commutative*

$$\begin{array}{ccc} \mathcal{P}(\mu) & \xrightarrow{\Phi} & \mathcal{J}(\mu) \\ {\scriptstyle\mathcal{T}}\Big\downarrow & & \Big\downarrow{\scriptstyle\mathcal{T}} \\ \mathcal{P}(\mu) & \xrightarrow{\Phi} & \mathcal{J}(\mu) \end{array} \tag{3.7}$$

This is consistent with the obvious periodicity $(T_1^{(1)})^L(p) = p$ of paths under the cyclic shift (2.3) and Remark 3.1. The time evolution of the angle variable (3.3) is linear. Thus the commutative diagram (3.7) transforms the nonlinear dynamics of paths into straight motions with various velocity vectors. The set $\mathcal{J}(\mu)$ is a tropical analogue of Jacobi variety in the theory of quasi-periodic solutions of soliton equations [3, 4]. The map Φ whose essential ingredient is the KKR bijection ϕ plays the role of the Abel–Jacobi map. Conceptually, the solution of the initial value problem is simply stated as $t^{\mathcal{N}}(p) = \Phi^{-1}\circ t^{\mathcal{N}}\circ\Phi(p)$ for any $t\in\mathcal{T}$, where t in the right hand side is linear. Practical calculations can be found in Examples 3.2 and 3.5. In Section 4, we shall present an explicit formula for the tropical Jacobi-inversion Φ^{-1} in terms of a tropical Riemann theta function. The result of this sort was first formulated and proved in [24, 20, 21] for $n=1$.

Example 3.1. Take an evolvable path

$$p = 211332111321133112221112 \in \left(B^{1,1}\right)^{\otimes 24}, \tag{3.8}$$

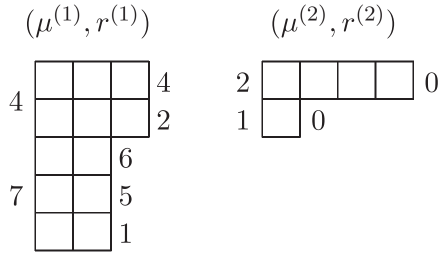
[8] We do not restrict t to a specific subgroup of $\mathcal{T}$.

whose energies are given as

$$E_1^{(1)} = 5, \quad E_2^{(1)} = 10, \quad E_{\geq 3}^{(1)} = 12, \quad E_1^{(2)} = 2, \quad E_2^{(2)} = 3, \quad E_3^{(2)} = 4, \quad E_{\geq 4}^{(2)} = 5.$$

From this and (2.13) we find $p \in \mathcal{P}(\mu)$ with $\mu = ((33222),(41))$. The cyclic shift $(T_1^{(1)})^j(p)$ is not highest for any j. However it can be made into a highest path p_+ and transformed to a rigged configuration as follows:

$$\begin{aligned} p_+ &= (T_1^{(1)})^3 (T_1^{(2)})^3 (p) \\ &= 1112211132211321133113322 \end{aligned} \quad \overset{\phi}{\longmapsto}$$



(3.9)

Thus action variable $(\mu^{(1)}, \mu^{(2)})$ indeed coincides with $\mu = ((33222),(41))$ obtained from the energy in agreement with Theorem 2.2. Although it is not necessary, we have exhibited the vacancy numbers on the left of $\mu^{(1)}$ and $\mu^{(2)}$ for convenience. We list the relevant data:

(ai)	$l_i^{(a)}$	$m_i^{(a)}$	$p_i^{(a)}$	$r_{i,\alpha}^{(a)}$	$\gamma_i^{(a)}$
(11)	3	2	4	2, 4	2
(12)	2	3	7	1, 5, 6	1
(21)	4	1	2	0	1
(22)	1	1	1	0	1

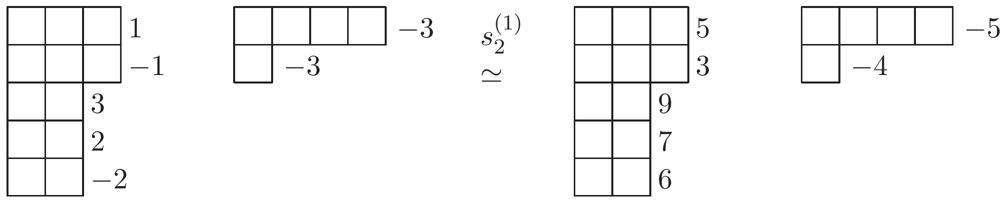
The index set H (2.6) corresponding to $\mu = (\mu^{(1)}, \mu^{(2)})$ in (3.9) is

$$H = \{(111), (112), (121), (122), (123), (211), (221)\}, \tag{3.10}$$

which has the cardinality $G = 7$. The quantity $\gamma_i^{(a)}$ is the order of symmetry which will be explained in Section 3.3. The matrix F (2.8) reads

$$\begin{aligned} F &= \begin{pmatrix} p_1^{(1)} + 6m_1^{(1)} & 4m_2^{(1)} & -3m_1^{(2)} & -m_2^{(2)} \\ 4m_1^{(1)} & p_2^{(1)} + 4m_2^{(1)} & -2m_1^{(2)} & -m_2^{(2)} \\ -3m_1^{(1)} & -2m_2^{(1)} & p_1^{(2)} + 8m_1^{(2)} & 2m_2^{(2)} \\ -m_1^{(1)} & -m_2^{(1)} & 2m_1^{(2)} & p_2^{(2)} + 2m_2^{(2)} \end{pmatrix} \\ &= \begin{pmatrix} 16 & 12 & -3 & -1 \\ 8 & 19 & -2 & -1 \\ -6 & -6 & 10 & 2 \\ -2 & -3 & 2 & 3 \end{pmatrix}. \end{aligned} \tag{3.11}$$

We understand that the riggings here are parts of extended one obeying (3.1). Then from (3.9) and (3.3), the extended rigged configuration for p (3.8) is the left hand side of



(3.12)

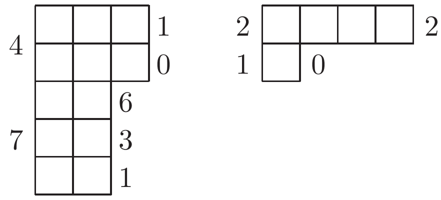
Here the transformation to the right hand side demonstrates an example of identification by $\mathcal{A}$. These two objects are examples of representative elements of the angle variable for p (3.8). We will calculate $(T_l^{(a)})^{1000}(p)$ in Example 3.5 using the reduced angle variable that will be introduced in the next subsection.

Example 3.2. Consider the evolvable path

$$p = 321113211222111223331111 \in \left(B^{1,1}\right)^{\otimes 24}, \tag{3.13}$$

which is same as in Example 1.1. The cyclic shift $(T_1^{(1)})^j(p)$ is not highest for any j. However it can be made into a highest path p_+ and transformed to a rigged configuration as

$$\begin{aligned} p_+ &= (T_1^{(1)})^{21} T_1^{(2)}(p) \\ &= 111221322111221332111331 \end{aligned} \quad \overset{\phi}{\longmapsto}$$

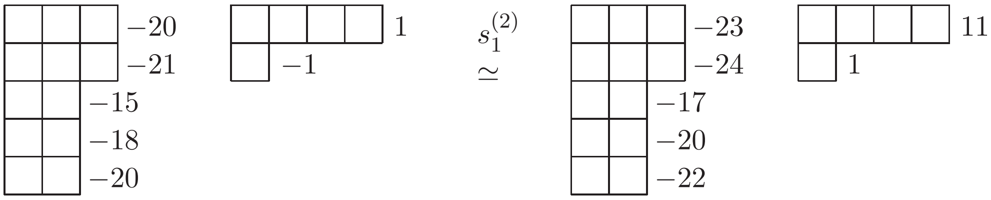


(3.14)

Thus p (3.13) belongs to the same level set $\mathcal{P}(\mu)$ with $\mu = ((33222),(41))$ as Example 3.1, hence the matrix F (2.8) is again given by (3.11). However, it has a different (trivial) order of symmetry $\gamma_i^{(a)}$ given in the following table:

(ai)	$l_i^{(a)}$	$m_i^{(a)}$	$p_i^{(a)}$	$r_{i,\alpha}^{(a)}$	$\gamma_i^{(a)}$
(11)	3	2	4	0, 1	1
(12)	2	3	7	1, 3, 6	1
(21)	4	1	2	2	1
(22)	1	1	1	0	1

Regard (3.14) as a part of extended rigged configuration as in Example 3.1. Then from (3.3), the extended rigged configuration for p (3.13) is the left hand side of



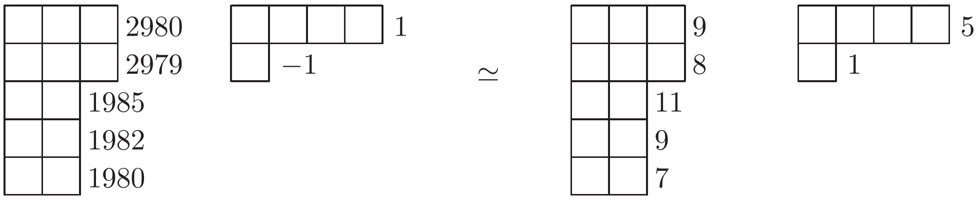
(3.15)

Here again the transformation to the right hand side demonstrates an example of identification by $\mathcal{A}$. These two objects are examples of representative elements of the angle variable for p (3.13).

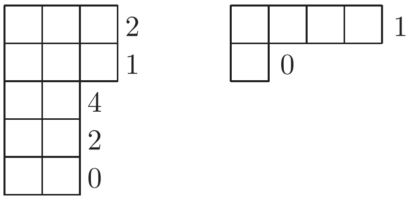
To illustrate the solution of the initial value problem along the inverse scheme (3.7), we derive

$$\left(T_3^{(1)}\right)^{1000}(p) = 221132211331111321322111. \tag{3.16}$$

By applying (3.3) to the left hand side of (3.15), the angle variable for $\left(T_3^{(1)}\right)^{1000}(p)$ is obtained as



where the equivalence is seen by using $\left(s_1^{(1)}\right)^{-358}\left(s_2^{(1)}\right)^{-136}\left(s_1^{(2)}\right)^{-117}\left(s_2^{(2)}\right)^{-86} \in \mathcal{A}$. The right hand side is the angle variable of $\left(T_1^{(1)}\right)^7 T_4^{(2)}(p'_+)$, where p'_+ is the highest path obtained by the KKR map



$$\overset{\phi^{-1}}{\longmapsto} \quad p'_+ = 1122111321122221331133211$$

Calculating $\left(T_1^{(1)}\right)^7 T_4^{(2)}(p'_+)$, one finds (3.16).

3.3 Decomposition into connected components

As remarked under Conjecture 2.1, the level set $\mathcal{P}(\mu)$ is a disjoint union of several connected components $\Sigma(p)$ under the time evolution $\mathcal{T}$. It is natural to decompose the commutative diagram (3.7) further into those connected components, i.e., $\mathcal{T}$-orbits, and seek the counterpart of $\Sigma(p)$ in $\mathcal{J}(\mu)$. To do this one needs a precise description of the internal symmetry of angle variables reflecting a certain commensurability of soliton configuration in a path. In addition, it is necessary to separate the angle variables into two parts, one recording the internal symmetry (denoted by $\boldsymbol{\lambda}$)[9] and the other accounting for the straight motions (denoted by ω). We call the latter part *reduced angle variables*. The point is that action of $\mathcal{T}$ becomes transitive by switching to the reduced angle variables from angle variables thereby allowing us to describe the connected components and their multiplicity explicitly. These results are generalizations of the $n = 1$ case [36].

Let $m \geq 1$ and $p \geq 0$.[10] We introduce the following set by imposing just a simple extra condition on $\tilde{\Lambda}(m,p)$ (3.2):

$$\Lambda(m,p) = \{(\lambda_\alpha)_{\alpha\in\mathbb{Z}} \mid \lambda_\alpha \in \mathbb{Z},\ \lambda_1 = 0,\ \lambda_\alpha \leq \lambda_{\alpha+1},\ \lambda_{\alpha+m} = \lambda_\alpha + p \ \text{ for all } \alpha\}. \tag{3.17}$$

This is a finite set with cardinality

$$|\Lambda(m,p)| = \binom{p+m-1}{m-1}. \tag{3.18}$$

Given a configuration μ, specify the data $m_i^{(a)}$, $l_i^{(a)}$, $p_i^{(a)}$, etc by (2.5)–(2.9). Set

$$X = X(\mu) = X^1 \times X^2, \qquad X^1 = \mathbb{Z}^g, \qquad X^2 = \prod_{(ai)\in\overline{H}} \Lambda\left(m_i^{(a)}, p_i^{(a)}\right),$$

and consider the splitting of extended rigged configurations into X^1 and X^2 as follows:

$$\begin{array}{ccccc} \tilde{\mathcal{J}}(\mu) & \longleftrightarrow & X^1 \times X^2 & & \\ \left(r_{i,\alpha}^{(a)}\right)_{(ai)\in\overline{H},\alpha\in\mathbb{Z}} & \longleftrightarrow & (\omega,\boldsymbol{\lambda}) & \omega_i^{(a)} = r_{i,1}^{(a)}, & \lambda_{i,\alpha}^{(a)} = r_{i,\alpha}^{(a)} - r_{i,1}^{(a)}, \end{array} \tag{3.19}$$

where $\omega = \left(\omega_i^{(a)}\right)_{(ai)\in\overline{H}}$ and $\boldsymbol{\lambda} = \left(\lambda_{i,\alpha}^{(a)}\right)_{(ai)\in\overline{H},\alpha\in\mathbb{Z}}$.

Recall that $\mathcal{T}$ and $\mathcal{A}$ are the abelian groups acting on $\tilde{\mathcal{J}}(\mu)$ as in (3.3)–(3.4). By the one to one correspondence (3.19), they also act on X commutatively as

$$T_l^{(a)}(\omega,\boldsymbol{\lambda}) = \left(\omega + h_l^{(a)}, \boldsymbol{\lambda}\right), \qquad h_l^{(a)} = \left(\delta_{ab}\min\left(l, l_j^{(b)}\right)\right)_{(bj)\in\overline{H}}, \tag{3.20}$$

[9]This $\boldsymbol{\lambda}$ may be viewed as an additional conserved quantity.
[10]This p is not a path of course and will set to be a vacancy number shortly.

$$s_i^{(a)}(\omega,\boldsymbol{\lambda}) = (\omega',\boldsymbol{\lambda}'), \qquad \begin{cases} \omega' &= \omega + \big(\lambda^{(b)}_{j,1+\delta_{ij}\delta_{ab}} + C_{ab}\min\big(l_i^{(a)}, l_j^{(b)}\big)\big)_{(bj)\in\overline{H}}, \\ \boldsymbol{\lambda}' &= \big(\lambda^{(b)}_{j,\alpha+\delta_{ij}\delta_{ab}} - \lambda^{(b)}_{j,1+\delta_{ij}\delta_{ab}}\big)_{(bj)\in\overline{H},\alpha\in\mathbb{Z}}. \end{cases} \tag{3.21}$$

In particular, $\mathcal{T}$ acts on X^1 transitively leaving X^2 unchanged.

From (3.5) and (3.19), the angle variables may also be viewed as elements of $X/\mathcal{A}$. It is the set of all $\mathcal{A}$-orbits, whose elements are written as $\mathcal{A}\cdot x$ with $x \in X$. Since $\mathcal{T}$ and $\mathcal{A}$ act on X commutatively, there is a natural action of $\mathcal{T}$ on $X/\mathcal{A}$. For $t \in \mathcal{T}$ and $y = \mathcal{A}\cdot x \in X/\mathcal{A}$, the action is given by $t\cdot y = \mathcal{A}\cdot(t\cdot x)$. For any representative element $(\omega,\boldsymbol{\lambda}) \in X/\mathcal{A}$ of an angle variable, we call the X^1 part ω a *reduced angle variable*. Under the simple bijective correspondence (3.19), the map (3.6) is rephrased as

$$\begin{array}{cccccccc} \Phi: & \mathcal{P}(\mu) & \longrightarrow & \mathcal{T}\times\mathcal{P}_+(\mu) & \xrightarrow{\mathrm{id}\times\iota} & \mathcal{T}\times X & \longrightarrow & X/\mathcal{A} \\ & p & \longmapsto & (t,p_+) & \longmapsto & \big(t,(\omega,\boldsymbol{\lambda})\big) & \longmapsto & \mathcal{A}\cdot\big(t\cdot(\omega,\boldsymbol{\lambda})\big), \end{array}$$

where we have used the same symbol Φ. The conjectural commutative diagram (3.7) becomes

$$\begin{array}{ccc} \mathcal{P}(\mu) & \xrightarrow{\Phi} & X/\mathcal{A} \\ {\scriptstyle\mathcal{T}}\big\downarrow & & \big\downarrow{\scriptstyle\mathcal{T}} \\ \mathcal{P}(\mu) & \xrightarrow{\Phi} & X/\mathcal{A} \end{array} \tag{3.22}$$

Now we are ready to describe the internal symmetry of angle variables and the resulting decomposition. We introduce a refinement of $\Lambda(m,p)$ (3.17) as follows:

$$\Lambda_\gamma(m,p) = \left\{\lambda \in \Lambda\left(\frac{m}{\gamma},\frac{p}{\gamma}\right) \Big| \lambda \notin \Lambda\left(\frac{m}{\gamma'},\frac{p}{\gamma'}\right) \text{ for any } \gamma' > \gamma\right\}, \tag{3.23}$$

where γ is a (not necessarily largest) common divisor of m and p. In other words $\Lambda_\gamma(m,p)$ is the set of all arrays $(\lambda_\alpha)_{\alpha\in\mathbb{Z}} \in \Lambda(m,p)$ that satisfy the reduced quasi-periodicity $\lambda_{\alpha+m/\gamma} = \lambda_\alpha + p/\gamma$ but do not satisfy the same relation when γ is replaced by a larger γ'. Such γ is called the *order of symmetry*. By the definition one has the disjoint union decomposition:

$$\Lambda(m,p) = \bigsqcup_\gamma \Lambda_\gamma(m,p),$$

where γ extends over all the common divisors of m and p. Taking the cardinality of this relation using (3.18) amounts to the identity

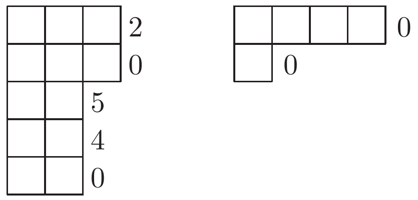
$$\binom{p+m-1}{m-1} = \sum_\gamma |\Lambda_\gamma(m,p)|. \tag{3.24}$$

Let $\boldsymbol{\gamma} = \big(\gamma_i^{(a)}\big)_{(ai)\in\overline{H}}$ be the array of order of symmetry for all blocks. Thus $\gamma_i^{(a)}$ is any common divisor of $m_i^{(a)}$ and $p_i^{(a)}$. We introduce the $g\times g$ matrix $F_{\boldsymbol{\gamma}}$ and the subsets $X^2_{\boldsymbol{\gamma}} \subset X^2$ and $X_{\boldsymbol{\gamma}} \subset X$ by

$$\begin{aligned} &F_{\boldsymbol{\gamma}} = \big(F_{ai,bj}/\gamma_j^{(b)}\big)_{(ai),(bj)\in\overline{H}}, \\ &X^2_{\boldsymbol{\gamma}} = \prod_{(ai)\in\overline{H}} \Lambda_{\gamma_i^{(a)}}\big(m_i^{(a)},p_i^{(a)}\big) \subset X^2, \qquad X_{\boldsymbol{\gamma}} = X^1\times X^2_{\boldsymbol{\gamma}} \subset X, \end{aligned} \tag{3.25}$$

where $F_{ai,bj}$ is specified in (2.8). The matrix F hence $F_{\boldsymbol{\gamma}}$ are positive definite if $p_i^{(a)} \ge 0$ for any $(a,i) \in \overline{H}$.

Example 3.3. Consider Example 3.1. Take the left hand side of (3.12) as a representative element of the angle variable of the path (3.8). Then the splitting (3.19) gives the reduced angle variable $\omega = {}^t(-1,-2,-3,-3)$ and $\boldsymbol{\lambda}$ is the quasi-periodic extension of the rigging in



(3.26)

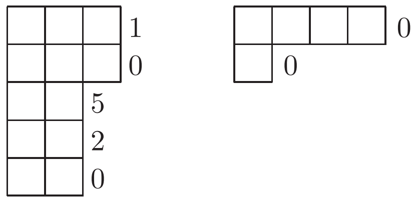
Thus $\boldsymbol{\lambda} = \big(\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_1^{(2)}, \lambda_2^{(2)}\big)$ reads

$$\begin{aligned} \lambda_1^{(1)} &= (\ldots,-4,-2,0,2,4,6,8,10,\ldots), & \lambda_2^{(1)} &= (\ldots,-7,-3,-2,0,4,5,7,11,12,\ldots),\\ \lambda_1^{(2)} &= (\ldots,-2,0,2,4,\ldots), & \lambda_2^{(2)} &= (\ldots,-1,0,1,2,\ldots), \end{aligned}$$

with $\lambda_{i,1}^{(a)} = 0$. There is order 2 symmetry $\gamma_1^{(1)} = 2$ in the block $(a,i) = (1,1)$ since $p_1^{(1)} = 4$ and $m_1^{(1)} = 2$ have a common divisor 2, and $\lambda_1^{(1)}$ satisfies $\lambda^{(1)}_{1,\alpha+m_1^{(1)}/2} = \lambda^{(1)}_{1,\alpha} + p_1^{(1)}/2$. Thus from (3.11), the matrix $F_{\boldsymbol{\gamma}}$ (3.25) becomes

$$F_{\boldsymbol{\gamma}} = F \cdot \mathrm{diag}\big(\tfrac{1}{2},1,1,1\big) = \begin{pmatrix} 8 & 12 & -3 & -1\\ 4 & 19 & -2 & -1\\ -3 & -6 & 10 & 2\\ -1 & -3 & 2 & 3 \end{pmatrix}. \tag{3.27}$$

Example 3.4. Consider Example 3.2. Take the left hand side of (3.15) as a representative element of the angle variable of the path (3.13). Then the splitting (3.19) gives the reduced angle variable $\omega = {}^t(-21,-20,1,-1)$ and $\boldsymbol{\lambda}$ is the quasi-periodic extension of the rigging in



Thus $\boldsymbol{\lambda} = \big(\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_1^{(2)}, \lambda_2^{(2)}\big)$ reads

$$\begin{aligned} \lambda_1^{(1)} &= (\ldots,-4,-3,0,1,4,5,8,9,\ldots), & \lambda_2^{(1)} &= (\ldots,-7,-5,-2,0,2,5,7,9,12,\ldots),\\ \lambda_1^{(2)} &= (\ldots,-2,0,2,4,\ldots), & \lambda_2^{(2)} &= (\ldots,-1,0,1,2,\ldots), \end{aligned}$$

with $\lambda_{i,1}^{(a)} = 0$. The order of symmetry is trivial in that $\forall \gamma_i^{(a)} = 1$. In this case one has $F_{\boldsymbol{\gamma}} = F$ (3.11).

An important property of the subset $X_{\boldsymbol{\gamma}} \subset X$ is that it is still invariant under the actions of both $\mathcal{T}$ and $\mathcal{A}$. Let $X_{\boldsymbol{\gamma}}/\mathcal{A}$ be the set of all $\mathcal{A}$-orbits. According to $X = \bigsqcup_{\boldsymbol{\gamma}} X_{\boldsymbol{\gamma}}$, we have the disjoint union decomposition:

$$X/\mathcal{A} = \bigsqcup_{\boldsymbol{\gamma}} (X_{\boldsymbol{\gamma}}/\mathcal{A}), \tag{3.28}$$

which induces the disjoint union decomposition of the level set $\mathcal{P}(\mu)$ according to (3.22). Writing the pre-image of $X_{\boldsymbol{\gamma}}/\mathcal{A}$ as $\mathcal{P}_{\boldsymbol{\gamma}}(\mu)$, one has $\mathcal{P}(\mu) = \sqcup_{\boldsymbol{\gamma}}\mathcal{P}_{\boldsymbol{\gamma}}(\mu)$ and the conjectural commutative diagram (3.22) splits into

$$\begin{array}{ccc} \mathcal{P}_{\boldsymbol{\gamma}}(\mu) & \xrightarrow{\Phi} & X_{\boldsymbol{\gamma}}/\mathcal{A} \\ \mathcal{T}\big\downarrow & & \big\downarrow\mathcal{T} \\ \mathcal{P}_{\boldsymbol{\gamma}}(\mu) & \xrightarrow{\Phi} & X_{\boldsymbol{\gamma}}/\mathcal{A} \end{array} \tag{3.29}$$

$\mathcal{P}_{\boldsymbol{\gamma}}(\mu)$ is the subset of the level set $\mathcal{P}(\mu)$ characterized by the order of symmetry $\boldsymbol{\gamma}$. We are yet to decompose it or equivalently $X_{\boldsymbol{\gamma}}/\mathcal{A}$ further into $\mathcal{T}$-orbits. Note that one can think of an $\mathcal{A}$-orbit $\mathcal{A}\cdot x$ either as an element of $X_{\boldsymbol{\gamma}}/\mathcal{A}$ or as a subset of $X_{\boldsymbol{\gamma}}$. Similarly a $\mathcal{T}$-orbit $\mathcal{T}\cdot(\mathcal{A}\cdot x)$ in $X_{\boldsymbol{\gamma}}/\mathcal{A}$ can either be regarded as an element of $(X_{\boldsymbol{\gamma}}/\mathcal{A})/\mathcal{T}$ or as a subset of $X_{\boldsymbol{\gamma}}/\mathcal{A}$. We adopt the latter interpretation. Then as we will see shortly in Proposition 3.1, the $\mathcal{T}$-orbits $\mathcal{T}\cdot(\mathcal{A}\cdot x)$ of $X_{\boldsymbol{\gamma}}/\mathcal{A}$ can be described explicitly in terms of $\mathbb{Z}^g/F_{\boldsymbol{\gamma}}\mathbb{Z}^g$, the set of integer points on the torus equipped with the following action of $\mathcal{T}$:

$$T_l^{(r)}(\mathcal{I}) = \mathcal{I} + h_l^{(r)} \mod F_{\boldsymbol{\gamma}}\mathbb{Z}^g \qquad \text{for } \mathcal{I} \in \mathbb{Z}^g/F_{\boldsymbol{\gamma}}\mathbb{Z}^g, \tag{3.30}$$

where $h_l^{(r)}$ is specified in (3.20). In what follows, we shall refer to $\mathbb{Z}^g/F_{\boldsymbol{\gamma}}\mathbb{Z}^g$ simply as torus.

Given any $x = (\omega, \boldsymbol{\lambda}) \in X_{\boldsymbol{\gamma}}$, consider the map

$$\begin{aligned} \chi:\ \mathcal{T}\cdot(\mathcal{A}\cdot x) &\longrightarrow \mathbb{Z}^g/F_{\boldsymbol{\gamma}}\mathbb{Z}^g \\ t\cdot(\mathcal{A}\cdot x) &\longmapsto \ t\cdot\omega \mod F_{\boldsymbol{\gamma}}\mathbb{Z}^g. \end{aligned} \tag{3.31}$$

Proposition 3.1 ([36]). *The map χ is well-defined, bijective and the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{T}\cdot(\mathcal{A}\cdot x) & \xrightarrow{\chi} & \mathbb{Z}^g/F_{\boldsymbol{\gamma}}\mathbb{Z}^g \\ \mathcal{T}\big\downarrow & & \big\downarrow\mathcal{T} \\ \mathcal{T}\cdot(\mathcal{A}\cdot x) & \xrightarrow{\chi} & \mathbb{Z}^g/F_{\boldsymbol{\gamma}}\mathbb{Z}^g \end{array}$$

Combining Proposition 3.1 with Conjecture 3.2 or its refined form (3.29), we obtain an explicit description of each connected component ($\mathcal{T}$-orbit) as a torus.

Conjecture 3.3. *For any path $p \in \mathcal{P}_{\boldsymbol{\gamma}}(\mu)$, the map $\Phi_\chi := \chi\circ\Phi$ gives a bijection between the connected component $\Sigma(p)$ and the torus $\mathbb{Z}^g/F_{\boldsymbol{\gamma}}\mathbb{Z}^g$ making the following diagram commutative:*

$$\begin{array}{ccc} \Sigma(p) & \xrightarrow{\Phi_\chi} & \mathbb{Z}^g/F_{\boldsymbol{\gamma}}\mathbb{Z}^g \\ \mathcal{T}\big\downarrow & & \big\downarrow\mathcal{T} \\ \Sigma(p) & \xrightarrow{\Phi_\chi} & \mathbb{Z}^g/F_{\boldsymbol{\gamma}}\mathbb{Z}^g \end{array} \tag{3.32}$$

The actions of $\mathcal{T}$ in Proposition 3.1 and Conjecture 3.3 are both transitive. Conjecture 3.3 is a principal claim in this paper. It says that the reduced angle variables live in the torus $\mathbb{Z}^g/F_{\boldsymbol{\gamma}}\mathbb{Z}^g$, where time evolutions of paths become straight motions. See Conjecture 3.6 for an analogous claim in a more general case than (2.14).

Remark 3.2. Due to the transitivity of $\mathcal{T}$-action, the commutative diagram (3.32) persists even if Φ_χ is redefined as $\Phi_\chi + c$ with any constant vector $c \in \mathbb{Z}^g$, meaning that the choice of the inverse image of $0 \in \mathbb{Z}^g/F_{\boldsymbol{\gamma}}\mathbb{Z}^g$ is at one's disposal. With this option in mind, an explicit way to construct the bijection Φ_χ is as follows. Fix an arbitrary highest path p_+ in $\Sigma(p)$ and set $\Phi_\chi(p_+) = 0$. For any $p' \in \Sigma(p)$ let t be an element of $\mathcal{T}$ such that $p' = t \cdot p_+$. One can always find such t since $\mathcal{T}$ acts on $\Sigma(p)$ transitively. If t is written as $t = \prod_{r,l} \big(T^{(r)}_l\big)^{d^{(r)}_l}$, then we set $\Phi_\chi(p') = \sum_{r,l} d^{(r)}_l h^{(r)}_l \mod F_{\boldsymbol{\gamma}}\,\mathbb{Z}^g$.

Remark 3.3. For $(a,i) \in \overline{H}$ we define $\eta^{(a)}_i = l^{(a)}_{i+1} + 1\,(i < g_a),\ = 1\,(i = g_a)$. Let $\overline{\mathcal{T}}$ be the free abelian group generated by all $T^{(a)}_{\eta^{(a)}_i}$'s. We let it act on the torus $\mathbb{Z}^g/F_{\boldsymbol{\gamma}}\,\mathbb{Z}^g$ by (3.30). This action is transitive because the g-dimensional lattice $\bigoplus_{(ai)\in\overline{H}} \mathbb{Z}h^{(a)}_{\eta^{(a)}_i} \subset \mathbb{Z}^g$ coincides with $\mathbb{Z}^g$ itself. If Conjecture 3.3 is valid, then as a subgroup of $\mathcal{T}$, $\overline{\mathcal{T}}$ also acts on $\Sigma(p)$ transitively. Hence the image of the bijection Φ_χ in the previous remark can be written as a linear combination of $h^{(r)}_{\eta^{(r)}_i}$ with $(r,i) \in \overline{H}$ only.

Example 3.5. As an application of the inverse scheme (3.32), we take the path p (3.8) in Example 3.1 and illustrate the solution of the initial value problem to derive

$$
\begin{aligned}
\big(T^{(1)}_2\big)^{1000}(p) &= 12221112211332111321131,\\
\big(T^{(1)}_3\big)^{1000}(p) &= 21332111221111222133111 3,\\
\big(T^{(2)}_1\big)^{1000}(p) &= 33111321133211221111122 2,\\
\big(T^{(2)}_2\big)^{1000}(p) &= 31113211321112221122113 3,\\
\big(T^{(2)}_3\big)^{1000}(p) &= 11322113211122113221331 1,\\
\big(T^{(2)}_4\big)^{1000}(p) &= 22111322111221132133311 1.
\end{aligned}
\tag{3.33}
$$

In Example 3.3, we have seen that the reduced angle variable of p is given by

$$
\omega = \begin{pmatrix} -1 \\ -2 \\ -3 \\ -3 \end{pmatrix} \in \mathbb{Z}^4/F_{\boldsymbol{\gamma}}\,\mathbb{Z}^4,
$$

where $F_{\boldsymbol{\gamma}}$ is specified in (3.27). The velocity vector $h^{(a)}_l$ (3.20) of the reduced angle variable corresponding to the time evolution $T^{(a)}_l$ is given by

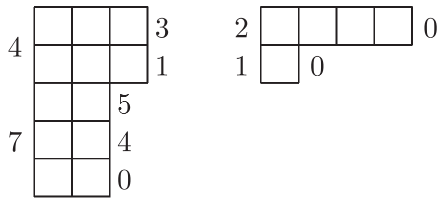
$$
h^{(1)}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad h^{(1)}_2 = 2h^{(1)}_1, \quad h^{(1)}_3 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \quad h^{(2)}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix},
$$
$$
h^{(2)}_2 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \quad h^{(2)}_3 = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 1 \end{pmatrix}, \quad h^{(2)}_4 = \begin{pmatrix} 0 \\ 0 \\ 4 \\ 1 \end{pmatrix}.
$$

The higher time evolutions $h^{(1)}_{l\geq 3}$ and $h^{(2)}_{l\geq 4}$ coincide with $h^{(1)}_3$ and $h^{(2)}_4$, respectively. Moreover from $h^{(1)}_2 = 2h^{(1)}_1$, one has $\big(T^{(1)}_2\big)^{1000}(p) = \big(T^{(1)}_1\big)^{2000}(p)$, and the latter is easily obtained since $T^{(1)}_1$ is just a cyclic shift (2.3), yielding $\big(T^{(1)}_2\big)^{1000}(p) = 12221112211332111321131$ in agreement with (3.33).

Next we consider $\big(T_3^{(1)}\big)^{1000}(p)$. The reduced angle variable for this path is

$$\omega + 1000h_3^{(1)} = \begin{pmatrix} 2999 \\ 1998 \\ -3 \\ -3 \end{pmatrix} \equiv \begin{pmatrix} 7 \\ 6 \\ 1 \\ 1 \end{pmatrix} \mod F_{\boldsymbol{\gamma}}\mathbb{Z}^4 = 6h_1^{(1)} + h_1^{(2)} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \mod F_{\boldsymbol{\gamma}}\mathbb{Z}^4.$$

This means $\big(T_3^{(1)}\big)^{1000}(p) = \big(T_1^{(1)}\big)^6 T_1^{(2)}(p')$, where p' is the path corresponding to ${}^t(1,0,0,0)$. In view of (3.26), this reduced angle variable corresponds to



$\stackrel{\phi^{-1}}{\longmapsto}$ $112211132211321133113221 = p'$.

From $T_1^{(2)}(p') = 1122111122221331113213321$, one gets $\big(T_1^{(1)}\big)^6 T_1^{(2)}(p')$ by a cyclic shift. The result yields $\big(T_3^{(1)}\big)^{1000}(p) = 213321112211112221331113$ in agreement with (3.33). Of course, the choice of the representative mod $F_{\boldsymbol{\gamma}}\mathbb{Z}^4$ is not unique.

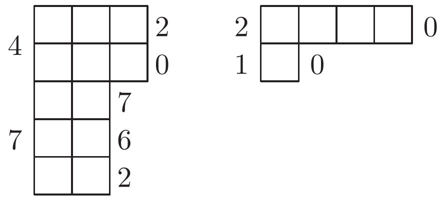
The procedure is parallel for $\big(T_l^{(2)}\big)^{1000}(p)$ with $l = 1, \ldots, 4$. The reduced angle variable of $\big(T_l^{(2)}\big)^{1000}(p)$ is

$$\omega + 1000h_l^{(2)} = \begin{pmatrix} -1 \\ -2 \\ 1000l - 3 \\ 997 \end{pmatrix} \overset{\bmod F_\gamma \mathbb{Z}^4}{\equiv} \begin{cases} 4h_1^{(1)} + 8h_1^{(2)} + {}^t(0,2,0,0), & l = 1, \\ 10h_1^{(1)} + h_1^{(2)} + {}^t(0,2,0,0), & l = 2, \\ 15h_1^{(1)} + 10h_1^{(2)} + {}^t(1,0,1,0), & l = 3, \\ 21h_1^{(1)} + 3h_1^{(2)} + {}^t(1,0,1,0), & l = 4. \end{cases}$$

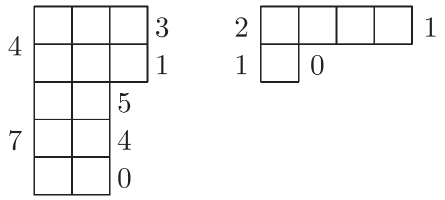
From this, one has

$$\begin{aligned} &\big(T_1^{(2)}\big)^{1000}(p) = \big(T_1^{(1)}\big)^4 \big(T_1^{(2)}\big)^8(p''), && \big(T_2^{(2)}\big)^{1000}(p) = \big(T_1^{(1)}\big)^{10} T_1^{(2)}(p''), \\ &\big(T_3^{(2)}\big)^{1000}(p) = \big(T_1^{(1)}\big)^{15} \big(T_1^{(2)}\big)^{10}(p'''), && \big(T_4^{(2)}\big)^{1000}(p) = \big(T_1^{(1)}\big)^{21} \big(T_1^{(2)}\big)^3(p'''). \end{aligned} \tag{3.34}$$

Here the paths p'' and p''' are those corresponding to the reduced angle variables ${}^t(0,2,0,0)$ and ${}^t(1,0,1,0)$, respectively. They are obtained as



$\stackrel{\phi^{-1}}{\longmapsto}$ $111222132111332113311122 = p''$,



$\stackrel{\phi^{-1}}{\longmapsto}$ $112211132211122133113321 = p'''$

In (3.34), the evolutions under $T_1^{(2)}$ is given by

$$T_1^{(2)}(p'') = 111222112211333111321132,$$

$$\big(T_1^{(2)}\big)^8(p'') = 1321133211221111112223311,$$
$$\big(T_1^{(2)}\big)^3(p''') = 1112211132211122211321333,$$
$$\big(T_1^{(2)}\big)^{10}(p''') = 1322133111132211132111221.$$

Applying the cyclic shift $T_1^{(1)}$ to these results, one can check that (3.34) leads to (3.33).

3.4 Bethe ansatz formula from size and number of connected components

The results in the previous section provide a beautiful interpretation of the character formula derived from the Bethe ansatz at $q = 0$ [19] in terms of the size and number of orbits in the periodic $A_n^{(1)}$ SCA. Let us first recall the character formula:

$$(x_1 + \cdots + x_{n+1})^L = \sum_{\mu} \Omega(\mu)\, x_1^{L-|\mu^{(1)}|} x_2^{|\mu^{(1)}|-|\mu^{(2)}|} \cdots x_{n+1}^{|\mu^{(n-1)}|-|\mu^{(n)}|}, \tag{3.35}$$

$$\Omega(\mu) = (\det F) \prod_{(ai)\in\overline{H}} \frac{1}{m_i^{(a)}} \binom{p_i^{(a)} + m_i^{(a)} - 1}{m_i^{(a)} - 1} \quad (\in \mathbb{Z}). \tag{3.36}$$

In (3.35), the sum is taken over n-tuple of Young diagrams $\mu = (\mu^{(1)}, \ldots, \mu^{(n)})$ *without any* constraint. All the quantities appearing in (3.36) are determined from μ by the left diagram in (2.5) and (2.6)–(2.8). The fact $\Omega(\mu) \in \mathbb{Z}$ can be easily seen by expanding $\det F$. It is vital that the binomial coefficient here is the extended one:

$$\binom{a}{b} = \frac{a(a-1)\cdots(a-b+1)}{b!} \qquad (a \in \mathbb{Z},\ b \in \mathbb{Z}_{\geq 0}),$$

which can be negative outside the range $0 \leq b \leq a$. Due to such negative contributions, the infinite sum (3.35) cancels out except leaving the finitely many positive contributions exactly when $L \geq |\mu^{(1)}| \geq \cdots \geq |\mu^{(n)}|$. For example when $L = 6$, $n = 2$, coefficients of monomials in the expansion (3.35) of $(x_1 + x_2 + x_3)^6$ are calculated as

$$\begin{aligned}
60\, x_1^3 x_2^2 x_1 :\ & \big(|\mu^{(1)}|, |\mu^{(2)}|\big) = (3, 1),\\
& \Omega((3),(1)) + \Omega((21),(1)) + \Omega((111),(1)) = 6 + 36 + 18 = 60,\\
0\, x_1^5 x_2^{-2} x_3^3 :\ & \big(|\mu^{(1)}|, |\mu^{(2)}|\big) = (1, 3),\\
& \Omega((1),(3)) + \Omega((1),(21)) + \Omega((1),(111)) = 6 + (-18) + 12 = 0.
\end{aligned}$$

In this way $\Omega(\mu)$ gives a decomposition of the multinomial coefficients according to μ. It is known [19, Lemma 3.7] that $\Omega(\mu) \geq 1$ if μ is a configuration, namely under the condition $p_i^{(a)} \geq 0$ for any $(a, i) \in \overline{H}$.

The formula (3.36) originates in the Bethe ansatz for the integrable vertex model associated with $U_q(A_n^{(1)})$ under the periodic boundary condition. In this context, μ specifies the string content. Namely, $\mu^{(a)}$ signifies that there are $m_i^{(a)}$ strings of color a and length $l_i^{(a)}$. Under such a string hypothesis, the Bethe equation becomes a linear congruence equation at $q = 0$ (3.42), and counting its off-diagonal solutions yields $\Omega(\mu)$. In this sense, the identity (3.35) implies a formal completeness of the Bethe ansatz and string hypothesis at $q = 0$. For more details, see Section 3.6, especially Theorem 3.3. A parallel story is known also at $q = 1$ [15] as mentioned under Theorem 2.2.

Back to our $A_n^{(1)}$ SCA, it is nothing but the integrable vertex model at $q = 0$, where $U_q\big(A_n^{(1)}\big)$ modules and row transfer matrices are effectively replaced by the crystals and time evolution operators, respectively. In view of this, it is natural to link the Bethe ansatz formula $\Omega(\mu)$ with the notions introduced in the previous subsection like level set, torus, connected components ($\mathcal{T}$-orbits) and so forth. This will be done in this subsection, providing a conceptual explanation of the earlier observations [25, 26]. Our main result is stated as

Theorem 3.1. *Assume the condition* (2.14) *and Conjecture* 3.3. *Then the Bethe ansatz formula* $\Omega(\mu)$ (3.36) *counts the number of paths in the level set* $\mathcal{P}(\mu)$ *as follows:*

$$|\mathcal{P}(\mu)| = \Omega(\mu) = \sum_{\boldsymbol{\gamma}} \underbrace{\det F_{\boldsymbol{\gamma}}}_{\text{size of a } \mathcal{T}\text{-orbit}} \underbrace{\prod_{(ai)\in\overline{H}} \frac{\left|\Lambda_{\gamma_i^{(a)}}\big(m_i^{(a)}, p_i^{(a)}\big)\right|}{m_i^{(a)}/\gamma_i^{(a)}}}_{\text{number of } \mathcal{T}\text{-orbits}}. \tag{3.37}$$

Here $\boldsymbol{\gamma} = \big(\gamma_i^{(a)}\big)_{(ai)\in\overline{H}}$ *and the sum extends over all the orders of symmetry, i.e., each* $\gamma_i^{(a)}$ *runs over all the common divisors of* $m_i^{(a)}$ *and* $p_i^{(a)}$.

The result (3.37) uncovers the SCA meaning of the Bethe ansatz formula (3.36). It consists of the contributions from sectors specified by the order of symmetry $\boldsymbol{\gamma}$. Each sector is an assembly of identical tori (connected components), therefore its contribution is factorized into its size and number (multiplicity).

For the proof we prepare a few facts. Note that $s_i^{(a)}$ (3.21) sending $\boldsymbol{\lambda}$ to $\boldsymbol{\lambda}'$ also defines an action of $\mathcal{A}$ on $X^2_{\boldsymbol{\gamma}}$ part alone.

Lemma 3.1. *The number of* $\mathcal{T}$*-orbits in* $X_{\boldsymbol{\gamma}}/\mathcal{A}$ *is* $|X^2_{\boldsymbol{\gamma}}/\mathcal{A}|$, *i.e., the number of* $\mathcal{A}$*-orbits in* $X^2_{\boldsymbol{\gamma}}$.

Proof. Let $y = \mathcal{A}\cdot x$ and $y' = \mathcal{A}\cdot x'$ be any two elements of $X_{\boldsymbol{\gamma}}/\mathcal{A}$. They belong to a common $\mathcal{T}$-orbit if and only if there exists $t \in \mathcal{T}$ such that $t\cdot y = y'$. It is equivalent to saying that there exist $t \in \mathcal{T}$ and $a \in \mathcal{A}$ such that $t\cdot(a\cdot x) = x'$. Write x, x' as $x = (\omega, \boldsymbol{\lambda})$ and $x' = (\omega', \boldsymbol{\lambda}')$.

Suppose $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ belong to a common $\mathcal{A}$-orbit in $X^2_{\boldsymbol{\gamma}}$. Then there exits $a \in \mathcal{A}$ such that $a\cdot\boldsymbol{\lambda} = \boldsymbol{\lambda}'$. Hence we have $a\cdot x = (\tilde{\omega}, \boldsymbol{\lambda}')$ with some $\tilde{\omega} \in X^1$. Since $\mathcal{T}$ acts on the X^1 part transitively and leaves the $X^2_{\boldsymbol{\gamma}}$ part untouched, there exits $t \in \mathcal{T}$ such that $t\cdot(a\cdot x) = x'$.

Suppose $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ belong to different $\mathcal{A}$-orbits. Then $a\cdot x$ and x' have different $X^2_{\boldsymbol{\gamma}}$ parts for any $a \in \mathcal{A}$. Hence for any $t \in \mathcal{T}$ and $a \in \mathcal{A}$, we have $t\cdot(a\cdot x) \neq x'$.

Thus y and y' belong to a common $\mathcal{T}$-orbit if and only if $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ belong to a common $\mathcal{A}$-orbit. The proof is completed. ■

Lemma 3.2. *The cardinality of the set* $X^2_{\boldsymbol{\gamma}}/\mathcal{A}$ *is given by*

$$|X^2_{\boldsymbol{\gamma}}/\mathcal{A}| = \prod_{(ai)\in\overline{H}} \frac{\left|\Lambda_{\gamma_i^{(a)}}\big(m_i^{(a)}, p_i^{(a)}\big)\right|}{m_i^{(a)}/\gamma_i^{(a)}}.$$

Proof. It is sufficient to show that each factor in the right hand side gives the number of $\mathcal{A}$-orbits for each (a,i) block. So we omit all the indices below and regard $\mathcal{A}$ as a free abelian group generated by a single element s.

Let $\lambda = (\lambda_\alpha)_{\alpha\in\mathbb{Z}}$ be any element of $\Lambda_\gamma(m,p)$. Then due to (3.21) we have $s^n\cdot\lambda = (\lambda_{\alpha+n} - \lambda_{1+n})_{\alpha\in\mathbb{Z}}$. By (3.23) and (3.17) this implies that $s^{m/\gamma}\cdot\lambda = \lambda$ and there is no $0 < k < m/\gamma$ such that $s^k\cdot\lambda = \lambda$. Hence the number of $\mathcal{A}$-orbits in $\Lambda_\gamma(m,p)$ is $|\Lambda_\gamma(m,p)|/(m/\gamma)$. ■

Lemma 3.3. *For any path* $p \in \mathcal{P}_{\boldsymbol{\gamma}}(\mu)$, *the size of each* $\mathcal{T}$*-orbit* $\Sigma(p)$ *is* $|\Sigma(p)| = \det F_{\boldsymbol{\gamma}}$.

Proof. This is due to Conjecture 3.3 and $\det F_{\boldsymbol{\gamma}} > 0$ under the assumption (2.14). See the remark under (3.25). ■

Now we are ready to give

Proof of Theorem 3.1.

$$
\begin{aligned}
|\mathcal{P}(\mu)| &\overset{(3.22),(3.28)}{=} \sum_{\boldsymbol{\gamma}} |X_{\boldsymbol{\gamma}}/\mathcal{A}| = \sum_{\boldsymbol{\gamma}} (\text{size of } \mathcal{T}\text{-orbits}) \times (\text{number of } \mathcal{T}\text{-orbits}) \\
&\overset{\text{Lemmas 3.1, 3.3}}{=} \sum_{\boldsymbol{\gamma}} (\det F_{\boldsymbol{\gamma}})\, |X^2_{\boldsymbol{\gamma}}/\mathcal{A}| \\
&\overset{(3.25),\,\text{Lemma 3.2}}{=} \sum_{\boldsymbol{\gamma}} \frac{\det F}{\prod_{(ai)\in\overline{H}} \gamma_i^{(a)}} \prod_{(ai)\in\overline{H}} \frac{|\Lambda_{\gamma_i^{(a)}}(m_i^{(a)}, p_i^{(a)})|}{m_i^{(a)}/\gamma_i^{(a)}} \\
&= \frac{\det F}{\prod_{(ai)\in\overline{H}} m_i^{(a)}} \sum_{\boldsymbol{\gamma}} \prod_{(ai)\in\overline{H}} |\Lambda_{\gamma_i^{(a)}}(m_i^{(a)}, p_i^{(a)})| \\
&= \frac{\det F}{\prod_{(ai)\in\overline{H}} m_i^{(a)}} \prod_{(ai)\in\overline{H}} \left(\sum_{\gamma_i^{(a)}} |\Lambda_{\gamma_i^{(a)}}(m_i^{(a)}, p_i^{(a)})| \right) \\
&\overset{(3.24)}{=} (\det F) \prod_{(ai)\in\overline{H}} \frac{1}{m_i^{(a)}} \binom{p_i^{(a)} + m_i^{(a)} - 1}{m_i^{(a)} - 1} = \Omega(\mu).
\end{aligned}
$$
■

One can give an explicit formula for $|\Lambda_\gamma(m,p)|$ by solving (3.24) by the Möbius inversion:

$$
|\Lambda_\gamma(m,p)| = \sum_\beta \mu\left(\frac{\beta}{\gamma}\right) \binom{\frac{p+m}{\beta} - 1}{\frac{m}{\beta} - 1}, \tag{3.38}
$$

where β runs over all the common divisors of m and p that is a multiple of γ. Here μ is the Möbius function in number theory [35] defined by $\mu(1) = 1, \mu(k) = 0$ if k is divisible by the square of an integer greater than one, and $\mu(k) = (-1)^j$ if k is the product of j distinct primes. (This μ should not be confused with the n-tuple of Young diagrams.)

Example 3.6. Let us consider the decomposition of the level set $\mathcal{P}(\mu)$ with $\mu = ((33222), (41))$, which is the same as Examples 3.1 and 3.2. From the tables therein and $\det F = 4656$, the formula (3.36) reads

$$
\begin{aligned}
\Omega(\mu) &= \det F \, \frac{1}{m_1^{(1)}} \binom{p_1^{(1)} + m_1^{(1)} - 1}{m_1^{(1)} - 1} \frac{1}{m_2^{(1)}} \binom{p_2^{(1)} + m_2^{(1)} - 1}{m_2^{(1)} - 1} \\
&= 4656 \cdot \frac{1}{2}\binom{5}{1} \cdot \frac{1}{3}\binom{9}{2} = 139680.
\end{aligned}
$$

On the other hand, (3.38) gives

$$
\begin{aligned}
|\Lambda_1(p_1^{(1)}, m_1^{(1)})| &= \mu\left(\frac{1}{1}\right) \binom{\frac{4+2}{1} - 1}{\frac{2}{1} - 1} + \mu\left(\frac{2}{1}\right) \binom{\frac{4+2}{2} - 1}{\frac{2}{2} - 1} = 5 - 1 = 4, \\
|\Lambda_2(p_1^{(1)}, m_1^{(1)})| &= \mu\left(\frac{2}{2}\right) \binom{\frac{4+2}{2} - 1}{\frac{2}{2} - 1} = 1, \\
|\Lambda_1(p_2^{(1)}, m_2^{(1)})| &= \mu\left(\frac{1}{1}\right) \binom{\frac{7+3}{1} - 1}{\frac{3}{1} - 1} = 36, \\
|\Lambda_1(p_1^{(2)}, m_1^{(2)})| &= |\Lambda_1(p_2^{(2)}, m_2^{(2)})| = 1.
\end{aligned}
$$

The formula (3.37) consists of the two terms corresponding to the order of symmetry $\boldsymbol{\gamma} = \bigl(\gamma_1^{(1)}, \gamma_2^{(1)}, \gamma_1^{(2)}, \gamma_2^{(2)}\bigr) = (1,1,1,1)$ and $(2,1,1,1)$. Let us write them as $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$, respectively. $F_{\boldsymbol{\gamma}_1} = F$ is given by (3.11) and $F_{\boldsymbol{\gamma}_2}$ by (3.27). Their determinants are $\det F_{\boldsymbol{\gamma}_1} = 4656$ and $\det F_{\boldsymbol{\gamma}_2} = 2328$. Thus (3.37) reads

$$|\mathcal{P}(\mu)| = \Omega(\mu) = 139680 = (\det F_{\boldsymbol{\gamma}_1}) \cdot 24 + (\det F_{\boldsymbol{\gamma}_2}) \cdot 12.$$

Accordingly the level set $\mathcal{P}(\mu)$ is decomposed into 36 tori as

$$\mathcal{P}(\mu) = 24\bigl(\mathbb{Z}^4/F_{\boldsymbol{\gamma}_1}\mathbb{Z}^4\bigr) \sqcup 12\bigl(\mathbb{Z}^4/F_{\boldsymbol{\gamma}_2}\mathbb{Z}^4\bigr).$$

The paths (3.8) and (3.13) in Examples 3.1 and 3.2 belong to one of the tori $\mathbb{Z}^4/F_{\boldsymbol{\gamma}_2}\mathbb{Z}^4$ and $\mathbb{Z}^4/F_{\boldsymbol{\gamma}_1}\mathbb{Z}^4$, respectively.

3.5 Dynamical period

Given a path $p \in \mathcal{P}_{\boldsymbol{\gamma}}(\mu)$, the *smallest* positive integer satisfying $(T_l^{(r)})^{\mathcal{N}}(p) = p$ is called the *dynamical period* of p under the time evolution $T_l^{(r)}$. Here we derive an explicit formula of the dynamical period as a simple corollary of Conjecture 3.3. It takes the precise account of the symmetry specified by $\boldsymbol{\gamma}$ and refines the earlier conjectures in [25, 26] for $n \geq 2$ which was obtained from the Bethe eigenvalues at $q = 0$.

For nonzero rational numbers $r_1, \ldots, r_s$, we define their least common multiple by

$$\mathrm{LCM}(r_1, \ldots, r_s) = \min\bigl(|\mathbb{Z} \cap r_1\mathbb{Z} \cap \cdots \cap r_s\mathbb{Z}| \setminus \{0\}\bigr).$$

Given (r, l) with $1 \leq r \leq n$ and $l \geq 1$, define $F_{\boldsymbol{\gamma}}[bj]$ (resp. $F[bj]$) to be the $g \times g$ matrix obtained from $F_{\boldsymbol{\gamma}}$ (3.25) (resp. F (2.8)) by replacing its (bj)th column by $h_l^{(r)}$ (3.20). Set

$$\mathcal{N}_l^{(r)} = \mathrm{LCM}\left(\frac{\det F_{\boldsymbol{\gamma}}}{\det F_{\boldsymbol{\gamma}}[bj]}\right)_{(bj)\in\overline{H}} = \mathrm{LCM}\left(\frac{\det F}{\gamma_j^{(b)} \det F[bj]}\right)_{(bj)\in\overline{H}}, \tag{3.39}$$

where the LCM should be taken over only those (bj) satisfying $\det F[bj] \neq 0$.

Theorem 3.2. *Under Conjecture* 3.3*, the dynamical period under $T_l^{(r)}$ is equal to $\mathcal{N}_l^{(r)}$* (3.39) *for all the paths in $\mathcal{P}_{\boldsymbol{\gamma}}(\mu)$.*

Proof. Under Conjecture 3.3, the relation $\bigl(T_l^{(r)}\bigr)^{\mathcal{N}}(p) = p$ is equivalent to

$$\mathcal{N} h_l^{(r)} \equiv 0 \mod F_{\boldsymbol{\gamma}}\mathbb{Z}^g \tag{3.40}$$

for any path $p \in \mathcal{P}_{\boldsymbol{\gamma}}(\mu)$. In other words there exists $z \in \mathbb{Z}^g$ such that $\mathcal{N} h_l^{(r)} = F_{\boldsymbol{\gamma}} z$. By Cramer's formula, the solution $z = (z_j^{(b)})_{(bj)\in\overline{H}}$ of this equation is given by $z_j^{(b)} = \mathcal{N}\frac{\det F_{\boldsymbol{\gamma}}[bj]}{\det F_{\boldsymbol{\gamma}}}$. The quantity $\mathcal{N}_l^{(r)}$ (3.39) is the smallest positive integer $\mathcal{N}$ that matches the condition $z \in \mathbb{Z}^g$ hence (3.40). ■

Remark 3.4. To the dynamical period under a composite time evolution $T = \prod_{r,l}\bigl(T_l^{(r)}\bigr)^{d_l^{(r)}}$, the same formula (3.39) applies by replacing the role of $h_l^{(r)}$ therein with $\sum_{r,l} d_l^{(r)} h_l^{(r)}$.

Remark 3.5. The formula (3.39) can be simplified when $n = 1$ (hence $r = 1$) and the order of symmetry is trivial, i.e., $\boldsymbol{\gamma} = (\gamma_i^{(1)})$ with $\forall\, \gamma_i^{(1)} = 1$ as explained in [24, equation (4.24)]. In particular if $l = \infty$ furthermore, one gets ($p_0^{(1)} = l_0^{(1)} = 0$)

$$\mathcal{N}_\infty^{(1)} = \mathrm{LCM}\left(\frac{p_j^{(1)} p_{j-1}^{(1)}}{(l_j^{(1)} - l_{j-1}^{(1)})p_{g_1}^{(1)}}\right)_{1\le j\le g_1},$$

which reproduces the result obtained in [40].

Example 3.7. Let us take the path p (3.8) and compute its dynamical period $\mathcal{N}_l^{(r)}$ under the time evolution $T_l^{(r)}$. The matrix $F_{\boldsymbol{\gamma}}$ is given in (3.27) and $\det F_{\boldsymbol{\gamma}} = 2328$. The relevant vectors $h_l^{(a)}$ are listed in Example 3.5. Consider $T_3^{(1)}$ for instance. Then, $h_3^{(1)} = {}^t(3,2,0,0)$ and

$$F_{\boldsymbol{\gamma}}[11] = \begin{pmatrix} 3 & 12 & -3 & -1 \\ 2 & 19 & -2 & -1 \\ 0 & -6 & 10 & 2 \\ 0 & -3 & 2 & 3 \end{pmatrix}, \qquad F_{\boldsymbol{\gamma}}[12] = \begin{pmatrix} 8 & 3 & -3 & -1 \\ 4 & 2 & -2 & -1 \\ -3 & 0 & 10 & 2 \\ -1 & 0 & 2 & 3 \end{pmatrix},$$

$$F_{\boldsymbol{\gamma}}[21] = \begin{pmatrix} 8 & 12 & 3 & -1 \\ 4 & 19 & 2 & -1 \\ -3 & -6 & 0 & 2 \\ -1 & -3 & 0 & 3 \end{pmatrix}, \qquad F_{\boldsymbol{\gamma}}[22] = \begin{pmatrix} 8 & 12 & -3 & 3 \\ 4 & 19 & -2 & 2 \\ -3 & -6 & 10 & 0 \\ -1 & -3 & 2 & 0 \end{pmatrix}$$

with $\det F_{\boldsymbol{\gamma}}[11] = 840$, $\det F_{\boldsymbol{\gamma}}[12] = 108$, $\det F_{\boldsymbol{\gamma}}[22] = 276$ and $\det F_{\boldsymbol{\gamma}}[22] = 204$. Thus (3.39) is calculated as

$$\begin{aligned}
\mathcal{N}_3^{(1)} &= \mathrm{LCM}\left(\frac{\det F_{\boldsymbol{\gamma}}}{\det F_{\boldsymbol{\gamma}}[11]}, \frac{\det F_{\boldsymbol{\gamma}}}{\det F_{\boldsymbol{\gamma}}[12]}, \frac{\det F_{\boldsymbol{\gamma}}}{\det F_{\boldsymbol{\gamma}}[21]}, \frac{\det F_{\boldsymbol{\gamma}}}{\det F_{\boldsymbol{\gamma}}[22]}\right) \\
&= \mathrm{LCM}\left(\frac{2328}{840}, \frac{2328}{108}, \frac{2328}{276}, \frac{2328}{204}\right) = \mathrm{LCM}\left(\frac{97}{35}, \frac{194}{9}, \frac{194}{23}, \frac{194}{17}\right) = 194.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\mathcal{N}_1^{(1)} &= \mathrm{LCM}\,(12, 24, 24, 24) = 24, \\
\mathcal{N}_2^{(1)} &= \mathrm{LCM}\,(6, 12, 12, 12) = 12, \\
\mathcal{N}_1^{(2)} &= \mathrm{LCM}\left(\frac{582}{23}, \frac{1164}{17}, \frac{1164}{65}, \frac{1164}{377}\right) = 1164, \\
\mathcal{N}_2^{(2)} &= \mathrm{LCM}\left(\frac{388}{29}, \frac{776}{13}, \frac{776}{141}, \frac{776}{197}\right) = 776, \\
\mathcal{N}_3^{(2)} &= \mathrm{LCM}\left(\frac{291}{32}, \frac{582}{11}, \frac{582}{179}, \frac{582}{107}\right) = 582, \\
\mathcal{N}_4^{(2)} &= \mathrm{LCM}\left(\frac{1164}{169}, \frac{2328}{49}, \frac{2328}{1009}, \frac{2328}{265}\right) = 2328.
\end{aligned}$$

Despite the nontrivial order of symmetry $\boldsymbol{\gamma} = (2,1,1,1)$, these final results coincide with the trivial case $\boldsymbol{\gamma} = (1,1,1,1)$ treated in Example 1.1. We have checked that these values agree with the actual dynamical periods of p by computer.

3.6 Relation to Bethe ansatz at $q = 0$

Let us quickly recall the relevant results from the Bethe ansatz at $q = 0$. For the precise definitions and statements, we refer to [19]. Consider the integrable $U_q\big(A_n^{(1)}\big)$ vertex model on a periodic chain of length L. If the quantum space is the L-fold tensor product of the vector representation, the Bethe equation takes the form:

$$\left(\frac{\sin\pi\big(u_i^{(a)}+\sqrt{-1}\hbar\delta_{a1}\big)}{\sin\pi\big(u_i^{(a)}-\sqrt{-1}\hbar\delta_{a1}\big)}\right)^L = -\prod_{b=1}^{n}\prod_{j=1}^{|\mu^{(b)}|}\frac{\sin\pi\big(u_i^{(a)}-u_j^{(b)}+\sqrt{-1}\hbar C_{ab}\big)}{\sin\pi\big(u_i^{(a)}-u_j^{(b)}-\sqrt{-1}\hbar C_{ab}\big)} \tag{3.41}$$

for $1 \le a \le n$ and $1 \le i \le \big|\mu^{(a)}\big|$. Here $L \ge \big|\mu^{(1)}\big| \ge \cdots \ge \big|\mu^{(n)}\big|$ specifies a sector (2.20) preserved by row transfer matrices. The parameter $\hbar$ is related to q by $q = e^{-2\pi\hbar}$.

Fix an n-tuple of Young diagrams, i.e., the string content $\mu = \big(\mu^{(1)}, \dots, \mu^{(n)}\big)$ as in (2.5). We keep the notations (2.6)–(2.9). By string solutions we mean the ones in which the unknowns $\big\{u_i^{(a)} \mid 1 \le a \le n, 1 \le i \le |\mu^{(a)}|\big\}$ are arranged as

$$\bigcup_{(ai\alpha)\in H}\ \bigcup_{u_{i,\alpha}^{(a)}\in\mathbb{R}} \big\{u_{i,\alpha}^{(a)} + \sqrt{-1}(i+1-2k)\hbar + \epsilon_{i\alpha k}^{(a)} \mid 1 \le k \le i\big\},$$

where $\epsilon_{i\alpha k}^{(a)}$ stands for a small deviation. $u_{i,\alpha}^{(a)}$ is the string center of the αth string of color a and length i. For a generic string solution, the Bethe equation is linearized at $q = 0$ into a logarithmic form called the *string center equation*:

$$\sum_{(bj\beta)\in H} A_{ai\alpha,bj\beta}\, u_{j,\beta}^{(b)} \equiv \frac{1}{2}\big(p_i^{(a)} + m_i^{(a)} + 1\big) \mod \mathbb{Z} \tag{3.42}$$

for $(ai\alpha) \in H$. Here the $G \times G$ coefficient matrix $A = (A_{ai\alpha,bj\beta})_{(ai\alpha),(bj\beta)\in H}$ is specified as

$$A_{ai\alpha,bj\beta} = \delta_{ab}\delta_{ij}\delta_{\alpha\beta}\big(p_i^{(a)} + m_i^{(a)}\big) + C_{ab}\min\big(l_i^{(a)}, l_j^{(b)}\big) - \delta_{ab}\delta_{ij}. \tag{3.43}$$

It is known [19] that A is positive definite if μ is a configuration, namely if (3.46) is satisfied.

There are a number of conditions which the solutions of the string center equation (3.42) are to satisfy or to be identified thereunder. First, the Bethe vector depends on $u_i^{(a)}$ only via $e^{2\pi\sqrt{-1}u_i^{(a)}}$. Therefore the string center should be understood as $u_{i,\alpha}^{(a)} \in \mathbb{R}/\mathbb{Z}$ rather than $\mathbb{R}$. Second, the original Bethe equation (3.41) is symmetric with respect to $u_i^{(a)}$ for $i = 1, \dots, \big|\mu^{(a)}\big|$, but their permutation does not lead to a new Bethe vector. Consequently, we should regard the string centers of the (a, i) block as

$$\big(u_{i,1}^{(a)}, u_{i,2}^{(a)}, \dots, u_{i,m_i^{(a)}}^{(a)}\big) \in (\mathbb{R}/\mathbb{Z})^{m_i^{(a)}}/\mathfrak{S}_{m_i^{(a)}}.$$

Last, we prohibit the collision of string centers $u_{i,\alpha}^{(a)} = u_{i,\beta}^{(a)}$ for $1 \le \alpha \ne \beta \le m_i^{(a)}$ for any $(a, i) \in \overline{H}$. This is a remnant of the well-known constraint on the Bethe roots so that the associated Bethe vector does not vanish. To summarize, we consider *off-diagonal solutions* $\mathbf{u} = \big(u_{i,\alpha}^{(a)}\big)_{(ai\alpha)\in H}$ to the string center equation (3.42) that live in

$$\big(u_{i,1}^{(a)}, u_{i,2}^{(a)}, \dots, u_{i,m_i^{(a)}}^{(a)}\big) \in \big((\mathbb{R}/\mathbb{Z})^{m_i^{(a)}} - \Delta_{m_i^{(a)}}\big)/\mathfrak{S}_{m_i^{(a)}} \quad \text{for each } (a, i) \in \overline{H},$$

where $\Delta_m = \{(v_1, \dots, v_m) \in (\mathbb{R}/\mathbb{Z})^m \mid v_\alpha = v_\beta \text{ for some } 1 \le \alpha \ne \beta \le m\}$. For simplicity we will often say ($q = 0$) *Bethe roots* to mean the off-diagonal solutions to the string center equation. Let $\mathcal{U}(\mu)$ be the set of the Bethe roots having the string content μ. The following result is derived by counting the Bethe roots of (3.42) by the Möbius inversion formula.

Theorem 3.3 ([19, Theorem 3.2]). *Under the condition* (3.46), *the formula* (3.36) *gives the number of Bethe roots, namely* $\Omega(\mu) = |\mathcal{U}(\mu)|$ *is valid.*

Given an extended rigged configuration $\mathbf{r} = \big(r^{(a)}_{i,\alpha}\big)_{(ai)\in\overline{H},\alpha\in\mathbb{Z}} \in \tilde{\mathcal{J}}(\mu)$, we define a map

$$\begin{array}{rccc} \Psi: & \tilde{\mathcal{J}}(\mu) & \longrightarrow & \mathcal{U}(\mu) \\ & \big(r^{(a)}_{i,\alpha}\big)_{(ai)\in\overline{H},\alpha\in\mathbb{Z}} & \longmapsto & \big(u^{(a)}_{i,\alpha}\big)_{(ai\alpha)\in H} \end{array}$$

by

$$\sum_{(bj\beta)\in H} A_{ai\alpha,bj\beta}\, u^{(b)}_{j,\beta} = \frac{1}{2}\big(p^{(a)}_i + m^{(a)}_i + 1\big) + r^{(a)}_{i,\alpha} + \alpha - 1 \quad \text{for} \ \ (ai\alpha)\in H. \tag{3.44}$$

Theorem 3.4. *The map* Ψ *induces the bijection between the set* $\mathcal{J}(\mu)$ *of angle variables and the set* $\mathcal{U}(\mu)$ *of Bethe roots.*

The proof is parallel with [24, Section 4.2] for the $n = 1$ case. The induced bijection will also be denoted by Ψ. It also induces the action of the time evolutions $\mathcal{T}$ on $\mathcal{U}(\mu)$ by $T^{(r)}_l(\mathbf{u}) = \Psi\big(T^{(r)}_l(\mathbf{r})\big)$ for $\mathbf{u} = \Psi(\mathbf{r})$. Under the condition (2.14), Conjecture 3.2 and Theorem 3.4 lead to the following commutative diagram among the level set, the angle variables and the Bethe roots:

$$\begin{array}{ccccc} \mathcal{P}(\mu) & \xrightarrow{\Phi} & \mathcal{J}(\mu) & \xrightarrow{\Psi} & \mathcal{U}(\mu) \\ {\scriptstyle\mathcal{T}}\big\downarrow & & {\scriptstyle\mathcal{T}}\big\downarrow & & {\scriptstyle\mathcal{T}}\big\downarrow \\ \mathcal{P}(\mu) & \xrightarrow{\Phi} & \mathcal{J}(\mu) & \xrightarrow{\Psi} & \mathcal{U}(\mu) \end{array} \tag{3.45}$$

Their cardinality is given by

$$|\mathcal{P}(\mu)| = |\mathcal{J}(\mu)| = |\mathcal{U}(\mu)| = \Omega(\mu)$$

with $\Omega(\mu)$ defined in (3.36). We will argue the time evolutions of the Bethe roots further in Section 4.2.

3.7 General case

Let $\mathcal{P}(\mu)$ be a level set. From μ, specify the data like $m^{(a)}_i$, $l^{(a)}_i$ and the vacancy number $p^{(a)}_i$ by (2.5)–(2.9). In Sections 3.2–3.5, we have considered the case $\forall\, p^{(a)}_i \geq 1$ (2.14). In this subsection we treat the general configuration, namely we assume

$$\mu = \big(\mu^{(1)}, \ldots, \mu^{(n)}\big) \ \text{ satisfies } \ p^{(a)}_i \geq 0 \ \text{ for all } (a,i)\in\overline{H}. \tag{3.46}$$

It turns out that the linearization scheme remains the same provided one discards some time evolutions and restricts the dynamics to a subgroup $\mathcal{T}'$ of $\mathcal{T}$. We begin by preparing some notations about partitions.

Let λ, ν be partitions or equivalently Young diagrams. We define $\lambda\cup\nu$ to be the partition whose parts are those of λ and ν, arranged in decreasing order [27]. For example, if $\lambda = (4221)$ and $\nu = (331)$, then $\lambda\cup\nu = (4332211)$.

We denote by $\Box^{(a)}_i$ the (a,i) block in the diagram $\mu^{(a)}$ in the sense of (2.5), and by $2\Box^{(a)}_i$ the corresponding block in $\mu^{(a)}\cup\mu^{(a)}$, which is a $\big(2m^{(a)}_i\big)\times l^{(a)}_i$ rectangle.

Let λ be a partition. We say that λ *covers* the block $\Box^{(a)}_i$ in $\mu^{(a)}$ if $\Box^{(a)}_i$ is inside the diagram λ when the two diagrams λ and $\mu^{(a)}$ are so placed as their top-left corners coincide.

Definition 3.1. The block $\square_i^{(a)}$ is *null* if $p_i^{(a)} = 0$.

Definition 3.2. The block $\square_i^{(a)}$ is *convex* if $\big(\mu^{(a-1)} \cup \mu^{(a+1)}\big)$ covers the block $2\square_i^{(a)}$ in $\mu^{(a)} \cup \mu^{(a)}$.

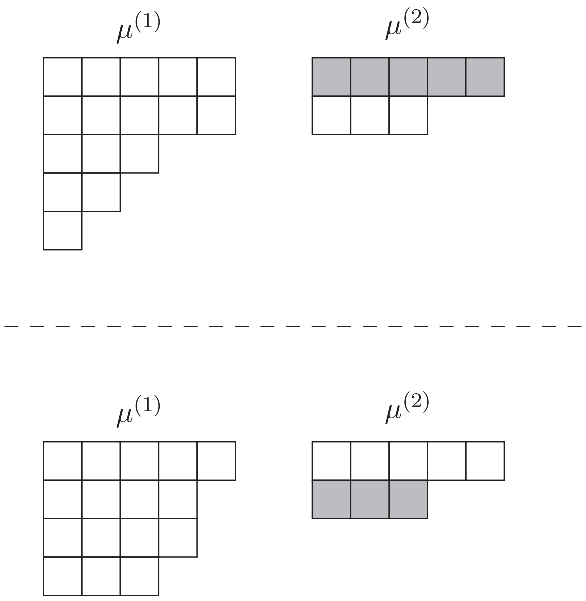
Here we interpret $\mu^{(0)} = (1^L)$ and $\mu^{(n+1)} = \varnothing$.

Say that the time evolution $T_l^{(a)}$ is *inadmissible* to μ if there exists $1 \le i \le g_a$ such that $\square_i^{(a)}$ is null and convex, and $l_i^{(a)} > l > l_{i+1}^{(a)}$. Otherwise we say that $T_l^{(a)}$ is *admissible* to μ. Here $l_{g_a+1}^{(a)}$ is to be understood as 0. Note that when $l_i^{(a)} - l_{i+1}^{(a)} = 1$, the block $\square_i^{(a)}$ does not cause an inadmissible time evolution even if it is null and convex. We will also say that $T_l^{(a)}$ is admissible or inadmissible to a path $p \in \mathcal{P}(\mu)$ depending on whether $T_l^{(a)}$ is admissible or inadmissible to μ.

As a generalization of Conjecture 2.1, we propose

Conjecture 3.4. *$T_l^{(r)}(\mathcal{P}(\mu)) = \mathcal{P}(\mu)$ holds if and only if $T_l^{(r)}$ is admissible to μ. Thus, $T_l^{(r)}(\mathcal{P}(\mu)) = \mathcal{P}(\mu)$ is valid for any r, l if and only if there is no block $\square_i^{(a)}$ that is null, convex and of length $l_i^{(a)}$ such that $l_i^{(a)} - l_{i+1}^{(a)} \ge 2$.*

Example 3.8. Consider the following configurations corresponding to the evolvable paths $p_I = 1211223331111122222233333$ (upper one) and $p_{II} = 1112221333211112222233333$ (lower one). Hatched blocks are null and convex. They show that $T_4^{(2)}$ is inadmissible to p_I, and $T_1^{(2)}$ and $T_2^{(2)}$ are inadmissible to p_{II}.



Let $\mathcal{T}'$ be the abelian group generated by all the time evolutions $T_l^{(a)}$ admissible to μ. Then Conjecture 3.4 implies that $\mathcal{T}'$ acts on the level set $\mathcal{P}(\mu)$. We define the connected component $\Sigma'(p)$ to be the $\mathcal{T}'$-orbit in $\mathcal{P}(\mu)$ that contains p.

As it turns out, the restriction of the dynamics from $\mathcal{T}$ to $\mathcal{T}'$ will be matched by introducing a sub-lattice $\mathbb{L}$ of $\mathbb{Z}^g$ as follows:

$$\mathbb{L} = \bigoplus_{(ai) \in \overline{H}} \mathbb{Z} h_{\xi_i^{(a)}}^{(a)} \subset \mathbb{Z}^g, \qquad \xi_i^{(a)} = \begin{cases} l_i^{(a)} & \text{if } \square_i^{(a)} \text{ is null and convex}, \\ l_{i+1}^{(a)} + 1 & \text{otherwise}. \end{cases}$$

See (3.20) for the definition of $h_l^{(a)}$. The subgroup $\mathcal{T}' \subset \mathcal{T}$ and $\xi_i^{(a)}$ here are analogues of $\overline{\mathcal{T}}$ and $\eta_i^{(a)}$ in Remark 3.3 in the present setting. It is easy to check

Lemma 3.4. *Let* $\mu = \left(\mu^{(1)}, \dots, \mu^{(n)}\right)$ *be a configuration, i.e.,* (3.46) *is satisfied. Suppose the* (a,i) *block of* μ *is null and convex. Then:*

(*i*) $p^{(a)}_{i+1} = 0$.

(*ii*) *For* $b = a \pm 1$ *there is no* $1 \le j \le g_b$ *such that* $l^{(a)}_i > l^{(b)}_j > l^{(a)}_{i+1}$.

Recall that the matrix $F_{\boldsymbol{\gamma}}$ is defined by (2.8) and (3.25).

Proposition 3.2. $F_{\boldsymbol{\gamma}}\mathbb{Z}^g$ *is a sub-lattice of* $\mathbb{L}$.

Proof. Let $\tilde{F}_a$ be the $g_a \times (g_{a-1} + g_a + g_{a+1})$ sub-matrix of $F_{\boldsymbol{\gamma}}$ for color a row and colors a, $a \pm 1$ column indices. Its (bj)th column ($b = a, a\pm 1$) is the array $\left(F_{ai,bj}/\gamma^{(b)}_j\right)_{1\le i\le g_a}$. Let $\bar{h}^{(a)}_{\xi^{(a)}_i} \in \mathbb{Z}^{g_a}$ be the sub-vector obtained from $h^{(a)}_{\xi^{(a)}_i}$ by extracting the color a rows. We are to show that for any a, all the column vectors of $\tilde{F}_a$ can be expressed as a linear combination of $\bar{h}^{(a)}_{\xi^{(a)}_i}$ with integer coefficients. For simplicity we demonstrate it by an example. The general case is similar.

Suppose $g_a = 7$ and the (a,i) block is null and convex if and only if $i = 1, 2, 6, 7$. Then the vectors $\bar{h}^{(a)}_{\xi^{(a)}_i}$ $(1 \le i \le 7)$ are given by the columns of the matrix in the left hand side of the following equation:

$$\begin{pmatrix} l_1 & l_2 & l_4+1 & l_5+1 & l_6+1 & l_6 & l_7 \\ l_2 & l_2 & l_4+1 & l_5+1 & l_6+1 & l_6 & l_7 \\ l_3 & l_3 & l_4+1 & l_5+1 & l_6+1 & l_6 & l_7 \\ l_4 & l_4 & l_4 & l_5+1 & l_6+1 & l_6 & l_7 \\ l_5 & l_5 & l_5 & l_5 & l_6+1 & l_6 & l_7 \\ l_6 & l_6 & l_6 & l_6 & l_6 & l_6 & l_7 \\ l_7 & l_7 & l_7 & l_7 & l_7 & l_7 & l_7 \end{pmatrix} \simeq \begin{pmatrix} l_1 & l_2 & 1 & & & & \\ l_2 & l_2 & 1 & & & & \\ l_3 & l_3 & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & l_6 & l_7 \\ & & & & & l_7 & l_7 \end{pmatrix}, \tag{3.47}$$

where $l_i = l^{(a)}_i$. The right hand side is obtained from the left (and vice versa) by elementary transformations adding integer multiple of one column to other columns.

By (*i*) of Lemma 3.4 we have $p^{(a)}_3 = 0$, and by (*ii*) of the same lemma any column of $\tilde{F}_a$ is an integer multiple (and possible integer addition to the 4-th and 5-th rows) of one of the following vectors:

$$\begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \\ l_5 \\ l_6 \\ l_7 \end{pmatrix} \text{for } l^{(b)}_j \ge l_1, \quad \begin{pmatrix} l_2 \\ l_2 \\ l_3 \\ l_4 \\ l_5 \\ l_6 \\ l_7 \end{pmatrix} \text{for } l^{(b)}_j = l_2, \quad \begin{pmatrix} l \\ l \\ l \\ x \\ y \\ l_6 \\ l_7 \end{pmatrix} \text{for } l_3 \ge l^{(b)}_j = l \ge l_6, \quad \begin{pmatrix} l_7 \\ l_7 \\ l_7 \\ l_7 \\ l_7 \\ l_7 \\ l_7 \end{pmatrix} \text{for } l^{(b)}_j = l_7,$$

where x and y are some integers. Clearly these vectors can be expressed by linear combinations of the columns of the matrices in (3.47) with integer coefficients. ■

Proposition 3.3. *The volume of the unit cell of the lattice* $\mathbb{L}$ *is equal to* $\prod \left(l^{(a)}_i - l^{(a)}_{i+1}\right)$, *where the product is taken over all the null and convex blocks* $\square^{(a)}_i$.

Proof. By using the above example we have

$$\begin{pmatrix} l_1 & l_2 & 1 & & & & \\ l_2 & l_2 & 1 & & & & \\ l_3 & l_3 & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & l_6 & l_7 \\ & & & & & l_7 & l_7 \end{pmatrix} \simeq \begin{pmatrix} l_1-l_2 & l_2-l_3 & 1 & & & & \\ & l_2-l_3 & 1 & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & l_6-l_7 & l_7 \\ & & & & & & l_7 \end{pmatrix}.$$

The general case is similar. ■

Let the abelian group $\mathcal{T}'$ act on the torus $\mathbb{L}/F_{\boldsymbol{\gamma}}\mathbb{Z}^g$ by (3.30). This action is transitive, since $\mathcal{T}'$ contains the free abelian subgroup generated by all $T^{(a)}_{\xi^{(a)}_i}$'s.

Conjecture 3.5. *Under the condition* (3.46)*,* $\Sigma'(p) \cap \mathcal{P}_+(\mu) \neq \varnothing$ *holds for any* $p \in \mathcal{P}(\mu)$*.*

Owing to this property, one can introduce Φ' by the same scheme as (3.6) with $\mathcal{T}$ replaced by $\mathcal{T}'$. Similarly, let $\chi' : \mathcal{T}' \cdot (\mathcal{A} \cdot x) \longrightarrow \mathbb{L}/F_{\boldsymbol{\gamma}}\mathbb{Z}^g$ be defined as in (3.31). As the generalization of Conjecture 3.3, we have

Conjecture 3.6. *For any path* $p \in \mathcal{P}_{\boldsymbol{\gamma}}(\mu)$*, the map* $\Phi'_\chi := \chi' \circ \Phi'$ *gives a bijection between the connected component* $\Sigma'(p)$ *and the torus* $\mathbb{L}/F_{\boldsymbol{\gamma}}\mathbb{Z}^g$ *making the following diagram commutative:*

$$\begin{array}{ccc} \Sigma'(p) & \xrightarrow{\Phi'_\chi} & \mathbb{L}/F_{\boldsymbol{\gamma}}\mathbb{Z}^g \\ {\scriptstyle \mathcal{T}'}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{T}'} \\ \Sigma'(p) & \xrightarrow{\Phi'_\chi} & \mathbb{L}/F_{\boldsymbol{\gamma}}\mathbb{Z}^g \end{array}$$

By Proposition 3.3, this conjecture implies the relation $|\Sigma'(p)| = \det F_{\boldsymbol{\gamma}} / \prod \big(l^{(a)}_i - l^{(a)}_{i+1}\big)$, where the product is taken over all null and convex blocks $\square^{(a)}_i$. Note that this factor is common to all the connected components in the level set. Thus Conjecture 3.6 also tells that

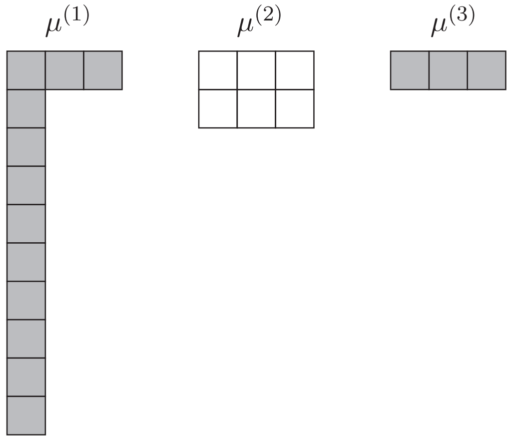
$$|\mathcal{P}(\mu)| = \frac{\Omega(\mu)}{\prod (l^{(a)}_i - l^{(a)}_{i+1})},$$

which refines [26, Conjecture 4.2].

Example 3.9. Consider the path $p_{III} = 121212343434121212$ of length $L = 18$. Its soliton content is given in the following table.

(ai)	$l^{(a)}_i$	$m^{(a)}_i$	$p^{(a)}_i$	$r^{(a)}_{i,\alpha}$	$\gamma^{(a)}_i$	$\xi^{(a)}_i$	${}^t h^{(a)}_{\xi^{(a)}_i}$
$(11)^*$	3	1	0	0	1	3	(3,1,0,0)
$(12)^*$	1	9	0	$0,\dots,0$	9	1	(1,1,0,0)
(21)	3	2	3	$0,0$	1	1	(0,0,1,0)
$(31)^*$	3	1	0	0	1	3	(0,0,0,1)

Its $(1,1)$, $(1,2)$, $(3,1)$ blocks marked by $*$ are null and convex. The soliton content is depicted as follows. (Null and convex blocks are hatched.)



$T_2^{(1)}$, $T_1^{(3)}$ and $T_2^{(3)}$ are inadmissible to p_{III}, while all the other $T_l^{(a)}$'s are admissible. The matrix F_γ for p_{III} is given by

$$F_\gamma = \begin{pmatrix} 6 & 2 & -6 & 0 \\ 2 & 2 & -2 & 0 \\ -3 & -1 & 15 & -3 \\ 0 & 0 & -6 & 6 \end{pmatrix}.$$

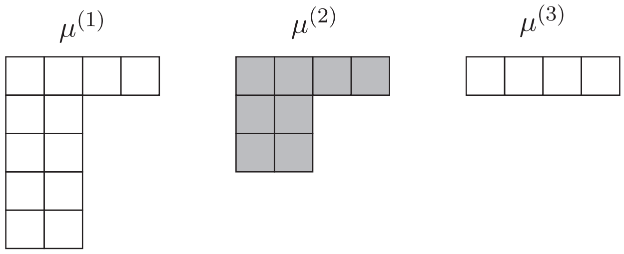
The cardinality of the connected component in the level set is calculated as

$$|\Sigma'(p_{III})| = \det F_\gamma / \bigl(l_1^{(1)} - l_2^{(1)}\bigr)\bigl(l_2^{(1)} - l_3^{(1)}\bigr)\bigl(l_1^{(3)} - l_2^{(3)}\bigr) = 432/(2 \cdot 1 \cdot 3) = 72.$$

Example 3.10. Consider the path $p_{IV} = 1122331142233444$ of length $L = 16$. Its soliton content is given in the following table.

(ai)	$l_i^{(a)}$	$m_i^{(a)}$	$p_i^{(a)}$	$r_{i,\alpha}^{(a)}$	$\gamma_i^{(a)}$	$\xi_i^{(a)}$	${}^t h^{(a)}_{\xi_i^{(a)}}$
(11)	4	1	0	0	1	3	(3,2,0,0,0)
(12)	2	4	2	1, 1, 0, 0	2	1	(1,1,0,0,0)
(21)*	4	1	0	0	1	4	(0,0,4,2,0)
(22)*	2	2	0	0, 0	2	2	(0,0,2,2,0)
(31)	4	1	0	0	1	1	(0,0,0,0,1)

Its $(2,1)$, $(2,2)$ blocks marked by $*$ are null and convex. The soliton content is depicted in the following with the null and convex blocks hatched, showing that $T_1^{(2)}$ and $T_3^{(2)}$ are inadmissible to p_{IV}.



The matrix F_γ for p_{IV} is given by

$$F_\gamma = \begin{pmatrix} 8 & 8 & -4 & -2 & 0 \\ 4 & 9 & -2 & -2 & 0 \\ -4 & -4 & 8 & 4 & -4 \\ -2 & -4 & 4 & 4 & -2 \\ 0 & 0 & -4 & -2 & 8 \end{pmatrix}$$

The cardinality of the connected component is calculated as

$$|\Sigma'(p_{IV})| = \det F_\gamma / \big(l_1^{(2)} - l_2^{(2)}\big)\big(l_2^{(2)} - l_3^{(2)}\big) = 2048/(2\cdot 2) = 512.$$

3.8 Summary of conjectures

Here is a summary of our conjectures. Each one is based on those in the preceding lines. Thus the principal ones are the linearizations.

Condition on soliton/string content μ	$\forall\, p_i^{(a)} \geq 1$ (2.14)	$\forall\, p_i^{(a)} \geq 0$ (3.46)
Stability of dynamics	Conjecture 2.1 $\mathcal{T}$	Conjecture 3.4 $\mathcal{T}'$
Intersection with highest paths	Conjecture 3.1	Conjecture 3.5
Linearization	Conjecture 3.2 $\simeq$ 3.3 Φ_χ	Conjecture 3.6 Φ'_χ
Relevant torus	$\mathbb{Z}^g/F_\gamma\mathbb{Z}^g$	$\mathbb{L}/F_\gamma\mathbb{Z}^g$

4 Tropical Riemann theta from combinatorial Bethe ansatz

4.1 From tropical tau function to tropical Riemann theta function

Let $\big(\mu^{(a)}, r^{(a)}\big)$ be the color a part of the rigged configuration $(\mu, \mathbf{r})$ depicted in the right diagram of (2.5). We keep the notations in (2.6)–(2.9). An explicit formula for the image path of the KKR map ϕ^{-1} (2.18) is known in terms of the *tropical tau function* [23]. It is related to the charge function on rigged configurations and is actually obtained from the tau function in the KP hierarchy [11] by the ultradiscretization with an elaborate adjustment of parameters from the KP and rigged configurations:

$$\begin{aligned}\tau_{k,d} = -\min\Bigg\{ &\frac{1}{2}\sum_{(ai\alpha),\,(bj\beta)} C_{ab}\min\big(l_i^{(a)}, l_j^{(b)}\big) N_{i,\alpha}^{(a)} N_{j,\beta}^{(b)} + \sum_{(ai\alpha)} r_{i,\alpha}^{(a)} N_{i\alpha}^{(a)} \\ &- k\sum_{(i\alpha)} N_{i,\alpha}^{(1)} + \sum_{(j\beta)} l_j^{(d)} N_{j,\beta}^{(d)} \Bigg\} \qquad (1 \leq d \leq n+1). \end{aligned} \tag{4.1}$$

Here the sums range over $(ai\alpha) \in H$ wherever $N_{i,\alpha}^{(a)}$ is involved. Thus in the second line of (4.1), $(i\alpha)$ runs over $1 \leq i \leq g_1$, $1 \leq \alpha \leq m_i^{(1)}$ and so does $(j\beta)$ over $1 \leq j \leq g_d$, $1 \leq \beta \leq m_j^{(d)}$. The last term $\sum_{(j\beta)} l_j^{(d)} N_{j,\beta}^{(d)}$ is to be understood as 0 when $d = n+1$. In (4.1), min is taken over $N_{i,\alpha}^{(a)} \in \{0,1\}$ for all $(ai\alpha) \in H$. Thus it consists of 2^G candidates. Obviously, $\tau_{k,d} \in \mathbb{Z}_{\geq 0}$ holds.

In the present case, the paths are taken from $(B^{1,1})^{\otimes L}$. We parameterize the set $B^{1,1} = \{1, 2, \ldots, n+1\}$ as $\{(x_1, x_2, \ldots, x_{n+1}) \in \{0,1\}^{n+1} \mid x_1 + \cdots + x_{n+1} = 1\}$.

Theorem 4.1 ([23]). *The image of the KKR map ϕ^{-1} is expressed as follows:*

$$\begin{aligned} \phi^{-1}:\ & (\mu, \mathbf{r}) \mapsto x_1 \otimes x_2 \otimes \cdots \otimes x_L \in (B^{1,1})^{\otimes L}, \\ & x_k = (x_{k,1}, x_{k,2}, \ldots, x_{k,n+1}) \in B^{1,1}, \\ & x_{k,a} = \tau_{k,a} - \tau_{k-1,a} - \tau_{k,a-1} + \tau_{k-1,a-1} \qquad (2 \leq a \leq n+1). \end{aligned} \tag{4.2}$$

In the context of the box-ball system, $x_{k,a}$ represents the number of balls of color a in the kth box from the left for $2 \le a \le n+1$. The remaining $x_{k,1}$ is determined from this by $x_{k,1} = 1 - x_{k,2} - \cdots - x_{k,n+1}$, which also takes values $x_{k,1} = 0, 1$ in the present case.

The following is a special case ($\forall\, \lambda_k = 1$) of [23, Proposition 5.1].

Proposition 4.1 ([23]). *The tropical tau function satisfies the tropical Hirota equation:*

$$\tau_{k-1,d} + \overline{\tau}_{k,d-1} = \max(\overline{\tau}_{k,d} + \tau_{k-1,d-1},\, \tau_{k,d} + \overline{\tau}_{k-1,d-1} - 1) \qquad (2 \le d \le n+1),$$

where $\overline{\tau}_{k,d}$ is obtained from $\tau_{k,d}$ by replacing $r^{(a)}_{i,\alpha}$ with $r^{(a)}_{i,\alpha} + l^{(a)}_i \delta_{a1}$.

Lemma 4.1. *Let $p, p' \in \big(B^{1,1}\big)^{\otimes L}$ be the highest paths whose rigged configurations are $(\mu, \mathbf{r})$ and $(\mu', \mathbf{r}')$, respectively. Then the rigged configuration of the highest path $p \otimes p' \in \big(B^{1,1}\big)^{\otimes 2L}$ is $(\mu, \mathbf{r}) \sqcup (\mu', \tilde{\mathbf{r}}')$, where $(\tilde{\mathbf{r}}') = \big(r^{(a)\prime}_{i,\alpha} + p^{(a)}_i\big)_{(ai\alpha)\in H}$ and $p^{(a)}_i$ is the vacancy number of $(\mu, \mathbf{r})$.*

In the lemma, $\sqcup$ stands for the sum (union) as the multisets of strings, namely, the rows of μ attached with riggings. More formally a string is a triple $\big(a, l^{(a)}_i, r^{(a)}_{i,\alpha}\big)$ consisting of its color a, length $l^{(a)}_i$ and rigging $r^{(a)}_{i,\alpha}$ that is labeled with H. Let $p \in \big(B^{1,1}\big)^{\otimes L}$ be the highest path corresponding to the rigged configuration $(\mu, \mathbf{r})$. From Lemma 4.1, the rigged configuration of the highest path $p^{\otimes M} \in \big(B^{1,1}\big)^{\otimes ML}$ is $\big(\mu, \mathbf{r}^1\big) \sqcup \big(\mu, \mathbf{r}^2\big) \sqcup \cdots \sqcup \big(\mu, \mathbf{r}^M\big)$, where $\mathbf{r}^k = \big(r^{(a)}_{i,\alpha} + (k-1)p^{(a)}_i\big)_{(ai\alpha)\in H}$. Pictorially, this corresponds to extending the right diagram in (2.5) vertically M times adding $p^{(a)}_i, 2p^{(a)}_i, \ldots, (M-1)p^{(a)}_i$ to the riggings in the (a,i) block for each (a,i). This reminds us of (3.1), and is in fact the origin of the extended rigged configuration in Section 3.1.

We proceed to the calculation of the tropical tau function $\tau^M_{k,d}$ associated with the above rigged configuration $\big(\mu, \mathbf{r}^1\big) \sqcup \big(\mu, \mathbf{r}^2\big) \sqcup \cdots \sqcup \big(\mu, \mathbf{r}^M\big)$. In (4.1), the variable $N^{(a)}_{i,\alpha}$ is to be replaced by the M replicas $N^{(a)}_{i,\alpha,1}, \ldots, N^{(a)}_{i,\alpha,M}$ to cope with the M-fold extension:

$$\begin{aligned} \tau^M_{k,d} = -\min\Bigg\{ &\frac{1}{2} \sum_{(ai\alpha),\,(bj\beta)} \sum_{1\le s,t\le M} C_{ab} \min\big(l^{(a)}_i, l^{(b)}_j\big) N^{(a)}_{i,\alpha,s} N^{(b)}_{j,\beta,t} \\ &+ \sum_{(ai\alpha)} \sum_{1\le s\le M} \big(r^{(a)}_{i,\alpha} + (s-1)p^{(a)}_i\big) N^{(a)}_{i,\alpha,s} \\ &- k \sum_{(i\alpha)} \sum_{1\le s\le M} N^{(1)}_{i,\alpha,s} + \sum_{(j\beta)} \sum_{1\le t\le M} l^{(d)}_j N^{(d)}_{j,\beta,t} \Bigg\}. \end{aligned} \tag{4.3}$$

All the summands here are invariant under permutations within $N^{(a)}_{i,\alpha,1}, \ldots, N^{(a)}_{i,\alpha,M}$ for each (a,i) except the replica symmetry breaking term $\sum_{(ai\alpha)} \sum_{1\le s\le M} (s-1)p^{(a)}_i N^{(a)}_{i,\alpha,s}$. Due to $p^{(a)}_i \ge 0$, one can reduce the minimizing variables $N^{(a)}_{i,\alpha,s} \in \{0,1\}$ to $n^{(a)}_{i,\alpha} \in \{0, 1, \ldots, M-1\}$ such that

$$N^{(a)}_{i,\alpha,1} = N^{(a)}_{i,\alpha,2} = \cdots = N^{(a)}_{i,\alpha,n^{(a)}_{i,\alpha}} = 1, \qquad N^{(a)}_{i,\alpha,n^{(a)}_{i,\alpha}+1} = \cdots = N^{(a)}_{i,\alpha,M} = 0. \tag{4.4}$$

As the result, (4.3) becomes

$$\begin{aligned} \tau^M_{k,d} = -\min\Bigg\{ &\frac{1}{2} \sum_{(ai\alpha),\,(bj\beta)} C_{ab} \min\big(l^{(a)}_i, l^{(b)}_j\big) n^{(a)}_{i,\alpha} n^{(b)}_{j,\beta} \\ &+ \sum_{(ai\alpha)} \left(r^{(a)}_{i,\alpha} n^{(a)}_{i,\alpha} + \frac{n^{(a)}_{i,\alpha}\big(n^{(a)}_{i,\alpha} - 1\big)}{2} p^{(a)}_i \right) - k \sum_{(i\alpha)} n^{(1)}_{i,\alpha} + \sum_{(j\beta)} l^{(d)}_j n^{(d)}_{j,\beta} \Bigg\}, \end{aligned}$$

where the minimum is now taken over $n^{(a)}_{i,\alpha} \in \{0, 1, \dots, M-1\}$ for all the blocks (a, i). The notation can be eased considerably by introducing a quadratic form of $\mathbf{n} = \big(n^{(a)}_{i,\alpha}\big)_{(ai\alpha)\in H}$ as follows:

$$\tau^M_{k,d} = -\min\left\{\frac{1}{2}{}^t\mathbf{n}B\mathbf{n} + {}^t\Big(\mathbf{r} - \frac{\mathbf{p}}{2} - k\mathbf{h}^{(1)}_1 + \mathbf{h}^{(d)}_\infty\Big)\mathbf{n}\right\} \qquad (1 \le d \le n+1, 1 \le k \le ML), \tag{4.5}$$

$$B = \big(\delta_{ab}\delta_{ij}\delta_{\alpha\beta}p^{(a)}_i + C_{ab}\min\big(l^{(a)}_i, l^{(b)}_j\big)\big)_{(ai\alpha),(bj\beta)\in H}, \tag{4.6}$$

$$\mathbf{r} = \big(r^{(a)}_{i,\alpha}\big)_{(ai\alpha)\in H}, \qquad \mathbf{p} = \big(p^{(a)}_i\big)_{(ai\alpha)\in H}, \tag{4.7}$$

$$\mathbf{h}^{(c)}_l = \big(\delta_{ac}\min\big(l, l^{(c)}_i\big)\big)_{(ai\alpha)\in H} \quad (1 \le c \le n, l \ge 1), \qquad \mathbf{h}^{(n+1)}_l = 0. \tag{4.8}$$

The G-dimensional vector $\mathbf{h}^{(c)}_l$ here should be distinguished from the g-dimensional one $h^{(c)}_l$ (3.20). It is the velocity vector of the time evolution $T^{(c)}_l$ in $\tilde{\mathcal{J}}(\mu)$ and $\mathcal{J}(\mu)$ in the light of (3.3). The matrix B is symmetric and positive definite. See for example [19, Lemma 3.8]. We call B (4.6) the tropical period matrix although a connection to a tropical curve is yet to be clarified. We also introduce the G-dimensional vector $\mathbf{1} = (1)_{(ai\alpha)\in H}$. Then from (2.7) and (4.8) we get

$$B\mathbf{1} = \left(\sum_{(bj\beta)\in H}\big(\delta_{ab}\delta_{ij}\delta_{\alpha\beta}p^{(a)}_i + C_{ab}\min\big(l^{(a)}_i, l^{(b)}_j\big)\big)\right)_{(ai\alpha)} = (L\delta_{a1})_{(ai\alpha)} = L\mathbf{h}^{(1)}_1, \tag{4.9}$$

$${}^t\mathbf{1}\mathbf{h}^{(c)}_\infty = \big|\mu^{(c)}\big|. \tag{4.10}$$

Example 4.1. With the ordering of indices in H (3.10), the tropical period matrix B (4.6) and the vectors $\mathbf{r}$, $\mathbf{p}$ (4.7) for the rigged configuration (3.9) are given as follows:

$$B = \begin{pmatrix} 10 & 6 & 4 & 4 & 4 & -3 & -1 \\ 6 & 10 & 4 & 4 & 4 & -3 & -1 \\ 4 & 4 & 11 & 4 & 4 & -2 & -1 \\ 4 & 4 & 4 & 11 & 4 & -2 & -1 \\ 4 & 4 & 4 & 4 & 11 & -2 & -1 \\ -3 & -3 & -2 & -2 & -2 & 10 & 2 \\ -1 & -1 & -1 & -1 & -1 & 2 & 3 \end{pmatrix}, \qquad \mathbf{r} = \begin{pmatrix} 4 \\ 2 \\ 6 \\ 5 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{p} = \begin{pmatrix} 4 \\ 4 \\ 7 \\ 7 \\ 7 \\ 2 \\ 1 \end{pmatrix}. \tag{4.11}$$

The velocity vector $\mathbf{h}^{(a)}_l$ (4.8) for the time evolution $T^{(a)}_l$ is specified as

$$\mathbf{h}^{(1)}_1 = \frac{1}{2}\mathbf{h}^{(1)}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{h}^{(1)}_{l\ge 3} = \begin{pmatrix} 3 \\ 3 \\ 2 \\ 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{h}^{(2)}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix},$$

$$\mathbf{h}^{(2)}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \qquad \mathbf{h}^{(2)}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 1 \end{pmatrix}, \qquad \mathbf{h}^{(2)}_{l\ge 4} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 4 \\ 1 \end{pmatrix}.$$

Now we take M to be even and make the shifts $\mathbf{n} \to \mathbf{n} + \frac{M}{2}\mathbf{1}$ and $k \to k' = k + \frac{ML}{2}$. Using (4.9), we find

$$\tau^M_{k',d} = -\min\left\{\frac{1}{2}{}^t\mathbf{n}B\mathbf{n} + {}^t\left(\mathbf{r} - \frac{\mathbf{p}}{2} - k\mathbf{h}^{(1)}_1 + \mathbf{h}^{(d)}_\infty\right)\mathbf{n}\right\} + u_M\,{}^t\mathbf{h}^{(d)}_\infty\mathbf{1} + v_M k' + w_M. \quad (4.12)$$

The first term is formally identical with (4.5) but now the minimum extends over $-\frac{M}{2} \le n^{(a)}_{i,\alpha} < \frac{M}{2}$. The scalars u_M, v_M, w_M are independent of k' and d, therefore these terms are irrelevant when taking the double difference $\tau^M_{k',a} - \tau^M_{k'-1,a} - \tau^M_{k',a-1} + \tau^M_{k'-1,a-1}$ as in (4.2).

Passing to the limit $M \to \infty$, we see that the relevant term in (4.12) tends to

$$\lim_{M:\,\mathrm{even}\to\infty}\left(\tau^M_{k',d} - u_M\,{}^t\mathbf{h}^{(d)}_\infty\mathbf{1} - v_M k' - w_M\right) = \Theta\left(\mathbf{r} - \frac{\mathbf{p}}{2} - k\mathbf{h}^{(1)}_1 + \mathbf{h}^{(d)}_\infty\right). \quad (4.13)$$

Here Θ denotes the tropical Riemann theta function

$$\Theta(\mathbf{z}) = -\min_{\mathbf{n}\in\mathbb{Z}^G}\left\{\frac{1}{2}{}^t\mathbf{n}B\mathbf{n} + {}^t\mathbf{z}\mathbf{n}\right\}, \quad (4.14)$$

which enjoys the quasi-periodicity

$$\Theta(\mathbf{z} + \mathbf{v}) = \Theta(\mathbf{z}) + {}^t\mathbf{v}B^{-1}\left(\mathbf{z} + \frac{\mathbf{v}}{2}\right) \qquad \text{for } \mathbf{v} \in B\mathbb{Z}^G. \quad (4.15)$$

In the context of the periodic box-ball system, the tropical Riemann theta function was firstly obtained in this way in [20] for rank $n = 1$ case. See [9, 10, 30] for an account from the tropical geometry point of view. Another remark is that Proposition 4.1 and (4.13) directly lead to the tropical Hirota equation for our Θ:

$$\begin{aligned}
&\Theta\big(\mathbf{J} + \mathbf{h}^{(1)}_\infty + \mathbf{h}^{(d-1)}_\infty\big) + \Theta\big(\mathbf{J} + \mathbf{h}^{(1)}_1 + \mathbf{h}^{(d)}_\infty\big)\\
&\quad = \max\big\{\Theta\big(\mathbf{J} + \mathbf{h}^{(1)}_\infty + \mathbf{h}^{(d)}_\infty\big) + \Theta\big(\mathbf{J} + \mathbf{h}^{(1)}_1 + \mathbf{h}^{(d-1)}_\infty\big),\\
&\qquad \Theta\big(\mathbf{J} + \mathbf{h}^{(1)}_1 + \mathbf{h}^{(1)}_\infty + \mathbf{h}^{(d-1)}_\infty\big) + \Theta\big(\mathbf{J} + \mathbf{h}^{(d)}_\infty\big) - 1\big\},
\end{aligned}$$

where we have set $\mathbf{J} = \mathbf{r} - \frac{\mathbf{p}}{2} - k\mathbf{h}^{(1)}_1$. For $n = 1$, see also [10].

From (4.2), we arrive at our main formula in this section.

Theorem 4.2. *The highest path $x_1 \otimes x_2 \otimes \cdots \otimes x_L \in \mathcal{P}_+(\mu)$ corresponding to the rigged configuration $(\mu, \mathbf{r})$ is given by $x_k = (x_{k,1}, x_{k,2}, \ldots, x_{k,n+1})$ with $x_{k,1} + \cdots + x_{k,n+1} = 1$ and*

$$\begin{aligned}
x_{k,a} = {}&\Theta\left(\mathbf{r} - \frac{\mathbf{p}}{2} - k\mathbf{h}^{(1)}_1 + \mathbf{h}^{(a)}_\infty\right) - \Theta\left(\mathbf{r} - \frac{\mathbf{p}}{2} - (k-1)\mathbf{h}^{(1)}_1 + \mathbf{h}^{(a)}_\infty\right) \quad (4.16)\\
&- \Theta\left(\mathbf{r} - \frac{\mathbf{p}}{2} - k\mathbf{h}^{(1)}_1 + \mathbf{h}^{(a-1)}_\infty\right) + \Theta\left(\mathbf{r} - \frac{\mathbf{p}}{2} - (k-1)\mathbf{h}^{(1)}_1 + \mathbf{h}^{(a-1)}_\infty\right) \quad (2 \le a \le n+1).
\end{aligned}$$

The data $\mathbf{r}$, $\mathbf{p}$ and $\mathbf{h}^{(c)}_l$ are determined from $(\mu, \mathbf{r})$ by (2.5), (2.7), (4.7) and (4.8).

Compared with the previous expression (4.2) with (4.1), the alternative formula (4.16) enjoys an extra symmetry under the shifts $\mathbf{r} \to \mathbf{r} + B\mathbb{Z}^G$ and $k \to k + L\mathbb{Z}$ due to the quasi-periodicity (4.15) and (4.9). In other words, (4.16) is defined for $\mathbf{r} \in \mathbb{Z}^G/B\mathbb{Z}^G$ and $k \in \mathbb{Z}/L\mathbb{Z}$. In view of this, it is natural to relate Theorem 4.2 with our periodic $A^{(1)}_n$ SCA.

Theorem 4.3. *Assume the condition* (2.14). *Let $p = x_1 \otimes x_2 \otimes \cdots \otimes x_L \in \mathcal{P}(\mu)$ be the path of the periodic $A^{(1)}_n$ SCA whose action variable is the quasi-periodic extension* (3.1) *of* $\mathbf{r} = \big(r^{(a)}_{i,\alpha}\big)_{(ai\alpha)\in H}$. *Under Conjecture* 3.2, *$x_k = (x_{k,1}, x_{k,2}, \ldots, x_{k,n+1})$ is given by* (4.16).

Note that this reduces a solution of the initial value problem $p \to T^{(r_1)}_{l_1} \cdots T^{(r_N)}_{l_N}(p)$ to a simple substitution $\mathbf{r} \to \mathbf{r} + \mathbf{h}^{(r_1)}_{l_1} + \cdots + \mathbf{h}^{(r_N)}_{l_N}$.

Remark 4.1. In addition to $\mathbf{r} \to \mathbf{r} + B\mathbb{Z}^G$, each term in (4.16) is invariant under $\mathbf{r} = \big(r^{(a)}_{i,\alpha}\big)_{(ai\alpha)\in H} \mapsto \big(r^{(a)}_{i,\sigma(\alpha)}\big)_{(ai\alpha)\in H}$ for any permutation $\sigma \in \mathfrak{S}_{m^{(a)}_i}$ that can depend on $(ai) \in \overline{H}$. Here $\mathfrak{S}_m$ denotes the symmetric group of degree m. In fact the set $\mathcal{J}(\mu)$ of angle variables is naturally described as

$$\mathcal{J}(\mu) \simeq \mathbb{Z}^G/B\mathbb{Z}^G/\prod_{(ai)\in\overline{H}} \mathfrak{S}_{m^{(a)}_i}. \tag{4.17}$$

To go to the right hand side, one just forgets the inequality $r^{(a)}_{i,\alpha} \le r^{(a)}_{i,\alpha+1}$ within each block $(ai) \in \overline{H}$ identifying all the re-orderings. Compare also the matrix B (4.6) with (3.4).

When $n = 1$, one can remove the assumption 'Under Conjecture 3.2' in Theorem 4.3 due to [24]. If further $\forall\, m^{(1)}_i = 1$, the formula (4.16) coincides with [20, equation (3.8)]. Theorem 4.3 here tells that it also holds throughout $m^{(1)}_i > 1$ by a natural extrapolation whose concrete form is given in (4.26) and (4.27).

The case $n = 1$ and general $m^{(1)}_i$ in (4.16) takes a different form from the corresponding formula in [21] expressed by a higher characteristic tropical Riemann theta function. This apparent difference in guises is caused by a different choices of effective minimizing variables as $n^{(a)}_{i,\alpha}$ in (4.4). In fact, introducing the M replicas $N^{(a)}_{i,\alpha,1}, \ldots, N^{(a)}_{i,\alpha,M}$ as in (4.3) is not the unique way of handling the tropical tau function for the M-fold extended rigged configuration $\big(\mu, \mathbf{r}^1\big) \sqcup \big(\mu, \mathbf{r}^2\big) \sqcup \cdots \sqcup \big(\mu, \mathbf{r}^M\big)$. Another natural option is to stick to the form (4.1) but extend $N^{(a)}_{i,\alpha}$ from $1 \le \alpha \le m^{(a)}_i$ to $1 \le \alpha \le M m^{(a)}_i$ assuming (3.1) for the rigging. Then the minimizing variable $n^{(a)}_i$ can be introduced simply via

$$N^{(a)}_{i,1} = N^{(a)}_{i,2} = \cdots = N^{(a)}_{i,n^{(a)}_i} = 1, \qquad N^{(a)}_{i,n^{(a)}_i+1} = \cdots = N^{(a)}_{i,Mm^{(a)}_i} = 0.$$

What makes the calculation tedious after this is that one has to classify $n^{(a)}_i$ by $\operatorname{mod} m^{(a)}_i$. This was done in [21] for $n = 1$. Here we omit the detail and only mention that the result is expressed in terms of a rational characteristic tropical Riemann theta function with the $g \times g$ reduced period matrix:

$$B^{\mathrm{red}} = \operatorname{diag}\big(m^{(a)}_i\big)_{(ai)\in\overline{H}} F = \big(\delta_{ab}\delta_{ij} m^{(a)}_i p^{(a)}_i + C_{ab} \min\big(l^{(a)}_i, l^{(b)}_j\big) m^{(a)}_i m^{(b)}_j\big)_{(ai),(bj)\in\overline{H}}.$$

For $n = 1$, this coincides with [21, equation (5.1)].

Let us proceed to an explicit Θ formula for a carrier of type $B^{1,l}$. Consider the highest path $p = x_1 \otimes x_2 \otimes \cdots \otimes x_L \in \mathcal{P}_+(\mu)$ in Theorem 4.2 expressed as (4.16). We consider the calculation of the time evolution $T^{(1)}_l(p) = p' = x'_1 \otimes \cdots \otimes x'_L$ according to (2.2). Locally it is depicted as

$$\begin{array}{ccccccc} & & x_k & & x_{k+1} & & \\ \cdots & y_{k-1} & + & y_k & + & & \cdots \\ & & x'_k & & x'_{k+1} & & \end{array}$$

From the proof of Lemma D.1, a carrier satisfying the periodic boundary condition $y_0 = y_L \in B^{1,l}$ and thereby inducing the time evolution $T^{(1)}_l$ can be constructed by $u^{1,l} \otimes p \simeq \hat{p} \otimes y_0$. Here $u^{1,l} \in B^{1,l}$ is the semistandard tableau of length l row shape whose entries are all 1. This fixes the carriers in the intermediate stage $y_1, y_2, \ldots, y_{L-1} \in B^{1,l}$ by $y_0 \otimes (x_1 \otimes \cdots \otimes x_k) \overset{\sim}{\mapsto} (x'_1 \otimes \cdots \otimes x'_k) \otimes y_k$ under the isomorphism $B^{1,l} \otimes \big(B^{1,1}\big)^{\otimes k} \simeq \big(B^{1,1}\big)^{\otimes k} \otimes B^{1,l}$.

Theorem 4.4. *The carrier $y_k \in B^{1,l}$ corresponding to (4.16) in the above sense is given by $y_k = (y_{k,1}, y_{k,2}, \ldots, y_{k,n+1})$ with $y_{k,1} + \cdots + y_{k,n+1} = l$ and*

$$\begin{aligned} y_{k,a} &= \Theta\left(\mathbf{r} - \frac{\mathbf{p}}{2} - k\mathbf{h}_1^{(1)} + \mathbf{h}_\infty^{(a)}\right) - \Theta\left(\mathbf{r} - \frac{\mathbf{p}}{2} - k\mathbf{h}_1^{(1)} + \mathbf{h}_l^{(1)} + \mathbf{h}_\infty^{(a)}\right) \\ &\quad - \Theta\left(\mathbf{r} - \frac{\mathbf{p}}{2} - k\mathbf{h}_1^{(1)} + \mathbf{h}_\infty^{(a-1)}\right) + \Theta\left(\mathbf{r} - \frac{\mathbf{p}}{2} - k\mathbf{h}_1^{(1)} + \mathbf{h}_l^{(1)} + \mathbf{h}_\infty^{(a-1)}\right) \quad (2 \le a \le n+1). \end{aligned} \tag{4.18}$$

This can be verified by modifying the derivation of Theorem 4.2 slightly by using [23, Theorem 2.1] and [18, Lemma 8.5]. The periodicity $y_0 = y_L$ is easily checked by (4.15). Naturally (4.18) gives the carrier for general paths in Theorem 4.3 under Conjecture 3.2. For $n = 1$ such a result for a higher spin case was obtained in [22] under the condition $\forall\, m_i^{(1)} = 1$. We will present a modest application of the formula (4.18) in Section 4.4.

4.2 Bethe vector at $q = 0$ from tropical Riemann theta function

We assume the condition (2.14) and Conjecture 3.2 in this subsection. Write the string center equation (3.44) simply as

$$A\mathbf{u} = \mathbf{c} + \mathbf{r} + \boldsymbol{\rho} \tag{4.19}$$

using the G-dimensional vectors $\mathbf{c} = \left(\frac{1}{2}\left(p_i^{(a)} + m_i^{(a)} + 1\right)\right)_{(ai\alpha)\in H}$ and $\boldsymbol{\rho} = (\alpha - 1)_{(ai\alpha)\in H}$. Note that the time evolution of the angle variable (3.3) is written as $T_l^{(r)}(\mathbf{r}) = \mathbf{r} + \mathbf{h}_l^{(r)}$ with $\mathbf{h}_l^{(r)}$ defined by (4.8). In view of (4.19), the time evolution of the Bethe roots introduced in (3.45) is expressed as $T_l^{(r)}(\mathbf{u}) = \mathbf{u} + A^{-1}\mathbf{h}_l^{(r)}$. At first sight, this appears contradictory, because $T_l^{(r)}$ is a transfer matrix at $q = 0$, which should leave the $q = 0$ Bethe eigenvectors invariant up to an overall scalar hence the relevant Bethe roots as well. The answer to this puzzle is that the path $p \in \mathcal{P}(\mu)$ that we are associating with $\mathbf{u}$ or $\mathbf{r}$ by $p = \Phi^{-1}(\mu, \mathbf{r})$ is a *monomial* in $(\mathbb{C}^{n+1})^{\otimes L}$, which is *not* a Bethe vector at $q = 0$ in general.

It is easy to remedy this. In fact, for each Bethe root $\mathbf{u}$ or equivalently the angle variable $\mathbf{r} = A\mathbf{u} - \mathbf{c} - \boldsymbol{\rho} \in \mathcal{J}(\mu)$, one can construct a vector $|\mathbf{r}\rangle \in (\mathbb{C}^{n+1})^{\otimes L}$ that possesses every aspect as a $q = 0$ Bethe vector as follows:

$$|\mathbf{r}\rangle = \sum_{\mathbf{s}\in\mathcal{J}(\mu)} c_{\mathbf{s},\mathbf{r}}\, \mathbf{v}(\mathbf{s}), \tag{4.20}$$

$$c_{\mathbf{s},\mathbf{r}} = \exp\left(-2\pi\sqrt{-1}\; {}^t\mathbf{s}\left(A^{-1}(\mathbf{r} + \mathbf{c} + \boldsymbol{\rho}) + \frac{\mathbf{1}}{2}\right)\right), \tag{4.21}$$

$$\mathbf{v}(\mathbf{s}) = \begin{pmatrix} x_{1,1}(\mathbf{s}) \\ \vdots \\ x_{1,n+1}(\mathbf{s}) \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} x_{L,1}(\mathbf{s}) \\ \vdots \\ x_{L,n+1}(\mathbf{s}) \end{pmatrix} \in \mathcal{P}(\mu) \subseteq \left(\mathbb{C}^{n+1}\right)^{\otimes L}.$$

Here $\mathbf{1} = (1)_{(ai\alpha)\in H}$ in (4.21), which has also appeared in (4.9). The component $x_{k,a}(\mathbf{s}) \in \{0, 1\}$ is specified by Theorem 4.3. Namely,

$$\begin{aligned} x_{k,a}(\mathbf{s}) &= \Theta\left(\mathbf{s} - \frac{\mathbf{p}}{2} - k\mathbf{h}_1^{(1)} + \mathbf{h}_\infty^{(a)}\right) - \Theta\left(\mathbf{s} - \frac{\mathbf{p}}{2} - (k-1)\mathbf{h}_1^{(1)} + \mathbf{h}_\infty^{(a)}\right) \\ &\quad - \Theta\left(\mathbf{s} - \frac{\mathbf{p}}{2} - k\mathbf{h}_1^{(1)} + \mathbf{h}_\infty^{(a-1)}\right) + \Theta\left(\mathbf{s} - \frac{\mathbf{p}}{2} - (k-1)\mathbf{h}_1^{(1)} + \mathbf{h}_\infty^{(a-1)}\right) \end{aligned}$$

for $2 \le a \le n+1$ and $x_{k,1}(\mathbf{s}) + \cdots + x_{k,n+1}(\mathbf{s}) = 1$. The vectors $\mathbf{p}$ and $\mathbf{h}_l^{(a)}$ are defined in (4.7) and (4.8). We embed $\mathcal{P}(\mu)$ into $(\mathbb{C}^{n+1})^{\otimes L}$ naturally and extend the time evolutions to the latter by $\mathbb{C}$-linearity. The vector $\mathbf{v}(\mathbf{s})$ here is the path corresponding to the angle variable $\mathbf{s} \in \mathcal{J}(\mu)$

in Theorem 4.3. It follows that $T_l^{(r)}(\mathbf{v}(\mathbf{s})) = \mathbf{v}\big(\mathbf{s}+\mathbf{h}_l^{(r)}\big)$. Then using $\mathcal{J}(\mu)+\mathbf{h}_l^{(r)} = \mathcal{J}(\mu)$, it is elementary to check

$$T_l^{(r)}(\,|\mathbf{r}\rangle\,) = c_{-\mathbf{h}_l^{(r)},\mathbf{r}}\,|\mathbf{r}\rangle, \tag{4.22}$$
$$c_{-\mathbf{h}_l^{(r)},\mathbf{r}} = \exp\left(2\pi\sqrt{-1}\,{}^t\mathbf{h}_l^{(r)}\left(A^{-1}(\mathbf{r}+\mathbf{c}+\boldsymbol{\rho})+\frac{\mathbf{1}}{2}\right)\right) = \exp\left(2\pi\sqrt{-1}\,{}^t\mathbf{h}_l^{(r)}\left(\mathbf{u}+\frac{\mathbf{1}}{2}\right)\right).$$

The last expression of the quantity $c_{-\mathbf{h}_l^{(r)},\mathbf{r}}$ reproduces the $q=0$ Bethe eigenvalue $\Lambda_l^{(r)} = \prod_{j\alpha}\big(-z_{j\alpha}^{(r)}\big)^{\min(j,l)}$ given in [26, equation (3)] with the string center $z_{j\alpha}^{(r)} = \exp\big(2\pi\sqrt{-1}u_{j\alpha}^{(r)}\big)$. Note further that the transition relation (4.20) is inverted as

$$\mathbf{v}(\mathbf{s}) = \frac{1}{|\mathcal{J}(\mu)|}\sum_{\mathbf{r}\in\mathcal{J}}\overline{c}_{\mathbf{s},\mathbf{r}}|\mathbf{r}\rangle,$$

where $\overline{c}_{\mathbf{s},\mathbf{r}}$ denotes the complex conjugate of $c_{\mathbf{s},\mathbf{r}}$. It follows that the space of the '$q=0$ Bethe vectors' $|\mathbf{r}\rangle$ coincides with the space spanned by the evolvable paths of the prescribed soliton content μ.

$$\bigoplus_{\mathbf{r}\in\mathcal{J}(\mu)}\mathbb{C}|\mathbf{r}\rangle = \bigoplus_{p\in\mathcal{P}(\mu)}\mathbb{C}\,p. \tag{4.23}$$

Thus the approach here bypasses a formidable task of computing the $q\to 0$ limit of the Bethe vectors directly, and leads to the joint eigenvectors $|\mathbf{r}\rangle$ of the commuting family of time evolutions. They form a basis of the space having the prescribed soliton content and possess the eigenvalues anticipated from the Bethe ansatz at $q=0$. These features together with Theorem 2.2 constitute the quantitative background of the identity (2.19).

4.3 Bethe eigenvalue at $q=0$ and dynamical period

Let us remark on the relation of the Bethe eigenvalue (4.22) to the dynamical period. In [25, 26], the formula (3.39) for the dynamical period of periodic $A_n^{(1)}$ SCA with $n>1$ without the order of symmetry $\boldsymbol{\gamma}$ was found by demanding the condition

$$\big(c_{-\mathbf{h}_l^{(r)},\mathbf{r}}\big)^{\mathcal{N}_l^{(r)\prime}} = \pm 1. \tag{4.24}$$

In fact, from the middle expression in (4.22) this condition is satisfied if

$$\mathcal{N}_l^{(r)\prime}\mathbf{h}_l^{(r)} = 0 \mod A\mathbb{Z}^G.$$

By the same argument as the proof of Theorem 3.2, one finds that the smallest positive integer satisfying this condition is

$$\mathcal{N}_l^{(r)\prime} = \mathrm{LCM}\left(\frac{\det A}{\det A[bj\beta]}\right)_{(bj\beta)\in H} = \mathrm{LCM}\left(\frac{\det F}{\det F[bj]}\right)_{(bj)\in\overline{H}},$$

where $A[bj\beta]$ denotes the $G\times G$ matrix obtained from A (3.43) by replacing its $(bj\beta)$th column by $\mathbf{h}_l^{(r)}$ (4.8). The LCM in the first (second) expression should be taken over only those $(bj\beta)$ such that $\det A[bj\beta]\neq 0$ ((bj) such that $\det F[bj]\neq 0$). The second equality is due to the structures of matrices A and F (2.8) and verified by a direct calculation. The expression $\mathcal{N}_l^{(r)\prime}$ derived in the heuristic approach [25, 26] captures the main structure of the full formula (3.39). However it neither fixes the sign in (4.24) nor takes the order of symmetry $\boldsymbol{\gamma}$ into account.

These shortcoming are fixed of course by refining the construction of joint eigenvectors along the connected components. Instead of trying to split the sum (4.20) into them, we simply introduce

$$|\omega, \boldsymbol{\lambda}\rangle = \sum_{\phi \in \mathbb{Z}^g / F_{\boldsymbol{\gamma}} \mathbb{Z}^g} \exp\big(-2\pi\sqrt{-1}\, {}^t\phi\, F_{\boldsymbol{\gamma}}^{-1}(\omega + d_{\boldsymbol{\lambda}})\big) \mathbf{v}(\phi, \boldsymbol{\lambda}) \in \big(\mathbb{C}^{n+1}\big)^{\otimes L} \tag{4.25}$$

for each angle variable $(\omega, \boldsymbol{\lambda}) \in X_{\boldsymbol{\gamma}}/\mathcal{A}$ having the order of symmetry $\boldsymbol{\gamma}$. Here $\mathbf{v}(\phi, \boldsymbol{\lambda}) \in \mathcal{P}_{\boldsymbol{\gamma}}(\mu)$ is the path corresponding to $(\phi, \boldsymbol{\lambda}) \in X_{\boldsymbol{\gamma}}/\mathcal{A}$, and $d_{\boldsymbol{\lambda}} \in \mathbb{Z}^g$ can be chosen arbitrarily. Then by noting $T_l^{(r)}\mathbf{v}(\phi, \boldsymbol{\lambda}) = \mathbf{v}(\phi + h_l^{(r)}, \boldsymbol{\lambda})$, it is straightforward to check

$$T_l^{(r)}(\,|\omega, \boldsymbol{\lambda}\rangle\,) = \exp\big(2\pi\sqrt{-1}\, {}^t h_l^{(r)}\, F_{\boldsymbol{\gamma}}^{-1}(\omega + d_{\boldsymbol{\lambda}})\big)|\omega, \boldsymbol{\lambda}\rangle,$$

This eigenvalue is indeed an $\mathcal{N}_l^{(r)}$th root of unity due to the proof of Theorem 3.2.

Given $\boldsymbol{\lambda}$, there are $\det F_{\boldsymbol{\gamma}}$ independent vectors $|\omega, \boldsymbol{\lambda}\rangle$ (4.25). On the other hand, the number of the choices of $\boldsymbol{\lambda} \in X^2_{\boldsymbol{\gamma}}/\mathcal{A}$ is given by Lemma 3.2. Obviously these vectors are all independent because the set of monomials involved in $|\omega, \boldsymbol{\lambda}\rangle$ and $|\omega', \boldsymbol{\lambda}'\rangle$ are the set of paths that are disjoint if $\boldsymbol{\lambda} \neq \boldsymbol{\lambda}'$. From this fact and (3.37), we obtain the refinement of (4.23) according to the order of symmetry $\boldsymbol{\gamma}$ and further (bit tautologically) according to the connected components:

$$\bigoplus_{(\omega, \boldsymbol{\lambda}) \in X_{\boldsymbol{\gamma}}/\mathcal{A}} \mathbb{C}|\omega, \boldsymbol{\lambda}\rangle = \bigoplus_{p \in \mathcal{P}_{\boldsymbol{\gamma}}(\mu)} \mathbb{C}\, p, \qquad \bigoplus_{\omega \in \mathbb{Z}^g / F_{\boldsymbol{\gamma}} \mathbb{Z}^g} \mathbb{C}|\omega, \boldsymbol{\lambda}\rangle = \bigoplus_{p \in \Sigma(p_0)} \mathbb{C}\, p,$$

where for example $p_0 = \Phi^{-1}((\omega = 0, \boldsymbol{\lambda})) \in \mathcal{P}_{\boldsymbol{\gamma}}(\mu)$.

4.4 Miscellaneous calculation of time average

As a modest application of the formulas by the tropical Riemann theta function, we first illustrate a calculation of some time average along the periodic box-ball system ($n = 1$). Analogous results will be stated for general n in the end. We use the terminology in the periodic box-ball system.

Let $p \in (B^{1,1})^{\otimes L}$ be the path with the angle variable $\mathbf{I}$. The time evolution $T_l^{(1)}$ will simply be denoted by T_l. Set $T_l^t(p) = x_1(t) \otimes \cdots \otimes x_L(t)$. For $n = 1$, one can label $x_k(t) = (x_{k,1}(t), x_{k,2}(t)) \in B^{1,1}$ just by $x_{k,2}(t)$, which from now on will simply be denoted by $x_k(t) (= 0, 1)$. It represents the number of balls in the kth box. Then (4.16) reads

$$\begin{aligned} x_k(t) = {} & \Theta(\mathbf{J} - k\mathbf{h}_1 + t\mathbf{h}_l) - \Theta(\mathbf{J} - (k-1)\mathbf{h}_1 + t\mathbf{h}_l) \\ & - \Theta(\mathbf{J} - k\mathbf{h}_1 + t\mathbf{h}_l + \mathbf{h}_\infty) + \Theta(\mathbf{J} - (k-1)\mathbf{h}_1 + t\mathbf{h}_l + \mathbf{h}_\infty) \end{aligned} \tag{4.26}$$

with $\mathbf{J} = \mathbf{I} - \frac{\mathbf{p}}{2}$. The notations (4.6)–(4.8) are simplified hereafter as

$$\begin{aligned} & B = (\delta_{ij} p_i + 2\min(l_i, l_j))_{(i\alpha),(j\beta) \in H}, \qquad H = \{(i, \alpha) \mid 1 \le i \le g, 1 \le \alpha \le m_i\}, \\ & \mathbf{p} = (p_i)_{(i\alpha) \in H}, \qquad p_i = L - 2\sum_{j=1}^{g} \min(l_i, l_j) m_j, \qquad \mathbf{h}_l = (\min(l, l_i))_{(i\alpha) \in H}. \end{aligned} \tag{4.27}$$

Here l_i, m_i are the shorthand of $l_i^{(1)}$, $m_i^{(1)}$ that specify the action variable (single partition) $\mu = \mu^{(1)}$ as in (2.5). The relation (4.9) reads

$$B\mathbf{h}_1 = L\mathbf{h}_1. \tag{4.28}$$

Note that the total number of balls at any time is $|\mu| = \sum_{i=1}^{g} l_i m_i$. This also follows immediately from (4.26) as

$$\begin{aligned}\sum_{k=1}^{L} x_k(t) &= \Theta(\mathbf{J} - L\mathbf{h}_1 + t\mathbf{h}_l) - \Theta(\mathbf{J} + t\mathbf{h}_l) - \Theta(\mathbf{J} - L\mathbf{h}_1 + t\mathbf{h}_l + \mathbf{h}_\infty) + \Theta(\mathbf{J} + t\mathbf{h}_l + \mathbf{h}_\infty) \\ &= -L{}^t\mathbf{h}_1 B^{-1}\left(\mathbf{J} + t\mathbf{h}_l - \frac{L\mathbf{h}_1}{2}\right) + L{}^t\mathbf{h}_1 B^{-1}\left(\mathbf{J} + t\mathbf{h}_l + \mathbf{h}_\infty - \frac{L\mathbf{h}_1}{2}\right) \\ &= L{}^t\mathbf{h}_1 B^{-1}\mathbf{h}_\infty = {}^t\mathbf{h}_1\mathbf{h}_\infty = |\mu|,\end{aligned}$$

where we have used (4.28) and the quasi-periodicity (4.15). We introduce the density of balls:

$$\rho = {}^t\mathbf{h}_1 B^{-1}\mathbf{h}_\infty = \frac{|\mu|}{L}.$$

Let $\mathcal{N}_l$ be the dynamical period under T_l. Thus $x_k(\mathcal{N}_l) = x_k(0)$ holds for any $1 \le k \le L$.

Proposition 4.2. *The time average of $x_k(t)$ under T_∞ over the period $\mathcal{N}_\infty$ is given by*

$$\frac{1}{\mathcal{N}_\infty}\sum_{t=0}^{\mathcal{N}_\infty - 1} x_k(t) = \rho. \tag{4.29}$$

Proof. Using (4.26) with $l = \infty$, we find that $\sum_{t=0}^{\mathcal{N}_\infty - 1} x_k(t)$ is equal to

$$\Theta(\mathbf{J} - k\mathbf{h}_1) - \Theta(\mathbf{J} - k\mathbf{h}_1 + \mathcal{N}_\infty\mathbf{h}_\infty) - \Theta(\mathbf{J} - (k-1)\mathbf{h}_1) + \Theta(\mathbf{J} - (k-1)\mathbf{h}_1 + \mathcal{N}_\infty\mathbf{h}_\infty).$$

Note that the assumption implies $\mathcal{N}_\infty\mathbf{h}_\infty \in B\mathbb{Z}^G$. Thus one can reduce this by applying the quasi-periodicity (4.15), obtaining $\mathcal{N}_\infty{}^t\mathbf{h}_\infty B^{-1}\mathbf{h}_1 = \mathcal{N}_\infty\rho$. ∎

Actually (4.29) follows at once without this sort of calculation if the left hand side is assumed to be independent of k. However, such homogeneity under spatial translation is not always valid for some time evolution T_l with $l < \infty$ and the initial condition that possess a special commensurability.

Denote the time average $\frac{1}{\mathcal{N}_l}\sum_{t=0}^{\mathcal{N}_l - 1} Q(t)$ of a quantity $Q(t)$ under T_l by $\langle Q\rangle_l$. Then a trivial corollary of Proposition 4.2 is

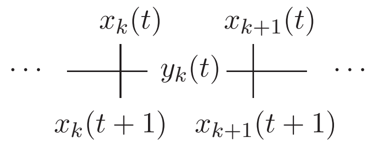
$$\left\langle \sum_{k=1}^{L} \omega^{k-1} x_k \right\rangle_\infty = \begin{cases} \rho L & \omega = 1, \\ \rho\frac{1-\omega^L}{1-\omega} & \omega \ne 1. \end{cases}$$

In particular, this average vanishes if ω is a nontrivial Lth root of unity.

Let us proceed to a less trivial example. In [22], the number of balls in the carrier for any time evolution T_l was expressed also in terms of the tropical Riemann theta function. The carrier that induces the time evolution $p \mapsto T_l(p)$ is the element $v_l \in B^{1,l}$ such that $v_l \otimes p \simeq T_l(p) \otimes v_l$ under the isomorphism of crystals $B^{1,l} \otimes \left(B^{1,1}\right)^{\otimes L} \simeq \left(B^{1,1}\right)^{\otimes L} \otimes B^{1,l}$. See the explanation around (2.2). There uniquely exists such v_l for any p with density $\rho < 1/2$ as shown in [24, Proposition 2.1]. Let us consider the intermediate stage of sending the carrier $y_0(t) := v_l$ to the right by repeated applications of combinatorial R:

$$\begin{aligned}&v_l \otimes x_1(t) \otimes \cdots \otimes x_k(t) \otimes x_{k+1}(t) \cdots \otimes x_L(t) \\ &\qquad \simeq x_1(t+1) \otimes \cdots \otimes x_k(t+1) \otimes y_k(t) \otimes x_{k+1}(t) \cdots \otimes x_L(t).\end{aligned}$$

This is depicted locally as



We identify $y_k(t) = (y_{k,1}(t), y_{k,2}(t)) \in B^{1,l}$ with the number of balls in the capacity l carrier. Namely, $y_{k,2}(t) \in \{0, 1, \dots, l\}$ will simply be denoted by $y_k(t)$. (Hence $y_{k,1}(t) = l - y_k(t)$.) Then for $x_k(t)$ given in (4.26), one has [22]

$$\begin{aligned} y_k(t) &= \Theta(\mathbf{J} - k\mathbf{h}_1 + t\mathbf{h}_l) - \Theta(\mathbf{J} - k\mathbf{h}_1 + (t+1)\mathbf{h}_l) \\ &\quad - \Theta(\mathbf{J} - k\mathbf{h}_1 + t\mathbf{h}_l + \mathbf{h}_\infty) + \Theta(\mathbf{J} - k\mathbf{h}_1 + (t+1)\mathbf{h}_l + \mathbf{h}_\infty). \end{aligned}$$

The preceding result (4.18) reduces to this upon setting $n = 1$, $a = 2$ and $\mathbf{r} - \frac{\mathbf{p}}{2} = \mathbf{J} + t\mathbf{h}_l$.

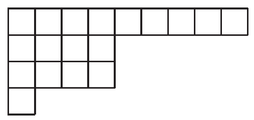
Proposition 4.3. *The time average of the number of balls in the carrier with capacity l at position k under T_l is given by*

$$\langle y_k \rangle_l = {}^t\mathbf{h}_l B^{-1} \mathbf{h}_\infty.$$

Proof. The proof is parallel with the one for Proposition 4.2 by using $\mathcal{N}_l \mathbf{h}_l \in B\mathbb{Z}^G$ and the quasi-periodicity (4.15). ■

Note that the result is independent of the order of symmetry $\boldsymbol{\gamma}$ as well as k.

Example 4.2. Take the path $p = 111222221111222222222111111111111112222111211$ of length $L = 45$ from [21, Example 5.2]. The action variable is



The data from (4.27) read

$$B = \begin{pmatrix} 27 & 8 & 8 & 2 \\ 8 & 27 & 8 & 2 \\ 8 & 8 & 27 & 2 \\ 2 & 2 & 2 & 39 \end{pmatrix}, \qquad \mathbf{p} = \begin{pmatrix} 9 \\ 19 \\ 19 \\ 37 \end{pmatrix}, \qquad \mathbf{h}_l = \begin{pmatrix} \min(l, 9) \\ \min(l, 4) \\ \min(l, 4) \\ \min(l, 1) \end{pmatrix}.$$

We list average $\langle y_k \rangle_l$ with the dynamical period $\mathcal{N}_l$.

l	1	2	3	4	5	6	7	8	∞
$\mathcal{N}_l$	45	1665	333	1665	1665	31635	3515	31635	6327
$\langle y_k \rangle_l$	$\rho = \frac{2}{5} = 0.4$	$\frac{147}{185}$	$\frac{44}{37}$	$\frac{293}{185}$	$\frac{6646}{3515}$	$\frac{1545}{703}$	$\frac{8804}{3515}$	$\frac{9883}{3515}$	$\frac{10962}{3515} = 3.11863\dots$

Now it requires little explanation to present the analogous result for $A^{(1)}_n$ case. Denote the time average $\frac{1}{\mathcal{N}^{(1)}_l} \sum_{t=0}^{\mathcal{N}^{(1)}_l - 1} Q(t)$ of a quantity $Q(t)$ under the time evolution $T^{(1)}_l$ by $\langle Q \rangle_l$. Here $\mathcal{N}^{(1)}_l$ is the dynamical period under $T^{(1)}_l$. Then, the same calculation as Proposition 4.3 using (4.18) leads to

Proposition 4.4. *Under Conjecture* 3.2, *the time average of the number of balls of color a in the carrier of type $B^{1,l}$ at position k is given by*

$$\langle y_{k,a} \rangle_l = {}^t\mathbf{h}^{(1)}_l B^{-1} \big(\mathbf{h}^{(a-1)}_\infty - \mathbf{h}^{(a)}_\infty \big) \qquad (2 \le a \le n+1).$$

Again this depends neither on k nor on the order of symmetry $\boldsymbol{\gamma}$. It reduces to Proposition 4.3 for $n = 1$ due to $\mathbf{h}_l^{(n+1)} = 0$. See (4.8). In particular at $l = 1$ we have

$$\langle y_{k,a}\rangle_1 = \frac{1}{L}{}^t\mathbf{1}\big(\mathbf{h}_\infty^{(a-1)} - \mathbf{h}_\infty^{(a)}\big) = \frac{|\mu^{(a-1)}| - |\mu^{(a)}|}{L}$$

by means of (4.9) and (4.10). From (2.20), this is the density of the color a balls in the path $\big(B^{1,1}\big)^{\otimes L}$, which is natural in view of (2.3). The average of the total number of balls is $\sum_{a=2}^{n+1}\langle y_{k,a}\rangle_l = {}^t\mathbf{h}_l^{(1)} B^{-1}\mathbf{h}_\infty^{(1)}$, which again reduces to Proposition 4.3 for $n = 1$.

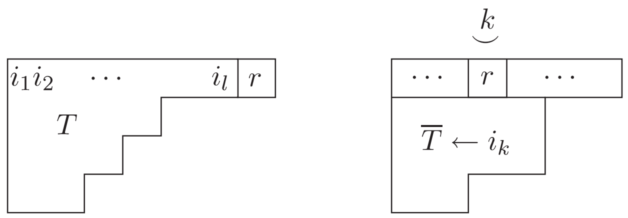
Example 4.3. Consider the paths in Example 1.1 ($n = 2$). The configuration μ is given in Example 3.2, which is the same as the one in Example 3.1. The matrix B and $\mathbf{h}_l^{(a)}$ are listed in Example 4.1, which says $\mathbf{h}_3^{(1)} = \mathbf{h}_\infty^{(1)}$. Denote the density by $\rho_a = \frac{|\mu^{(a-1)}|-|\mu^{(a)}|}{L}$. The average $\langle y_{k,a}\rangle_l$ is shown in the following table with the dynamical period given in the end of Example 1.1.

l	1	2	∞
$\mathcal{N}_l^{(1)}$	24	12	194
$\langle y_{k,2}\rangle_l$	$\rho_2 = \frac{7}{24} = 0.291\dots$	$\frac{7}{12} = 0.583\dots$	$\frac{155}{194} = 0.798\dots$
$\langle y_{k,3}\rangle_l$	$\rho_3 = \frac{5}{24} = 0.208\dots$	$\frac{5}{12} = 0.416\dots$	$\frac{109}{194} = 0.561\dots$

A Row and column insertions

Let T be a semistandard tableau. The row insertion of the number r into T is denoted by $T \leftarrow r$ and defined recursively as follows:

$T \leftarrow r \;=$

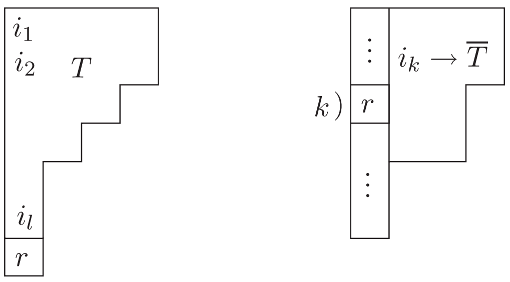


if $i_l \le r$ — finished; if $i_{k-1} \le r < i_k$ — to be continued

Here we have denoted the entries in the first row of T by $i_1 \le i_2 \le \cdots \le i_l$, and the other part of T by $\overline{T}$. The finished case includes the situation $T = \varnothing$. In the to be continued case, r bumps out the smallest number i_k that is larger than r (row bumping). The tableau $T \leftarrow r$ is obtained by repeating the row bumping until finished.

The column insertion of the number r into T is denoted by $r \rightarrow T$ and defined recursively as follows:

$r \rightarrow T \;=$



if $i_l < r$ — finished; if $i_{k-1} < r \le i_k$ — to be continued

Here we have denoted the entries of the first column of T by $i_1 < i_2 < \cdots < i_l$, and the other part of T by $\overline{T}$. The finished case includes the situation $T = \varnothing$. In the to be continued case, r bumps out the smallest number i_k that is not less than r (column bumping). The tableau $r \to T$ is obtained by repeating the column bumping until finished.

B Proof of Theorem 2.1

In this appendix, we use the notions from crystal theory. See [14, 13, 31] and [24, Section 2.3] for the notations φ_i, ε_i, the Kashiwara operator $\tilde{e}_i$ and the Weyl group simple reflection S_i. The crystal $B^{1,l}$ can be parameterized as $B^{1,l} = \{u = (u_1, \ldots, u_{n+1}) \in (\mathbb{Z}_{\geq 0})^{n+1} \mid u_1 + \cdots + u_{n+1} = l\}$, where u_i denotes the number of letter i contained in a length l row shape semistandard tableau u [31, 8]. The cyclic automorphism σ acts as $\sigma((u_1, \ldots, u_{n+1})) = (u_2, \ldots, u_{n+1}, u_1)$, or equivalently $u_i \mapsto u_{i+1}$. The automorphism σ acts on a tensor product component-wise. The indices are to be considered in $\mathbb{Z}/(n+1)\mathbb{Z}$. Recall that $B = (B^{1,1})^{\otimes L}$ and B_1 is defined in (2.15). We invoke the following factorization property of the combinatorial R into simple reflections.

Theorem B.1 ([8, Theorem 2]). *For any path $p \in B$ and $u = (u_1, u_2, \ldots, u_{n+1}) \in B^{1,l}$ with $u_1 \gg u_2, \ldots, u_{n+1}$, let $B^{1,l} \otimes B \ni u \otimes p \simeq \tilde{p} \otimes \tilde{u} \in B \otimes B^{1,l}$ be the image of the isomorphism. Then $\tilde{u} \otimes \tilde{p}$ is expressed by a product of simple reflections as*

$$\tilde{u} \otimes \tilde{p} = (\sigma \otimes \sigma) S_2 \cdots S_n S_{n+1}(u \otimes p). \tag{B.1}$$

Of course the actual image $\tilde{p} \otimes \tilde{u}$ is obtained by swapping the two tensor components of (B.1).

For l sufficiently large, fix the element $u = (u_1, \ldots, u_{n+1}) \in B^{1,l}$ according to $p \in B_1$ by

$$\begin{aligned} &u_i = \varphi_i(S_{i+1} S_{i+2} \cdots S_{n+1}(p)) \qquad (2 \leq i \leq n+1), \\ &u_1 = l - (u_2 + \cdots + u_{n+1}). \end{aligned} \tag{B.2}$$

This is possible, i.e., $u_1 \geq 0$, if l is sufficiently large. We are going to show that substitution of (B.2) into (B.1) leads to

$$\tilde{u} = u, \qquad \tilde{p} = \sigma S_2 S_3 \cdots S_{n+1}(p). \tag{B.3}$$

Since p is $T^{(1)}_\infty$-evolvable by the assumption of Theorem 2.1, $\tilde{u} = u$ means that u (B.2) is a proper carrier and

$$T^{(1)}_\infty(p) = \sigma S_2 S_3 \cdots S_{n+1}(p). \tag{B.4}$$

Let r_i be the Weyl group simple reflection acting on B component-wise. Then, by using $r_i^2 = \mathrm{id}$ and $\sigma r_2 r_3 \cdots r_{n+1} = \mathrm{id}$, the formula (B.4) is rewritten as

$$T^{(1)}_\infty(p) = \overline{K}_2 \overline{K}_3 \cdots \overline{K}_{n+1}(p), \quad \overline{K}_a = r_{n+1} \cdots r_{a+1} r_a S_a r_{a+1} \cdots r_n r_{n+1}.$$

It is straightforward to check that the gauge transformed simple reflection $\overline{K}_a$ is equal to K_a defined by the procedure (i)–(iv) explained before Theorem 2.1 by using the signature rule explained in [24, Section 2.3].

It remains to verify (B.3). We illustrate the calculation for $n = 3$, for the general case is completely parallel. Then (B.2) reads (note $S_4 = S_0, \varphi_4 = \varphi_0, \varepsilon_4 = \varepsilon_0$, etc.)

$$u_4 = \varphi_0(p), \qquad u_3 = \varphi_3(S_0(p)), \qquad u_2 = \varphi_2(S_3 S_0(p)).$$

We set

$$u^{(1)} \otimes p^{(1)} = S_0(u \otimes p), \qquad u^{(2)} \otimes p^{(2)} = S_3 S_0(u \otimes p), \qquad u^{(3)} \otimes p^{(3)} = S_2 S_3 S_0(u \otimes p),$$

$$\alpha_4 = \varepsilon_0(p), \qquad \alpha_3 = \varepsilon_3(p^{(1)}), \qquad \alpha_2 = \varepsilon_2(p^{(2)}).$$

We know u_1 is sufficiently large and from $p \in B_1$, $\alpha_4 \geq u_4$ is valid. To first compute $S_0(u \otimes p)$, we consider the 0-signature:

$$u \otimes p = (u_1, u_2, u_3, u_4) \otimes p \quad 0: \overbrace{- - \cdots - -}^{u_1}\overbrace{+++}^{u_4}\overbrace{----}^{\alpha_4}\overbrace{+++}^{u_4}.$$

Thus S_0 actually acts as $S_0 = \tilde{e}_0^{u_1-u_4} \otimes \tilde{e}_0^{\alpha_4-u_4} = \tilde{e}_0^{u_1-u_4} \otimes S_0$, leading to

$$u^{(1)} \otimes p^{(1)} = (u_4, u_2, u_3, u_1) \otimes S_0(p) \quad 3: \overbrace{- - \cdots - -}^{u_1}\overbrace{+++}^{u_3}\overbrace{----}^{\alpha_3}\overbrace{+++}^{u_3},$$

where we have depicted the 3-signature on account of $\varphi_3(p^{(1)}) = \varphi_3(S_0(p)) = u_3$. From $p \in B_1$ we see that $\alpha_3 = \varepsilon_3(p^{(1)}) = \varepsilon_3(S_0(p)) \geq \varphi_3(S_0(p)) = u_3$. Thus the next S_3 actually acts as $S_3 = \tilde{e}_3^{u_1-u_3} \otimes \tilde{e}_3^{\alpha_3-u_3} = \tilde{e}_3^{u_1-u_3} \otimes S_3$, leading to

$$u^{(2)} \otimes p^{(2)} = (u_4, u_2, u_1, u_3) \otimes S_3S_0(p) \quad 2: \overbrace{- - \cdots - -}^{u_1}\overbrace{+++}^{u_2}\overbrace{----}^{\alpha_2}\overbrace{+++}^{u_2},$$

where we have depicted the 2-signature on account of $\varphi_2(p^{(2)}) = \varphi_2(S_3S_0(p)) = u_2$. From $p \in B_1$ we see that $\alpha_2 = \varepsilon_2(p^{(2)}) = \varepsilon_2(S_3S_0(p)) \geq \varphi_2(S_3S_0(p)) = u_2$. Thus the next S_2 actually acts as $S_2 = \tilde{e}_2^{u_1-u_2} \otimes \tilde{e}_2^{\alpha_2-u_2} = \tilde{e}_2^{u_1-u_2} \otimes S_2$, leading to

$$u^{(3)} \otimes p^{(3)} = (u_4, u_1, u_2, u_3) \otimes S_2S_3S_0(p).$$

Thus (B.1) gives

$$\tilde{u} \otimes \tilde{p} = \sigma((u_4, u_1, u_2, u_3)) \otimes \sigma S_2S_3S_0(p) = (u_1, u_2, u_3, u_4) \otimes \sigma S_2S_3S_0(p),$$

yielding (B.3).

C KKR bijection

C.1 General remarks

The original KKR bijection [15, 16] is the one between rigged configurations and Littlewood–Richardson tableaux. Its ultimate generalization in type $A_n^{(1)}$ corresponding to $B^{r_1,l_1}\otimes\cdots\otimes B^{r_1,l_L}$ is available in [18]. In the simple setting of this paper, the Littlewood–Richardson tableaux are in one to one correspondence with highest paths in $B = (B^{1,1})^{\otimes L}$. The KKR bijection in this paper means the one (2.18) between rigged configurations and those highest paths. See [23, Appendix C] for an exposition in a slightly more general setting $B^{1,l_1} \otimes \cdots \otimes B^{1,l_L}$.

Recall that a rigged configuration $(\mu, \mathbf{r})$ is a multiset of strings, where a string is a triple $\big(a, l_i^{(a)}, r_{i,\alpha}^{(a)}\big)$ consisting of color, length and rigging. See the right diagram in (2.5). A string $\big(a, l_i^{(a)}, r_{i,\alpha}^{(a)}\big)$ is *singular* if $r_{i,\alpha}^{(a)} = p_i^{(a)}$, namely if the rigging attains the allowed maximum (2.17). We also recall that one actually has to attach the data L with $(\mu, \mathbf{r})$ to specify the vacancy numbers $p_i^{(a)}$ (2.7). Thus we write a rigged configuration as $(\mu, \mathbf{r})_L$. We regard a highest path $b_1 \otimes \cdots \otimes b_L \in (B^{1,1})^{\otimes L}$ as a word $b_1b_2\ldots b_L \in \{1, 2, \ldots, n+1\}^L$. The algorithms explained below obviously satisfy the property (2.20).

C.2 Algorithm for ϕ

Given a highest path $b_1 \dots b_L$, we construct the rigged configuration $\phi(b_1 \dots b_L) = (\mu, \mathbf{r})_L$ inductively with respect to L. When $L = 0$, we understand that $\phi(\cdot)$ is the array of empty partitions. Suppose that $\phi(b_1 \dots b_L) = (\mu, \mathbf{r})_L$ has been obtained. Denote $b_{L+1} \in \{1, \dots, n+1\}$ simply by d. We are to construct $(\mu', \mathbf{r}')_{L+1} = \phi(b_1 \dots b_L d)$ from $(\mu, \mathbf{r})_L$ and d. If $d = 1$, then $(\mu', \mathbf{r}')_{L+1} = (\mu, \mathbf{r})_{L+1}$ meaning that nothing needs to be done except increasing the vacancy numbers $p_i^{(1)}$ by one. (Recall in (2.7) that only $p_i^{(1)}$ depends on L.) Suppose $d \geq 2$.

(i) Set $\ell^{(d)} = \infty$. For $c = d-1, d-2, \dots, 1$ in this order, proceed as follows. Find the color c singular string whose length $\ell^{(c)}$ is largest within the condition $\ell^{(c)} \leq \ell^{(c+1)}$. If there are more than one such strings, pick any one of them. If there is no such string with color c, set $\ell^{(c)} = 0$. Denote these selected strings by $\left(c, \ell^{(c)}, r_*^{(c)}\right)$ with $c = d-1, d-2, \dots, 1$, where it is actually void when $\ell^{(c)} = 0$.

(ii) Replace the selected string $\left(c, \ell^{(c)}, r_*^{(c)}\right)$ by $\left(c, \ell^{(c)}+1, r_\bullet^{(c)}\right)$ for all $c = d-1, d-2, \dots, 1$ leaving the other strings unchanged. Here the new rigging $r_\bullet^{(c)}$ is to be fixed so that the enlarged string $\left(c, \ell^{(c)}+1, r_\bullet^{(c)}\right)$ becomes singular with respect to the resulting new configuration μ' and $L+1$.

The algorithm is known to be well-defined and the resulting object gives the sought rigged configuration $(\mu', \mathbf{r}')_{L+1} = \phi(b_1 \dots b_L d)$.

C.3 Algorithm for ϕ^{-1}

Given a rigged configuration $(\mu, \mathbf{r})_L$, we construct a highest path $b_1 \dots b_L = \phi^{-1}((\mu, \mathbf{r})_L)$ inductively with respect to L. We are to determine $d(= b_L) \in \{1, \dots, n+1\}$ and $(\mu', \mathbf{r}')_{L-1}$ such that $\phi^{-1}((\mu, \mathbf{r})_L) = \phi^{-1}((\mu', \mathbf{r}')_{L-1})\, d$.

(i) Set $\ell^{(0)} = 1$. For $c = 1, 2, \dots, n$ in this order, proceed as follows until stopped. Find the color c singular string whose length $\ell^{(c)}$ is smallest within the condition $\ell^{(c-1)} \leq \ell^{(c)}$. If there are more than one such strings, pick any one of them. If there is no such string with color c, set $d = c$ and stop. If $c = n$ and such a color n string still exists, set $d = n+1$ and stop. Denote these selected strings by $\left(c, \ell^{(c)}, r_*^{(c)}\right)$ with $c = 1, 2, \dots, d-1$.

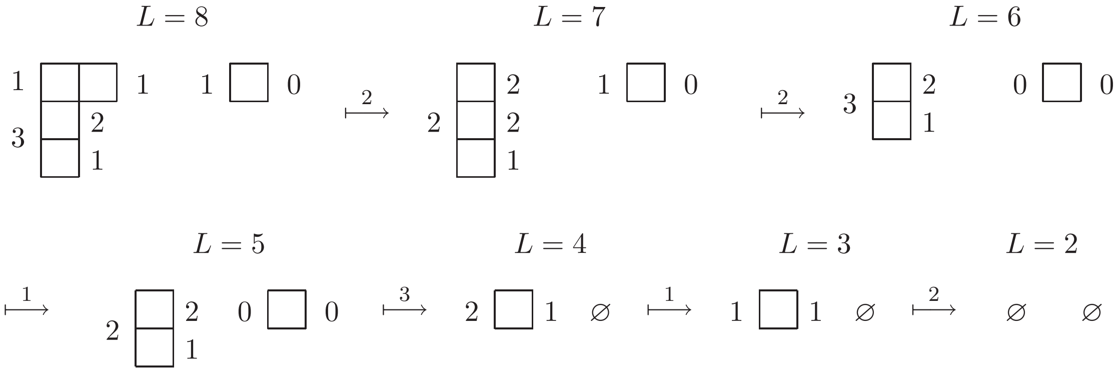
(ii) Replace the selected string $\left(c, \ell^{(c)}, r_*^{(c)}\right)$ by $\left(c, \ell^{(c)}-1, r_\bullet^{(c)}\right)$ for all $c = 1, 2, \dots, d-1$ leaving the other strings unchanged. When $\ell^{(c)} = 1$, this means that the length one string is to be eliminated. The new rigging $r_\bullet^{(c)}$ is to be fixed so that the shortened string $\left(c, \ell^{(c)}-1, r_\bullet^{(c)}\right)$ becomes singular with respect to the resulting new configuration μ' and $L-1$.

For empty rigged configuration, we understand that $\phi^{-1}((\emptyset, \emptyset)_L) = \phi^{-1}((\emptyset, \emptyset)_{L-1})\, 1 = \cdots = \overbrace{11\dots1}^{L}$. The algorithm is known well-defined and ends up with the empty rigged configuration at $L = 0$. The resulting sequence gives the sought highest path $b_1 \dots b_L = \phi^{-1}((\mu, \mathbf{r})_L)$.

Example C.1. Let us demonstrate ϕ^{-1} along an $L = 8$ example.

$$
\begin{array}{rl}
1 & \square\square\ \ 1 \\
3 & \square\ \ 2 \\
 & \square\ \ 1
\end{array}
\qquad
\begin{array}{rl}
1 & \square\ \ 0
\end{array}
\qquad \overset{\phi^{-1}}{\longmapsto} \qquad 11213122. \tag{C.1}
$$

This is the bottom right case of Example 2.4. The numbers on the left of Young diagrams are vacancy numbers, which are subsidiary data but convenient to keep track of.



Here we have exhibited the number d in the algorithm (i) over the arrows. After this, we only get 1 twice. Thus we get the sequence $\stackrel{2}{\mapsto}\stackrel{2}{\mapsto}\stackrel{1}{\mapsto}\stackrel{3}{\mapsto}\stackrel{1}{\mapsto}\stackrel{2}{\mapsto}\stackrel{1}{\mapsto}\stackrel{1}{\mapsto}$ to reach the trivial rigged configuration. Reading these numbers backwards we obtain (C.1).

D Proof of Theorem 2.2

For $p = b_1 \otimes \cdots \otimes b_L \in B$, define another version of energy $\hat{E}^{(r)}_l(p) = e_1 + \cdots + e_L$ with the diagram (2.2) by taking $v = u^{r,l}$, the highest element of $B^{r,l}$. $u^{r,l}$ is the semistandard tableau whose entries in the jth row are all j. For example, $u^{2,3} = \begin{smallmatrix}111\\222\end{smallmatrix}$. Set

$$\hat{\mathcal{P}}(\mu) = \left\{ p \in \mathcal{P} \mid \hat{E}^{(a)}_l(p) = \sum_{i=1}^{g_a} \min\left(l, l^{(a)}_i\right) m^{(a)}_i \right\}, \tag{D.1}$$

$$\hat{\mathcal{P}}_+(\mu) = \left\{ p \in \hat{\mathcal{P}}(\mu) \mid p : \text{highest} \right\}. \tag{D.2}$$

These definitions resemble (2.11) and (2.12). The differences are that here enters $\hat{E}^{(r)}_l$ instead of $E^{(r)}_l$, and there is no requirement that p be evolvable in (D.1).

Theorem D.1 ([33]). *The KKR bijection* (2.18) *can be restricted into a finer bijection:*

$$\phi : \ \hat{\mathcal{P}}_+(\mu) \to \mathrm{RC}(\mu).$$

Lemma D.1. *The equality $E^{(r)}_l(p) = \hat{E}^{(r)}_l(p)$ is valid for any $p \in \mathcal{P}_+(\mu)$.*

Proof. Take $p \in \mathcal{P}_+(\mu)$. Then by Lemma 4.1, the rigged configuration for $p \otimes p$ has the configuration obtained from μ by replacing $m^{(a)}_i$ with $2m^{(a)}_i$. Thus from (D.1) and Theorem D.1, one has $\hat{E}^{(r)}_l(p\otimes p) = 2\hat{E}^{(r)}_l(p)$. On the other hand, define $v^{r,l} \in B^{r,l}$ (and $\hat{p}$) by $u^{r,l}\otimes p \simeq \hat{p}\otimes v^{r,l}$. Then, using Lemma 4.1 in this paper and Lemma 8.5 in [18], one can show $v^{r,l} \otimes p \simeq p' \otimes v^{r,l}$ for some $p' \in B$. Since p is evolvable, we can claim that $v^{r,l}$ works as a carrier to define $E^{(r)}_l(p)$. Then from the relation $u^{r,l} \otimes p \otimes p \simeq \hat{p} \otimes v^{r,l} \otimes p \simeq \hat{p} \otimes p' \otimes v^{r,l}$, the energy $\hat{E}^{(r)}_l$ associated to $p \otimes p$ is given by $\hat{E}^{(r)}_l(p \otimes p) = \hat{E}^{(r)}_l(p) + E^{(r)}_l(p)$. Comparing this with $\hat{E}^{(r)}_l(p \otimes p) = 2\hat{E}^{(r)}_l(p)$, we get $E^{(r)}_l(p) = \hat{E}^{(r)}_l(p)$. ■

Lemma D.2. *There is a natural inclusion $\mathcal{P}_+(\mu) \hookrightarrow \hat{\mathcal{P}}_+(\mu)$.*

Proof. Due to Lemma D.1, the difference of $\hat{\mathcal{P}}_+(\mu)$ (D.2) and $\mathcal{P}_+(\mu)$ (2.12) is only the extra condition that p be evolvable in (2.11). ■

We expect that actually $\mathcal{P}_+(\mu) = \hat{\mathcal{P}}_+(\mu)$ holds, namely, all the highest paths are evolvable. However we do not need this fact in this paper.

Proof of Theorem 2.2. Combine Theorem D.1 and Lemma D.2. ■

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