On a Nonlocal Ostrovsky–Whitham Type Dynamical System, Its Riemann Type Inhomogeneous Regularizations and Their Integrability*

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Abstract. Short-wave perturbations in a relaxing medium, governed by a special reduction of the Ostrovsky evolution equation, and later derived by Whitham, are studied using the gradient-holonomic integrability algorithm. The bi-Hamiltonicity and complete integrability of the corresponding dynamical system is stated and an infinite hierarchy of commuting to each other conservation laws of dispersive type are found. The well defined regularization of the model is constructed and its Lax type integrability is discussed. A generalized hydrodynamical Riemann type system is considered, infinite hierarchies of conservation laws, related compatible Poisson structures and a Lax type representation for the special case $N = 3$ are constructed.

Key words: generalized Riemann type hydrodynamical equations; Whitham type dynamical systems; Hamiltonian systems; Lax type integrability; gradient-holonomic algorithm

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1 Introduction

Many important problems of propagating waves in nonlinear media with distributed parameters, for instance, invisible non-dissipative dark matter, playing a decisive role [9, 10] in the formation of large scale structure in the Universe like galaxies, clusters of galaxies, super-clusters, can be described by means of evolution differential equations of special type. It is also well known [2,

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17, 25, 28] that shortwave perturbations in a relaxing one dimensional medium can be described by means of some reduction of the Ostrovsky equations, coinciding with the Whitham type evolution equation

$$du/dt = 2uu_x + \int_{\mathbb{R}} \mathcal{K}(x,s)u_s ds, \tag{1.1}$$

discussed first in [28]. Here the kernel $\mathcal{K} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ depends on the medium elasticity properties with spatial memory and can, in general, be a function of the pressure gradient $u_x \in C^\infty(\mathbb{R};\mathbb{R})$, evolving with respect to equation (1.1). In particular, if the nonlinear medium is endowed still with spatial memory properties, that is the wave amplitude depends on the orbit, swept by its front, the propagation of the corresponding wave can be modeled by means of the so called generalized Ostrovsky evolution equations [20]. Namely, if to put $\mathcal{K}(x,s) = \frac{1}{2}|x-s|$, $x,s \in \mathbb{R}$, then equation (1.1) can be reduced to

$$du/dt = 2uu_x + \partial^{-1}u,$$

which was, in particular, studied before in [17, 21, 22].

Since some media possess elasticity properties depending strongly on the spatial pressure gradient u_x, $x \in \mathbb{R}$, the corresponding Whitham type kernel looks like

$$\mathcal{K}(x,s) := -\theta(x-s)u_s \tag{1.2}$$

for $x,s \in \mathbb{R}$, naturally modeling the relaxing spatial memory effects. The resulting equation (1.1) with the kernel (1.2) becomes

$$du/dt = 2uu_x - \partial^{-1}u_x^2 := K[u], \tag{1.3}$$

which appears to possess very interesting mathematical properties. The latter will be the main topic of the next sections below.

Owing to the results, obtained before in [25, 26], the dynamical system (1.3) appeared to be a Lax type integrable bi-Hamiltonian flow, but with ill posed temporal evolution. As it was demonstrated in [26], a suitable finite-dimensional reduction scheme, if applied to the corresponding hierarchy of conservation laws for constructing explicit solutions to the Ostrovsky–Whitham type nonlinear dynamical system (1.3) by means of quadratures, meets some technical problems. Some of these integrability aspects were before presented in [2], where a suitable well posed regularization of the equation (1.3) in the form

$$\left.\begin{aligned} u_t &= 2uu_x - v \\ v_t &= 2uv_x \end{aligned}\right\} := K[u,v] \tag{1.4}$$

for treating this nonlocality problem was proposed.

Below the well posed integrability problem for the Ostrovsky–Whitham type nonlinear and nonlocal dynamical system (1.4) will be reanalyzed in detail making use of this regularization scheme. The corresponding implectic structures and Lax type representations are found by means of the differential-geometric tools, devised and extended in [7, 8, 15, 24]. A natural Riemann type generalization of the dynamical system (1.4) is proposed, owing to a recent observation by D. Holm and M. Pavlov:

$$D_t^N u = 0, \qquad N \in \mathbb{Z}_+, \tag{1.5}$$

which at $N = 2$ is exactly equivalent to the system (1.4). The integrability properties of equation (1.5) at $N = 3$ were analyzed in detail, the conservation laws, corresponding compatible implectic structures and Lax type representation are constructed.

It is worth to mention that the obtained in this work Lax type pair (3.6) for the regularized dynamical system (1.4) was found first in work [4]. It coincides with those found later in [23], making use of a very special bi-Lagrangian representation of the dynamical system (1.4). But the existence of the singular co-implectic structure (3.14) in these references was not stated. A detailed analysis of the relationships between solutions of dynamical systems (1.3) and (1.4), based on a reciprocal transformation, suggested by M. Pavlov in [23], was presented recently in [27]. Mention also work [14], where the geometric aspects of the equation like (2.1) were studied.

Note also here that theory of integrable homogenous hydrodynamic type systems with distinct characteristic velocities was constructed by S.P. Tsarev. In this paper we consider the first example in a literature of nonhomogeneous integrable hydrodynamic type systems with a sole characteristic velocity. Such a theory does not exist at this moment.

2 A regularization scheme and the geometric integrability problem

Define a smooth periodic function $v \in C^{\infty}_{2\pi}(\mathbb{R};\mathbb{R})$, such that

$$v := \partial^{-1} u_x^2$$

for any $x, t \in \mathbb{R}$, where the function $u \in C^{\infty}_{2\pi}(\mathbb{R};\mathbb{R})$ solves equation (1.3). Then it is easy to state that the following regularized nonlinear dynamical system

$$\left.\begin{aligned} u_t &= 2uu_x - v \\ v_t &= 2uv_x \end{aligned}\right\} := K[u,v] \tag{2.1}$$

of hydrodynamic type, which was introduced before in [6], studied in [2, 4, 12, 13, 19] and analyzed as a Gurevich–Zybin system in [23], and is already well defined on the extended 2π-periodic functional space $\mathcal{M} := C^{\infty}_{2\pi}(\mathbb{R};\mathbb{R}^2)$ and equivalent on the functional submanifold $\mathcal{M}_{\mathrm{red}} := \{(u,v) \in \mathcal{M} : v_x - u_x^2 = 0\}$ to that given by expression (1.3), as it was mentioned in [2] and discussed recently in [27]. The system (2.1) can be rewritten as the following set of equations

$$\begin{aligned} &u_t = 2uu_x - v, \qquad v_t = 2uv_x, \\ &u_x = w, \ v_x = u_x w, \\ &w_t = v_x + 2uw_x, \end{aligned} \tag{2.2}$$

which is equivalent to a set of differential two-forms

$$\begin{aligned} \{\alpha\} := \big\{ &\alpha^{(1)} = du \wedge dx + 2udu \wedge dt - vdx \wedge dt, \ \alpha^{(2)} = dv \wedge dx + 2udv \wedge dt, \\ &\alpha^{(3)} = du \wedge dt - wdx \wedge dt, \ \alpha^{(4)} = dv \wedge dt - wdu \wedge dt, \\ &\alpha^{(5)} = dw \wedge dx + dv \wedge dt + 2udw \wedge dt \big\}. \end{aligned} \tag{2.3}$$

This set of two-forms generates the closed ideal $\mathcal{I}(\alpha)$, since

$$\begin{aligned} &d\,\alpha^{(1)} = -\alpha^{(2)} \wedge dt, \qquad d\alpha^{(2)} = 2du \wedge \alpha^{(4)}, \qquad d\alpha^{(3)} = -\alpha^{(5)} \wedge dt, \\ &d\alpha^{(4)} = -dw \wedge \alpha^{(3)} - wdt \wedge \alpha^{(5)}, \qquad d\alpha^{(5)} = -2dw \wedge \alpha^{(3)} - 2wdt \wedge \alpha^{(5)}. \end{aligned}$$

The set of differential forms (2.3), being integrable, defines the integral submanifold $\bar{M}$ by means of the condition $\mathcal{I}(\alpha) = 0$. Making now use of the differential-geometric method devised in [11, 18, 24] and extending algorithmically the approach of [15], we will look for a reduced upon the integral submanifold $\bar{M}$ connection one-form Γ, belonging to some not yet determined its holonomy Lie algebra $\mathcal{G}$. This 1-form can be represented as follows:

$$\Gamma = \mathcal{A}(u,v,w)dx + \mathcal{B}(u,v,w)dt, \tag{2.4}$$

where the elements $\mathcal{A}, \mathcal{B} \in \mathcal{G}$ satisfy determining equations

$$\begin{aligned}\Omega = {} & \frac{\partial \mathcal{A}}{\partial u} du \wedge dx + \frac{\partial \mathcal{A}}{\partial v} dv \wedge dx + \frac{\partial \mathcal{A}}{\partial w} dw \wedge dx + \frac{\partial \mathcal{B}}{\partial u} du \wedge dt \\ & + \frac{\partial \mathcal{B}}{\partial v}\, dv \wedge dt + \frac{\partial \mathcal{B}}{\partial w}\, dw \wedge dt + [\mathcal{A}, \mathcal{B}] dx \wedge dt \\ \Rightarrow \quad & g_1(du \wedge dx + 2u du \wedge dt - v dx \wedge dt) + g_2(dv \wedge dx + 2u dv \wedge dt) \\ & + g_3(du \wedge dt - w dx \wedge dt) + g_4(dv \wedge dt - w du \wedge dt) \\ & + g_5(dw \wedge dx + 2u dw \wedge dt + dv \wedge dt) \in \mathcal{I}(\alpha) \otimes \mathcal{G}\end{aligned} \tag{2.5}$$

for some $\mathcal{G}$-valued functions $g_1, \ldots, g_5 \in \mathcal{G}$ on M. From (2.5) one finds that

$$\begin{aligned} & \frac{\partial \mathcal{A}}{\partial u} = g_1, \qquad \frac{\partial \mathcal{A}}{\partial v} = g_2, \qquad \frac{\partial \mathcal{A}}{\partial w} = g_5, \\ & \frac{\partial \mathcal{B}}{\partial u} = 2u g_1 + g_3 - w g_4, \qquad \frac{\partial \mathcal{B}}{\partial v} = 2u g_2 + g_4 + g_5, \\ & \frac{\partial \mathcal{B}}{\partial w} = 2u g_5, \qquad [\mathcal{A}, \mathcal{B}] = -v g_1 - w g_3. \end{aligned} \tag{2.6}$$

Thereby, from the obtained set of relationships (2.6) one can find that

$$\begin{aligned} & \mathcal{B} = 2u\mathcal{A} + \mathcal{C}(u,v), \qquad g_4 = \frac{\partial \mathcal{C}}{\partial v} - \frac{\partial \mathcal{A}}{\partial w}, \qquad g_3 = 2\mathcal{A} + \frac{\partial \mathcal{C}}{\partial u} + w\frac{\partial \mathcal{C}}{\partial v} - w\frac{\partial \mathcal{A}}{\partial w}, \\ & [\mathcal{A}, \mathcal{C}] = -v\frac{\partial \mathcal{A}}{\partial u} - 2w\mathcal{A} - w\frac{\partial \mathcal{C}}{\partial u} - w^2\frac{\partial \mathcal{C}}{\partial v} + w^2\frac{\partial \mathcal{A}}{\partial w}, \end{aligned}$$

serving for final searching for connection (2.4).

3 The bi-Hamiltonian structure and Lax-type representation

Consider the following polynomial expansion of the element $\mathcal{A}(u,v;w) \in \mathcal{G}$ with respect to the variable w:

$$\mathcal{A} = \mathcal{A}_0(u,v) + \mathcal{A}_1(u,v)w + \mathcal{A}_2(u,v)w^2$$

and substitute it into the last equation of (2.6). As a result we obtain:

$$\begin{aligned} & [\mathcal{A}_0, C] = -v\frac{\partial \mathcal{A}_0}{\partial u}, \qquad [\mathcal{A}_1, C] = -v\frac{\partial \mathcal{A}_1}{\partial u} - 2\mathcal{A}_0 - \frac{\partial C}{\partial u}, \\ & [\mathcal{A}_2, C] = -v\frac{\partial \mathcal{A}_2}{\partial u} - \frac{\partial C}{\partial v} - \mathcal{A}_1, \end{aligned} \tag{3.1}$$

or

$$\mathcal{A}_1 = [C, \mathcal{A}_2] - v\frac{\partial \mathcal{A}_2}{\partial u} - \frac{\partial C}{\partial v}. \tag{3.2}$$

which can be substituted into the second equation of (3.1):

$$[[C,\mathcal{A}_2],C] - 2v\left[\frac{\partial \mathcal{A}_2}{\partial u}, C\right] - \left[\frac{\partial C}{\partial v}, C\right] = -v\left[\frac{\partial C}{\partial u}, \mathcal{A}_2\right] - v^2\frac{\partial^2 \mathcal{A}_2}{\partial u^2} - v\frac{\partial^2 C}{\partial u \partial v} - 2\mathcal{A}_0 - \frac{\partial C}{\partial u}.$$

Thus, recalling (3.1) and (3.2), we have that

$$2\mathcal{A}_0 = [C, [C, \mathcal{A}_2]] + 2v\left[\frac{\partial \mathcal{A}_2}{\partial u}, C\right] + \left[\frac{\partial C}{\partial v}, C\right] - v\left[\frac{\partial C}{\partial u}, \mathcal{A}_2\right] - v^2\frac{\partial^2 \mathcal{A}_2}{\partial u^2} - v\frac{\partial^2 C}{\partial u \partial v} - \frac{\partial C}{\partial u},$$
$$[\mathcal{A}_0, C] = -v\frac{\partial \mathcal{A}_0}{\partial u}, \qquad \mathcal{A}_1 = [C, \mathcal{A}_2] - v\frac{\partial \mathcal{A}_2}{\partial u} - \frac{\partial C}{\partial v}. \tag{3.3}$$

Now we will assume that the element $C := C_0$ is constant and the elements $\mathcal{A}_0$ and $\mathcal{A}_2$ are linear with respect to variables u and v, that is

$$\mathcal{A}_0 = \mathcal{A}_0^{(0)} + \mathcal{A}_0^{(1)}u + \mathcal{A}_0^{(2)}v, \qquad \mathcal{A}_2 = \mathcal{A}_2^{(0)} + \mathcal{A}_2^{(1)}u + \mathcal{A}_2^{(2)}v.$$

Whence and from (3.3) one gets:

$$2\mathcal{A}_0^{(0)} = [C_0, [C_0, \mathcal{A}_2^{(0)}]], \qquad [\mathcal{A}_0^{(1)}, C_0] = 0, \qquad [\mathcal{A}_0^{(2)}, C_0] = -\mathcal{A}_0^{(1)},$$
$$2\mathcal{A}_0^{(1)} = [C_0, [C_0, \mathcal{A}_2^{(1)}]], \qquad 2\mathcal{A}_0^{(2)} = [C_0, [C_0, \mathcal{A}_2^{(2)}]] + 2[\mathcal{A}_2^{(1)}, C_0]. \tag{3.4}$$

To solve the algebraic system (3.4) we need to calculate [24] the corresponding holonomy Lie algebra of the connection (2.4). As a result of simple, but slightly cumbersome calculations, we derive that elements $\mathcal{A}_2^{(j)}$, $j = 0, \ldots, 2$, and C_0 belong to the Lie algebra $sl(2;\mathbb{C})$, whose basis L_0, L_+ and L_- can be taken to satisfy the following canonical commutation relations:

$$[L_0, L_\pm] = \pm L_\pm, \qquad [L_+, L_-] = 2L_0.$$

Thereby, making use of the standard determining expansions

$$\mathcal{A}_2^{(j)} = \sum_\pm c_\pm^{(j)} L_\pm + c_0^{(j)} L_0, \qquad C_0 = \sum_\pm k_\pm L_\pm + k_0 L_0, \tag{3.5}$$

where $j = 0, \ldots, 2$, and substituting (3.5) into (3.4), we obtain some relationships on values $c_\pm^{(j)}, c_0^{(j)} \in \mathbb{C}$, $j = 0, \ldots, 2$, and $k_\pm, k_0 \in \mathbb{C}$. Resolving by means of simple but slightly cumbersome calculations these relationships, we find the searched for basic elements $\mathcal{A}$ and $\mathcal{B}$ of the connection Γ, depending on a spectral parameter $\lambda \in \mathbb{C}$, thereby giving rise to the corresponding Lax type commutative spectral representation for dynamical system (2.1) in the following (2×2)-matrix form:

$$\frac{df}{dx} = \ell[u, v; \lambda]f, \qquad \frac{df}{dt} = p(\ell)f, \qquad p(\ell) := 2u\ell[u, v; \lambda] + q, \tag{3.6}$$
$$\ell[u, v; \lambda] := \begin{pmatrix} -\lambda u_x & -v_x \\ \lambda^2 & \lambda u_x \end{pmatrix}, \qquad q := \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}, \qquad p(\ell) = \begin{pmatrix} -2\lambda u_x u & -2v_x u \\ \lambda + 2\lambda^2 u & 2\lambda u_x u \end{pmatrix},$$

defining the generalized time-independent spectrum $\mathrm{Spec}(\ell) \subset \mathbb{C}$: $\lambda \in \mathrm{Spec}(\ell)$, if the corresponding solution $f \in L_\infty(\mathbb{R}; \mathbb{C}^2)$. It is worth to remark here that the Lax type representation (3.6), found for the dynamical system (2.1), is not unique. Moreover, making use of other imbeddings of the connection form (2.4) into a suitable holonomy Lie algebra $\mathcal{G}$, one can construct different Lax type representations, which could appear to be more useful for finding exact solutions to dynamical system (2.1) by means of, for instance, the inverse spectral transform method.

The standard Riccati equation, derived from (3.6), allows to obtain right away an infinite hierarchy of local conservation laws:

$$\hat{\gamma}_{-1} := \int_0^{2\pi} \sqrt{u_x^2 - v_x}\, dx, \qquad \hat{\gamma}_0 := \int_0^{2\pi} \frac{(u_x v_{xx} - v_x u_{xx})}{2v_x\sqrt{u_x^2 - v_x}} dx, \qquad \ldots, \tag{3.7}$$

and so on. All of conservation laws (3.7) except γ_{-1}, are singular at the Cauchy condition (2.2). This means that we need to construct other hierarchy of polynomial conservation laws regular on the functional submanifold

$$\mathcal{M}_{\rm red} := \big\{(u,v) \in \mathcal{M} : u_x^2 - v_x = 0,\ x \in \mathbb{R}/2\pi\mathbb{Z}\big\}. \tag{3.8}$$

The latter exists owing to the results of [23, 24]. The simplest way to search for them consists in determining the bi-Hamiltonian structure of flow (2.1). As it is easy to check, dynamical system (2.1) is canonically Hamiltonian, that is

$$\frac{d}{dt}(u,v)^{\intercal} := -\hat{\vartheta}\,\mathrm{grad}\,\hat{H}_{\vartheta} = \hat{K}[u,v],$$

where the corresponding co-symplectic structure $\hat{\vartheta} : T^*(\mathcal{M}) \to T(\mathcal{M})$ is canonical, equals

$$\hat{\vartheta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.9}$$

and satisfies the Noether equation

$$L_{\hat{K}}\hat{\vartheta} = 0 = d\hat{\vartheta}/dt - \hat{\vartheta}\hat{K}'^{,*} - \hat{K}'\hat{\vartheta}.$$

To prove this, it is enough to find by means of the small parameter method, devised before in [24] a non-symmetric ($\varphi' \neq \varphi'^{,*}$) solution $\varphi \in T(\mathcal{M})$ to the following Lie–Lax equation:

$$d\varphi/dt + \hat{K}'^{,*}\varphi = \mathrm{grad}\,L \tag{3.10}$$

for some suitably chosen smooth functional $L \in \mathcal{D}(M)$. As a result of easy calculations one obtains that

$$\varphi = (-v, 0)^{\intercal}, \qquad L = -\int_0^{2\pi} uv dx. \tag{3.11}$$

Making use of (3.11) and the classical Legendrian relationship for the suitable Hamiltonian function

$$H := (\varphi, \hat{K}) - L, \tag{3.12}$$

and the corresponding symplectic structure

$$\hat{\vartheta}^{-1} := \varphi' - \varphi'^{,*} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.13}$$

one obtains the implectic structure (3.9) and the corresponding non-singular Hamilton function

$$\hat{H}_{\vartheta} := \int_0^{2\pi} (v^2/2 + v_x u^2) dx.$$

It is here worth to mention that the determining Lie–Lax equation (3.10) possesses still another solution

$$\varphi = \left(\frac{u_x}{2}, -\frac{u_x^2}{2v_x}\right), \qquad L = \frac{1}{4}\int_0^{2\pi} uv_x dx,$$

giving rise, owing to formulas (3.13) and (3.12) to the new co-implectic (singular "symplectic") structure

$$\hat{\eta}^{-1} := \varphi' - \varphi'^{,*} = \begin{pmatrix} \partial & -\partial u_x v_x^{-1} \\ -u_x v_x^{-1}\partial & \frac{1}{2}(u_x^2 v_x^{-2}\partial + \partial u_x^2 v_x^{-2}) \end{pmatrix} \tag{3.14}$$

and the Hamiltonian functional

$$\hat{H}_\eta := \frac{1}{2}\int_0^{2\pi}(uv_x - vu_x)dx.$$

The co-implectic structure (3.14) is, evidently, singular since $\hat{\eta}^{-1}(u_x, v_x)^\intercal = 0$. Remark also that, owing to the general symplectic theory results [1, 8, 11, 15, 16, 18, 24] for nonlinear dynamical systems on smooth functional manifolds, operator (3.14) defines on the manifold $\mathcal{M}$ a closed differential two-form. Thereby it is *a priori* co-symplectic, satisfying on $\mathcal{M}$ the standard Jacobi brackets condition. Moreover, the implectic structure $\hat{\eta} : T^*(\mathcal{M}) \to T^*(\mathcal{M})$ satisfies the determining Noether equation

$$L_{\hat{K}}\hat{\eta} = 0 = d\hat{\eta}/dt - \hat{\eta}\hat{K}'^{,*} - \hat{K}'\hat{\eta},$$

whose solutions can also be obtained by means of the small parameter method, devised before in [16, 24]. As a result, the second implectic operator has the form

$$\hat{\eta} := \left(\begin{array}{cc} \partial^{-1} & 2u_x\partial^{-1} \\ 2\partial^{-1}u_x & 2v_x\partial^{-1} + 2\partial^{-1}v_x \end{array} \right), \tag{3.15}$$

giving rise to a new infinite hierarchy of polynomial conservation laws

$$\hat{\gamma}_n := \int_0^1 d\lambda \langle (\hat{\vartheta}^{-1}\hat{\eta})^n \operatorname{grad} \hat{H}_\vartheta[u\lambda\}, u\rangle \tag{3.16}$$

for all $n \in \mathbb{Z}_+$.

In particular, one can easily observe that there hold representations

$$\frac{d}{dt}(u, v)^\intercal = -\hat{\eta} \operatorname{grad} \hat{H}_\eta, \qquad \frac{d}{dx}(u, v)^\intercal = -\hat{\vartheta} \operatorname{grad} \hat{H}_\eta,$$

where

$$\hat{H}_\eta := \frac{1}{2}\int_0^{2\pi}(uv_x - vu_x)dx.$$

Thereby, one can formulate the following proposition.

Proposition 1. *The Riemann type hydrodynamical system* (2.1) *is a Lax type integrable bi-Hamiltonian flow on the functional manifold* $\mathcal{M}$. *The corresponding implectic pairs are compatible and given by matrix operators* (3.9) *and* (3.15)*, the Lax type representation is presented in the differential matrix form* (3.6)*.*

Now, making use of (3.16), one can apply the standard reduction procedure upon the corresponding finite dimensional functional subspaces $\mathcal{M}^{2n} \subset \mathcal{M}$, $n \in \mathbb{Z}_+$, and obtain a large set of exact solutions of special quasi-periodic and solitonic type to dynamical system (2.1) upon the functional submanifold $\mathcal{M}_{\mathrm{red}}$, if the Cauchy data are taken to satisfy constraint (3.8). Here we need to mention that a general solution to the system (2.1), obtained in [23, 27], is presented in an unwieldy involved form, almost completely not feasible for practical applications.

4 A Riemann type hydrodynamical generalization

It is here interesting to mention (owing to recent observations by D. Holm for $N = 2$ and for arbitrary $N \in \mathbb{Z}_+$ by M. Pavlov) that the dynamical system (2.1) can be equivalently rewritten up to the time rescaling as

$$D_t^2 u = 0, \qquad D_t := \partial/\partial t + u\partial, \tag{4.1}$$

under the flow velocity condition $dx/dt := u$, which is a partial case [5] of the generalized Riemann type hydrodynamic system

$$D_t^N u = 0 \tag{4.2}$$

for any integer $N \in \mathbb{Z}_+$. If $N = 3$, having defined the new variables $v := D_t u$, $z := D_t v$, one easily obtains the new dynamical system

$$\left.\begin{aligned} u_t &= v - uu_x \\ v_t &= z - uv_x \\ z_t &= -uz_x \end{aligned}\right\} := K[u, v, z] \tag{4.3}$$

of hydrodynamical type, which proves also to possess infinite hierarchies of polynomial conservation laws.

As we are interested first in the conservation laws for the system (4.3), the following proposition holds.

Proposition 2. *Let $H(\lambda) := \int_0^{2\pi} h(x;\lambda)dx \in D(\mathcal{M})$ be an almost everywhere smooth functional on the manifold $\mathcal{M}$, depending parametrically on $\lambda \in \mathbb{C}$, and whose density satisfies the differential condition*

$$h_t = \lambda(uh)_x \tag{4.4}$$

for all $t \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ on the solution set of equation (4.1)*. Then the following iterative differential relationship*

$$(f/h)_t = \lambda(uf/h)_x \tag{4.5}$$

holds, if a smooth function $f \in C^\infty(\mathbb{R};\mathbb{R})$ (parametrically depending on $\lambda \in \mathbb{C}$) satisfies for all $t \in \mathbb{R}$ the linear equation

$$f_t = 2\lambda u_x f + \lambda u f_x. \tag{4.6}$$

Proof. We have from (4.4)–(4.6) that

$$\begin{aligned}(f/h)_t &= f_t/h - fh_t/h^2 = f_t/h - \lambda f u_x/h - \lambda f u h_x/h^2 = f_t/h + \lambda f u(1/h)_x - \lambda u_x f/h \\ &= \lambda(uf)_x/h + \lambda uf(1/h)_x = \lambda(uf/h)_x,\end{aligned}$$

proving the proposition. ■

The obvious generalization of the previous proposition is read as follows.

Proposition 3. *If a smooth function $h \in C^\infty(\mathbb{R};\mathbb{R})$ satisfies the relationships*

$$h_t = ku_x h + uh_x,$$

where $k \in \mathbb{R}$, then

$$H = \int_0^{2\pi} h^{1/k} dx$$

is a conservation law for the Riemann type hydrodynamical system (2.1)*.*

The following polynomial dispersionless functionals, constructed by means of Proposition 3, are conserved with respect to the flow (4.3):

$$
\begin{aligned}
H_n^{(1)} &:= \int_0^{2\pi} dx z^n \left(vu_x - v_x u - \frac{n+2}{n+1} z \right), \\
H^{(4)} &:= \int_0^{2\pi} dx \big[-7v_x v^2 u + z\big(6zu + 2v_x u^2 - 3v^2 - 4vuu_x\big)\big], \\
H^{(5)} &:= \int_0^{2\pi} dx \big(z^2 u_x - 2zvv_x\big), \qquad H^{(6)} := \int_0^{2\pi} dx \big(z_z v^3 + 3z^2 v_x u + z^3\big), \\
H^{(7)} &:= \int_0^{2\pi} dx \big(z_x v^3 + 3z^2 vu_x - 3z^3\big), \\
H^{(8)} &:= \int_0^{2\pi} dx z \big(6z^2 u + 3zv_x u^2 - 3zv^2 - 4zvu_x - 2v_x v^2 u + 2v^3 u_x\big), \\
H^{(9)} &:= \int_0^{2\pi} dx \big[1001 v_x v^4 u + \big(1092 z^2 u^2 + 364 z v_x u^3 - \\
&\qquad - 1092 z v^2 u - 728 z v u_x u^2 - 364 v_x v^2 u^2 + 273 v^4 + 728 v^3 u_x u\big)\big], \\
H_n^{(2)} &:= \int_0^{2\pi} dx z_x v z^n, \qquad H_n^{(3)} := \int_0^{2\pi} dx z_x \big(v^2 - 2zu\big)^n,
\end{aligned}
$$

where $n \in \mathbb{Z}_+$. In particular, as $n = 1, 2, \ldots$, from (4.3) one obtains that

$$
\begin{aligned}
H_0^{(2)} &:= \int_0^{2\pi} dx z_x v, \qquad H_1^{(2)} := \int_0^{2\pi} dx z_x z v, \qquad \ldots, \\
H_1^{(3)} &:= \int_0^{2\pi} dx z_x \big(v^2 - 2uz\big), \qquad H_2^{(3)} := \int_0^{2\pi} dx z_x \big(v^4 + 4z^2 u^2 - 4zv^2 u\big), \qquad \ldots,
\end{aligned}
$$

and so on. Similarly one can construct also infinite hierarchies of conservation laws for the hydrodynamical system (4.3), which are both non-polynomial and dispersive:

$$
\begin{aligned}
H_1^{(1/4)} &= \int_0^{2\pi} dx \big(-2u_{xx} u_x z_x + u_{xx} v_x^2 + 2u_x^2 z_{xx} - u_x v_{xx} v_x + 3v_{xx} z_x - 3v_x z_{xx}\big)^{1/4}, \\
H_2^{(1/3)} &= \int_0^{2\pi} dx (-v_{xx} z_x + v_x z_{xx})^{1/3}, \\
H_3^{(1/3)} &= \int_0^{2\pi} dx (v_{xx} u_x - v_x u_{xx} - z_{xx})^{1/3}, \\
H_1^{(1/2)} &= \int_0^{2\pi} dx \big[-2vu_x z_x + v_x^2 + z(-u_x v_x + 3z_x)\big]^{1/2}, \\
H_2^{(1/2)} &= \int_0^{2\pi} dx \big(8u_x^3 z_x - 3u_x^2 v_x^2 - 18u_x v_x z_x + 6v_x^3 + 9z_x\big)^{1/2}, \\
H_1^{(1/5)} &= \int_0^{2\pi} dx \big(-2u_{xxx} u_x z_x + u_{xxx} v_x^2 + 6u_{xx}^2 z_x - 6u_{xx} u_x z_{xx} \\
&\qquad - 3u_{xx} v_{xx} v_x + 2u_x^2 z_{xxx} - u_x v_{xxx} v_x + 3u_x v_{xx}^2 + 3v_{xxx} z_x - 3v_x z_{xxx}\big)^{1/5}, \\
H_3^{(1/3)} &= \int_0^{2\pi} dx \big[k_1 u(-v_{xx} z_x + v_x z_{xx}) + k_1 v(u_{xx} z_x - u_x z_{xx}) \\
&\qquad + z(k_2 u_{xx} v_x - k_2 u_x v_{xx} + k_1 z_{xx} + k_2 z_{xx}) + k_3(-3u_x v_x z_x + v_x^3 + 3z_x^2)\big]^{1/3}, \qquad \ldots,
\end{aligned}
$$

and so on, where $k_j \in \mathbb{R}$, $j = 1, 2, 3$, are arbitrary real numbers. The problem which remains still open consists in proving, if any, that the generalized hydrodynamical system (4.3) is a Lax type integrable bi-Hamiltonian flow on the periodic functional manifold $\mathcal{M} := C^{(\infty)}(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^3)$, as it was proved above for the system (4.2) at $N = 2$. This problem will be analyzed in the Section below.

5 The Hamiltonian analysis

Consider the system (4.3) as a nonlinear dynamical system

$$\left.\begin{array}{l} u_t = v - uu_x \\ v_t = z - uv_x \\ z_t = -uz_x \end{array}\right\} := K[u,v,z], \tag{5.1}$$

on the 2π-periodic smooth functional manifold $\mathcal{M}$ and analyze it from the Hamiltonian point of view. To tackle with this problem, it is enough to construct [7, 11, 24] exact non-symmetric solutions to the Lie–Lax equation

$$d\varphi/dt + K'^{,*}\varphi = \operatorname{grad} L, \qquad \varphi' \neq \varphi'^{,*}, \tag{5.2}$$

for some functional $L \in D(\mathcal{M})$, where $\varphi \in T^*(\mathcal{M})$ is, in general, a quasi-local vector, such that the system (4.3) allows the following Hamiltonian representation:

$$K[u,v,z] = -\eta \operatorname{grad} H[u,v,z], \qquad H = (\varphi, K) - L, \qquad \eta^{-1} = \varphi' - \varphi'^{,*}.$$

As a test solution to (5.2) one can take the one

$$\varphi = \left(u_x/2, 0, -z_x^{-1}u_x^2/2 + z_x^{-1}v_x\right)^\intercal, \qquad L = \frac{1}{2}\int_0^{2\pi}(2z + vu_x)dx,$$

which gives rise to the following co-implectic operator:

$$\eta^{-1} := \varphi' - \varphi'^{,*} = \begin{pmatrix} \partial & 0 & -\partial u_x z_x^{-1} \\ 0 & 0 & \partial z_x \\ -u_x z_x^{-1}\partial & z_x\partial & \begin{array}{l}\frac{1}{2}(u_x^2 z_x^{-2}\partial + \partial u_x^2 z_x^{-2}) \\ -\,(v_x z_x^{-2}\partial + \partial v_x z_x^{-2})\end{array} \end{pmatrix}. \tag{5.3}$$

This expression is not strictly invertible, as its kernel possesses the translation vector field $d/dx : \mathcal{M} \to T(\mathcal{M})$ with components $(u_x, v_x, z_x)^\intercal \in T(\mathcal{M})$, that is $\eta^{-1}(u_x, v_x, z_x)^\intercal = 0$.

Nonetheless, upon formal inverting the operator expression (5.3), we obtain by means of simple enough, but slightly cumbersome, direct calculations, that the Hamiltonian function equals

$$H := \int_0^{2\pi} dx(u_x v - z). \tag{5.4}$$

and the implectic η-operator looks as

$$\eta := \begin{pmatrix} \partial^{-1} & u_x\partial^{-1} & 0 \\ \partial^{-1}u_x & v_x\partial^{-1} + \partial^{-1}v_x & \partial^{-1}z_x \\ 0 & z_x\partial^{-1} & 0 \end{pmatrix}. \tag{5.5}$$

The same way, representing the Hamiltonian function (5.4) in the scalar form

$$H = (\psi, (u_x, v_x, z_x)^\intercal), \qquad \psi = \frac{1}{2}\left(-v, u + \cdots, -\frac{1}{\sqrt{z}}\partial^{-1}\sqrt{z}\right)^\intercal, \tag{5.6}$$

one can construct a second implectic (co-symplectic) operator $\vartheta : T^*(\mathcal{M}) \to T(\mathcal{M})$, looking up to $O(\mu^2)$ terms, as follows:

$$\vartheta = \begin{pmatrix} \mu\left(\frac{(u^{(1)})^2}{z^{(1)}}\partial + \partial\frac{(u^{(1)})^2}{z^{(1)}}\right) & \begin{matrix} 1 + \frac{2\mu}{3}\left(\frac{u^{(1)}v^{(1)}}{z^{(1)}}\partial\right. \\ \left. + 2\partial\frac{u^{(1)}v^{(1)}}{z^{(1)}}\right) \end{matrix} & \frac{2\mu}{3}\left(\partial\frac{(v^{(1)})^2}{z^{(1)}} + \partial u^{(1)}\right) \\ \begin{matrix} -1 + \frac{2\mu}{3}\left(\partial\frac{u^{(1)}v^{(1)}}{z^{(1)}}\right. \\ \left. + 2\frac{u^{(1)}v^{(1)}}{z^{(1)}}\partial\right) \end{matrix} & \begin{matrix} \frac{2\mu}{3}\left(\frac{(v^{(1)})^2}{z^{(1)}}\partial + \partial\frac{(v^{(1)})^2}{z^{(1)}}\right) \\ + \frac{2\mu}{3}\left(u^{(1)}\partial + \partial u^{(1)}\right) \end{matrix} & 2\mu\partial v^{(1)} \\ \frac{2\mu}{3}\left(\frac{(v^{(1)})^2}{z^{(1)}}\partial + u^{(1)}\partial\right) & 2\mu v^{(1)}\partial & \mu\left(\partial z^{(1)} + z^{(1)}\partial\right) \end{pmatrix} + O(\mu^2), \quad (5.7)$$

where we put, by definition, $\vartheta^{-1} := (\psi' - \psi'^{,*})$, $u := \mu u^{(1)}$, $v := \mu v^{(1)}$, $z := \mu z^{(1)}$ as $\mu \to 0$, and whose exact form needs some additional simple enough but cumbersome calculations, which will be presented in a work under preparation.

The operator (5.7) satisfies the Hamiltonian vector field condition:

$$(u_x, v_x, z_x)^\intercal = -\vartheta \operatorname{grad} H,$$

following easily from (5.6).

Now having applied to the pair of implectic operators the gradient-holonomic scheme [11, 16, 24] of constructing a Lax type representation for the dynamical system (5.1) we obtain by means of slightly cumbersome and tedious calculations the following compatible Lax type representation:

$$\begin{gathered} f_x = \ell[u, v; \lambda]f, \qquad f_t = p(\ell)f, \qquad p(\ell) := -u\ell[u, v; \lambda] + q(\lambda), \\ \ell[u, v, z; \lambda = \begin{pmatrix} \lambda u_x & -v_x & z_x \\ 3\lambda^2 & -2\lambda u_x & \lambda v_x \\ \lambda^2 r[u, v, z] & -3\lambda & \lambda u_x \end{pmatrix}, \qquad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ p(\ell) = \begin{pmatrix} -\lambda u u_x & u v_x & -u z_x \\ -3u\lambda^2 + \lambda & 2\lambda u u_x & -\lambda u v_x \\ -\lambda^2 r[u, v, z]u & 1 + 3u\lambda & -\lambda u u_x \end{pmatrix}, \end{gathered} \quad (5.8)$$

where $f \in C^\infty(\mathbb{R}; \mathbb{C}^3)$, $\lambda \in \mathbb{C}\backslash\{0\}$ is a spectral parameter and $r : \mathcal{M} \to \mathbb{R}$ is a smooth mapping, satisfying the differential equation

$$D_t r + u_x r = 6.$$

The latter possesses a wide set $\mathcal{R}$ of different solutions, amongst which there are the following:

$$\begin{aligned} r \in \mathcal{R} := \Bigg\{ & \left[\left(6xv - 3u^2\right)/z\right]_x, \ 3\left(2v_x - u_x^2\right)z_x^{-1}, \ \frac{2u_x^3 - 6u_x v_x + 9z_x}{2u_x z_x - v_x^2}, \\ & \left(v_x v^3 - 3u_x v^2 z + u z_x\left(uz - v^2\right) + 6vz^2\right)z^{-3} \Bigg\}. \end{aligned} \quad (5.9)$$

Thereby, the following proposition holds.

Proposition 4. *The generalized Riemann type hydrodynamical equation* (4.2) *at* $N = 2$ *and* $N = 3$ *is equivalent to Lax type integrable bi-Hamiltonian dynamical systems* (2.1) *and* (5.1), *whose Hamiltonian structures and Lax type representations are given by expressions* (3.13), (3.15), (3.6), *and* (5.5), (5.7), (5.8), (5.9), *respectively.*

Note here that only the third element from the set (5.9) allows the reduction $z = 0$ to the case $N = 2$. Concerning the case $N = 4$ and the general case $N \in \mathbb{Z}_+$, applying successively the method devised above, one can obtain [3] for the Riemann type hydrodynamical system (5.1) both infinite hierarchies of dispersive and dispersionless conservation laws, their symplectic structures and the related Lax type representations, which is a topic of the next work under preparation.

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