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Multisymplectic Lagrangian and Hamiltonian Formalisms of Classical Field Theories

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Abstract. This review paper is devoted to presenting the standard *multisymplectic formulation* for describing geometrically classical field theories, both the regular and singular cases. First, the main features of the Lagrangian formalism are revisited and, second, the Hamiltonian formalism is constructed using *Hamiltonian sections*. In both cases, the variational principles leading to the Euler–Lagrange and the Hamilton–De Donder–Weyl equations, respectively, are stated, and these field equations are given in different but equivalent geometrical ways in each formalism. Finally, both are unified in a new formulation (which has been developed in the last years), following the original ideas of Rusk and Skinner for mechanical systems.

Key words: classical field theories; Lagrangian and Hamiltonian formalisms; fiber bundles; multisymplectic manifolds

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1 Introduction

In recent years much work has been done with the aim of establishing the suitable geometrical structures for describing classical field theories.

There are different kinds of geometrical models for making a covariant description of classical field theories described by first-order Lagrangians. For instance, we have the so-called *k-symplectic formalism* which uses the *k-symplectic forms* introduced by Awane [4, 5, 6], and which coincides with the *polysymplectic formalism* described by Günther [46] (see also [84]). A natural extension of this is the *k-cosymplectic formalism*, which is the generalization to field theories of the cosymplectic description of non-autonomous mechanical systems [75, 76]. Furthermore, there are the polysymplectic formalisms developed by Sardanashvily et al. [39, 91] and Kanatchikov [52], which are based on the use of vector-valued forms on fiber bundles, and which are different descriptions of classical field theories than the polysymplectic one proposed by Günther. In addition, soldering forms on linear frame bundles are also polysymplectic forms, and their study and applications to field theory constitute the k-symplectic geometry developed by Norris [85, 86, 87]. There also exists the formalism based on using *Lepagean forms*, used for describing certain kinds of equivalent Lagrangian models with non-equivalent Hamiltonian descriptions [64, 65, 66, 67]. Finally, a new geometrical framework for field theories based on the use of Lie algebroids has been developed in recent works [72, 82, 83].

In this work, we consider only the *multisymplectic* models [18, 41, 43, 68, 79], first introduced by Tulczyjew and other authors [37, 40, 60, 61]. They arise from the study of multisymplectic manifolds and their properties (see [14, 15] for general references, and Appendix A.1 for a brief review); in particular, those concerning the behavior of multisymplectic Lagrangian and Hamiltonian systems.

The usual way of working with field theories consists in stating their Lagrangian formalism [3, 11, 17, 26, 27, 37, 39, 40, 92], and jet bundles are the appropriate domain for doing so. The construction of this formalism for regular and singular theories is reviewed in Section 2.

The Hamiltonian description presents different kinds of problems. For instance, the choice of the multimomentum bundle for developing the theory is not unique [29, 30], and different kinds of Hamiltonian systems can be defined, depending on this choice and on the way of introducing the physical content (the "Hamiltonian") [23, 25, 47, 48, 78, 88]. Here we present one of the most standard ways of defining Hamiltonian systems, which is based on using *Hamiltonian sections* [16]; although this construction can also be done taking *Hamiltonian densities* [16, 39, 79, 91]. In particular, the construction of Hamiltonian systems which are the Hamiltonian counterpart of Lagrangian systems is carried out by using the *Legendre map* associated with the Lagrangian system, and this problem has been studied by different authors in the (*hyper*) *regular* case [16, 92], and in the *singular* (*almost-regular*) case [39, 69, 91]. In Section 3 we review some of these constructions.

Another subject of interest in the geometrical description of classical field theories concerns the field equations. In the multisymplectic models, both in the Lagrangian and Hamiltonian formalisms, these equations can be derived from a suitable variational principle: the so-called *Hamilton principle* in the Lagrangian formalism and *Hamilton–Jacobi principle* in the Hamiltonian formulation [3, 23, 26, 30, 37, 40], and the field equations are usually written by using the multisymplectic form in order to characterize the critical sections which are solutions of the problem. In addition, these critical sections can be thought of as being the integral manifolds of certain kinds of integrable multivector fields or Ehresmann connections, defined in the bundles where the formalism is developed, and satisfying a suitable geometric equation which is the intrinsic formulation of the systems of partial differential equations locally describing the field [26, 27, 28, 69, 92]. All these aspects are discussed in Sections 2 and 3 (furthermore, a quick review on multivector fields and connections is given in Appendix A.2). Moreover, multivector fields are also used in order to state generalized Poisson brackets in the Hamiltonian formalism of field theories [34, 50, 51, 52, 88].

In ordinary mechanics there is also a unified formulation of Lagrangian and Hamiltonian formalisms [94], which is based on the use of the *Whitney sum* of the tangent and cotangent bundles (the *velocity* and *momentum phase spaces* of the system). This formalism has been generalized for non-autonomous mechanics [7, 20, 45] and recently for classical field theories [24, 71]. The main features of this formulation are explained in Section 4.

Finally, an example showing the application of these formalisms is analyzed in Section 5. A last section is devoted to make a discussion about the current status on the research on different topics concerning the multisymplectic approach to classical field theories.

We ought to point out that there are also geometric frameworks for describing the non-covariant or space-time formalism of field theories, where the use of *Cauchy surfaces* is the fundamental tool [42, 44, 74]. Nevertheless we do not consider these topics in this survey.

As a review paper, this work recovers results and contributions from several previous papers, such as [16, 24, 26, 27, 28, 30, 47, 69, 71, 88], among others.

In this paper, manifolds are real, paracompact, connected and C^∞, maps are C^∞, and sum over crossed repeated indices is understood.

2 Lagrangian formalism

2.1 Lagrangian systems

A *classical field theory* is described by the following elements: First, we have the *configuration fibre bundle* $\pi\colon E \to M$, with $\dim M = m$ and $\dim E = n + m$, where M is an oriented manifold with volume form $\omega \in \Omega^m(M)$. $\pi^1\colon J^1\pi \to E$ is the first-order jet bundle of local sections of π, which is also a bundle over M with projection $\bar{\pi}^1 = \pi \circ \pi^1\colon J^1\pi \longrightarrow M$, and $\dim J^1\pi = nm + n + m$. We denote by (x^ν, y^A, v^A_ν) ($\nu = 1,\dots,m$; $A = 1,\dots,n$) natural coordinates in $J^1\pi$ adapted to the bundle structure and such that $\omega = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m \equiv \mathrm{d}^m x$. Second, we give the *Lagrangian density*, which is a $\bar{\pi}^1$-semibasic m-form on $J^1\pi$ and hence it can be expressed as $\mathcal{L} = \pounds(\bar{\pi}^{1*}\omega)$, where $\pounds \in \mathrm{C}^\infty(J^1\pi)$ is the *Lagrangian function* associated with $\mathcal{L}$ and ω.

The bundle $J^1\pi$ is endowed with a canonical structure, $\mathcal{V} \in \Omega^1(J^1\pi) \otimes \Gamma(J^1\pi, \mathrm{V}(\pi^1)) \otimes \Gamma(J^1\pi, \bar{\pi}^{1^*}\mathrm{T}M)$, which is called the *vertical endomorphism* [26, 37, 40, 92] (here $\mathrm{V}(\pi^1)$ denotes the vertical subbundle with respect to the projection π^1, and $\Gamma(J^1\pi, \mathrm{V}(\pi^1))$ the set of sections in the corresponding bundle). Then the *Poincaré–Cartan m and $(m+1)$-forms* associated with $\mathcal{L}$ are defined as

$$\Theta_{\mathcal{L}} := i(\mathcal{V})\mathcal{L} + \mathcal{L} \in \Omega^m(J^1\pi), \qquad \Omega_{\mathcal{L}} := -\mathrm{d}\Theta_{\mathcal{L}} \in \Omega^{m+1}(J^1\pi).$$

We have the following local expressions (where $\mathrm{d}^{m-1}x_\alpha \equiv i\left(\frac{\partial}{\partial x^\alpha}\right)\mathrm{d}^m x$):

$$\begin{aligned}
\Theta_{\mathcal{L}} &= \frac{\partial \pounds}{\partial v^A_\nu}\mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu - \left(\frac{\partial \pounds}{\partial v^A_\nu} v^A_\nu - \pounds\right)\mathrm{d}^m x, \\
\Omega_{\mathcal{L}} &= -\frac{\partial^2 \pounds}{\partial v^B_\nu \partial v^A_\alpha}\mathrm{d}v^B_\nu \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\alpha - \frac{\partial^2 \pounds}{\partial y^B \partial v^A_\alpha}\mathrm{d}y^B \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\alpha \\
&\quad + \frac{\partial^2 \pounds}{\partial v^B_\nu \partial v^A_\alpha} v^A_\alpha \mathrm{d}v^B_\nu \wedge \mathrm{d}^m x + \left(\frac{\partial^2 \pounds}{\partial y^B \partial v^A_\alpha} v^A_\alpha - \frac{\partial \pounds}{\partial y^B} + \frac{\partial^2 \pounds}{\partial x^\alpha \partial v^B_\alpha}\right)\mathrm{d}y^B \wedge \mathrm{d}^m x. \qquad (1)
\end{aligned}$$

Definition 1. $(J^1\pi, \Omega_{\mathcal{L}})$ is said to be a *Lagrangian system*. The Lagrangian system and the Lagrangian function are said to be *regular* if $\Omega_{\mathcal{L}}$ is a multisymplectic $(m+1)$-form (i.e., 1-non-degenerate) [16, 26]. Elsewhere they are *singular* (or *non-regular*).

The regularity condition is locally equivalent to $\det\left(\frac{\partial^2 \pounds}{\partial v^A_\alpha \partial v^B_\nu}(\bar{y})\right) \neq 0$, $\forall\, \bar{y} \in J^1\pi$. We must point out that, in field theories, the notion of regularity is not uniquely defined (for other approaches see, for instance, [9, 21, 22, 64, 66, 67]).

2.2 Lagrangian field equations

The Lagrangian field equations can be derived from a variational principle. In fact:

Definition 2. Let $(J^1\pi, \Omega_{\mathcal{L}})$ be a Lagrangian system. Let $\Gamma(M, E)$ be the set of sections of π. Consider the map

$$\begin{aligned}
\mathbf{L}\colon\ &\Gamma(M, E) \longrightarrow \mathbb{R}, \\
&\phi \mapsto \int_M (j^1\phi)^*\Theta_{\mathcal{L}},
\end{aligned}$$

where the convergence of the integral is assumed. The variational problem for this Lagrangian system is the search of the critical (or stationary) sections of the functional $\mathbf{L}$, with respect to the variations of ϕ given by $\phi_t = \sigma_t \circ \phi$, where $\{\sigma_t\}$ is a local one-parameter group of any compact-supported $Z \in \mathfrak{X}^{\mathrm{V}(\pi)}(E)$ (the module of π-vertical vector fields in E), that is:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_M \left(j^1\phi_t\right)^*\Theta_{\mathcal{L}} = 0.$$

This is the *Hamilton principle* of the Lagrangian formalism.

The Hamilton principle is equivalent to find a distribution $\mathcal{D}$ in $J^1\pi$ such that:

1. $\mathcal{D}$ is m-dimensional.
2. $\mathcal{D}$ is $\bar{\pi}^1$-transverse.
3. $\mathcal{D}$ is integrable (that is, *involutive*).
4. The integral manifolds of $\mathcal{D}$ are the canonical liftings to $J^1\pi$ of the critical sections of the Hamilton principle.

A distribution $\mathcal{D}$ satisfying 1 and 2 is associated with a connection in the bundle $\bar{\pi}^1\colon J^1\pi \to M$ (integrable if 3 holds), whose local expression is

$$\nabla = \mathrm{d}x^\mu \otimes \left(\frac{\partial}{\partial x^\nu} + F^A_\nu \frac{\partial}{\partial y^A} + G^A_{\nu\rho} \frac{\partial}{\partial v^A_\rho} \right). \tag{2}$$

Furthermore, these kinds of integrable distributions and the corresponding connections are associated with classes of integrable (i.e., non-vanishing, locally decomposable and involutive) $\bar{\pi}^1$-transverse m-multivector fields in $J^1\pi$ (see Appendix A.2). If 2 holds, the local expression in natural coordinates of an element of one of these classes is

$$\mathcal{X} = \bigwedge_{\nu=1}^{m} f \left(\frac{\partial}{\partial x^\nu} + F^A_\nu \frac{\partial}{\partial y^A} + G^A_{\nu\rho} \frac{\partial}{\partial v^A_\rho} \right), \qquad (f \in \mathrm{C}^\infty(J^1\pi) \text{ non-vanishing}). \tag{3}$$

If, in addition, the integral sections are holonomic (that is, they are canonical liftings of sections of $\pi\colon E \to M$), then the integrable connections and their associated classes of multivector fields are called *holonomic*. To be holonomic is equivalent to be integrable and *semi-holonomic*, that is, $F^A_\nu = v^A_\nu$ in the above local expressions. Then:

Theorem 1. *Let $(J^1\pi, \Omega_\mathcal{L})$ be a Lagrangian system. The following assertions on a section $\phi \in \Gamma(M, E)$ are equivalent:*

1. *ϕ is a critical section for the variational problem posed by the Hamilton principle.*
2. *$(j^1\phi)^*\, i(X)\Omega_\mathcal{L} = 0$, $\forall\, X \in \mathfrak{X}(J^1\pi)$ (see [40]).*
3. *If $(U; x^\nu, y^A, v^A_\nu)$ is a natural system of coordinates in $J^1\pi$, then $j^1\phi = \big(x^\nu, y^A(x^\eta), \frac{\partial y^A}{\partial x^\nu}(x^\eta)\big)$ in U satisfies the Euler–Lagrange equations (see [26, 40])*

$$\frac{\partial \pounds}{\partial y^A} \circ j^1\phi - \frac{\partial}{\partial x^\mu} \left(\frac{\partial \pounds}{\partial v^A_\mu} \circ j^1\phi \right) = 0. \tag{4}$$

4. *$j^1\phi$ is an integral section of a class of holonomic multivector fields $\{\mathcal{X}_\mathcal{L}\} \subset \mathfrak{X}^m(J^1\pi)$ satisfying (see [27]):*

$$i(\mathcal{X}_\mathcal{L})\Omega_\mathcal{L} = 0, \qquad \forall\, \mathcal{X}_\mathcal{L} \in \{\mathcal{X}_\mathcal{L}\}. \tag{5}$$

5. *$j^1\phi$ is an integral section of a holonomic connection $\nabla_\mathcal{L}$ in $J^1\pi$ satisfying (see [69]):*

$$i(\nabla_\mathcal{L})\Omega_\mathcal{L} = (m-1)\Omega_\mathcal{L}. \tag{6}$$

Proof. See [26, 27, 37, 40, 69, 92].

($1 \Longleftrightarrow 2$) Let $Z \in \mathfrak{X}^{\mathrm{V}(\pi)}(E)$ be a compact-supported vector field, and $V \subset M$ an open set such that ∂V is a $(m-1)$-dimensional manifold and that $\bar{\tau}(\mathrm{supp}\,(Z)) \subset V$. We denote by

$j^1Z \in \mathfrak{X}(J^1\pi)$ the canonical lifting of Z to $J^1\pi$; and, if $Z \in \mathfrak{X}^{\mathrm{V}(\pi)}(E)$, then $j^1Z \in \mathfrak{X}^{\mathrm{V}(\bar{\pi}^1)}(J^1\pi)$ (see [26] for the details). Therefore

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\int_M (j^1\phi_t)^*\Theta_{\mathcal{L}} &= \frac{d}{dt}\Big|_{t=0}\int_V (j^1\phi_t)^*\Theta_{\mathcal{L}} = \frac{d}{dt}\Big|_{t=0}\int_V [j^1(\sigma_t\circ\phi)]^*\Theta_{\mathcal{L}} \\
&= \frac{d}{dt}\Big|_{t=0}\int_V (j^1\phi)^*[(j^1\sigma_t)^*\Theta_{\mathcal{L}}] = \int_V (j^1\phi)^*\left(\lim_{t\to 0}\frac{(j^1\sigma_t)^*\Theta_{\mathcal{L}}-\Theta_{\mathcal{L}}}{t}\right) \\
&= \int_V (j^1\phi)^*\,\mathrm{L}(j^1Z)\Theta_{\mathcal{L}} = \int_V (j^1\phi)^*[i(j^1Z)\mathrm{d}\Theta_{\mathcal{L}} + \mathrm{d}\,i(j^1Z)\Theta_{\mathcal{L}}] \\
&= -\int_V (j^1\phi)^*[i(j^1Z)\Omega_{\mathcal{L}} - \mathrm{d}\,i(j^1Z)\Theta_{\mathcal{L}}] \\
&= -\int_V (j^1\phi)^*\,i(j^1Z)\Omega_{\mathcal{L}} + \int_V \mathrm{d}[(j^1\phi)^*\,i(Z)\Theta_{\mathcal{L}}] \\
&= -\int_V (j^1\phi)^*\,i(j^1Z)\Omega_{\mathcal{L}} + \int_{\partial V} (j^1\phi)^*\,i(j^1Z)\Theta_{\mathcal{L}} \\
&= -\int_V (j^1\phi)^*\,i(j^1Z)\Omega_{\mathcal{L}},
\end{aligned}$$

as a consequence of Stoke's theorem and the hypothesis made on the supports of the vertical fields. Thus, by the fundamental theorem of the variational calculus we conclude that $\frac{d}{dt}\big|_{t=0}\int_V (j^1\phi_t)^*\Theta_{\mathcal{L}} = 0$ if, and only if, $(j^1\phi)^*\,i(j^1Z)\Omega_{\mathcal{L}} = 0$, for every compact-supported $Z \in \mathfrak{X}^{\mathrm{V}(\pi)}(E)$. However, as compact-supported vector fields generate locally the $\mathrm{C}^\infty(E)$-module of vector fields in E, it follows that the last equality holds for every $Z \in \mathfrak{X}^{\mathrm{V}(\pi)}(E)$.

Now, suppose $\phi \in \Gamma(M,E)$ is a critical section; that is, $(j^1\phi)^*\,i(j^1Z)\Omega_{\mathcal{L}} = 0$, for every $Z \in \mathfrak{X}^{\mathrm{V}(\pi)}(E)$, and consider $X \in \mathfrak{X}(J^1E)$, which can be written as $X = X_\phi + X_v$ where X_ϕ is tangent to the image of $j^1\phi$ and X_v is $\bar{\pi}^1$-vertical, both in the points of the image of $j^1\phi$. However, $X_v = (X_v - j^1(\pi^1_*X_v)) + j^1(\pi^1_*X_v)$, where $j^1(\pi^1_*X_v)$ is understood as the prolongation of a vector field which coincides with $\pi^1_*X_v$ on the image of ϕ. Observe that $\pi^1_*(X_v - j^1(\pi^1_*X_v)) = 0$ on the points of the image of $j^1\phi$. Therefore

$$(j^1\phi)^*\,i(X)\Omega_{\mathcal{L}} = (j^1\phi)^*\,i(X_\phi)\Omega_{\mathcal{L}} + (j^1\phi)^*\,i(X_v - j^1(\pi^1_*X_v))\Omega_{\mathcal{L}} + (j^1\phi)^*\,i(j^1(\pi^1_*X_v))\Omega_{\mathcal{L}}.$$

However, $(j^1\phi)^*\,i(X_\phi)\Omega_{\mathcal{L}} = 0$, because X_ϕ is tangent to the image of $j^1\phi$, hence $\Omega_{\mathcal{L}}$ acts on linearly dependent vector fields. Nevertheless, $(j^1\phi)^*\,i(X_v - j^1(\pi^1_*X_v))\Omega_{\mathcal{L}} = 0$, because $X_v - j^1(\pi^1_*X_v)$ is π^1-vertical and $\Omega_{\mathcal{L}}$ vanishes on these vector fields, when it is restricted to $j^1\phi$. Therefore, as ϕ is stationary and $\pi^1_*X_v \in \mathfrak{X}^{\mathrm{V}(\pi)}(E)$, we have

$$\int_M (j^1\phi)^*\,i(X)\Omega_{\mathcal{L}} = \int_M (j^1\phi)^*\,i(j^1(\pi^1_*X_v))\Omega_{\mathcal{L}} = 0.$$

The converse is a consequence of the first paragraph, since the condition $(j^1\phi)^*\,i(X)\Omega_{\mathcal{L}} = 0$, $\forall\, X \in \mathfrak{X}(J^1\pi)$, holds, in particular, for j^1Z, for every $Z \in \mathfrak{X}^{\mathrm{V}(\pi)}(E)$.

($2 \Leftrightarrow 3$) If $X = \alpha^\nu\frac{\partial}{\partial x^\nu} + \beta^A\frac{\partial}{\partial y^A} + \gamma^A_\nu\frac{\partial}{\partial v^A_\nu} \in \mathfrak{X}(J^1\pi)$, taking into account the local expression (1) of $\Omega_{\mathcal{L}}$, we have

$$\begin{aligned}
i(X)\Omega_{\mathcal{L}} = (-1)^\eta\alpha^\nu\Bigg[&\frac{\partial^2\pounds}{\partial v^B_\mu\partial v^A_\eta}\mathrm{d}v^B_\mu\wedge\mathrm{d}y^A\wedge\mathrm{d}^{m-2}x_{\eta\nu} + \frac{\partial^2\pounds}{\partial y^B\partial v^A_\eta}\mathrm{d}y^B\wedge\mathrm{d}y^A\wedge\mathrm{d}^{m-2}x_{\eta\nu} \\
&- \frac{\partial^2\pounds}{\partial v^B_\mu\partial v^A_\eta}v^A_\eta\mathrm{d}v^B_\mu\wedge\mathrm{d}^{m-1}x_\nu - \left(\frac{\partial^2\pounds}{\partial y^B\partial v^A_\eta}v^A_\eta - \frac{\partial\pounds}{\partial y^B} + \frac{\partial^2\pounds}{\partial x^\eta\partial v^B_\eta}\right)\mathrm{d}y^B\wedge\mathrm{d}^{m-1}x_\nu\Bigg] \\
&+ \beta^A\left[\frac{\partial^2\pounds}{\partial v^B_\mu\partial v^A_\eta}\mathrm{d}v^B_\mu\wedge\mathrm{d}^{m-1}x_\eta + \left(\frac{\partial^2\pounds}{\partial y^A\partial v^B_\eta} - \frac{\partial^2\pounds}{\partial y^B\partial v^A_\eta}\right)\mathrm{d}y^B\wedge\mathrm{d}^{m-1}x_\eta\right.
\end{aligned}$$

$$+\left(\frac{\partial^2 \pounds}{\partial y^A \partial v^B_\eta} v^B_\eta - \frac{\partial \pounds}{\partial y^A} + \frac{\partial^2 \pounds}{\partial x^\eta \partial v^A_\eta}\right) \mathrm{d}^m x\Bigg]$$
$$+ \gamma^A_\nu \left[-\frac{\partial^2 \pounds}{\partial v^A_\nu \partial v^B_\eta} \mathrm{d}y^B \wedge \mathrm{d}^{m-1}x_\eta + \frac{\partial^2 \pounds}{\partial v^A_\nu \partial v^B_\eta} v^B_\eta \mathrm{d}^m x\right]$$

but if $\phi = (x^\mu, y^A(x^\eta))$, then $j^1\phi = (x^\mu, y^A(x^\eta), v^A(x^\eta)) = \left(x^\mu, y^A(x^\eta), \frac{\partial y^A}{\partial x^\mu}(x^\eta)\right)$, and hence

$$(j^1\phi)^*\, i(X)\Omega_{\mathcal{L}} = (-1)^{\eta+\nu}\alpha^\eta \left[\frac{\partial}{\partial x^\mu}\left(\frac{\partial \pounds}{\partial v^A_\mu} \circ j^1\phi\right) - \frac{\partial \pounds}{\partial y^A} \circ j^1\phi\right] \frac{\partial (y^A \circ \phi)}{\partial x^\eta} \mathrm{d}^m x$$
$$+ \beta^A \left[\frac{\partial}{\partial x^\mu}\left(\frac{\partial \pounds}{\partial v^A_\mu} \circ j^1\phi\right) - \frac{\partial \pounds}{\partial y^A} \circ j^1\phi\right] \mathrm{d}^m x,$$

and, as this holds for every $X \in \mathfrak{X}(J^1\pi)$, we conclude that $(j^1\phi)^*\, i(X)\Omega_{\mathcal{L}} = 0$ if, and only if, the Euler–Lagrange equations (4) hold for ϕ.

($3 \Leftrightarrow 4$) Using the local expressions (1) of Ω_h and (3) for $\mathcal{X}_{\mathcal{L}}$, and taking $f = 1$ as a representative of the class $\{\mathcal{X}_{\mathcal{L}}\}$, from the equation (5), we obtain that

$$0 = \left(F^B_\mu - v^B_\mu\right) \frac{\partial^2 \pounds}{\partial v^A_\nu \partial v^B_\mu}, \tag{7}$$

$$0 = \frac{\partial \pounds}{\partial y^A} - \frac{\partial^2 \pounds}{\partial x^\mu \partial v^A_\mu} - \frac{\partial^2 \pounds}{\partial y^B \partial v^A_\mu} F^B_\mu - \frac{\partial^2 \pounds}{\partial v^B_\nu \partial v^A_\mu} G^B_{\nu\mu} + \frac{\partial^2 \pounds}{\partial y^A \partial v^B_\mu} \left(F^B_\mu - v^B_\mu\right), \tag{8}$$

but, if $X_{\mathcal{L}}$ is holonomic, it is semiholonomic and then $F^B_\mu = v^B_\mu$. Therefore the equations (7) are identities, and the equations (8) are

$$0 = \frac{\partial \pounds}{\partial y^A} - \frac{\partial^2 \pounds}{\partial x^\mu \partial v^A_\mu} - \frac{\partial^2 \pounds}{\partial y^B \partial v^A_\mu} v^B_\mu - \frac{\partial^2 \pounds}{\partial v^B_\nu \partial v^A_\mu} G^B_{\nu\mu}\ . \tag{9}$$

Now, for a section $\phi = (x^\mu, y^A(x^\eta))$, if $j^1\phi = \left(x^\mu, y^A(x^\eta), \frac{\partial y^A}{\partial x^\mu}(x^\eta)\right)$ is an integral section of $X_{\mathcal{L}}$, then $G^A_{\nu\mu} = \frac{\partial^2 y^A}{\partial x^\nu \partial x^\mu}$, and the equations (9) are equivalent to the Euler–Lagrange equations for ϕ.

($3 \Leftrightarrow 5$) The proof is like in the above item: using the local expressions (1) of $\Omega_{\mathcal{L}}$ and (2) for $\nabla_{\mathcal{L}}$, we prove that the equation (6) holds for an integrable connection if, and only if, the Euler–Lagrange equations (4) hold for its integral sections. ■

Semi-holonomic (but not necessarily integrable) locally decomposable multivector fields and connections which are solution to the Lagrangian equations (5) and (6) respectively are called *Euler–Lagrange multivector fields* and *connections* for $(J^1\pi, \Omega_{\mathcal{L}})$.

If $(J^1\pi, \Omega_{\mathcal{L}})$ is regular, Euler–Lagrange m-multivector fields and connections exist in $J^1\pi$, although they are not necessarily integrable. If $(J^1\pi, \Omega_{\mathcal{L}})$ is singular, in the most favourable cases, Euler–Lagrange multivector fields and connections only exist in some submanifold $S \hookrightarrow J^1\pi$, which can be obtained after applying a suitable constraint algorithm (see [70]).

3 Hamiltonian formalism

3.1 Multimomentum bundles. Legendre maps

As we have pointed out in the introduction, the construction of the Hamiltonian formalism of field theories is more involved than the Lagrangian formulation. In fact, there are different bundles where the Hamiltonian formalism can be developed (see, for instance, [29], and references therein). Here we take one of the most standard choices.

First, $\mathcal{M}\pi \equiv \Lambda^m_2 \mathrm{T}^*E$, is the bundle of m-forms on E vanishing by the action of two π-vertical vector fields (so $\dim \mathcal{M}\pi = nm+n+m+1$), and is diffeomorphic to the set $\mathrm{Aff}(J^1\pi, \Lambda^m\mathrm{T}^*M)$, made of the affine maps from $J^1\pi$ to $\Lambda^m\mathrm{T}^*M$ (the multicotangent bundle of M of order m [15]) [16, 30]. It is called the *extended multimomentum bundle*, and its canonical submersions are denoted

$$\kappa\colon\ \mathcal{M}\pi \to E; \qquad \bar\kappa = \pi\circ\kappa\colon\ \mathcal{M}\pi \to M.$$

As $\mathcal{M}\pi$ is a subbundle of $\Lambda^m\mathrm{T}^*E$, then $\mathcal{M}\pi$ is endowed with a canonical form $\Theta \in \Omega^m(\mathcal{M}\pi)$ (the "tautological form"), which is defined as follows: let $(x,\alpha) \in \Lambda^m_2\mathrm{T}^*E$, with $x\in E$ and $\alpha \in \Lambda^m_2\mathrm{T}^*_xE$; then, for every $X_1,\dots,X_m \in \mathrm{T}_{(x,\alpha)}(\mathcal{M}\pi)$,

$$\Theta((x,\alpha);X_1,\dots,X_m) := \alpha(x;\mathrm{T}_{(x,\alpha)}\kappa(X_1),\dots,\mathrm{T}_{(q,\alpha)}\kappa(X_m)).$$

Then we define the multisymplectic form $\Omega := -\mathrm{d}\Theta \in \Omega^{m+1}(\mathcal{M}\pi)$. They are known as the *multimomentum Liouville m and $(m+1)$-forms*

If we introduce natural coordinates (x^ν, y^A, p^ν_A, p) in $\mathcal{M}\pi$ adapted to the bundle $\pi\colon E\to M$, and such that $\omega = \mathrm{d}^m x$, the local expressions of these forms are

$$\Theta = p^\nu_A \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu + p\mathrm{d}^m x, \qquad \Omega = -\mathrm{d}p^\nu_A \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu - \mathrm{d}p\wedge \mathrm{d}^m x.$$

Now we denote by $J^1\pi^*$ the quotient $\mathcal{M}\pi/\pi^*\Lambda^m\mathrm{T}^*M$, with $\dim J^1\pi^* = nm+n+m$. We have the natural submersions

$$\tau\colon\ J^1\pi^* \to E; \qquad \bar\tau = \pi\circ\tau\colon\ J^1\pi^* \to M.$$

Furthermore, the natural submersion $\mu\colon \mathcal{M}\pi \to J^1\pi^*$ endows $\mathcal{M}\pi$ with the structure of an affine bundle over $J^1\pi^*$, with $(\pi\circ\tau)^*\Lambda^m\mathrm{T}^*M$ as the associated vector bundle. $J^1\pi^*$ is usually called the *restricted multimomentum bundle* associated with the bundle $\pi\colon E\to M$.

Natural coordinates in $J^1\pi^*$ (adapted to the bundle $\pi\colon E\to M$) are denoted by (x^ν, y^A, p^ν_A).

Definition 3. Let $(J^1\pi, \Omega_{\mathcal{L}})$ be a Lagrangian system. The *extended Legendre map* associated with $\mathcal{L}$, $\widetilde{\mathcal{FL}}\colon J^1\pi \to \mathcal{M}\pi$, is defined by

$$(\widetilde{\mathcal{FL}}(\bar y))(Z_1,\dots,Z_m) := (\Theta_{\mathcal{L}})_{\bar y}(\bar Z_1,\dots,\bar Z_m),$$

where $Z_1,\dots,Z_m \in \mathrm{T}_{\pi^1(\bar y)}E$, and $\bar Z_1,\dots,\bar Z_m \in \mathrm{T}_{\bar y}J^1\pi$ are such that $\mathrm{T}_{\bar y}\pi^1\bar Z_\alpha = Z_\alpha$.

The *restricted Legendre map* associated with $\mathcal{L}$ is $\mathcal{FL} := \mu\circ\widetilde{\mathcal{FL}}\colon J^1\pi \to J^1\pi^*$.

In natural coordinates we have:

$$\begin{aligned}
&\widetilde{\mathcal{FL}}^* x^\alpha = x^\alpha, && \widetilde{\mathcal{FL}}^* y^A = y^A, && \widetilde{\mathcal{FL}}^* p^\alpha_A = \frac{\partial \pounds}{\partial v^A_\alpha}, && \widetilde{\mathcal{FL}}^* p = \pounds - v^A_\alpha \frac{\partial \pounds}{\partial v^A_\alpha},\\
&\mathcal{FL}^* x^\alpha = x^\alpha, && \mathcal{FL}^* y^A = y^A, && \mathcal{FL}^* p^\alpha_A = \frac{\partial \pounds}{\partial v^A_\alpha}.
\end{aligned}$$

Then, observe that $\widetilde{\mathcal{FL}}^*\Theta = \Theta_{\mathcal{L}}$, and $\widetilde{\mathcal{FL}}^*\Omega = \Omega_{\mathcal{L}}$.

Definition 4. $(J^1\pi, \Omega_{\mathcal{L}})$ is *regular* (*hyper-regular*) if $\mathcal{FL}$ is a local (global) diffeomorphism. Elsewhere it is *singular*. (This definition is equivalent to that given above.)

$(J^1\pi, \Omega_{\mathcal{L}})$ is *almost-regular* if

1. $\mathcal{P} := \mathcal{FL}(J^1\pi)$ is a closed submanifold of $J^1\pi^*$ (natural embedding $\jmath_0\colon \mathcal{P}\hookrightarrow J^1\pi^*$).
2. $\mathcal{FL}$ is a submersion onto its image.
3. The fibres $\mathcal{FL}^{-1}(\mathcal{FL}(\bar y))$, $\forall\, \bar y \in J^1\pi$, are connected submanifolds of $J^1\pi$.

3.2 The (hyper)regular case

In the Hamiltonian formalism of field theories, there are different ways of introducing the physical information (the "Hamiltonian"). For instance, we can use connections in the multimomentum bundles in order to obtain a covariant definition of the so-called *Hamiltonian densities* (see, for instance, [16, 39, 79, 91]).

Nevertheless, the simplest way of defining (regular) Hamiltonian systems in field theory consists in considering the bundle $\bar{\tau}\colon J^1\pi^* \to M$ and then giving sections $h\colon J^1\pi^* \to \mathcal{M}\pi$ of the projection μ, which are called *Hamiltonian sections* and carry the physical information of the system. Then we can define the differentiable forms

$$\Theta_h := h^*\Theta \in \mathit{\Omega}^m(J^1\pi^*), \qquad \Omega_h := -\mathrm{d}\Theta_h = h^*\Omega \in \mathit{\Omega}^{m+1}(J^1\pi^*)$$

which are the *Hamilton–Cartan m and $(m+1)$ forms* of $J^1\pi^*$ associated with the Hamiltonian section h. The couple $(J^1\pi^*, \Omega_h)$ is said to be a *Hamiltonian system.*

In a local chart of natural coordinates, a Hamiltonian section is specified by a *local Hamiltonian function* $\mathrm{h} \in \mathrm{C}^\infty(U)$, $U \subset J^1\pi^*$, such that $h(x^\nu, y^A, p^\nu_A) \equiv (x^\nu, y^A, p^\nu_A, p = -\mathrm{h}(x^\gamma, y^B, p^\eta_B))$. Then, the local expressions of the Hamilton–Cartan forms associated with h are

$$\Theta_h = p^\nu_A \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu - \mathrm{h}\mathrm{d}^m x, \qquad \Omega_h = -\mathrm{d}p^\nu_A \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu + \mathrm{d}\mathrm{h} \wedge \mathrm{d}^m x. \tag{10}$$

Notice that Ω_h is 1-nondegenerate; that is, a multisymplectic form (as a simple calculation in coordinates shows).

Now we want to associate Hamiltonian systems to the Lagrangian ones. First we consider the hyper-regular case (the regular case is analogous, but working locally).

If $(J^1\pi, \Omega_\mathcal{L})$ is a hyper-regular Lagrangian system, then we have the diagram

$$\begin{tikzcd} & \mathcal{M}\pi \arrow[d, shift right, "\mu"'] \\ J^1\pi \arrow[ur, "\widetilde{\mathcal{FL}}"] \arrow[r, "\mathcal{FL}"] & J^1\pi^* \arrow[u, shift right, "h"'] \end{tikzcd}$$

It is proved [16] that $\tilde{\mathcal{P}} := \widetilde{\mathcal{FL}}(J^1\pi)$ is a 1-codimensional imbedded submanifold of $\mathcal{M}\pi$ ($\tilde{\jmath}_0\colon \tilde{\mathcal{P}} \hookrightarrow \mathcal{M}\pi$ denotes is the natural embedding), which is transverse to μ, and is diffeomorphic to $J^1\pi^*$. This diffeomorphism is μ^{-1}, when μ is restricted to $\tilde{\mathcal{P}}$, and also coincides with the map $h := \widetilde{\mathcal{FL}} \circ \mathcal{FL}^{-1}$, when it is restricted onto its image (which is just $\tilde{\mathcal{P}}$). Thus h and $(J^1\pi^*, \Omega_h)$ are the *Hamiltonian section* and the *Hamiltonian system associated with the hyper-regular Lagrangian system* $(J^1\pi, \Omega_\mathcal{L})$, respectively.

Locally, the Hamiltonian section $h(x^\nu, y^A, p^\nu_A) = (x^\nu, y^A, p^\nu_A, p = -\mathrm{h}(x^\gamma, y^B, p^\gamma_B))$ is specified by the *local Hamiltonian function*

$$\mathrm{h} = p^\nu_A (\mathcal{FL}^{-1})^* v^A_\nu - (\mathcal{FL}^{-1})^* \pounds .$$

Then we have the local expressions (10) for the corresponding Hamilton–Cartan forms and, of course, $\mathcal{FL}^*\Theta_h = \Theta_\mathcal{L}$, and $\mathcal{FL}^*\Omega_h = \Omega_\mathcal{L}$.

The Hamiltonian field equations can also be derived from a variational principle. In fact:

Definition 5. Let $(J^1\pi^*, \Omega_h)$ be a Hamiltonian system. Let $\Gamma(M, J^1\pi^*)$ be the set of sections of $\bar{\tau}$. Consider the map

$$\begin{aligned} \mathbf{H}\colon\ & \Gamma(M, J^1\pi^*) \longrightarrow \mathbb{R}, \\ & \psi \mapsto \int_M \psi^*\Theta_h, \end{aligned}$$

where the convergence of the integral is assumed. The variational problem for this Hamiltonian system is the search for the critical (or stationary) sections of the functional $\mathbf{H}$, with respect to the variations of ψ given by $\psi_t = \sigma_t \circ \psi$, where $\{\sigma_t\}$ is the local one-parameter group of any compact-supported $Z \in \mathfrak{X}^{\mathrm{V}(\bar{\tau})}(J^1\pi^*)$ (the module of $\bar{\tau}$-vertical vector fields in $J^1\pi^*$), that is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int_M \psi_t^* \Theta_h = 0.$$

This is the so-called *Hamilton–Jacobi principle* of the Hamiltonian formalism.

The Hamilton–Jacobi principle is equivalent to find distributions $\mathcal{D}$ of $J^1\pi^*$ such that:

1. $\mathcal{D}$ is m-dimensional.
2. $\mathcal{D}$ is $\bar{\tau}$-transverse.
3. $\mathcal{D}$ is integrable (that is, *involutive*).
4. The integral manifolds of $\mathcal{D}$ are the critical sections of the Hamilton–Jacobi principle.

As in the Lagrangian formalism, $\mathcal{D}$ are associated with classes of integrable and $\bar{\tau}$-transverse m-multivector fields $\{\mathcal{X}\} \subset \mathfrak{X}^m(J^1\pi^*)$ or, what is equivalent, with connections in the bundle $\bar{\pi}\colon J^1\pi \to M$, whose expressions are

$$\mathcal{X} = \bigwedge_{\nu=1}^{m} f \left(\frac{\partial}{\partial x^\nu} + F_\nu^A \frac{\partial}{\partial y^A} + G_{A\nu}^\rho \frac{\partial}{\partial p_A^\rho} \right), \qquad (f \in \mathrm{C}^\infty(J^1\pi^*) \text{ non-vanishing}), \tag{11}$$

$$\nabla = \mathrm{d}x^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + F_\mu^A \frac{\partial}{\partial y^A} + G_{A\mu}^\rho \frac{\partial}{\partial p_A^\rho} \right). \tag{12}$$

Then we have:

Theorem 2. *The following assertions on a section $\psi \in \Gamma(M, J^1\pi^*)$ are equivalent:*

1. *ψ is a critical section for the variational problem posed by the Hamilton–Jacobi principle.*
2. $\psi^*\, i(X)\Omega_h = 0$, $\forall\, X \in \mathfrak{X}(J^1\pi^*)$.
3. *If $(U; x^\nu, y^A, p_A^\nu)$ is a natural system of coordinates in $J^1\pi^*$, then ψ satisfies the Hamilton–De Donder–Weyl equations in U*

$$\frac{\partial(y^A \circ \psi)}{\partial x^\nu} = \frac{\partial \mathrm{h}}{\partial p_A^\nu} \circ \psi, \qquad \frac{\partial(p_A^\nu \circ \psi)}{\partial x^\nu} = -\frac{\partial \mathrm{h}}{\partial y^A} \circ \psi. \tag{13}$$

4. *ψ is an integral section of a class of integrable and $\bar{\tau}$-transverse multivector fields $\{\mathcal{X}_h\} \subset \mathfrak{X}^m(J^1\pi^*)$ satisfying that*

$$i(\mathcal{X}_h)\Omega_h = 0, \qquad \forall\, \mathcal{X}_h \in \{\mathcal{X}_h\}. \tag{14}$$

5. *ψ is an integral section of an integrable connection ∇_h in $J^1\pi^*$ satisfying the equation*

$$i(\nabla_h)\Omega_h = (m-1)\Omega_h. \tag{15}$$

Proof. This proof is taken from [23, 28], and [30].

$(1 \Leftrightarrow 2)$ Let $Z \in \mathfrak{X}^{\mathrm{V}(\bar{\tau})}(J^1\pi^*)$ be a compact-supported vector field, and $V \subset M$ an open set such that ∂V is a $(m-1)$-dimensional manifold and that $\bar{\tau}(\mathrm{supp}\,(Z)) \subset V$. Then

$$\frac{d}{dt}\Big|_{t=0} \int_M \psi_t^* \Theta_h = \frac{d}{dt}\Big|_{t=0} \int_V \psi_t^* \Theta_h = \frac{d}{dt}\Big|_{t=0} \int_V \psi^*(\sigma_t^* \Theta_h) = \int_V \psi^* \left(\lim_{t \to 0} \frac{\sigma_t^* \Theta_h - \Theta_h}{t} \right)$$

$$
\begin{aligned}
&= \int_V \psi^* \,\mathrm{L}(Z)\Theta_h = \int_V \psi^*(i(Z)\mathrm{d}\Theta_h + \mathrm{d}\, i(Z)\Theta_h) \\
&= -\int_V \psi^*(i(Z)\Omega_h - \mathrm{d}\, i(Z)\Theta_h) = -\int_V \psi^*\, i(Z)\Omega_h + \int_V \mathrm{d}[\psi^*\, i(Z)\Theta_h] \\
&= -\int_V \psi^*\, i(Z)\Omega_h + \int_{\partial V} \psi^*\, i(Z)\Theta_h = -\int_V \psi^*\, i(Z)\Omega_h,
\end{aligned}
$$

as a consequence of Stoke's theorem and the hypothesis made on the supports of the vertical fields. Thus, by the fundamental theorem of the variational calculus we conclude that $\frac{d}{dt}\big|_{t=0} \int_V \psi_t^*\Theta_h = 0$ if, and only if, $\psi^*\, i(Z)\Omega_h = 0$, for every compact-supported $Z \in \mathfrak{X}^{\mathrm{V}(\bar{\tau})}(J^1\pi^*)$. However, as compact-supported vector fields generate locally the $\mathrm{C}^\infty(J^1\pi^*)$-module of vector fields in $J^1\pi^*$, it follows that the last equality holds for every $Z \in \mathfrak{X}^{\mathrm{V}(\bar{\tau})}(J^1\pi^*)$.

Now, if $\mathrm{p} \in \mathrm{Im}\,\psi$, then $\mathrm{T}_\mathrm{p} J^1\pi^* = \mathrm{V}_\mathrm{p}(\bar{\tau}) \oplus \mathrm{T}_\mathrm{p}(\mathrm{Im}\,\psi)$. So if $X \in \mathfrak{X}(J^1\pi^*)$, then

$$
X_\mathrm{p} = (X_\mathrm{p} - \mathrm{T}_\mathrm{p}(\psi \circ \bar{\tau})(X_\mathrm{p})) + \mathrm{T}_\mathrm{p}(\psi \circ \bar{\tau})(X_\mathrm{p}) \equiv X_\mathrm{p}^V + X_\mathrm{p}^\psi ,
$$

and therefore

$$
\psi^*\, i(X)\Omega_h = \psi^*\, i(X^V)\Omega_h + \psi^*\, i(X^\psi)\Omega_h = \psi^*\, i(X^\psi)\Omega_h = 0,
$$

since $\psi^*\, i(X^V)\Omega_h = 0$, by the conclusion in the above paragraph. Furthermore, $X_\mathrm{p}^\psi \in \mathrm{T}_\mathrm{p}(\mathrm{Im}\,\psi)$, and $\dim(\mathrm{Im}\,\psi) = m$, being $\Omega_h \in \Omega^{m+1}(J^1\pi^*)$. Hence we conclude that $\psi^*\, i(X)\Omega_h = 0$, for every $X \in \mathfrak{X}(J^1\pi^*)$.

The converse is obvious taking into account the reasoning of the first paragraph, since the condition $\psi^*\, i(X)\Omega_h = 0$, $\forall\, X \in \mathfrak{X}(J^1\pi^*)$, holds, in particular, for every $Z \in \mathfrak{X}^{\mathrm{V}(\bar{\tau})}(J^1\pi^*)$.

($2 \Leftrightarrow 3$) If $X = \alpha^\nu \frac{\partial}{\partial x^\nu} + \beta^A \frac{\partial}{\partial y^A} + \gamma_A^\nu \frac{\partial}{\partial p_A^\nu} \in \mathfrak{X}(J^1\pi^*)$, taking into account the local expression (10) of Ω_h, we have

$$
\begin{aligned}
i(X)\Omega_h = (-1)^\eta \alpha^\eta &\left(\mathrm{d}p_A^\nu \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-2}x_{\eta\nu} - \frac{\partial \mathrm{h}}{\partial p_A^\nu} \mathrm{d}p_A^\nu \wedge \mathrm{d}^{m-1}x_\eta \right) \\
&+ \beta^A \left(\mathrm{d}p_A^\nu \wedge \mathrm{d}^{m-1}x_\nu + \frac{\partial \mathrm{h}}{\partial y^A}\mathrm{d}^m x \right) + \gamma_A^\nu \left(-\mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu + \frac{\partial \mathrm{h}}{\partial p_A^\nu} \wedge \mathrm{d}^m x \right)
\end{aligned}
$$

but if $\psi = (x^\nu, y^A(x^\eta), p_A^\nu(x^\eta))$, then

$$
\begin{aligned}
\psi^*\, i(X)\Omega_h = (-1)^{\eta+\nu}\alpha^\eta &\left(\frac{\partial(y^A \circ \psi)}{\partial x^\nu} - \frac{\partial \mathrm{h}}{\partial p_A^\nu}\Big|_\psi \right) \frac{\partial(p_A^\nu \circ \psi)}{\partial x^\eta} \mathrm{d}^m x \\
&+ \beta^A \left(\frac{\partial(p_A^\nu \circ \psi)}{\partial x^\nu} + \frac{\partial \mathrm{h}}{\partial y^A}\Big|_\psi \right) \mathrm{d}^m x + \gamma_A^\nu \left(-\frac{\partial(y^A \circ \psi)}{\partial x^\nu} + \frac{\partial \mathrm{h}}{\partial p_A^\nu}\Big|_\psi \right) \mathrm{d}^m x,
\end{aligned}
$$

and, as this holds for every $X \in \mathfrak{X}(J^1\pi^*)$, we conclude that $\psi^*\, i(X)\Omega_h = 0$ if, and only if, the Hamilton–De Donder–Weyl equations (13) hold for ψ.

($3 \Leftrightarrow 4$) Using the local expressions (10) of Ω_h and (11) for $\mathcal{X}_h$, and taking $f = 1$ as a representative of the class $\{\mathcal{X}_h\}$, the equation (14), in coordinates, is

$$
F_\nu^A = \frac{\partial \mathrm{h}}{\partial p_A^\nu}, \qquad G_{A\nu}^\nu = -\frac{\partial \mathrm{h}}{\partial y^A}.
$$

This result allows us to assure the local existence of (classes of) multivector fields satisfying the desired conditions. The corresponding global solutions are then obtained using a partition of unity subordinated to a covering of $J^1\pi^*$ made of local natural charts. Now, if $\psi(x) = (x^\nu, y^A(x^\gamma), p_A^\nu(x^\gamma))$ is an integral section of $\mathcal{X}_h$, then

$$
\frac{\partial(y^A \circ \psi)}{\partial x^\nu} = F_\nu^A \circ \psi, \qquad \frac{\partial(p_A^\rho \circ \psi)}{\partial x^\nu} = G_{A\nu}^\rho \circ \psi.
$$

Thus, combining both expressions we obtain the Hamilton–De Donder–Weyl equations (13) for ψ.

$(3 \Leftrightarrow 5)$ The proof is like in the above item: using the local expressions (10) of Ω_h and (12) for ∇_h, we prove that the equation (15) holds for an integrable connection if, and only if, the Hamilton–De Donder–Weyl equations (13) hold for its integral sections. ∎

The $\bar{\tau}$-transverse locally decomposable multivector fields and connections which are solution to the Hamiltonian equations (14) and (15) respectively (but not necessarily integrable) are called *Hamilton–De Donder–Weyl multivector fields* and *connections* for $(J^1\pi^*, \Omega_h)$.

Hence, the existence of Hamilton–De Donder–Weyl multivector fields and connections for $(J^1\pi^*, \Omega_h)$ is assured, although they are not necessarily integrable.

Finally, we can establish the equivalence between the Lagrangian and Hamiltonian formalisms in the hyper-regular case:

Theorem 3. (equivalence theorem for sections) *Let $(J^1\pi, \Omega_{\mathcal{L}})$ be a hyper-regular Lagrangian system, and $(J^1\pi^*, \Omega_h)$ the associated Hamiltonian system.*

If a section $\phi \in \Gamma(M, E)$ is a solution to the Lagrangian variational problem (Hamilton principle), then the section $\psi = \mathcal{FL} \circ j^1\phi \in \Gamma(M, J^1\pi^*)$ is a solution to the Hamiltonian variational problem (Hamilton–Jacobi principle).

Conversely, if $\psi \in \Gamma(M, J^1\pi^*)$ is a solution to the Hamiltonian variational problem, then the section $\phi = \tau \circ \psi \in \Gamma(M, E)$ is a solution to the Lagrangian variational problem.

Proof. This proof is taken from [28] and [30].

Bearing in mind the diagram

$$\begin{tikzcd}
J^1\pi \arrow[rr, "\mathcal{FL}"] \arrow[dr, "\pi^1"] & & J^1\pi^* \arrow[dl, "\tau"'] \\
& E \arrow[d, shift left, "\pi"] & \\
& M \arrow[u, shift left, "\phi"] \arrow[uul, "j^1\phi"] \arrow[uur, "\psi"'] &
\end{tikzcd}$$

if ϕ is a solution to the Lagrangian variational problem then $(j^1\phi)^*\, i(X)\Omega_{\mathcal{L}} = 0$, for every $X \in \mathfrak{X}(J^1\pi)$ (Theorem 1, item 2); therefore, as $\mathcal{FL}$ is a local diffeomorphism,

$$\begin{aligned}
0 &= (j^1\phi)^*\, i(X)\Omega_{\mathcal{L}} = (j^1\phi)^*\, i(X)(\mathcal{FL}^*\Omega_h) \\
&= (j^1\phi)^*\mathcal{FL}^*(i(\mathcal{FL}_*^{-1}X)\Omega_h) = (\mathcal{FL} \circ j^1\phi)^*\, i(X')\Omega_h),
\end{aligned}$$

which holds for every $X' \in \mathfrak{X}(J^1\pi^*)$ and thus, by the item 2 of Theorem 2, $\psi \equiv \mathcal{FL} \circ j^1\phi$ is a solution to the Hamiltonian variational problem.

Conversely, let $\psi \in \Gamma(M, J^1\pi^*)$ be a solution to the Hamiltonian variational problem. Reversing the above reasoning we obtain that $(\mathcal{FL}^{-1} \circ \psi)^*\, i(X)\Omega_{\mathcal{L}} = 0$, for every $X \in \mathfrak{X}(J^1\pi)$, and hence $\sigma \equiv \mathcal{FL}^{-1} \circ \psi \in \Gamma(M, J^1E)$ is a critical section for the Lagrangian variational problem. Then, as we are in the hyper-regular case, σ must be an holonomic section, $\sigma = j^1\phi$ [27, 69, 92], and since the above diagram is commutative, $\phi = \tau^1 \circ \psi \in \Gamma(M, E)$. ∎

The equivalence between the Lagrangian and the Hamiltonian formalisms can be stated also in terms of multivector fields and connections (see [28]).

3.3 The almost-regular case

Now, consider the almost-regular case. Let $\tilde{\mathcal{P}} := \widetilde{\mathcal{F}\mathcal{L}}(J^1\pi)$, $\mathcal{P} := \mathcal{F}\mathcal{L}(J^1\pi)$ (the natural projections are denoted by $\tau_0^1 \colon \mathcal{P} \to E$ and $\bar{\tau}_0^1 := \pi \circ \tau_0^1 \colon \mathcal{P} \to M$), and assume that $\mathcal{P}$ is a fibre bundle over E and M. Denote by $\tilde{\jmath}_0 \colon \tilde{\mathcal{P}} \hookrightarrow \mathcal{M}\pi$ the natural imbedding, and by $\widetilde{\mathcal{F}\mathcal{L}}_0$ and $\mathcal{F}\mathcal{L}_0$ the restrictions of $\widetilde{\mathcal{F}\mathcal{L}}$ and $\mathcal{F}\mathcal{L}$ to their images, respectively. So, we have the diagram

$$
\begin{tikzcd}
 & \tilde{\mathcal{P}} \arrow[rr, "\tilde{\jmath}_0"] \arrow[d, shift left, "\tilde{\mu}"] & & \mathcal{M}\pi \arrow[d, "\mu"] \\
J^1\pi \arrow[ur, "\widetilde{\mathcal{F}\mathcal{L}}_0"] \arrow[r, "\mathcal{F}\mathcal{L}_0"] & \mathcal{P} \arrow[u, shift left, "h_{\mathcal{P}}"] \arrow[rr, "\jmath_0"] \arrow[dr, "\bar{\tau}_0"'] & & J^1\pi^* \arrow[dl, "\bar{\tau}"] \\
 & & M &
\end{tikzcd}
$$

Now, it can be proved that the μ-transverse submanifold $\tilde{\mathcal{P}}$ is diffeomorphic to $\mathcal{P}$ [69]. This diffeomorphism is denoted $\tilde{\mu} \colon \tilde{\mathcal{P}} \to \mathcal{P}$, and it is just the restriction of the projection μ to $\tilde{\mathcal{P}}$. Then, taking $h_{\mathcal{P}} := \tilde{\mu}^{-1}$, we define the Hamilton–Cartan forms

$$\Theta_h^0 = (\tilde{\jmath}_0 \circ h_{\mathcal{P}})^*\Theta \in \mathit{\Omega}^m(\mathcal{P}), \qquad \Omega_h^0 = -\mathrm{d}\Theta_h^0(\tilde{\jmath}_0 \circ h_{\mathcal{P}})^*\Omega \in \mathit{\Omega}^{m+1}(\mathcal{P}),$$

which verify that $\mathcal{F}\mathcal{L}_0^*\Omega_h^0 = \Omega_{\mathcal{L}}$. Then $h_{\mathcal{P}}$ is also called a *Hamiltonian section*, and $(\mathcal{P}, \Omega_h^0)$ is the *Hamiltonian system* associated with the almost-regular Lagrangian system $(J^1\pi, \Omega_{\mathcal{L}})$. In general, Ω_h^0 is a pre-multisymplectic form and $(\mathcal{P}, \Omega_h^0)$ is the *Hamiltonian system associated with the almost-regular Lagrangian system* $(J^1\pi, \Omega_{\mathcal{L}})$.

In this framework, the *Hamilton–Jacobi principle* for $(\mathcal{P}, \Omega_h^0)$ is stated like above, and the critical sections $\psi_0 \in \Gamma(M, \mathcal{P})$ can be characterized in an analogous way than in Theorem 2.

If Ω_h^0 is a pre-multisymplectic form, Hamilton–De Donder–Weyl multivector vector fields and connections only exist, in the most favourable cases, in some submanifold $S \hookrightarrow J^1\pi$, and they are not necessarily integrable. As in the Lagrangian case, S can be obtained after applying the suitable constraint algorithm [70]. Then, the equivalence theorem follows in an analogous way than above.

It is important to point out that the analysis of the Hamiltonian description of non-regular field theories is far to be completed and, in fact, there is a lot of topics under discussion. For instance, there are some kinds of singular Lagrangian systems for which the construction of the associated Hamiltonian formalism (following the procedure that we have presented here) is ambiguous and, in order to overcome this trouble, a different notion of regularity must be done, which involve the use of *Lepagean forms* [64, 66, 67]. Neverthelees, the analysis of this and other problems exceeds the scope of this work.

4 Unified Lagrangian–Hamiltonian formalism

4.1 Geometric framework

The *extended* and the *restricted jet-multimomentum bundles* are

$$\mathcal{W} := J^1\pi \times_E \mathcal{M}\pi, \qquad \mathcal{W}_r := J^1\pi \times_E J^1\pi^*,$$

with natural coordinates $(x^\alpha, y^A, v^A_\alpha, p^\alpha_A, p)$ and $(x^\alpha, y^A, v^A_\alpha, p^\alpha_A)$. We have natural projections (submersions) $\mu_{\mathcal{W}} \colon \mathcal{W} \to \mathcal{W}_r$, and

$$
\begin{array}{llll}
\rho_1 \colon \mathcal{W} \to J^1\pi, & \rho_2 \colon \mathcal{W} \to \mathcal{M}\pi, & \rho_E \colon \mathcal{W} \to E, & \rho_M \colon \mathcal{W} \to M, \\
\rho_1^r \colon \mathcal{W}_r \to J^1\pi, & \rho_2^r \colon \mathcal{W}_r \to J^1\pi^*, & \rho_E^r \colon \mathcal{W}_r \to E, & \rho_M^r \colon \mathcal{W}_r \to M.
\end{array} \qquad (16)
$$

Definition 6. The *coupling m-form* in $\mathcal{W}$, denoted by $\mathcal{C}$, is an m-form along ρ_M which is defined as follows: for every $\bar{y} \in J^1_y E$, with $\bar{\pi}^1(\bar{y}) = \pi(y) = x \in E$, and $\mathbf{p} \in \mathcal{M}_y\pi$, let $w \equiv (\bar{y}, \mathbf{p}) \in \mathcal{W}_y$, then

$$\mathcal{C}(w) := (\mathrm{T}_x\phi)^*\mathbf{p},$$

where $\phi\colon M \to E$ satisfies that $j^1\phi(x) = \bar{y}$. Then, we denote by $\hat{\mathcal{C}} \in \mathit{\Omega}^m(\mathcal{W})$ the ρ_M-semibasic form associated with $\mathcal{C}$.

The *canonical m-form* $\Theta_{\mathcal{W}} \in \mathit{\Omega}^m(\mathcal{W})$ is defined as $\Theta_{\mathcal{W}} := \rho_2^*\Theta$, and is ρ_E-semibasic. The *canonical* $(m+1)$-*form* is the pre-multisymplectic form $\Omega_{\mathcal{W}} := -\mathrm{d}\Theta_{\mathcal{W}} = \rho_1^*\Omega \in \mathit{\Omega}^{m+1}(\mathcal{W})$.

There exists $\hat{C} \in \mathrm{C}^\infty(\mathcal{W})$ such that $\hat{\mathcal{C}} = \hat{C}(\rho_M^*\omega)$, and $\hat{\mathcal{C}}(w) = (p + p_A^\alpha v_\alpha^A)\mathrm{d}^m x$.

Local expressions of $\Theta_{\mathcal{W}}$ and $\Omega_{\mathcal{W}}$ are the same than for Θ and Ω.

Let $\hat{\mathcal{L}} := \rho_1^*\mathcal{L} \in \mathit{\Omega}^m(\mathcal{W})$, and $\hat{\mathcal{L}} = \hat{L}(\rho_M^*\omega)$, with $\hat{L} = \rho_1^*L \in \mathrm{C}^\infty(\mathcal{W})$. We define the *Hamiltonian submanifold* $\jmath_0\colon \mathcal{W}_0 \hookrightarrow \mathcal{W}$ by

$$\mathcal{W}_0 := \{w \in \mathcal{W} \mid \hat{\mathcal{L}}(w) = \hat{\mathcal{C}}(w)\}.$$

The constraint function defining $\mathcal{W}_0$ is

$$\hat{C} - \hat{L} = p + p_A^\alpha v_\alpha^A - \hat{L}\big(x^\nu, y^B, v_\nu^B\big) = 0.$$

There are projections which are the restrictions to $\mathcal{W}_0$ of the projections (16), as it is shown in the following diagram:

$$\begin{tikzcd}
 & J^1\pi & \\
\mathcal{W}_0 \arrow[ur, "\rho_1^0"] \arrow[r, "\jmath_0"] \arrow[dr, "\rho_2^0"] \arrow[ddr, "\hat{\rho}_2^0"'] & \mathcal{W} \arrow[u, "\rho_1"] \arrow[r, "\mu_{\mathcal{W}}"] \arrow[d, "\rho_2"] & \mathcal{W}_r \arrow[ul, "\rho_1^r"'] \arrow[dl, "\rho_2^r"'] \arrow[ddl, "\hat{\rho}_2^r"] \\
 & \mathcal{M}\pi \arrow[d, "\mu"] & \\
 & J^1\pi^* &
\end{tikzcd}$$

$(x^\alpha, y^A, v_\alpha^A, p_A^\alpha)$ are local coordinates in $\mathcal{W}_0$, and

$$\rho_1^0(x^\alpha, y^A, v_\alpha^A, p_A^\alpha) = (x^\alpha, y^A, v_\alpha^A), \qquad \jmath_0(x^\alpha, y^A, v_\alpha^A, p_A^\alpha) = (x^\alpha, y^A, v_\alpha^A, p_A^\alpha, L - v_\alpha^A p_A^\alpha),$$
$$\hat{\rho}_2^0(x^\alpha, y^A, v_\alpha^A, p_A^\alpha) = (x^\alpha, y^A, p_A^\alpha), \qquad \rho_2^0(x^\alpha, y^A, v_\alpha^A, p_A^\alpha) = (x^\alpha, y^A, p_A^\alpha, L - v_\alpha^A p_A^\alpha).$$

It is proved that $\mathcal{W}_0$ is a 1-codimensional $\mu_{\mathcal{W}}$-transversal submanifold of $\mathcal{W}$, diffeomorphic to $\mathcal{W}_r$. As a consequence, $\mathcal{W}_0$ induces a *Hamiltonian section* of $\mu_{\mathcal{W}}$, $\hat{h}\colon \mathcal{W}_r \to \mathcal{W}$, which is locally specified by giving the local *Hamiltonian function* $\hat{H} = -\hat{L} + p_A^\alpha v_\alpha^A$; that is, $\hat{h}(x^\alpha, y^A, v_\alpha^A, p_A^\alpha) = (x^\alpha, y^A, v_\alpha^A, p_A^\alpha, -\hat{H})$. From $\hat{h}$ we recover a Hamiltonian section $\tilde{h}\colon \mathcal{P} \to \mathcal{M}\pi$ defined by $\tilde{h}([\mathbf{p}]) = (\rho_2 \circ \hat{h})[(\rho_2^r)^{-1}(\jmath([\mathbf{p}]))]$, $\forall\, [\mathbf{p}] \in \mathcal{P}$. (See the diagram.)

$$\begin{tikzcd}
\tilde{\mathcal{P}} \arrow[r, "\tilde{\jmath}"] \arrow[d, shift left, "\tilde{\mu}"] & \mathcal{M}\pi \arrow[d, "\mu"] & \mathcal{W} \arrow[l, "\rho_2"'] \\
\mathcal{P} \arrow[u, shift left, "\tilde{\mu}^{-1}"] \arrow[ur, "\tilde{h}"'] \arrow[r, "\jmath"'] & J^1\pi^* & \mathcal{W}_r \arrow[l, "\rho_2^r"] \arrow[u, "\hat{h}"']
\end{tikzcd}$$

(For hyper-regular systems we have $\tilde{\mathcal{P}} = \mathcal{M}\pi$ and $\mathcal{P} = J^1\pi^*$.)

We define the forms $\Theta_0 := \jmath_0^*\Theta_{\mathcal{W}} = \rho_2^{0*}\Theta \in \Omega^m(\mathcal{W}_0)$, and $\Omega_0 := \jmath_0^*\Omega_{\mathcal{W}} = \rho_2^{0*}\Omega \in \Omega^{m+1}(\mathcal{W}_0)$, whose local expressions are

$$\Theta_0 = (L - p_A^\alpha v_\alpha^A)\mathrm{d}^m x + p_A^\alpha \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\alpha,$$
$$\Omega_0 = \mathrm{d}(p_A^\alpha v_\alpha^A - L) \wedge \mathrm{d}^m x - \mathrm{d}p_A^\alpha \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\alpha,$$

$(\mathcal{W}_0, \Omega_0)$ (equiv. $(\mathcal{W}_r, \hat{h}^*\Omega_0)$) is a pre-multisymplectic Hamiltonian system.

4.2 Field equations

A *Lagrange–Hamilton problem* consists in finding sections $\psi_0 \in \Gamma(M, \mathcal{W}_0)$ such that

$$\psi_0^*\, i(Y_0)\Omega_0 = 0, \qquad \forall\, Y_0 \in \mathfrak{X}(\mathcal{W}_0). \tag{17}$$

Taking $Y_0 \in \mathfrak{X}^{\mathrm{V}(\hat{\rho}_2^0)}(\mathcal{W}_0)$ we get the *first constraint submanifold* $\jmath_1 \colon \mathcal{W}_1 \hookrightarrow \mathcal{W}_0$,

$$\mathcal{W}_1 = \{(\bar{y}, \mathbf{p}) \in \mathcal{W}_0 \mid i(V_0)(\Omega_0)_{(\bar{y},\mathbf{p})} = 0, \text{ for every } V_0 \in \mathrm{V}(\hat{\rho}_2^0)\},$$

and sections solution to (17) take values on it. $\mathcal{W}_1$ is defined by $p_A^\alpha = \frac{\partial L}{\partial v_\alpha^A}$, hence

$$\mathcal{W}_1 = \{(\bar{y}, \widetilde{\mathcal{F}\mathcal{L}}(\bar{y})) \in \mathcal{W} \mid \bar{y} \in J^1\pi\}\ ,$$

and $\mathcal{W}_1$ is diffeomorphic to $J^1\pi$.

Theorem 4. (see diagram (18)) *If $\psi_0 \colon M \to \mathcal{W}_0$ is a section fulfilling equation* (17)*, then $\psi_0 = (\psi_{\mathcal{L}}, \psi_{\mathcal{H}}) = (\psi_{\mathcal{L}}, \widetilde{\mathcal{F}\mathcal{L}} \circ \psi_{\mathcal{L}})$, where $\psi_{\mathcal{L}} = \rho_1^0 \circ \psi_0$, and:*

1. *$\psi_{\mathcal{L}}$ is the canonical lift of the projected section $\phi = \rho_E^0 \circ \psi_0 \colon M \to E$ (that is, $\psi_{\mathcal{L}}$ is a holonomic section).*
2. *$\psi_{\mathcal{L}} = j^1\phi$ is a solution to the Lagrangian problem, and $\mu \circ \psi_{\mathcal{H}} = \mu \circ \widetilde{\mathcal{F}\mathcal{L}} \circ \psi_{\mathcal{L}} = \mathcal{F}\mathcal{L} \circ j^1\phi$ is a solution to the Hamiltonian problem.*

   Conversely, for every section $\phi \colon M \to E$ such that $j^1\phi$ is a solution to the Lagrangian problem (and hence $\mathcal{F}\mathcal{L} \circ j^1\phi$ is a solution to the Hamiltonian problem) we have that $\psi_0 = (j^1\phi, \widetilde{\mathcal{F}\mathcal{L}} \circ j^1\phi)$, is a solution to (17)*.*

$$\begin{tikzcd}
 & & \mathcal{W} \arrow[ddll, "\rho_1"'] \arrow[dr, "\rho_2"] & & \\
 & & \mathcal{W}_0 \arrow[u, "\jmath_0"] \arrow[r, "\rho_2^0"] \arrow[dll, "\rho_1^0"'] & \mathcal{M}\pi & \\
J^1\pi \arrow[drr, "\pi^1"] & & \mathcal{W}_1 \arrow[u, "\jmath_1"] \arrow[ll, "\rho_1^1"'] \arrow[r, "\rho_2^1"] \arrow[d, "\rho_E^1"'] & J^1\pi^* \arrow[dl, "\tau^1"'] & \mathcal{M}\pi \\
 & & E & & \\
 & & M \arrow[u, "\phi"] \arrow[uu, bend right, "\psi_1"'] \arrow[uuu, bend right=40, "\psi_0"'] \arrow[uull, "\psi_{\mathcal{L}} = j^1\phi"] \arrow[uurr, "\psi_{\mathcal{H}} = \widetilde{\mathcal{F}\mathcal{L}} \circ j^1\phi"'] & &
\end{tikzcd} \tag{18}$$

Proof. This proof is taken from [24]. See also [71].

1. Taking $\left\{\frac{\partial}{\partial p_A^\alpha}\right\}$ as a local basis for the ρ_1^0-vertical vector fields, and a section ψ_0, we have

$$\begin{aligned}
&i\left(\frac{\partial}{\partial p_A^\alpha}\right)\Omega_0 = v_\alpha^A \mathrm{d}^m x - \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\alpha \quad \Longrightarrow \\
&0 = \psi_0^*\left[i\left(\frac{\partial}{\partial p_A^\alpha}\right)\Omega_0\right] = \left(v_\alpha^A(x) - \frac{\partial y^A}{\partial x^\alpha}\right)\mathrm{d}^m x,
\end{aligned}$$

and thus the holonomy condition appears naturally within the unified formalism, and it is not necessary to impose it by hand to ψ_0. Thus we have that $\psi_0 = \left(x^\alpha, y^A, \frac{\partial y^A}{\partial x^\alpha}, \frac{\partial L}{\partial v_\alpha^A}\right)$, since ψ_0 takes values in $\mathcal{W}_1$, and hence it is of the form $\psi_0 = (j^1\phi, \widetilde{\mathcal{FL}} \circ j^1\phi)$, for $\phi = (x^\alpha, y^A) = \rho_E^0 \circ \psi_0$.

2. Since sections $\psi_0 \colon M \to \mathcal{W}_0$ solution to (17) take values in $\mathcal{W}_1$, we can identify them with sections $\psi_1 \colon M \to \mathcal{W}_1$. These sections ψ_1 verify, in particular, that $\psi_1^*\, i(Y_1)\Omega_1 = 0$ holds for every $Y_1 \in \mathfrak{X}(\mathcal{W}_1)$. Obviously $\psi_0 = \jmath_1 \circ \psi_1$. Moreover, as $\mathcal{W}_1$ is the graph of $\widetilde{\mathcal{FL}}$, denoting by $\rho_1^1 = \rho_1^0 \circ \jmath_1 \colon \mathcal{W}_1 \to J^1\pi$ the diffeomorphism which identifies $\mathcal{W}_1$ with $J^1\pi$, if we define $\Omega_1 = \jmath_1^*\Omega_0$, we have that $\Omega_1 = \rho_1^{1*}\Omega_{\mathcal{L}}$. In fact; as $(\rho_1^1)^{-1}(\bar{y}) = (\bar{y}, \widetilde{\mathcal{FL}}(\bar{y}))$, for every $\bar{y} \in J^1\pi$, then $(\rho_0^2 \circ \jmath_1 \circ (\rho_1^1)^{-1})(\bar{y}) = \widetilde{\mathcal{FL}}(\bar{y}) \in \mathcal{M}\pi$, and hence

$$\Omega_{\mathcal{L}} = \left(\rho_0^2 \circ \jmath_1 \circ (\rho_1^1)^{-1}\right)^*\Omega = \left[\left((\rho_1^1)^{-1}\right)^* \circ \jmath_1^* \circ \rho_0^{2*}\right]\Omega = \left[\left((\rho_1^1)^{-1}\right)^* \circ \jmath_1^*\right]\Omega_0 = \left((\rho_1^1)^{-1}\right)^*\Omega_1.$$

Now, let $X \in \mathfrak{X}(J^1\pi)$. We have

$$\begin{aligned}
(j^1\phi)^*\, i(X)\Omega_{\mathcal{L}} &= (\rho_1^0 \circ \psi_0)^*\, i(X)\Omega_{\mathcal{L}} = (\rho_1^0 \circ \jmath_1 \circ \psi_1)^*\, i(X)\Omega_{\mathcal{L}} \\
&= (\rho_1^1 \circ \psi_1)^*\, i(X)\Omega_{\mathcal{L}} = \psi_1^*\, i((\rho_1^1)_*^{-1}X)(\rho_1^{1*}\Omega_{\mathcal{L}}) = \psi_1^*\, i(Y_1)\Omega_1 \\
&= \psi_1^*\, i(Y_1)(\jmath_1^*\Omega_0) = (\psi_1^* \circ \jmath_1^*)\, i(Y_0)\Omega_0 = \psi_0^*\, i(Y_0)\Omega_0, \qquad (19)
\end{aligned}$$

where $Y_0 \in \mathfrak{X}(\mathcal{W}_0)$ is such that $Y_0 = \jmath_{1*}Y_1$. But as $\psi_0^*\, i(Y_0)\Omega_0 = 0$, for every $Y_0 \in \mathfrak{X}(\mathcal{W}_0)$, then we conclude that $(j^1\phi)^*\, i(X)\Omega_{\mathcal{L}} = 0$, for every $X \in \mathfrak{X}(J^1\pi)$.

Conversely, let $j^1\phi \colon M \to J^1\pi$ such that $(j^1\phi)^*\, i(X)\Omega_{\mathcal{L}} = 0$, for every $X \in \mathfrak{X}(J^1\pi)$, and define $\psi_0 \colon M \to \mathcal{W}_0$ as $\psi_0 := (j^1\phi, \widetilde{\mathcal{FL}} \circ j^1\phi)$ (observe that ψ_0 takes its values in $\mathcal{W}_1$). Taking into account that, on the points of $\mathcal{W}_1$, every $Y_0 \in \mathfrak{X}(\mathcal{W}_0)$ splits into $Y_0 = Y_0^1 + Y_0^2$, with $Y_0^1 \in \mathfrak{X}(\mathcal{W}_0)$ tangent to $\mathcal{W}_1$, and $Y_0^2 \in \mathfrak{X}^{\mathrm{V}(\rho_1^0)}(\mathcal{W}_0)$, we have that

$$\psi_0^*\, i(Y_0)\Omega_0 = \psi_0^*\, i(Y_0^1)\Omega_0 + \psi_0^*\, i(Y_0^2)\Omega_0 = 0,$$

because for Y_0^1, the same reasoning as in (19) leads to

$$\psi_0^*\, i(Y_0^1)\Omega_0 = (j^1\phi)^*\, i(X_0^1)\Omega_{\mathcal{L}} = 0,$$

(where $X_0^1 = (\rho_1^1)_*^{-1}Y_0^1$) and, as $j^1\phi$ is a holonomic section for Y_0^2, following also the same reasoning as in (19), a local calculus gives

$$\psi_0^*\, i(Y_0^2)\Omega_0 = (j^1\phi)^*\left[\left(f_A^\alpha(x)\left(v_\alpha^A - \frac{\partial y^A}{\partial x^\alpha}\right)\right)\mathrm{d}^m x\right] = 0.$$

The result for the sections $\mathcal{FL} \circ j^1\phi$ is a direct consequence of the *equivalence Theorem* 3 between the Lagrangian and Hamiltonian formalisms. ■

Thus, equation (17) gives equations of three different classes:

1. Algebraic equations, determining $\mathcal{W}_1 \hookrightarrow \mathcal{W}_0$, where the sections solution take their values. These are the *primary Hamiltonian constraints*, and generate, by $\hat{\rho}_2^0$ projection, the primary constraints of the Hamiltonian formalism for singular Lagrangians.

2. Differential equations, forcing the sections solution ψ_0 to be holonomic.
3. The Euler–Lagrange equations.

Field equations in the unified formalism can also be stated in terms of multivector fields and connections in $\mathcal{W}_0$. In fact, the problem of finding sections solution to (17) can be formulated equivalently as follows: finding a distribution D_0 of $\mathrm{T}(\mathcal{W}_0)$ such that it is integrable (that is, *involutive*), m-dimensional, ρ_M^0-transverse, and the integral manifolds of D_0 are the sections solution to the above equations. (Note that we do not ask them to be lifting of π-sections; that is, the holonomic condition). This is equivalent to stating that the sections solution to this problem are the integral sections of one of the following equivalent elements:

- A class of integrable and ρ_M^0-transverse m-multivector fields $\{X_0\} \subset \mathfrak{X}^m(\mathcal{W}_0)$ satisfying that
$$i(X_0)\Omega_0 = 0, \qquad \text{for every } X_0 \in \{X_0\}.$$
- An integrable connection ∇_0 in $\rho_M^0 \colon \mathcal{W}_0 \to M$ such that
$$i(\nabla_0)\Omega_0 = (m-1)\Omega_0.$$

Locally decomposable and ρ_M^0-transverse multivector fields and orientable connections which are solutions of these equations are called *Lagrange–Hamiltonian multivector fields* and *jet fields* for $(\mathcal{W}_0, \Omega_0)$. Euler–Lagrange and Hamilton–De Donder–Weyl multivector fields can be recovered from these Lagrange–Hamiltonian multivector fields (see [24]).

5 Example

As an example of application of these formalisms we consider a classical system which has been taken from [24]: minimal surfaces (in $\mathbb{R}^3$). Other examples of application of the multisymplectic formalism are explained in detail in [39, 43, 91] as well as in many other references (see, for instance, [16, 25, 26, 27, 28, 30, 69, 71] and quoted references).

5.1 Geometric elements. Lagrangian and Hamiltonian formalisms

The problem consists in looking for mappings $\varphi\colon U \subset \mathbb{R}^2 \to \mathbb{R}$ such that their graphs have minimal area as sets of $\mathbb{R}^3$, and satisfy certain boundary conditions.

For this model, we have that $M = \mathbb{R}^2$, $E = \mathbb{R}^2 \times \mathbb{R}$, and

$$J^1\pi = \pi^*\mathrm{T}^*\mathbb{R}^2 \otimes \mathbb{R} = \pi^*\mathrm{T}^*M = \pi^*\mathrm{T}^*\mathbb{R}^2, \qquad \mathcal{M}\pi = \pi^*(\mathrm{T}M \times_M E),$$
$$J^1\pi^* = \pi^*\mathrm{T}M = \pi^*\mathrm{T}\mathbb{R}^2.$$

The coordinates in $J^1\pi$, $J^1\pi^*$ and $\mathcal{M}\pi$ are denoted (x^1, x^2, y, v_1, v_2), (x^1, x^2, y, p^1, p^2), and $(x^1, x^2, y, p^1, p^2, p)$ respectively. If $\omega = \mathrm{d}x^1 \wedge \mathrm{d}x^2$, the Lagrangian density is

$$\mathcal{L} = \left[1 + (v_1)^2 + (v_2)^2\right]^{1/2} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \equiv \pounds \mathrm{d}x^1 \wedge \mathrm{d}x^2,$$

and the Poincaré–Cartan forms are

$$\Theta_{\mathcal{L}} = \frac{v_1}{\pounds}\mathrm{d}y \wedge \mathrm{d}x^2 - \frac{v_2}{\pounds}\mathrm{d}y \wedge \mathrm{d}x^1 + \pounds\left(1 - \left(\frac{v_1}{\pounds}\right)^2 - \left(\frac{v_2}{\pounds}\right)^2\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2,$$
$$\Omega_{\mathcal{L}} = -\mathrm{d}\left(\frac{v_1}{\pounds}\right) \wedge \mathrm{d}y \wedge \mathrm{d}x^2 + \mathrm{d}\left(\frac{v_2}{\pounds}\right) \wedge \mathrm{d}y \wedge \mathrm{d}x^1$$

$$- \mathrm{d}\left[\pounds \left(1 - \left(\frac{v_1}{\pounds}\right)^2 - \left(\frac{v_2}{\pounds}\right)^2 \right) \right] \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2.$$

The Euler–Lagrange equation of the problem are

$$\begin{aligned}
0 &= \left(\frac{\partial p^2}{\partial x^2} + \frac{\partial p^1}{\partial x^1}\right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 = \left[\frac{\partial}{\partial x^1}\left(\frac{v_1}{\pounds}\right) + \frac{\partial}{\partial x^2}\left(\frac{v_2}{\pounds}\right)\right] \mathrm{d}x^1 \wedge \mathrm{d}x^2 \\
&= \frac{1}{\pounds^3}\left[\left(1 + \left(\frac{\partial y}{\partial x^1}\right)^2\right) \frac{\partial^2 y}{\partial x^2 \partial x^2} + \left(1 + \left(\frac{\partial y}{\partial x^2}\right)^2\right) \frac{\partial^2 y}{\partial x^1 \partial x^1} \right. \\
&\quad \left. - 2 \frac{\partial y}{\partial x^1} \frac{\partial y}{\partial x^2} \frac{\partial^2 y}{\partial x^1 \partial x^2} \right] \mathrm{d}x^1 \wedge \mathrm{d}x^2,
\end{aligned} \tag{20}$$

and the associated Euler–Lagrange m-vector fields and connections which are the solutions to the Lagrangian problem are

$$\begin{aligned}
X_{\mathcal{L}} &= f \left(\frac{\partial}{\partial x^1} + v_1 \frac{\partial}{\partial y} + \frac{\partial v_1}{\partial x^1} \frac{\partial}{\partial v_1} + \frac{\partial v_2}{\partial x^1} \frac{\partial}{\partial v_2} \right) \wedge \left(\frac{\partial}{\partial x^2} + v_2 \frac{\partial}{\partial y} + \frac{\partial v_1}{\partial x^2} \frac{\partial}{\partial v_1} + \frac{\partial v_2}{\partial x^2} \frac{\partial}{\partial v_2} \right), \\
\nabla_{\mathcal{L}} &= \mathrm{d}x^1 \otimes \left(\frac{\partial}{\partial x^1} + v_1 \frac{\partial}{\partial y} + \frac{\partial v_1}{\partial x^1} \frac{\partial}{\partial v_1} + \frac{\partial v_2}{\partial x^1} \frac{\partial}{\partial v_2} \right) \\
&\quad + \mathrm{d}x^2 \otimes \left(\frac{\partial}{\partial x^2} + v_2 \frac{\partial}{\partial y} + \frac{\partial v_1}{\partial x^2} \frac{\partial}{\partial v_1} + \frac{\partial v_2}{\partial x^2} \frac{\partial}{\partial v_2} \right).
\end{aligned}$$

The Legendre maps are given by

$$\begin{aligned}
\mathcal{FL}(x^1, x^2, y, v_1, v_2) &= \left(x^1, x^2, y, \frac{v_1}{\pounds}, \frac{v_2}{\pounds} \right), \\
\widetilde{\mathcal{FL}}(x^1, x^2, y, v_1, v_2) &= \left(x^1, x^2, y, \frac{v_1}{\pounds}, \frac{v_2}{\pounds}, \pounds - \frac{(v_1)^2}{\pounds} - \frac{(v_2)^2}{\pounds} \right),
\end{aligned}$$

and then $\mathcal{L}$ is hyperregular. The Hamiltonian function is $\mathrm{h} = -[1 - (p^1)^2 - (p^2)^2]^{1/2}$, and so the Hamilton–Cartan forms are

$$\begin{aligned}
\Theta_h &= p^1 \mathrm{d}y \wedge \mathrm{d}x^2 - p^2 \mathrm{d}y \wedge \mathrm{d}x^1 - \mathrm{h}\mathrm{d}x^1 \wedge \mathrm{d}x^2, \\
\Omega_h &= -\mathrm{d}p^1 \wedge \mathrm{d}y \wedge \mathrm{d}x^2 + \mathrm{d}p^2 \wedge \mathrm{d}y \wedge \mathrm{d}x^1 + \mathrm{d}\mathrm{h} \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2.
\end{aligned}$$

The Hamilton–De Donder–Weyl equations of the problem are

$$\frac{\partial y}{\partial x^1} = -\frac{p^1}{\mathrm{h}}, \qquad \frac{\partial y}{\partial x^2} = -\frac{p^2}{\mathrm{h}}, \qquad \frac{\partial p^1}{\partial x^1} = -\frac{\partial p^2}{\partial x^2}, \tag{21}$$

and the corresponding Hamilton–De Donder–Weyl m-vector fields and connections which are the solutions to the Hamiltonian problem are

$$\begin{aligned}
X_h &= f \left(\frac{\partial}{\partial x^1} - \frac{p^1}{\mathrm{h}} \frac{\partial}{\partial y} + \frac{\partial p^1}{\partial x^1} \frac{\partial}{\partial p^1} + \frac{\partial p^2}{\partial x^1} \frac{\partial}{\partial p^2} \right) \wedge \left(\frac{\partial}{\partial x^2} - \frac{p^2}{\mathrm{h}} \frac{\partial}{\partial y} + \frac{\partial p^1}{\partial x^2} \frac{\partial}{\partial p^1} + \frac{\partial p^2}{\partial x^2} \frac{\partial}{\partial p^2} \right), \\
\nabla_h &= \mathrm{d}x^1 \otimes \left(\frac{\partial}{\partial x^1} - \frac{p^1}{\mathrm{h}} \frac{\partial}{\partial y} + \frac{\partial p^1}{\partial x^1} \frac{\partial}{\partial p^1} + \frac{\partial p^2}{\partial x^1} \frac{\partial}{\partial p^2} \right) \\
&\quad + \mathrm{d}x^2 \otimes \left(\frac{\partial}{\partial x^2} - \frac{p^2}{\mathrm{h}} \frac{\partial}{\partial y} + \frac{\partial p^1}{\partial x^2} \frac{\partial}{\partial p^1} + \frac{\partial p^2}{\partial x^2} \frac{\partial}{\partial p^2} \right).
\end{aligned}$$

5.2 Unified formalism

For the unified formalism we have

$$\mathcal{W} = \pi^*\mathrm{T}^*M \times_E \pi^*(\mathrm{T}M \times_M E), \qquad \mathcal{W}_r = \pi^*\mathrm{T}^*M \times_E \pi^*\mathrm{T}M = \pi^*(\mathrm{T}^*M \times_M \mathrm{T}M).$$

If $w = (x^1, x^2, y, v_1, v_2, p^1, p^2, p) \in \mathcal{W}$, the coupling form is $\hat{\mathcal{C}} = (p^1 v_1 + p^2 v_2 + p)\mathrm{d}x^1 \wedge \mathrm{d}x^2$; therefore

$$\mathcal{W}_0 = \left\{(x^1, x^2, y, v_1, v_2, p^1, p^2, p) \in \mathcal{W} \mid [1 + (v_1)^2 + (v_2)^2]^{1/2} - p^1 v_1 - p^2 v_2 - p = 0\right\},$$

and we have the forms

$$\begin{aligned}\Theta_0 &= \left(\left[1 + (v_1)^2 + (v_2)^2\right]^{1/2} - p^1 v_1 - p^2 v_2\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2 - p^2\mathrm{d}y \wedge \mathrm{d}x_1 + p^1\mathrm{d}y \wedge \mathrm{d}x_2,\\ \Omega_0 &= -\mathrm{d}\left(\left[1 + (v_1)^2 + (v_2)^2\right]^{1/2} - p^1 v_1 - p^2 v_2\right) \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2\\ &\quad + \mathrm{d}p^2 \wedge \mathrm{d}y \wedge \mathrm{d}x_1 - \mathrm{d}p^1 \wedge \mathrm{d}y \wedge \mathrm{d}x_2.\end{aligned}$$

Taking first $\hat{\rho}_2^0$-vertical vector fields $\frac{\partial}{\partial v_\alpha}$ we obtain

$$0 = i\left(\frac{\partial}{\partial v_\alpha}\right)\Omega_0 = \left(p^\alpha - \frac{v_\alpha}{\pounds}\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2,$$

which determines the submanifold $\mathcal{W}_1 = \mathrm{graph}\,\widetilde{\mathcal{FL}}$ (diffeomorphic to $J^1\pi$), and reproduces the expression of the Legendre map. Now, taking ρ_1^0-vertical vector fields $\frac{\partial}{\partial p^\alpha}$, the contraction $i\left(\frac{\partial}{\partial p^\alpha}\right)\Omega_0$ gives, for $\alpha = 1, 2$, $v_1\mathrm{d}x^1 \wedge \mathrm{d}x^2 - \mathrm{d}y \wedge \mathrm{d}x^2$ and $v_2\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}y \wedge \mathrm{d}x^1$ respectively, so that, for a section $\psi_0 = (x^1, x^2, y(x^1, x^2), v_1(x^1, x^2), v_2(x^1, x^2), p^1(x^1, x^2), p^2(x^1, x^2))$ taking values in $\mathcal{W}_1$, we have that the condition $\psi_0^*\left[i\left(\frac{\partial}{\partial p^\alpha}\right)\Omega_0\right] = 0$ leads to

$$\left(v_1 - \frac{\partial y}{\partial x^1}\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2 = 0, \qquad \left(v_2 - \frac{\partial y}{\partial x^2}\right)\mathrm{d}x^1 \wedge \mathrm{d}x^2 = 0,$$

which are the holonomy condition. Finally, taking the vector field $\frac{\partial}{\partial y}$ we have

$$i\left(\frac{\partial}{\partial y}\right)\Omega_0 = -\mathrm{d}p^2 \wedge \mathrm{d}x^1 + \mathrm{d}p^1 \wedge \mathrm{d}x^2$$

and, for a section ψ_0 fulfilling the former conditions, the equation $0 = \psi_0^*\left[i\left(\frac{\partial}{\partial y}\right)\Omega_0\right]$ leads to the Euler–Lagrange equations (20). Now, bearing in mind the expressions of h and the Legendre map, from the Euler–Lagrange equations we get the Hamilton–De Donder–Weyl equations (21).

The m-vector fields and connections which are the solutions to the problem in the unified formalism are

$$\begin{aligned}X_0 &= f\left(\frac{\partial}{\partial x^1} + v_1\frac{\partial}{\partial y} + \frac{\partial v_1}{\partial x^1}\frac{\partial}{\partial v_1} + \frac{\partial v_2}{\partial x^1}\frac{\partial}{\partial v_2} + \frac{\partial p^1}{\partial x^1}\frac{\partial}{\partial p^1} + \frac{\partial p^2}{\partial x^1}\frac{\partial}{\partial p^2}\right)\\ &\quad \wedge \left(\frac{\partial}{\partial x^2} + v_2\frac{\partial}{\partial y} + \frac{\partial v_1}{\partial x^2}\frac{\partial}{\partial v_1} + \frac{\partial v_2}{\partial x^2}\frac{\partial}{\partial v_2} + \frac{\partial p^1}{\partial x^2}\frac{\partial}{\partial p^1} + \frac{\partial p^2}{\partial x^2}\frac{\partial}{\partial p^2}\right),\\ \nabla_0 &= \mathrm{d}x^1 \otimes \left(\frac{\partial}{\partial x^1} + v_1\frac{\partial}{\partial y} + \frac{\partial v_1}{\partial x^1}\frac{\partial}{\partial v_1} + \frac{\partial v_2}{\partial x^1}\frac{\partial}{\partial v_2} + \frac{\partial p^1}{\partial x^1}\frac{\partial}{\partial p^1} + \frac{\partial p^2}{\partial x^1}\frac{\partial}{\partial p^2}\right)\\ &\quad + \mathrm{d}x^2 \otimes \left(\frac{\partial}{\partial x^2} + v_2\frac{\partial}{\partial y} + \frac{\partial v_1}{\partial x^2}\frac{\partial}{\partial v_1} + \frac{\partial v_2}{\partial x^2}\frac{\partial}{\partial v_2} + \frac{\partial p^1}{\partial x^2}\frac{\partial}{\partial p^1} + \frac{\partial p^2}{\partial x^2}\frac{\partial}{\partial p^2}\right),\end{aligned}$$

(f being a non-vanishing function) where the coefficients $\frac{\partial v_\alpha}{\partial x^\nu} = \frac{\partial^2 y}{\partial x^\nu \partial x^\alpha}$ are related by the Euler–Lagrange equations, and the coefficients $\frac{\partial p^\alpha}{\partial x^\nu}$ are related by the Hamilton–De Donder–Weyl equations (the third one). From these expressions we recover the Euler–Lagrange m-vector fields and connections which are the solutions to the Lagrangian problem, and the Hamilton–De Donder–Weyl m-vector fields and connections which are the solutions to the Hamiltonian problem obtained in the above paragraph.

6 Discussion and outlook

Multisymplectic geometry and its application to describe classical field theories have been fields of increasing interest in the last years. A lot of well-known results in the realm of symplectic geometry and symplectic mechanics have been generalized also for the multisymplectic case, but there are many other problems which remain open. Next we review some of these results and problems, and their current status.

A fundamental result in symplectic geometry is the Darboux theorem. The analogous result also holds in some particular cases of multisymplectic forms (for instance, for volume forms). Nevertheless, in the general case, a multisymplectic manifold does not admit a system of Darboux coordinates for the multisymplectic form. In fact this is a problem arising from linear algebra: the classification of skew-symmetric tensors of degree greater than two is still an open problem. The kind of multisymplectic manifolds admitting Darboux coordinates has been identified [73], and they are those being locally multisymplectomorphic to bundles of forms (see also [33] for another approach to this problem).

Another interesting subject concerns to the definition of *Poisson brackets* in multisymplectic manifolds. This is a relevant point, for instance, for the further quantization of classical field theories. This problem has been studied in the realm of polysymplectic manifolds [50, 51] and for the multisymplectic case some recent contributions are [34, 35, 36]. However, the problem is not completely solved satisfactorily, and the research on this topic is still open.

In the same way, approaches for generalizing symplectic integrators to this geometric framework (i.e., the so-called *multisymplectic integrators*) have been studied in recent years, and numerical methods have been developed for solving the field equations, which are based on the use of these multisymplectic integrators [77, 79]. Research on this topic is in progress.

Another field of increasing interest in the last years is the study of systems in classical field theories with nonholonomic constraints. This is a meeting topic between honholonomic mechanics and classical field theories. The construction of the Lagrangian and Hamiltonian formalism, as well as other problems such as the study of symmetries and reduction have been analyzed for the k-symplectic formulation [72] and for the multisymplectic models in several works [10, 95, 96, 97, 98].

Further developments have not been achieved. For instance, the generalization of the Marsden–Weinstein reduction theorem [80] to the multisymplectic framework. Concerning reduction theory in general, only partial results about reduction by foliations are currently being studied [49]. The corresponding reduction theorem has been stated and proved for the k-symplectic formulation [84], but the theory of reduction of multisymplectic Lagrangian and Hamiltonian systems under the action of groups of symmetries is still under research, and only partial results have been achieved [17, 18, 19, 81].

The problem of quantization of classical field theories is another relevant topic to be developed. There are several works due to Kanatchikov devoted to geometric (pre)quantization of polysymplectic field theories [53, 54, 55, 56, 57, 58, 59], some attempts for the k-symplectic case [12, 89], and other different approaches for the quantization of fields, in general (see, for instance, [8, 90]). Nevertheless, the study of the geometric structures and obstructions to perform the geometric quantization program for covariant multisymplectic field theories is open to further research.

As a final remark, many of the subjects that we have presented in this work have been studied also for higher-order field theories (see, for instance, [1, 2, 31, 32, 38, 62, 63, 92, 93]). One of the problems of the first multisymplectic models for these theories was that the definition of the corresponding multisymplectic structure (the Poincaré–Cartan form) was ambiguous. This trouble have been solved recently [13]. But, in general, the problem of stating complete and satisfactory geometrical models for the Lagrangian and Hamiltonian formalisms of these kinds of theories, as well as other related topics (symmetries, constraint algorithms for the singular cases, quantization, ...) are under development.

One can expect to see more work on all these subjects in the future.

A Appendix

A.1 Multisymplectic manifolds

Definition 7. Let $\mathcal{M}$ be a differentiable manifold, and $\Omega \in \mathit{\Omega}^k(\mathcal{M})$ $(1 < k \leq \dim \mathcal{M})$.

Ω is a *multisymplectic form*, and then $(\mathcal{M}, \Omega)$ is a *multisymplectic manifold*, if

1. $\Omega \in Z^k(\mathcal{M})$ (it is closed).
2. Ω is 1-nondegenerate; that is, for every $p \in \mathcal{M}$ and $X_p \in \mathrm{T}_p\mathcal{M}$, $i(X_p)\Omega_p = 0 \Leftrightarrow X_p = 0$.

If Ω is closed and 1-degenerate then it is a *pre-multisymplectic form*, and $(\mathcal{M}, \Omega)$ is a *pre-multisymplectic manifold*.

Multisymplectic manifolds of degree $k = 2$ are the usual symplectic manifolds, and manifolds with a distinguished volume form are multisymplectic manifolds of degree its dimension. Other examples of multisymplectic manifolds are provided by compact semisimple Lie groups equipped with the canonical cohomology 3-class, symplectic 6-dimensional Calabi–Yau manifolds with the canonical 3-class, etc. There are no multisymplectic manifolds of degrees 1 or $\dim \mathcal{M} - 1$ because $\ker \Omega$ is nonvanishing in both cases.

Another very important kind of multisymplectic manifold is the *multicotangent bundle* of a manifold Q, $\Lambda^k(\mathrm{T}^*Q)$, that is, the bundle of k-forms in Q. This bundle is endowed with a canonical k-form $\Theta \in \mathit{\Omega}^k(\Lambda^k(\mathrm{T}^*Q)$, and then $\Omega := -\mathrm{d}\Theta \in \mathit{\Omega}^{k+1}(\Lambda^k(\mathrm{T}^*Q)$ is a 1-nondegenerate form. Then the couple $(\Lambda^k(\mathrm{T}^*Q), \Omega)$ is a multisymplectic manifold.

A local classification of multisymplectic forms can be done only for particular cases [73, 33].

A.2 Multivector fields

See [27] for details. Let $\mathcal{M}$ be a n-dimensional differentiable manifold. Sections of $\Lambda^m(\mathrm{T}\mathcal{M})$ are called *m-multivector fields* in $\mathcal{M}$ (they are the contravariant skew-symmetric tensors of order m in $\mathcal{M}$). We denote by $\mathfrak{X}^m(\mathcal{M})$ the set of m-multivector fields in $\mathcal{M}$. Then, $\mathcal{X} \in \mathfrak{X}^m(\mathcal{M})$ is *locally decomposable* if, for every $p \in \mathcal{M}$, there is an open neighbourhood $U_p \subset \mathcal{M}$ and $X_1, \ldots, X_m \in \mathfrak{X}(U_p)$ such that $\mathcal{X}|_{U_p} = X_1 \wedge \ldots \wedge X_m$.

A non-vanishing $\mathcal{X} \in \mathfrak{X}^m(\mathcal{M})$ and a m-dimensional distribution $\mathcal{D} \subset \mathrm{T}\mathcal{M}$ are *locally associated* if there exists a connected open set $U \subseteq \mathcal{M}$ such that $\mathcal{X}|_U$ is a section of $\Lambda^m\mathcal{D}|_U$. If $\mathcal{X}, \mathcal{X}' \in \mathfrak{X}^m(\mathcal{M})$ are non-vanishing multivector fields locally associated with the same distribution $\mathcal{D}$, on the same connected open set U, then there exists a non-vanishing function $f \in \mathrm{C}^\infty(U)$ such that $\mathcal{X}'|_U = f\mathcal{X}$. This fact defines an equivalence relation in the set of non-vanishing m-multivector fields in $\mathcal{M}$, whose equivalence classes will be denoted by $\{\mathcal{X}\}_U$. Then there is a one-to-one correspondence between the m-dimensional orientable distributions $\mathcal{D}$ in

$\mathrm{T}\mathcal{M}$ and the equivalence classes $\{\mathcal{X}\}_{\mathcal{M}}$ of non-vanishing, locally decomposable m-multivector fields in $\mathcal{M}$.

A non-vanishing, locally decomposable multivector field $\mathcal{X} \in \mathfrak{X}^m(\mathcal{M})$ is said to be *integrable* (resp. *involutive*) if its associated distribution is integrable (resp. involutive). If $\mathcal{X} \in \mathfrak{X}^m(\mathcal{M})$ is integrable (resp. involutive), then so is every other in its equivalence class $\{\mathcal{X}\}$, and all of them have the same integral manifolds. Moreover, *Frobenius theorem* allows us to say that a non-vanishing and locally decomposable multivector field is integrable if, and only if, it is involutive.

If $\pi\colon \mathcal{M} \to M$ is a fiber bundle, we are interested in the case where the integral manifolds of integrable multivector fields in $\mathcal{M}$ are sections of π. Thus, $\mathcal{X} \in \mathfrak{X}^m(\mathcal{M})$ is said to be *π-transverse* if, at every point $y \in \mathcal{M}$, $(i(\mathcal{X})(\pi^*\beta))_y \neq 0$, for every $\beta \in \Omega^m(M)$ with $\omega(\pi(y)) \neq 0$. Then, if $\mathcal{X} \in \mathfrak{X}^m(\mathcal{M})$ is integrable, it is π-transverse if, and only if, its integral manifolds are local sections of $\pi\colon \mathcal{M} \to M$. Finally, it is clear that classes of locally decomposable and π-transverse multivector fields $\{\mathcal{X}\} \subseteq \mathfrak{X}^m(\mathcal{M})$ are in one-to-one correspondence with orientable Ehresmann connection forms ∇ in $\pi\colon \mathcal{M} \to M$. This correspondence is characterized by the fact that the horizontal subbundle associated with ∇ is the distribution associated with $\{\mathcal{X}\}$. In this correspondence, classes of integrable locally decomposable and π-transverse m multivector fields correspond to flat orientable Ehresmann connections.

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