Symmetry, Integrability and Geometry: Methods and Applications SIGMA 5 (2009), 097, 22 pages

Indefinite Affine Hyperspheres Admitting a Pointwise Symmetry. Part 2*

Christine SCHARLACH

Technische Universität Berlin, Fak. II, Inst. f. Mathematik, MA 8-3, 10623 Berlin, Germany
E-mail: *schar@math.tu-berlin.de*
URL: http://www.math.tu-berlin.de/~schar/

Received May 08, 2009, in final form October 06, 2009; Published online October 19, 2009
doi:10.3842/SIGMA.2009.097

Abstract. An affine hypersurface M is said to admit a pointwise symmetry, if there exists a subgroup G of $\mathrm{Aut}(T_pM)$ for all $p \in M$, which preserves (pointwise) the affine metric h, the difference tensor K and the affine shape operator S. Here, we consider 3-dimensional indefinite affine hyperspheres, i.e. $S = H\mathit{Id}$ (and thus S is trivially preserved). In Part 1 we found the possible symmetry groups G and gave for each G a canonical form of K. We started a classification by showing that hyperspheres admitting a pointwise $\mathbb{Z}_2 \times \mathbb{Z}_2$ resp. $\mathbb{R}$-symmetry are well-known, they have constant sectional curvature and Pick invariant $J < 0$ resp. $J = 0$. Here, we continue with affine hyperspheres admitting a pointwise $\mathbb{Z}_3$- or $SO(2)$-symmetry. They turn out to be warped products of affine spheres ($\mathbb{Z}_3$) or quadrics ($SO(2)$) with a curve.

Key words: affine hyperspheres; indefinite affine metric; pointwise symmetry; affine differential geometry; affine spheres; warped products

2000 Mathematics Subject Classification: 53A15; 53B30

1 Introduction

Let M^n be a connected, oriented manifold. Consider an immersed hypersurface with relative normalization, i.e., an immersion $\varphi\colon M^n \to \mathbb{R}^{n+1}$ together with a transverse vector field ξ such that $D\xi$ has its image in $\varphi_* T_x M$. Equi-affine geometry studies the properties of such immersions under equi-affine transformations, i.e. volume-preserving linear transformations ($SL(n+1,\mathbb{R})$) and translations.

In the theory of nondegenerate equi-affine hypersurfaces there exists a canonical choice of transverse vector field ξ (unique up to sign), called the affine (Blaschke) normal, which induces a connection ∇, a nondegenerate symmetric bilinear form h and a 1-1 tensor field S by

$$D_X Y = \nabla_X Y + h(X,Y)\xi, \tag{1}$$

$$D_X \xi = -SX, \tag{2}$$

for all $X, Y \in \mathcal{X}(M)$. The connection ∇ is called the induced affine connection, h is called the affine metric or Blaschke metric and S is called the affine shape operator. In general ∇ is not the Levi-Civita connection $\hat{\nabla}$ of h. The difference tensor K is defined as

$$K(X,Y) = \nabla_X Y - \hat{\nabla}_X Y, \tag{3}$$

for all $X, Y \in \mathcal{X}(M)$. Moreover the form $h(K(X,Y),Z)$ is a symmetric cubic form with the property that for any fixed $X \in \mathcal{X}(M)$, $\operatorname{trace} K_X$ vanishes. This last property is called the

*This paper is a contribution to the Special Issue "Élie Cartan and Differential Geometry". The full collection is available at http://www.emis.de/journals/SIGMA/Cartan.html apolarity condition. The difference tensor K, together with the affine metric h and the affine shape operator S are the most fundamental algebraic invariants for a nondegenerate affine hypersurface (more details in Section 2). We say that M^n is indefinite, definite, etc. if the affine metric h is indefinite, definite, etc. (Because the affine metric is a multiple of the Euclidean second fundamental form, a positive definite hypersurface is locally strongly convex.) For details of the basic theory of nondegenerate affine hypersurfaces we refer to [7] and [10].

Here we will restrict ourselves to the case of affine hyperspheres, i.e. the shape operator will be a (constant) multiple of the identity ($S = H Id$). Geometrically this means that all affine normals pass through a fixed point or they are parallel. There are many affine hyperspheres, even in the two-dimensional case only partial classifications are known. This is due to the fact that affine hyperspheres reduce to the study of the Monge-Ampère equations. Our question is the following: *What can we say about a three-dimensional affine hypersphere which admits a pointwise G-symmetry, i.e. there exists a non-trivial subgroup G of the isometry group such that for every $p \in M$ and every $L \in G$:*

$$K(LX_p, LY_p) = L(K(X_p, Y_p)) \qquad \forall\, X_p, Y_p \in T_pM.$$

We have motivated this question in Part 1 [13] (see also [1, 14, 8]). A classification of 3-dimensional positive definite affine hyperspheres admitting pointwise symmetries was obtained in [14]. We continue the classification in the indefinite case. We can assume that the affine metric has index two, i.e. the corresponding isometry group is the (special) Lorentz group $\mathrm{SO}(1,2)$. In Part 1, it turns out that a $\mathrm{SO}(1,2)$-stabilizer of a nontrivial traceless cubic form is isomorphic to either $\mathrm{SO}(2)$, $\mathrm{SO}(1,1)$, $\mathbb{R}$, the group S_3 of order 6, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_2$ or it is trivial. We have shown that hyperspheres admitting a pointwise $\mathbb{Z}_2 \times \mathbb{Z}_2$- resp. $\mathbb{R}$-symmetry are well-known, they have constant sectional curvature and Pick invariant $J < 0$ resp. $J = 0$.

In the following we classify the indefinite affine hyperspheres which admit a pointwise $\mathbb{Z}_3$-, $\mathrm{SO}(2)$- or $\mathrm{SO}(1,1)$-symmetry. They turn out to be warped products of affine spheres ($\mathbb{Z}_3$) or quadrics ($SO(2)$, $SO(1,1)$) with a curve. As a result we get a new composition method. Since the methods for the proofs for $SO(1,1)$ are similar to those for $\mathbb{Z}_3$- or $\mathrm{SO}(2)$-symmetry (and as long) we will omit them here. Both methods and results are different in case of S_3-symmetry and will be published elsewhere. The paper is organized as follows:

We will state the basic formulas of (equi-)affine hypersurface-theory needed in the further classification in Section 2. In Section 3, we show that in case of $\mathrm{SO}(2)$-, S_3- or $\mathbb{Z}_3$-symmetry we can extend the canonical form of K (cf. [13]) locally. Thus we can obtain information about the coefficients of K and ∇ from the basic equations of Gauss, Codazzi and Ricci (cf. Section 4). In Section 5 we show, that in case of $\mathbb{Z}_3$- or $\mathrm{SO}(2)$-symmetry it follows that the hypersurface admits a warped product structure $\mathbb{R} \times_{e^f} N^2$. Then we classify such hyperspheres by showing how they can be constructed starting from 2-dimensional positive definite affine spheres resp. quadrics (cf. Theorems 1–8). We end in Section 6 by stating the classification results in case of $SO(1,1)$-symmetry (cf. Theorems 9–16).

The classification can be seen as a generalization of the well known Calabi product of hyperbolic affine spheres [2, 6] and of the constructions for affine spheres considered in [5]. The following natural question for a (de)composition theorem, related to these constructions, gives another motivation for studying 3-dimensional hypersurfaces admitting a pointwise symmetry:

(De)composition Problem. *Let M^n be a nondegenerate affine hypersurface in $\mathbb{R}^{n+1}$. Under what conditions do there exist affine hyperpsheres M_1^r in $\mathbb{R}^{r+1}$ and M_2^s in $\mathbb{R}^{s+1}$, with $r+s = n-1$, such that $M = I \times_{f_1} M_1 \times_{f_2} M_2$, where $I \subset \mathbb{R}$ and f_1 and f_2 depend only on I (i.e. M admits a warped product structure)? How can the original immersion be recovered starting from the immersion of the affine spheres?*

Of course the first dimension in which the above problem can be considered is three and our results provide an answer in that case.

2 Basics of affine hypersphere theory

First we recall the definition of the affine normal ξ (cf. [10]). In equi-affine hypersurface theory on the ambient space $\mathbb{R}^{n+1}$ a fixed volume form det is given. A transverse vector field ξ induces a volume form θ on M by $\theta(X_1, \dots, X_n) = \det(\varphi_* X_1, \dots, \varphi_* X_n, \xi)$. Also the affine metric h defines a volume form ω_h on M, namely $\omega_h = |\det h|^{1/2}$. Now the affine normal ξ is uniquely determined (up to sign) by the conditions that $D\xi$ is everywhere tangential (which is equivalent to $\nabla\theta = 0$) and that

$$\theta = \omega_h. \tag{4}$$

Since we only consider 3-dimensional indefinite hyperspheres, i.e.

$$S = H\,Id, \qquad H = \text{const}, \tag{5}$$

we can fix the orientation of the affine normal ξ such that the affine metric has signature one. Then the sign of H in the definition of an affine hypersphere is an invariant.

Next we state some of the fundamental equations, which a nondegenerate hypersurface has to satisfy, see also [10] or [7]. These equations relate S and K with amongst others the curvature tensor R of the induced connection ∇ and the curvature tensor $\hat{R}$ of the Levi-Civita connection $\widehat{\nabla}$ of the affine metric h. There are the Gauss equation for ∇, which states that:

$$R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY,$$

and the Codazzi equation

$$(\nabla_X S)Y = (\nabla_Y S)X.$$

Also we have the total symmetry of the affine cubic form

$$C(X, Y, Z) = (\nabla_X h)(Y, Z) = -2h(K(X, Y), Z). \tag{6}$$

The fundamental existence and uniqueness theorem, see [3] or [4], states that given h, ∇ and S such that the difference tensor is symmetric and traceless with respect to h, on a simply connected manifold M an affine immersion of M exists if and only if the above Gauss equation and Codazzi equation are satisfied.

From the Gauss equation and Codazzi equation above the Codazzi equation for K and the Gauss equation for $\widehat{\nabla}$ follow:

$$(\widehat{\nabla}_X K)(Y, Z) - (\widehat{\nabla}_Y K)(X, Z) = \tfrac{1}{2}(h(Y, Z)SX - h(X, Z)SY - h(SY, Z)X + h(SX, Z)Y),$$

and

$$\hat{R}(X, Y)Z = \tfrac{1}{2}(h(Y, Z)SX - h(X, Z)SY + h(SY, Z)X - h(SX, Z)Y) - [K_X, K_Y]Z$$

If we define the Ricci tensor of the Levi-Civita connection $\widehat{\nabla}$ by:

$$\widehat{\mathrm{Ric}}(X, Y) = \mathrm{trace}\{Z \mapsto \hat{R}(Z, X)Y\}. \tag{7}$$

and the Pick invariant by:

$$J = \frac{1}{n(n-1)} h(K, K), \tag{8}$$

then from the Gauss equation we get for the scalar curvature $\hat{\kappa} = \frac{1}{n(n-1)}(\sum_{i,j} h^{ij}\widehat{\mathrm{Ric}}_{ij})$:

$$\hat{\kappa} = H + J. \tag{9}$$

For an affine hypersphere the Gauss and Codazzi equations have the form:

$$R(X,Y)Z = H(h(Y,Z)X - h(X,Z)Y), \tag{10}$$
$$(\nabla_X H)Y = (\nabla_Y H)X, \qquad \text{i.e.} \quad H = \text{const}, \tag{11}$$
$$(\widehat{\nabla}_X K)(Y,Z) = (\widehat{\nabla}_Y K)(X,Z), \tag{12}$$
$$\hat{R}(X,Y)Z = H(h(Y,Z)X - h(X,Z)Y) - [K_X, K_Y]Z. \tag{13}$$

Since H is constant, we can rescale φ such that $H \in \{-1, 0, 1\}$.

3 A local frame for pointwise SO(2)-, S_3- or $\mathbb{Z}_3$-symmetry

Let M^3 be a hypersphere admitting a SO(2)-, S_3- or $\mathbb{Z}_3$-symmetry. According to [13, Theorem 4, 2.–4.] there exists for every $p \in M^3$ an ONB $\{\mathbf{t}, \mathbf{v}, \mathbf{w}\}$ of T_pM^3 such that

$$\begin{aligned} K(\mathbf{t},\mathbf{t}) &= -2a_4\mathbf{t}, & K(\mathbf{t},\mathbf{v}) &= a_4\mathbf{v}, & K(\mathbf{t},\mathbf{w}) &= a_4\mathbf{w}, \\ K(\mathbf{v},\mathbf{v}) &= -a_4\mathbf{t} + a_6\mathbf{v}, & K(\mathbf{v},\mathbf{w}) &= -a_6\mathbf{w}, & K(\mathbf{w},\mathbf{w}) &= -a_4\mathbf{t} - a_6\mathbf{v}, \end{aligned}$$

where $a_4 > 0$ and $a_6 = 0$ in case of $SO(2)$-symmetry, $a_4 = 0$ and $a_6 > 0$ for S_3, and $a_4 > 0$ and $a_6 > 0$ for $\mathbb{Z}_3$.

We would like to extend the ONB locally. It is well known that $\widehat{\mathrm{Ric}}$ (cf. (7)) is a symmetric operator and we compute (some of the computations in this section are done with the CAS Mathematica[1]):

Lemma 1. *Let $p \in M^3$ and $\{\mathbf{t}, \mathbf{v}, \mathbf{w}\}$ the basis constructed earlier. Then*

$$\begin{aligned} \widehat{\mathrm{Ric}}(\mathbf{t},\mathbf{t}) &= -2(H - 3a_4^2), & \widehat{\mathrm{Ric}}(\mathbf{t},\mathbf{v}) &= 0, \\ \widehat{\mathrm{Ric}}(\mathbf{t},\mathbf{w}) &= 0, & \widehat{\mathrm{Ric}}(\mathbf{v},\mathbf{v}) &= 2(H - a_4^2 + a_6^2), \\ \widehat{\mathrm{Ric}}(\mathbf{v},\mathbf{w}) &= 0, & \widehat{\mathrm{Ric}}(\mathbf{w},\mathbf{w}) &= 2(H - a_4^2 + a_6^2). \end{aligned}$$

Proof. The proof is a straight-forward computation using the Gauss equation (13). It follows e.g. that

$$\begin{aligned} \hat{R}(\mathbf{t},\mathbf{v})\mathbf{t} &= H\mathbf{v} - K_{\mathbf{t}}(a_4\mathbf{v}) + K_{\mathbf{v}}(-2a_4\mathbf{t}) = H\mathbf{v} - a_4^2\mathbf{v} - 2a_4^2\mathbf{v} = (H - 3a_4^2)\mathbf{v}, \\ \hat{R}(\mathbf{t},\mathbf{w})\mathbf{t} &= H\mathbf{w} - K_{\mathbf{t}}(a_4\mathbf{w}) + K_{\mathbf{w}}(-2a_4\mathbf{t}) = H\mathbf{w} - a_4^2\mathbf{w} - 2a_4^2\mathbf{w} = (H - 3a_4^2)\mathbf{w}, \\ \hat{R}(\mathbf{t},\mathbf{v})\mathbf{w} &= -K_{\mathbf{t}}(-a_6\mathbf{w}) + K_{\mathbf{v}}(a_4\mathbf{w}) = 0. \end{aligned}$$

From this it immediately follows that

$$\widehat{\mathrm{Ric}}(\mathbf{t},\mathbf{t}) = -2(H - 3a_4^2)$$

and

$$\widehat{\mathrm{Ric}}(\mathbf{t},\mathbf{w}) = 0.$$

The other equations follow by similar computations. ∎

[1] See Appendix or `http://www.math.tu-berlin.de/~schar/IndefSym_typ234.html`.

We want to show that the basis we have constructed, at each point p, can be extended differentiably to a neighborhood of the point p such that, at every point, K with respect to the frame $\{T, V, W\}$ has the previously described form.

Lemma 2. *Let M^3 be an affine hypersphere in $\mathbb{R}^4$ which admits a pointwise* $\mathrm{SO}(2)$*-, S_3- or $\mathbb{Z}_3$-symmetry. Let $p \in M$. Then there exists an orthonormal frame $\{T, V, W\}$ defined in a neighborhood of the point p such that K is given by:*

$$\begin{aligned} &K(T,T) = -2a_4T, && K(T,V) = a_4V, && K(T,W) = a_4W,\\ &K(V,V) = -a_4T + a_6V, && K(V,W) = -a_6W, && K(W,W) = -a_4T - a_6V, \end{aligned}$$

where $a_4 > 0$ and $a_6 = 0$ in case of $\mathrm{SO}(2)$*-symmetry, $a_4 = 0$ and $a_6 > 0$ in case of S_3-symmetry, and $a_4 > 0$ and $a_6 > 0$ in case of $\mathbb{Z}_3$-symmetry.*

Proof. First we want to show that at every point the vector $\mathbf{t}$ is uniquely defined (up to sign) and differentiable. We introduce a symmetric operator $\hat{A}$ by:

$$\widehat{\mathrm{Ric}}(Y, Z) = h(\hat{A}Y, Z).$$

Clearly $\hat{A}$ is a differentiable operator on M. Since $2(H - 3a_4^2) \neq 2(H - a_4^2 + a_6^2)$, the operator has two distinct eigenvalues. A standard result then implies that the eigen distributions are differentiable. We take T a local unit vector field spanning the 1-dimensional eigen distribution, and local orthonormal vector fields $\tilde{V}$ and $\tilde{W}$ spanning the second eigen distribution. If $a_6 = 0$, we can take $V = \tilde{V}$ and $W = \tilde{W}$.

As T is (up to sign) uniquely determined, for $a_6 \neq 0$ there exist differentiable functions a_4, c_6 and c_7, $c_6^2 + c_7^2 \neq 0$, such that

$$\begin{aligned} &K(T,T) = -2a_4T, && K(\tilde{V},\tilde{V}) = -a_4T + c_6\tilde{V} + c_7\tilde{W},\\ &K(T,\tilde{V}) = a_4\tilde{V}, && K(\tilde{V},\tilde{W}) = c_7\tilde{V} - c_6\tilde{W},\\ &K(T,\tilde{W}) = a_4\tilde{W}, && K(\tilde{W},\tilde{W}) = -a_4T - c_6\tilde{V} - c_7\tilde{W}. \end{aligned}$$

As we have shown in [13], in the proof of Theorem 2 (Case 2), we can always rotate $\tilde{V}$ and $\tilde{W}$ such that we obtain the desired frame. ∎

Remark 1. It actually follows from the proof of the previous lemma that the vector field T is (up to sign) invariantly defined on M, and therefore the function a_4, too. Since the Pick invariant (8) $J = \frac{1}{3}(-5a_4^2 + 2a_6^2)$, the function a_6 also is invariantly defined on the affine hypersphere M^3.

4 Gauss and Codazzi for pointwise SO(2)-, S_3- or $\mathbb{Z}_3$-symmetry

In this section we always will work with the local frame constructed in the previous lemma. We denote the coefficients of the Levi-Civita connection with respect to this frame by:

$$\begin{aligned} &\widehat{\nabla}_T T = a_{12}V + a_{13}W, && \widehat{\nabla}_T V = a_{12}T - b_{13}W, && \widehat{\nabla}_T W = a_{13}T + b_{13}V,\\ &\widehat{\nabla}_V T = a_{22}V + a_{23}W, && \widehat{\nabla}_V V = a_{22}T - b_{23}W, && \widehat{\nabla}_V W = a_{23}T + b_{23}V,\\ &\widehat{\nabla}_W T = a_{32}V + a_{33}W, && \widehat{\nabla}_W V = a_{32}T - b_{33}W, && \widehat{\nabla}_W W = a_{33}T + b_{33}V. \end{aligned}$$

We will evaluate first the Codazzi and then the Gauss equations ((12) and (13)) to obtain more information.

Lemma 3. *Let M^3 be an affine hypersphere in $\mathbb{R}^4$ which admits a pointwise* $\mathrm{SO}(2)$*-, S_3- or $\mathbb{Z}_3$-symmetry and $\{T, V, W\}$ the corresponding ONB. If the symmetry group is*

SO(2), *then* $0 = a_{12} = a_{13} = a_{23} = a_{32}$, $a_{33} = a_{22}$ *and*
$T(a_4) = -4a_{22}a_4$, $0 = V(a_4) = W(a_4)$,

$\mathbf{S_3}$**,** *then* $0 = a_{12} = a_{13}$, $a_{23} = -3b_{13} = -a_{32}$, $a_{33} = a_{22}$ *and*
$T(a_6) = -a_{22}a_6$, $V(a_6) = 3b_{33}a_6$, $W(a_6) = -3b_{23}a_6$,

$\mathbb{Z}_3$ **and** $\mathbf{a_6^2 \neq 4a_4^2}$**,** *then* $0 = a_{12} = a_{13} = a_{23} = a_{32}$, $a_{33} = a_{22}$, $b_{13} = 0$,
$T(a_4) = -4a_{22}a_4$, $0 = V(a_4) = W(a_4)$, *and*
$T(a_6) = -a_{22}a_6$, $V(a_6) = 3b_{33}a_6$, $W(a_6) = -3b_{23}a_6$,

$\mathbb{Z}_3$ **and** $\mathbf{a_6 = 2a_4}$**,** *then* $a_{12} = 2a_{22} = -2a_{33} = -b_{33}$,
$a_{13} = -2a_{23} = -2a_{32} = b_{23}$, $b_{13} = 0$, *and*
$T(a_4) = 0$, $V(a_4) = -4a_{22}a_4$, $W(a_4) = 4a_{23}a_4$.

Proof. An evaluation of the Codazzi equations (12) with the help of the CAS Mathematica leads to the following equations (they relate to eq1–eq6 and eq8–eq9 in the Mathematica notebook):

$$V(a_4) = -2a_{12}a_4, \qquad T(a_4) = -4a_{22}a_4 + a_{12}a_6, \qquad 0 = 4a_{23}a_4 + a_{13}a_6, \tag{14}$$
$$W(a_4) = -2a_{13}a_4, \qquad 0 = 4a_{32}a_4 + a_{13}a_6, \qquad T(a_4) = -4a_{33}a_4 - a_{12}a_6, \tag{15}$$
$$T(a_6) - V(a_4) = 3a_{12}a_4 - a_{22}a_6, \qquad 0 = a_{13}a_4 + (a_{23} + 3b_{13})a_6, \tag{16}$$
$$W(a_4) = (a_{23} + a_{32})a_6, \qquad W(a_6) = (-a_{23} + 3a_{32})a_4 - b_{23}a_6,$$
$$V(a_6) = (-a_{22} + a_{33})a_4 + 3b_{33}a_6, \tag{17}$$
$$T(a_6) = -a_{12}a_4 - a_{33}a_6, \qquad W(a_4) = -3a_{13}a_4 + (-a_{32} + 3b_{13})a_6, \tag{18}$$
$$V(a_4) = (-a_{22} + a_{33})a_6, \qquad W(a_6) = (3a_{23} - a_{32})a_4 - 3b_{23}a_6, \tag{19}$$
$$0 = (a_{23} - a_{32})a_4, \tag{20}$$
$$W(a_4) = -a_{13}a_4 + (a_{32} - 3b_{13})a_6. \tag{21}$$

From the first equation of (15) (we will use the notation (15).1) and (17).1 resp. (14).3 and (15).2 we get:

$$0 = 2a_{13}a_4 + (a_{23} + a_{32})a_6, \qquad 0 = 2(a_{23} + a_{32})a_4 + a_{13}a_6. \tag{22}$$

From (19).1) and (14).1 resp. (14).2 and (15).3 we get:

$$0 = -2a_{12}a_4 + 2(a_{22} - a_{33})a_6, \qquad 0 = 2(-a_{22} + a_{33})a_4 + a_{12}a_6. \tag{23}$$

We consider first the case, that $\mathbf{a_6^2 \neq 4a_4^2}$. Then we obtain from the foregoing equations that $a_{13} = 0$, $a_{32} = -a_{23}$, $a_{12} = 0$ and $a_{33} = a_{22}$. Furthermore it follows from (14).1 that $V(a_4) = 0$, from (14).2 that $T(a_4) = -4a_{22}a_4$ and from (14).3 that $a_{23}a_4 = 0$. Equation (15).1 becomes $W(a_4) = 0$, equation (16).2 $T(a_6) = -a_{22}a_6$ and (16).3 $(a_{23} + 3b_{13})a_6 = 0$. Finally equation (17).2 resp. 3 gives $W(a_6) = -3b_{23}a_6$ and $V(a_6) = 3b_{33}a_6$.

In case of $SO(2)$-symmetry ($a_4 > 0$ and $a_6 = 0$) it follows that $a_{23} = 0$ and thus the statement of the theorem.

In case of S_3-symmetry ($a_4 = 0$ and $a_6 > 0$) it follows that $a_{23} = -3b_{13}$ and thus the statement of the theorem.

In case of $\mathbb{Z}_3$-symmetry ($a_4 > 0$ and $a_6 > 0$) it follows that $a_{23} = 0$ and $b_{13} = 0$ and thus the statement of the theorem.

In case that $a_6 = \pm 2a_4$ ($\neq 0$), we can choose V, W such that $\mathbf{a_6 = 2a_4}$. Now equations (20), (14).3 and (16).3 lead to $a_{23} = a_{32}$, $a_{13} = -2a_{23}$ and $b_{13} = 0$. A combination of (14).2 and (15).3 gives $a_{12} = (a_{22} - a_{33})$, and then by equations (16).2, (14).1 and (14).2 that $a_{33} = -a_{22}$. Thus $T(a_4) = 0$ by (14).2, $V(a_4) = -4a_{22}a_4$ by (14).1 and $W(a_4) = 4a_{22}a_4$ by (15).1. Finally (17).2 and (15).1 resp. (17).3 and (14).1 imply that $b_{23} = -a_{23}$ resp. $b_{33} = -a_{22}$. ■

An evaluation of the Gauss equations (13) with the help of the CAS Mathematica leads to the following:

Lemma 4. *Let M^3 be an affine hypersphere in $\mathbb{R}^4$ which admits a pointwise* SO(2)-, S_3- *or* $\mathbb{Z}_3$*-symmetry and $\{T, V, W\}$ the corresponding ONB. Then*

$$T(a_{22}) = -a_{22}^2 + a_{23}^2 + H - 3a_4^2, \tag{24}$$

$$T(a_{23}) = -2a_{22}a_{23}, \tag{25}$$

$$W(a_{22}) + V(a_{23}) = 0, \tag{26}$$

$$W(a_{23}) - V(a_{22}) = 0, \tag{27}$$

$$V(b_{13}) - T(b_{23}) = a_{22}b_{23} + (a_{23} + b_{13})b_{33}, \tag{28}$$

$$T(b_{33}) - W(b_{13}) = (a_{23} + b_{13})b_{23} - a_{22}b_{33}, \tag{29}$$

$$V(b_{33}) - W(b_{23}) = -a_{22}^2 - a_{23}^2 + 2a_{23}b_{13} + b_{23}^2 + b_{33}^2 + H + a_4^2 + 2a_6^2. \tag{30}$$

If the symmetry group is $\mathbb{Z}_3$, then $a_6^2 \neq 4a_4^2$.

Proof. The equations relate to eq11–eq13 and eq16 in the Mathematica notebook. If $a_6^2 = 4a_4^2 (\neq 0)$, then we obtain by equations eq11.1 and eq12.3 resp. eq15.3 and eq12.3 that $2V(a_{22}) = -4a_{22}^2 - H + 3a_4^2$ resp. $2W(a_{23}) = 4a_{23}^2 + H - 3a_4^2$, thus $V(a_{22}) - W(a_{23}) = -2a_{22}^2 - 2a_{23}^2 - H + 3a_4^2$. This gives a contradiction to eq13.3, namely $V(a_{22}) - W(a_{23}) = -2a_{22}^2 - 2a_{23}^2 - H - 9a_4^2$. ■

5 Pointwise $\mathbb{Z}_3$- or SO(2)-symmetry

The following methods only work in the case of $\mathbb{Z}_3$- or SO(2)-symmetry, therefore the case of S_3-symmetry will be considered elsewhere. As the vector field T is globally defined, we can define the distributions $L_1 = \operatorname{span}\{T\}$ and $L_2 = \operatorname{span}\{V, W\}$. In the following we will investigate these distributions. For the terminology we refer to [9].

Lemma 5. *The distribution L_1 is autoparallel (totally geodesic) with respect to $\widehat{\nabla}$.*

Proof. From $\widehat{\nabla}_T T = a_{12}V + a_{13}W = 0$ (cf. Lemmas 3 and 4) the claim follows immediately. ■

Lemma 6. *The distribution L_2 is spherical with mean curvature normal $U_2 = a_{22}T$.*

Proof. For $U_2 = a_{22}T \in L_1 = L_2^{\perp}$ we have $h(\widehat{\nabla}_{E_a} E_b, T) = h(E_a, E_b)h(U_2, T)$ for $E_a, E_b \in \{V, W\}$, and $h(\widehat{\nabla}_{E_a} U_2, T) = h(E_a(a_{22})T + a_{22}\widehat{\nabla}_{E_a} T, T) = 0$ (cf. Lemma 3 and (26), (27)). ■

Remark 2. a_{22} is independent of the choice of ONB $\{V, W\}$. It therefore is a globally defined function on M^3.

We introduce a coordinate function t by $\frac{\partial}{\partial t} := T$. Using the previous lemma, according to [12], we get:

Lemma 7. *(M^3, h) admits a warped product structure $M^3 = I \times_{e^f} N^2$ with $f : I \to \mathbb{R}$ satisfying*

$$\frac{\partial f}{\partial t} = a_{22}. \tag{31}$$

Proof. Proposition 3 in [12] gives the warped product structure with warping function $\lambda_2 : I \to \mathbb{R}$. If we introduce $f = \ln \lambda_2$, following the proof we see that $a_{22}T = U_2 = -\operatorname{grad}(\ln \lambda_2) = -\operatorname{grad} f$. ■

Lemma 8. *The curvature of N^2 is ${}^N K(N^2) = e^{2f}(H + 2a_6^2 + a_4^2 - a_{22}^2)$.*

Proof. From Proposition 2 in [12] we get the following relation between the curvature tensor $\hat{R}$ of the warped product M^3 and the curvature tensor $\tilde{R}$ of the usual product of pseudo-Riemannian manifolds ($X, Y, Z \in \mathcal{X}(M)$ resp. their appropriate projections):

$$\begin{aligned}\hat{R}(X,Y)Z = \tilde{R}(X,Y)Z &+ h(Y,Z)(\widehat{\nabla}_X U_2 - h(X,U_2)U_2) - h(\widehat{\nabla}_X U_2 - h(X,U_2)U_2, Z)Y\\ &- h(X,Z)(\widehat{\nabla}_Y U_2 - h(Y,U_2)U_2) + h(\widehat{\nabla}_Y U_2 - h(Y,U_2)U_2, Z)X\\ &+ h(U_2,U_2)(h(Y,Z)X - h(X,Z)Y).\end{aligned}$$

Now $\tilde{R}(X,Y)Z = {}^N\hat{R}(X,Y)Z$ for all $X, Y, Z \in TN^2$ and otherwise zero (cf. [11, page 89], Corollary 58) and $K(N^2) = K(V,W) = \frac{h(-\hat{R}(V,W)V,W)}{h(V,V)h(W,W)-h(V,W)^2}$ (cf. [11, page 77], the curvature tensor has the opposite sign). Since $h(X,Y) = e^{2f}\,{}^N h(X,Y)$ for $X, Y \in TN^2$, it follows that

$${}^N K(N^2) = e^{2f} h(-{}^N\hat{R}(V,W)V,W).$$

Finally we obtain by the Gauss equation (13) the last ingredient for the computation: $\hat{R}(V,W)V = -(H + 2a_6^2 + a_4^2)W$ (cf. the Mathematica notebook). ■

Summarized we have obtained the following structure equations (cf. (1), (2) and (3)), where $a_6 = 0$ in case of SO(2)-symmetry resp. $b_{13} = 0$ in case of $\mathbb{Z}_3$-symmetry:

$$D_T T = -2a_4 T - \xi, \tag{32}$$
$$D_T V = +a_4 V - b_{13} W, \tag{33}$$
$$D_T W = +b_{13} V + a_4 W, \tag{34}$$
$$D_V T = +(a_{22} + a_4)V, \tag{35}$$
$$D_W T = +(a_{22} + a_4)W, \tag{36}$$
$$D_V V = +a_6 V - b_{23} W + (a_{22} - a_4)T + \xi, \tag{37}$$
$$D_V W = +b_{23} V - a_6 W, \tag{38}$$
$$D_W V = -(b_{33} + a_6)W, \tag{39}$$
$$D_W W = +(b_{33} - a_6)V + (a_{22} - a_4)T + \xi, \tag{40}$$
$$D_X \xi = -HX. \tag{41}$$

The Codazzi and Gauss equations ((12) and (13)) have the form (cf. Lemmas 3 and 4):

$$T(a_4) = -4a_{22}a_4, \qquad 0 = V(a_4) = W(a_4), \tag{42}$$
$$T(a_6) = -a_{22}a_6, \qquad V(a_6) = 3b_{33}a_6, \qquad W(a_6) = -3b_{23}a_6, \tag{43}$$
$$T(a_{22}) = -a_{22}^2 + H - 3a_4^2, \qquad V(a_{22}) = 0, \qquad W(a_{22}) = 0, \tag{44}$$
$$V(b_{13}) - T(b_{23}) = a_{22}b_{23} + b_{13}b_{33}, \tag{45}$$
$$T(b_{33}) - W(b_{13}) = b_{13}b_{23} - a_{22}b_{33}, \tag{46}$$
$$V(b_{33}) - W(b_{23}) = -a_{22}^2 + b_{23}^2 + b_{33}^2 + H + a_4^2 + 2a_6^2, \tag{47}$$

where $a_6 = 0$ in case of SO(2)-symmetry resp. $b_{13} = 0$ in case of $\mathbb{Z}_3$-symmetry.

Our first goal is to find out how N^2 is immersed in $\mathbb{R}^4$, i.e. to find an immersion independent of t. A look at the structure equations (32)–(41) suggests to start with a linear combination of T and ξ.

We will solve the problem in two steps. First we look for a vector field X with $D_T X = \alpha X$ for some function α: We define $X := AT + \xi$ for some function A on M^3. Then $D_T X = \alpha X$ iff $\alpha = -A$ and $\frac{\partial}{\partial t} A = -A^2 + 2a_4 A + H$, and $A := a_{22} - a_4$ solves the latter differential equation.

Next we want to multiply X with some function β such that $D_T(\beta X) = 0$: We define a positive function β on $\mathbb{R}$ as the solution of the differential equation:

$$\tfrac{\partial}{\partial t}\beta = (a_{22} - a_4)\beta \tag{48}$$

with initial condition $\beta(t_0) > 0$. Then $D_T(\beta X) = 0$ and by (35), (41) and (36) we get (since β, a_{22} and a_4 only depend on t):

$$D_T(\beta((a_{22} - a_4)T + \xi)) = 0, \tag{49}$$

$$D_V(\beta((a_{22} - a_4)T + \xi)) = \beta(a_{22}^2 - a_4^2 - H)V, \tag{50}$$

$$D_W(\beta((a_{22} - a_4)T + \xi)) = \beta(a_{22}^2 - a_4^2 - H)W. \tag{51}$$

To obtain an immersion we need that $\nu := a_{22}^2 - a_4^2 - H$ vanishes nowhere, but we only get:

Lemma 9. *The function $\nu = a_{22}^2 - a_4^2 - H$ is globally defined, $\frac{\partial}{\partial t}(e^{2f}\nu) = 0$ and ν vanishes identically or nowhere on $\mathbb{R}$.*

Proof. Since $0 = \frac{\partial}{\partial t}{}^N K(N^2) = \frac{\partial}{\partial t}(e^{2f}(2a_6^2 - \nu))$ (Lemma 8) and $\frac{\partial}{\partial t}(e^{2f}2a_6^2) = 0$ (cf. (43) and (31)), we get that $\frac{\partial}{\partial t}(e^{2f}\nu) = 0$. Thus $\frac{\partial}{\partial t}\nu = -2(\frac{\partial}{\partial t}f)\nu = -2a_{22}\nu$. ■

5.1 The first case: $\nu \neq 0$ on M^3

We may, by translating f, i.e. by replacing N^2 with a homothetic copy of itself, assume that $e^{2f}\nu = \varepsilon_1$, where $\varepsilon_1 = \pm 1$.

Lemma 10. *$\varPhi := \beta((a_{22} - a_4)T + \xi)\colon M^3 \to \mathbb{R}^4$ induces a proper affine sphere structure, say $\tilde{\phi}$, mapping N^2 into a 3-dimensional linear subspace of $\mathbb{R}^4$. $\tilde{\phi}$ is part of a quadric iff $a_6 = 0$.*

Proof. By (50) and (51) we have $\varPhi_*(E_a) = \beta\nu E_a$ for $E_a \in \{V, W\}$. A further differentiation, using (37) (β and ν only depend on t), gives:

$$\begin{aligned} D_V\varPhi_*(V) &= \beta\nu D_V V = \beta\nu((a_{22} - a_4)T + a_6V - b_{23}W + \xi) \\ &= a_6\varPhi_*(V) - b_{23}\varPhi_*(W) + \nu\varPhi = a_6\varPhi_*(V) - b_{23}\varPhi_*(W) + \varepsilon_1 e^{-2f}\varPhi. \end{aligned}$$

Similarly, we obtain the other derivatives, using (38)–(40), thus:

$$D_V\varPhi_*(V) = a_6\varPhi_*(V) - b_{23}\varPhi_*(W) + e^{-2f}\varepsilon_1\varPhi, \tag{52}$$

$$D_V\varPhi_*(W) = b_{23}\varPhi_*(V) - a_6\varPhi_*(W), \tag{53}$$

$$D_W\varPhi_*(V) = -(b_{33} + a_6)\varPhi_*(W), \tag{54}$$

$$D_W\varPhi_*(W) = (b_{33} - a_6)\varPhi_*(V) + e^{-2f}\varepsilon_1\varPhi, \tag{55}$$

$$D_{E_a}\varPhi = \beta e^{-2f}\varepsilon_1 E_a. \tag{56}$$

The foliation at $f = f_0$ gives an immersion of N^2 to M^3, say π_{f_0}. Therefore, we can define an immersion of N^2 to $\mathbb{R}^4$ by $\tilde{\phi} := \varPhi \circ \pi_{f_0}$, whose structure equations are exactly the equations above when $f = f_0$. Hence, we know that $\tilde{\phi}$ maps N^2 into $\operatorname{span}\{\varPhi_*(V), \varPhi_*(W), \varPhi\}$, an affine hyperplane of $\mathbb{R}^4$ and $\frac{\partial}{\partial t}\varPhi = 0$ implies $\varPhi(t, v, w) = \tilde{\phi}(v, w)$.

We can read off the coefficients of the difference tensor $K^{\tilde{\phi}}$ of $\tilde{\phi}$ (cf. (1) and (3)): $K^{\tilde{\phi}}(\tilde{V}, \tilde{V}) = a_6\tilde{V}$, $K^{\tilde{\phi}}(\tilde{V}, \tilde{W}) = -a_6\tilde{W}$,$K^{\tilde{\phi}}(\tilde{W}, \tilde{W}) = -a_6\tilde{V}$, and see that $\operatorname{trace}(K^{\tilde{\phi}})_X$ vanishes. The affine metric introduced by this immersion corresponds with the metric on N^2. Thus $\varepsilon_1\tilde{\phi}$ is the affine normal of $\tilde{\phi}$ and $\tilde{\phi}$ is a proper affine sphere with mean curvature ε_1. Finally the vanishing of the difference tensor characterizes quadrics. ■

Our next goal is to find another linear combination of T and ξ, this time only depending on t. (Then we can express T in terms of ϕ and some function of t.)

Lemma 11. *Define $\delta := HT + (a_{22} + a_4)\xi$. Then there exist a constant vector $C \in \mathbb{R}^4$ and a function $a(t)$ such that*

$$\delta(t) = a(t)C.$$

Proof. Using (35) resp. (36) and (41) we obtain that $D_V\delta = 0 = D_W\delta$. Hence δ depends only on the variable t. Moreover, we get by (32), (44), (42) and (41) that

$$\begin{aligned}\frac{\partial}{\partial t}\delta &= D_T(HT + (a_{22} + a_4)\xi)\\ &= H(-2a_4T - \xi) + (-a_{22}^2 + H - 3a_4^2 - 4a_{22}a_4)\xi - (a_{22} + a_4)HT\\ &= -(3a_4 + a_{22})(HT + (a_{22} + a_4)\xi) = -(3a_4 + a_{22})\delta.\end{aligned}$$

This implies that there exists a constant vector C in $\mathbb{R}^4$ and a function $a(t)$ such that $\delta(t) = a(t)C$. ∎

Notice that for an improper affine hypersphere ($H = 0$) ξ is constant and parallel to C. Combining $\tilde{\phi}$ and δ we obtain for T (cf. Lemmas 10 and 11) that

$$T(t, v, w) = -\frac{a}{\nu}C + \frac{1}{\beta\nu}(a_{22} + a_4)\tilde{\phi}(v, w). \tag{57}$$

In the following we will use for the partial derivatives the abbreviation $\varphi_x := \frac{\partial}{\partial x}\varphi$, $x = t, v, w$.

Lemma 12.

$$\varphi_t = -\frac{a}{\nu}C + \frac{\partial}{\partial t}\left(\frac{1}{\beta\nu}\right)\tilde{\phi}, \qquad \varphi_v = \frac{1}{\beta\nu}\tilde{\phi}_v, \qquad \varphi_w = \frac{1}{\beta\nu}\tilde{\phi}_w.$$

Proof. As by (48) and Lemma 9 $\frac{\partial}{\partial t}\frac{1}{\beta\nu} = \frac{1}{\beta\nu}(a_{22} + a_4)$, we obtain the equation for $\varphi_t = T$ by (57). The other equations follow from (50) and (51). ∎

It follows by the uniqueness theorem of first order differential equations and applying a translation that we can write

$$\varphi(t, v, w) = \tilde{a}(t)C + \frac{1}{\beta\nu}(t)\tilde{\phi}(v, w)$$

for a suitable function $\tilde{a}$ depending only on the variable t. Since C is transversal to the image of $\tilde{\phi}$ (cf. Lemmas 10 and 11, $\nu \not\equiv 0$), we obtain that after applying an equiaffine transformation we can write: $\varphi(t, v, w) = (\gamma_1(t), \gamma_2(t)\phi(v, w))$, in which $\tilde{\phi}(v, w) = (0, \phi(v, w))$. Thus we have proven the following:

Theorem 1. *Let M^3 be an indefinite affine hypersphere of $\mathbb{R}^4$ which admits a pointwise $\mathbb{Z}_3$- or $SO(2)$-symmetry. Let $a_{22}^2 - a_4^2 \neq H$ for some $p \in M^3$. Then M^3 is affine equivalent to*

$$\varphi : \ I \times N^2 \to \mathbb{R}^4 : \ (t, v, w) \mapsto (\gamma_1(t), \gamma_2(t)\phi(v, w)),$$

where $\phi : N^2 \to \mathbb{R}^3$ is a (positive definite) elliptic or hyperbolic affine sphere and $\gamma : I \to \mathbb{R}^2$ is a curve. Moreover, if M^3 admits a pointwise $SO(2)$-symmetry then N^2 is either an ellipsoid or a two-sheeted hyperboloid.

We want to investigate the conditions imposed on the curve γ. For this we compute the derivatives of φ:

$$\begin{aligned}
&\varphi_t = (\gamma_1', \gamma_2'\phi), && \varphi_v = (0, \gamma_2\phi_v), && \varphi_w = (0, \gamma_2\phi_w),\\
&\varphi_{tt} = (\gamma_1'', \gamma_2''\phi), && \varphi_{tv} = (0, \gamma_2'\phi_v), && \varphi_{tw} = (0, \gamma_2'\phi_w),\\
&\varphi_{vv} = (0, \gamma_2\phi_{vv}), && \varphi_{vw} = (0, \gamma_2\phi_{vw}), && \varphi_{ww} = (0, \gamma_2'\phi_{ww}).
\end{aligned} \tag{58}$$

Furthermore we have to distinguish if M^3 is proper ($H = \pm 1$) or improper ($H = 0$).

First we consider the case that M^3 is proper, i.e. $\xi = -H\varphi$. An easy computation shows that the condition that ξ is a transversal vector field, namely $0 \neq \det(\varphi_t, \varphi_v, \varphi_w, \xi) = -\gamma_2^2(\gamma_1\gamma_2' - \gamma_1'\gamma_2)\det(\phi_v, \phi_w, \phi)$, is equivalent to $\gamma_2 \neq 0$ and $\gamma_1\gamma_2' - \gamma_1'\gamma_2 \neq 0$. To check the condition that ξ is the Blaschke normal (cf. (4)), we need to compute the Blaschke metric h, using (1), (58), (52)–(55) and the notation $r, s \in \{v, w\}$ and g for the Blaschke metric of ϕ:

$$\begin{aligned}
&\varphi_{tt} = \cdots \varphi_t + \frac{\gamma_1'\gamma_2'' - \gamma_1''\gamma_2'}{H(\gamma_1\gamma_2' - \gamma_1'\gamma_2)}\xi, \qquad \varphi_{tr} = \mathrm{tang},\\
&\varphi_{rs} = \mathrm{tang} - \frac{\gamma_1'\gamma_2}{H(\gamma_1\gamma_2' - \gamma_1'\gamma_2)}\varepsilon_1 g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial s}\right)\xi.
\end{aligned}$$

We obtain that

$$\det h = h_{tt}(h_{vv}h_{ww} - h_{vw}^2) = \frac{\gamma_1'\gamma_2'' - \gamma_1''\gamma_2'}{H^3(\gamma_1\gamma_2' - \gamma_1'\gamma_2)^3}(\gamma_1')^2\gamma_2^2 \det g.$$

Thus

$$\gamma_2^4(\gamma_1\gamma_2' - \gamma_1'\gamma_2)^2 \det(\phi_v, \phi_w, \phi)^2 = \left|\frac{\gamma_1'\gamma_2'' - \gamma_1''\gamma_2'}{(\gamma_1\gamma_2' - \gamma_1'\gamma_2)^3}(\gamma_1')^2\gamma_2^2 \det g\right|$$

is equivalent to (4). Since ϕ is a definite proper affine sphere with normal $-\varepsilon_1\phi$, we can again use (4) to obtain

$$\xi = -H\varphi \iff \gamma_2^2|\gamma_1\gamma_2' - \gamma_1'\gamma_2|^5 = |\gamma_1'\gamma_2'' - \gamma_1''\gamma_2'|(\gamma_1')^2 \neq 0.$$

From the computations above (g is positive definite) also it follows that φ is indefinite iff either

$$\begin{aligned}
&H \operatorname{sign}(\gamma_1\gamma_2' - \gamma_1'\gamma_2) = \operatorname{sign}(\gamma_1'\gamma_2'' - \gamma_1''\gamma_2') = \operatorname{sign}(\gamma_1'\gamma_2\varepsilon_1) \qquad \text{or}\\
&-H \operatorname{sign}(\gamma_1\gamma_2' - \gamma_1'\gamma_2) = \operatorname{sign}(\gamma_1'\gamma_2'' - \gamma_1''\gamma_2') = \operatorname{sign}(\gamma_1'\gamma_2\varepsilon_1).
\end{aligned}$$

Next we consider the case that M^3 is improper, i.e. ξ is constant. By Lemma 11 ξ is parallel to C and thus transversal to ϕ. Hence we can apply an affine transformation to obtain $\xi = (1, 0, 0, 0)$. An easy computation shows that the condition that ξ is a transversal vector field, namely $0 \neq \det(\varphi_t, \varphi_v, \varphi_w, \xi) = -\gamma_2^2\gamma_2' \det(\phi_v, \phi_w, \phi)$, is equivalent to $\gamma_2 \neq 0$ and $\gamma_2' \neq 0$. To check the condition that ξ is the Blaschke normal (cf. (4)) we need to compute the Blaschke metric h, using (1), (58), (52)–(55) and the notation $r, s \in \{v, w\}$ and g for the Blaschke metric of ϕ:

$$\varphi_{tt} = \cdots \varphi_t - \frac{\gamma_1'\gamma_2'' - \gamma_1''\gamma_2'}{\gamma_2'}\xi, \qquad \varphi_{tr} = \mathrm{tang}, \qquad \varphi_{rs} = \mathrm{tang} + \frac{\gamma_1'\gamma_2}{\gamma_2'}\varepsilon_1 g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial s}\right)\xi.$$

We obtain that

$$\det h = h_{tt}(h_{vv}h_{ww} - h_{vw}^2) = -\frac{\gamma_1'\gamma_2'' - \gamma_1''\gamma_2'}{(\gamma_2')^3}(\gamma_1')^2\gamma_2^2 \det g.$$

Thus (4) is equivalent to

$$\gamma_2^4(\gamma_2')^2\det(\phi_v,\phi_w,\phi)^2 = \left|\frac{\gamma_1'\gamma_2''-\gamma_1''\gamma_2'}{(\gamma_2')^3}(\gamma_1')^2\gamma_2^2\det g\right|.$$

Since ϕ is a definite proper affine sphere with normal $-\varepsilon_1\phi$, we can again use (4) to obtain

$$\xi = (1,0,0,0) \Longleftrightarrow \gamma_2^2|\gamma_2'|^5 = |\gamma_1'\gamma_2''-\gamma_1''\gamma_2'|(\gamma_1')^2 \neq 0.$$

From the computations above also it follows that φ is indefinite iff either

$$-\operatorname{sign}(\gamma_2') = \operatorname{sign}(\gamma_1'\gamma_2''-\gamma_1''\gamma_2') = \operatorname{sign}(\gamma_1'\gamma_2\varepsilon_1) \qquad \text{or}$$
$$\operatorname{sign}(\gamma_2') = \operatorname{sign}(\gamma_1'\gamma_2''-\gamma_1''\gamma_2') = \operatorname{sign}(\gamma_1'\gamma_2\varepsilon_1).$$

So we have seen under which conditions we can construct a 3-dimensional indefinite affine hypersphere out of an affine sphere:

Theorem 2. *Let $\phi : N^2 \to \mathbb{R}^3$ be a positive definite elliptic or hyperbolic affine sphere (with mean curvature $\varepsilon_1 = \pm 1$), and let $\gamma = (\gamma_1,\gamma_2) : I \to \mathbb{R}^2$ be a curve. Define $\varphi : I \times N^2 \to \mathbb{R}^4$ by $\varphi(t,v,w) = (\gamma_1(t),\gamma_2(t)\phi(v,w))$.*

(i) If γ satisfies $\gamma_2^2|\gamma_1\gamma_2'-\gamma_1'\gamma_2|^5 = \operatorname{sign}(\gamma_1'\gamma_2\varepsilon_1)(\gamma_1'\gamma_2''-\gamma_1''\gamma_2')(\gamma_1')^2 \neq 0$, then φ defines a 3-dimensional indefinite proper affine hypersphere.

(ii) If γ satisfies $\gamma_2^2|\gamma_2'|^5 = \operatorname{sign}(\gamma_1'\gamma_2\varepsilon_1)(\gamma_1'\gamma_2''-\gamma_1''\gamma_2')(\gamma_1')^2 \neq 0$, then φ defines a 3-dimensional indefinite improper affine hypersphere.

Now we are ready to check the symmetries.

Theorem 3. *Let $\phi : N^2 \to \mathbb{R}^3$ be a positive definite elliptic or hyperbolic affine sphere (with mean curvature $\varepsilon_1 = \pm 1$), and let $\gamma = (\gamma_1,\gamma_2) : I \to \mathbb{R}^2$ be a curve such that $\varphi(t,v,w) = (\gamma_1(t),\gamma_2(t)\phi(v,w))$ defines a 3-dimensional indefinite affine hypersphere. Then $\varphi(N^2\times I)$ admits a pointwise $\mathbb{Z}_3$- or $SO(2)$-symmetry.*

Proof. We already have shown that φ defines a 3-dimensional indefinite proper resp. improper affine hypersphere. To prove the symmetry we need to compute K. By assumption, ϕ is an affine sphere with Blaschke normal $\xi^\phi = -\varepsilon_1\phi$. For the structure equations (1) we use the notation $\phi_{rs} = {}^\phi\Gamma_{rs}^u\phi_u - g_{rs}\varepsilon_1\phi$, $r,s,u \in \{v,w\}$. Furthermore we introduce the notation $\alpha = \gamma_1\gamma_2' - \gamma_1'\gamma_2$. Note that $\alpha' = \gamma_1\gamma_2'' - \gamma_1''\gamma_2$. If φ is proper, using (58), we get the structure equations (1) for φ:

$$\varphi_{tt} = \frac{\alpha'}{\alpha}\varphi_t + \frac{\gamma_1'\gamma_2''-\gamma_1''\gamma_2'}{H\alpha}\xi, \qquad \varphi_{tr} = \frac{\gamma_2'}{\gamma_2}\varphi_r,$$
$$\varphi_{rs} = {}^\phi\Gamma_{rs}^u\varphi_u - g_{rs}\varepsilon_1\frac{\gamma_1\gamma_2}{\alpha}\varphi_t - g_{rs}\varepsilon_1\frac{\gamma_1'\gamma_2}{H\alpha}\xi.$$

We compute K using (6) and obtain:

$$(\nabla_{\varphi_t}h)(\varphi_r,\varphi_s) = \left(\left(\frac{\gamma_1\gamma_2}{\alpha}\right)'\frac{\alpha}{\gamma_1\gamma_2} - 2\frac{\gamma_2'}{\gamma_2}\right)h(\varphi_r,\varphi_s),$$
$$(\nabla_{\varphi_r}h)(\varphi_t,\varphi_t) = 0,$$

implying that K_{φ_t} restricted to the space spanned by φ_v and φ_w is a multiple of the identity. Taking T in direction of φ_t, we see that φ_v and φ_w are orthogonal to T. Thus we can construct an ONB $\{T,V,W\}$ with V,W spanning $\operatorname{span}\{\varphi_v,\varphi_w\}$ such that $a_1 = 2a_4$, $a_2 = a_3 = a_5 = 0$. By the considerations in [13, Section 4] we see that φ admits a pointwise $\mathbb{Z}_3$- or $SO(2)$-symmetry. If φ is improper, the proof runs completely analogous. ■

5.2 The second case: $\nu \equiv 0$ and $H \neq 0$ on M^3

Next, we consider the case that $H = a_{22}^2 - a_4^2$ and $H \neq 0$ on M^3. It follows that $a_{22} \neq \pm a_4$ on M^3.

We already have seen that M^3 admits a warped product structure. The map Φ we have constructed in Lemma 10 will not define an immersion (cf. (50) and (51)). Anyhow, for a fixed point t_0, we get from (37)–(40), (50) and (51), using the notation $\tilde{\xi} = (a_{22} - a_4)T + \xi$:

$$D_V V = a_6 V - b_{23} W + \tilde{\xi}, \qquad D_V W = b_{23} V - a_6 W,$$
$$D_W V = -(b_{33} + a_6)W, \qquad D_W W = (b_{33} - a_6)V + \tilde{\xi}, \qquad D_{E_a}\tilde{\xi} = 0, \qquad E_a \in \{V, W\}.$$

Thus, if v and w are local coordinates which span the second distribution L_2, then we can interpret $\varphi(t_0, v, w)$ as a positive definite improper affine sphere in a 3-dimensional linear subspace.

Moreover, we see that this improper affine sphere is a paraboloid provided that $a_6(t_0, v, w)$ vanishes identically. From the differential equations (43) determining a_6, we see that this is the case exactly when a_6 vanishes identically, i.e. when M^3 admits a pointwise $SO(2)$-symmetry.

After applying a translation and a change of coordinates, we may assume that

$$\varphi(t_0, v, w) = (v, w, f(v, w), 0),$$

with affine normal $\tilde{\xi}(t_0, v, w) = (0, 0, 1, 0)$. To obtain T at t_0, we consider (35) and (36) and get that

$$D_{E_a}(T - (a_{22} + a_4)\varphi) = 0, \qquad E_a, E_b \in \{V, W\}.$$

Evaluating at $t = t_0$, this means that there exists a constant vector C, transversal to $\operatorname{span}\{V, W, \xi\}$, such that $T(t_0, v, w) = (a_{22}+a_4)(t_0)\varphi(t_0, v, w)+C$. Since $a_{22}+a_4 \neq 0$ everywhere, we can write:

$$T(t_0, v, w) = \alpha_1(v, w, f(v, w), \alpha_2), \tag{59}$$

where $\alpha_1, \alpha_2 \neq 0$ and we applied an equiaffine transformation so that $C = (0, 0, 0, \alpha_1\alpha_2)$. To obtain information about $D_T T$ we have that $D_T T = -2a_4 T - \xi$ (cf. (32)) and $\xi = \tilde{\xi} - (a_{22} - a_4)T$ by the definition of $\tilde{\xi}$. Also we know that $\tilde{\xi}(t_0, v, w) = (0, 0, 1, 0)$ and by (49)–(51) that $D_X(\beta\tilde{\xi}) = 0$, $X \in \mathcal{X}(M)$. Taking suitable initial conditions for the function β ($\beta(t_0) = 1$), we get that $\beta\tilde{\xi} = (0, 0, 1, 0)$ and finally the following vector valued differential equation:

$$D_T T = (a_{22} - 3a_4)T - \frac{1}{\beta}(0, 0, 1, 0).$$

Solving this differential equation, taking into account the initial conditions (59) at $t = t_0$, we get that there exist functions δ_1 and δ_2 depending only on t such that

$$T(t, u, v) = (\delta_1(t)v, \delta_1(t)w, \delta_1(t)(f(v, w) + \delta_2(t)), \alpha_2\delta_1(t)),$$

where $\delta_1(t_0) = \alpha_1$, $\delta_2(t_0) = 0$, $\delta_1'(t) = (a_{22} - 3a_4)\delta_1(t)$ and $\delta_2'(t) = \delta_1^{-1}(t)\beta^{-1}(t)$. As $T(t, v, w) = \frac{\partial\varphi}{\partial t}(t, v, w)$ and $\varphi(t_0, v, w) = (v, w, f(v, w), 0)$ it follows by integration that

$$\varphi(t, v, w) = (\gamma_1(t)v, \gamma_1(t)w, \gamma_1(t)f(v, w) + \gamma_2(t), \alpha_2(\gamma_1(t) - 1)),$$

where $\gamma_1'(t) = \delta_1(t)$, $\gamma_1(t_0) = 1$, $\gamma_2(t_0) = 0$ and $\gamma_2'(t) = \delta_1(t)\delta_2(t)$. After applying an affine transformation we have shown:

Theorem 4. *Let M^3 be an indefinite proper affine hypersphere of $\mathbb{R}^4$ which admits a pointwise $\mathbb{Z}_3$- or $SO(2)$-symmetry. Let $H = a_{22}^2 - a_4^2 (\neq 0)$ on M^3. Then M^3 is affine equivalent with*

$$\varphi: \ I \times N^2 \to \mathbb{R}^4: \ (t,v,w) \mapsto (\gamma_1(t)v, \gamma_1(t)w, \gamma_1(t)f(v,w) + \gamma_2(t), \gamma_1(t)),$$

where $\psi : N^2 \to \mathbb{R}^3 : (v,w) \mapsto (v,w,f(v,w))$ is a positive definite improper affine sphere with affine normal $(0,0,1)$ and $\gamma : I \to \mathbb{R}^2$ is a curve. Moreover, if M^3 admits a pointwise $SO(2)$-symmetry then N^2 is an elliptic paraboloid.

We want to investigate the conditions imposed on the curve γ. For this we compute the derivatives of φ:

$$\begin{aligned}
&\varphi_t = (\gamma_1' v, \gamma_1' w, \gamma_1' f(v,w) + \gamma_2', \gamma_1'),\\
&\varphi_v = (\gamma_1, 0, \gamma_1 f_v, 0), \qquad \varphi_w = (0, \gamma_1, \gamma_1 f_w, 0),\\
&\varphi_{tt} = (\gamma_1'' v, \gamma_1'' w, \gamma_1'' f(v,w) + \gamma_2'', \gamma_1''),\\
&\varphi_{tv} = \tfrac{\gamma_1'}{\gamma_1}\varphi_v, \qquad \varphi_{tw} = \tfrac{\gamma_1'}{\gamma_1}\varphi_w,\\
&\varphi_{vv} = (0,0,f_{vv}\gamma_1, 0), \qquad \varphi_{vw} = (0,0,\gamma_1 f_{vw}, 0), \qquad \varphi_{ww} = (0,0,\gamma_1 f_{ww}, 0).
\end{aligned} \tag{60}$$

M^3 is a proper hypersphere, i.e. $\xi = -H\varphi$. An easy computation shows that the condition that ξ is a transversal vector field, namely $0 \neq \det(\varphi_t, \varphi_v, \varphi_w, \xi) = -H\gamma_1^2(\gamma_1\gamma_2' - \gamma_1'\gamma_2)$, is equivalent to $\gamma_1 \neq 0$ and $\gamma_1\gamma_2' - \gamma_1'\gamma_2 \neq 0$. Since $(0,0,1,0) = \frac{\gamma_1}{\gamma_1\gamma_2' - \gamma_1'\gamma_2}\varphi_t - \frac{\gamma_1'}{\gamma_1\gamma_2' - \gamma_1'\gamma_2}\varphi$, we have the following structure equations:

$$\begin{aligned}
&\varphi_{tt} = \left(\frac{\gamma_1''}{\gamma_1'} + \frac{\gamma_1'\gamma_2'' - \gamma_1''\gamma_2'}{\gamma_1'} \frac{\gamma_1}{\gamma_1\gamma_2' - \gamma_1'\gamma_2}\right)\varphi_t + \frac{\gamma_1'\gamma_2'' - \gamma_1''\gamma_2'}{\gamma_1\gamma_2' - \gamma_1'\gamma_2}\frac{1}{H}\xi,\\
&\varphi_{tr} = \frac{\gamma_1'}{\gamma_1}\varphi_r, \qquad \varphi_{rs} = \frac{\gamma_1^2}{\gamma_1\gamma_2' - \gamma_1'\gamma_2} f_{rs}\varphi_t + \frac{\gamma_1\gamma_1'}{\gamma_1\gamma_2' - \gamma_1'\gamma_2} f_{rs}\frac{1}{H}\xi.
\end{aligned} \tag{61}$$

We obtain:

$$\det h = h_{tt}(h_{vv}h_{ww} - h_{vw}^2) = \frac{\gamma_1'\gamma_2'' - \gamma_1''\gamma_2'}{H^3(\gamma_1\gamma_2' - \gamma_1'\gamma_2)^3}\gamma_1^2(\gamma_1')^2(f_{vv}f_{ww} - f_{vw}^2).$$

Since ψ is a positive definite improper affine sphere with affine normal $(0,0,1)$, we get by (4) that $f_{vv}f_{ww} - f_{vw}^2 = 1$. Now (4) (for ξ) is equivalent to

$$\gamma_1^4(\gamma_1\gamma_2' - \gamma_1'\gamma_2)^2 = \left|\frac{\gamma_1'\gamma_2'' - \gamma_1''\gamma_2'}{(\gamma_1\gamma_2' - \gamma_1'\gamma_2)^3}\right|\gamma_1^2(\gamma_1')^2.$$

It follows that

$$\xi = -H\varphi \Longleftrightarrow \gamma_1^2|\gamma_1\gamma_2' - \gamma_1'\gamma_2|^5 = |\gamma_1'\gamma_2'' - \gamma_1''\gamma_2'|(\gamma_1')^2 \neq 0.$$

From the computations above also it follows that φ is indefinite iff either

$$\begin{aligned}
&\operatorname{sign}(\gamma_1'\gamma_2'' - \gamma_1''\gamma_2') = \operatorname{sign}(H(\gamma_1\gamma_2' - \gamma_1'\gamma_2)) = -\operatorname{sign}(\gamma_1\gamma_1') \qquad \text{or}\\
&\operatorname{sign}(\gamma_1'\gamma_2'' - \gamma_1''\gamma_2') = -\operatorname{sign}(H(\gamma_1\gamma_2' - \gamma_1'\gamma_2)) = -\operatorname{sign}(\gamma_1\gamma_1').
\end{aligned}$$

So we have seen under which conditions we can construct a 3-dimensional indefinite affine hypersphere out of an affine sphere:

Theorem 5. *Let $\psi : N^2 \to \mathbb{R}^3 : (v,w) \mapsto (v,w,f(v,w))$ be a positive definite improper affine sphere with affine normal $(0,0,1)$, and let $\gamma : I \to \mathbb{R}^2$ be a curve. Define $\varphi : I \times N^2 \to \mathbb{R}^4$ by $\varphi(t,v,w) = (\gamma_1(t)v, \gamma_1(t)w, \gamma_1(t)f(v,w) + \gamma_2(t), \gamma_1(t))$. If $\gamma = (\gamma_1, \gamma_2)$ satisfies $\gamma_1^2|\gamma_1\gamma_2' - \gamma_1'\gamma_2|^5 = -\operatorname{sign}(\gamma_1\gamma_1')(\gamma_1'\gamma_2'' - \gamma_1''\gamma_2')(\gamma_1')^2 \neq 0$, then φ defines a 3-dimensional indefinite proper affine hypersphere.*

Now we are ready to check the symmetries.

Theorem 6. *Let $\psi : N^2 \to \mathbb{R}^3 : (v,w) \mapsto (v,w,f(v,w))$ be a positive definite improper affine sphere with affine normal $(0,0,1)$, and let $\gamma : I \to \mathbb{R}^2$ be a curve such that $\varphi(t,v,w) = (\gamma_1(t)v, \gamma_1(t)w, \gamma_1(t)f(v,w) + \gamma_2(t), \gamma_1(t))$ defines a 3-dimensional indefinite proper affine hypersphere. Then $\varphi(N^2 \times I)$ admits a pointwise $\mathbb{Z}_3$- or $SO(2)$-symmetry.*

Proof. We already have shown that φ defines a 3-dimensional indefinite proper affine hypersphere with affine normal $\xi = -H\varphi$. To prove the symmetry we need to compute K. We get the induced connection and the affine metric from the structure equations (61). We compute K using (6) and obtain:

$$
\begin{aligned}
&(\nabla_{\varphi_t} h)(\varphi_r, \varphi_s) = \left(\frac{\partial}{\partial t} \ln\left(\frac{\gamma_1\gamma_1'}{\gamma_1\gamma_2' - \gamma_1'\gamma_2}\right) - 2\frac{\gamma_1'}{\gamma_1}\right) h(\varphi_r, \varphi_s),\\
&(\nabla_{\varphi_r} h)(\varphi_t, \varphi_t) = 0,
\end{aligned}
$$

implying that K_{φ_t} restricted to the space spanned by φ_v and φ_w is a multiple of the identity. Taking T in direction of φ_t, we see that φ_v and φ_w are orthogonal to T. Thus we can construct an ONB $\{T, V, W\}$ with V, W spanning $\operatorname{span}\{\varphi_v, \varphi_w\}$ such that $a_1 = 2a_4$, $a_2 = a_3 = a_5 = 0$. By the considerations in [13, Section 4] we see that φ admits a pointwise $\mathbb{Z}_3$- or $SO(2)$-symmetry. ∎

5.3 The third case: $\nu \equiv 0$ and $H = 0$ on M^3

The final cases now are that $\nu \equiv 0$ and $H = 0$ on the whole of M^3 and hence $a_{22} = \pm a_4$.

First we consider the case that $a_{22} = a_4 =: a > 0$. Again we use that M^3 admits a warped product structure and we fix a parameter t_0. At the point t_0, we have by (37)–(41):

$$
\begin{aligned}
&D_V V = +a_6 V - b_{23} W + \xi,\\
&D_V W = +b_{23} V - a_6 W,\\
&D_W V = -(b_{33} + a_6) W,\\
&D_W W = +(b_{33} - a_6) V + \xi,\\
&D_X \xi = 0.
\end{aligned}
$$

Thus, if v and w are local coordinates which span the second distribution L_2, then we can interpret $\varphi(t_0, v, w)$ as a positive definite improper affine sphere in a 3-dimensional linear subspace.

Moreover, we see that this improper affine sphere is a paraboloid provided that $a_6(t_0, v, w)$ vanishes identically. From the differential equations (43) determining a_6, we see that this is the case exactly when a_6 vanishes identically, i.e. when M^3 admits a pointwise $SO(2)$-symmetry.

After applying an affine transformation and a change of coordinates, we may assume that

$$\varphi(t_0, v, w) = (v, w, f(v,w), 0), \tag{62}$$

with affine normal $\xi(t_0, v, w) = (0,0,1,0)$, actually

$$\xi(t, v, w) = (0,0,1,0)$$

(ξ is constant on M^3 by assumption). Furthermore we obtain by (35) and (36), that $D_U T = 2aU$ for all $U \in L_2$. We define $\delta := T - 2a\varphi$, which is transversal to $\operatorname{span}\{V, W, \xi\}$. Since a is independent of v and w (cf. (42)), $D_U \delta = 0$, and we can assume that

$$T(t_0, v, w) - 2a(t_0)\varphi(t_0, v, w) = (0,0,0,1). \tag{63}$$

We can integrate (42) ($T(a) = -4a^2$) and we take $a = \frac{1}{4t}$, $t > 0$. Thus (32) becomes $D_T T = -\frac{1}{2t}T - \xi$ and we obtain the following linear second order ordinary differential equation:

$$\frac{\partial^2}{\partial t^2}\varphi + \frac{1}{2t}\frac{\partial}{\partial t}\varphi = -\xi. \tag{64}$$

The general solution is $\varphi(t,v,w) = -\frac{t^2}{3}\xi + 2\sqrt{t}A(v,w) + B(v,w)$. The initial conditions (62) and (63) imply that $A(v,w) = \left(\frac{v}{2\sqrt{t_0}}, \frac{w}{2\sqrt{t_0}}, \frac{f(v,w)}{2\sqrt{t_0}} + \frac{2}{3}t_0^{3/2}, \sqrt{t_0}\right)$ and $B(v,w) = (0,0,-t_0^2,-2t_0)$. Obviously we can translate B to zero. Furthermore we can translate the affine sphere and apply an affine transformation to obtain $A(v,w) = \frac{1}{2\sqrt{t_0}}(v,w,f(v,w),1)$. After a change of coordinates we get:

$$\varphi(t,v,w) = (tv, tw, tf(v,w) - ct^4, t), \qquad c,t > 0. \tag{65}$$

Next we consider the case that $-a_{22} = a_4 =: a > 0$. Again we use that M^3 admits a warped product structure and we fix a parameter t_0. A look at (37)–(41) suggests to define $\tilde{\xi} = -2aT + \xi$, then we get at the point t_0:

$$\begin{aligned}
D_V V &= +a_6 V - b_{23} W + \tilde{\xi},\\
D_V W &= +b_{23} V - a_6 W,\\
D_W V &= -(b_{33} + a_6) W,\\
D_W W &= +(b_{33} - a_6) V + \tilde{\xi},\\
D_U \tilde{\xi} &= 0.
\end{aligned}$$

Thus, if v and w are local coordinates which span the second distribution L_2, then we can interpret $\varphi(t_0,v,w)$ as a positive definite improper affine sphere in a 3-dimensional linear subspace.

Moreover, we see that this improper affine sphere is a paraboloid provided that $a_6(t_0,v,w)$ vanishes identically. From the differential equations (43) determining a_6, we see that this is the case exactly when a_6 vanishes identically, i.e. when M^3 admits a pointwise $SO(2)$-symmetry.

After applying an affine transformation and a change of coordinates, we may assume that

$$\varphi(t_0,v,w) = (v,w,f(v,w),0), \tag{66}$$

with affine normal

$$\tilde{\xi}(t_0,v,w) = (0,0,1,0). \tag{67}$$

We have considered $\tilde{\xi}$ before. We can solve (48) ($\frac{\partial}{\partial t}\beta = -2a\beta$) explicitly by $\beta = c\frac{1}{\sqrt{a}}$ (cf. (42)) and get by (49)–(51) that $D_X(\frac{1}{\sqrt{a}}\tilde{\xi}) = 0$. Thus $\frac{1}{\sqrt{a}}(-2aT + \xi) =: C$ for a constant vector C, i.e. $T = -\frac{1}{2a}(\sqrt{a}C - \xi)$. Notice that by (41) ξ is a constant vector, too. We can choose $a = \frac{1}{4|t|}$, $t < 0$ (cf. (42)), and we obtain the ordinary differential equation:

$$\frac{\partial}{\partial t}\varphi = -\sqrt{|t|}C - 2t\xi, \qquad t < 0. \tag{68}$$

The solution (after a translation) with respect to the initial condition (66) is $\varphi(t,v,w) = \frac{2}{3}|t|^{\frac{3}{2}}C - t^2\xi + (v,w,f(v,w),0)$. Notice that C is a multiple of $\tilde{\xi}$ and hence by (67) a constant multiple of $(0,0,1,0)$. Furthermore ξ is transversal to the space spanned by $\varphi(t_0,v,w)$. So we get after an affine transformation and a change of coordinates:

$$\varphi(t,v,w) = (v,w,f(v,w) + ct^3, t^4), \qquad c,t > 0. \tag{69}$$

Combining both results (65) and (69) we have:

Theorem 7. *Let M^3 be an indefinite improper affine hypersphere of $\mathbb{R}^4$ which admits a pointwise $\mathbb{Z}_3$- or $SO(2)$-symmetry. Let $a_{22}^2 = a_4^2$ on M^3. Then M^3 is affine equivalent with either*

$$\varphi : \ I \times N^2 \to \mathbb{R}^4 : \ (t, v, w) \mapsto (tv, tw, tf(v, w) - ct^4, t), \qquad (a_{22} = a_4) \qquad or$$
$$\varphi : \ I \times N^2 \to \mathbb{R}^4 : \ (t, v, w) \mapsto (v, w, f(v, w) + ct^3, t^4), \qquad (-a_{22} = a_4),$$

where $\psi : N^2 \to \mathbb{R}^3 : (v, w) \mapsto (v, w, f(v, w))$ is a positive definite improper affine sphere with affine normal $(0, 0, 1)$ and $c, t \in \mathbb{R}^+$. Moreover, if M^3 admits a pointwise $SO(2)$-symmetry then N^2 is an elliptic paraboloid.

The computations for the converse statement can be done completely analogous to the previous cases, they even are simpler (the curve is given parametrized).

Theorem 8. *Let $\psi : N^2 \to \mathbb{R}^3 : (v, w) \mapsto (v, w, f(v, w))$ be a positive definite improper affine sphere with affine normal $(0, 0, 1)$. Define $\varphi(t, v, w) = (tv, tw, tf(v, w) - ct^4, t)$ or $\varphi(t, v, w) = (v, w, f(v, w) + ct^3, t^4)$, where $c, t \in \mathbb{R}^+$. Then φ defines a 3-dimensional indefinite improper affine hypersphere, which admits a pointwise $\mathbb{Z}_3$- or $SO(2)$-symmetry.*

6 Pointwise SO(1, 1)-symmetry

Let M^3 be a hypersphere admitting a SO(1, 1)-symmetry. We only state the classification results. The proofs are done quite similar, using a lightvector-frame instead of an orthonormal one, and will appear elsewhere. We denote a lightvector-frame by $\{E, V, F\}$, where E and F are lightvectors and V is spacelike (cf. [13]).

Lemma 13. *Let M^3 be an affine hypersphere in $\mathbb{R}^4$ which admits a pointwise* SO(1, 1)*-symmetry. Let $p \in M$. Then there exists a lightvector-frame $\{E, V, F\}$ defined in a neighborhood of the point p and a positive function b_4 such that K is given by:*

$$K(V, V) = -2b_4 V, \qquad K(V, E) = b_4 E, \qquad K(V, F) = b_4 F,$$
$$K(E, E) = 0, \qquad K(E, F) = b_4 V, \qquad K(F, F) = 0.$$

In the following we denote the coefficients of the Levi-Civita connection with respect to this frame by:

$$\widehat{\nabla}_E E = a_{11} E + b_{11} V, \qquad \widehat{\nabla}_E V = a_{12} E - b_{11} F, \qquad \widehat{\nabla}_E F = -a_{12} V - a_{11} F,$$
$$\widehat{\nabla}_V E = a_{21} E + b_{21} V, \qquad \widehat{\nabla}_V V = a_{22} E - b_{21} F, \qquad \widehat{\nabla}_V F = -a_{22} V - a_{21} F,$$
$$\widehat{\nabla}_F E = a_{31} E + b_{31} V, \qquad \widehat{\nabla}_F V = a_{32} E - b_{31} F, \qquad \widehat{\nabla}_F F = -a_{32} V - a_{31} F.$$

Similar as before, it turns out that the vector field V is globally defined, and we can define the distributions $L_1 = \mathrm{span}\{V\}$ and $L_2 = \mathrm{span}\{E, F\}$. Again, L_1 is autoparallel with respect to $\widehat{\nabla}$, and L_2 is spherical with mean curvature normal $-a_{12}V$. We introduce a coordinate function v by $\frac{\partial}{\partial v} := V$.

Lemma 14. *The function $\nu = b_4^2 - a_{12}^2 - H$ is globally defined, $\frac{\partial}{\partial v}(e^{2f}\nu) = 0$ and ν vanishes identically or nowhere on $\mathbb{R}$.*

Again we have to distinguish three cases.

6.1 The first case: $\nu \neq 0$ on M^3

Theorem 9. *Let M^3 be an indefinite affine hypersphere of $\mathbb{R}^4$ which admits a pointwise $SO(1,1)$-symmetry. Let $b_4^2 - a_{12}^2 \neq H$ for some $p \in M^3$. Then M^3 is affine equivalent to*

$$\varphi: \ I \times N^2 \to \mathbb{R}^4: \ (v,x,y) \mapsto (\gamma_1(v), \gamma_2(v)\phi(x,y)),$$

where $\phi : N^2 \to \mathbb{R}^3$ is a one-sheeted hyperboloid and $\gamma : I \to \mathbb{R}^2$ is a curve.

Theorem 10. *Let $\phi : N^2 \to \mathbb{R}^3$ be a one-sheeted hyperboloid and let $\gamma : I \to \mathbb{R}^2$ be a curve. Define $\varphi : I \times N^2 \to \mathbb{R}^4$ by $\varphi(v,x,y) = (\gamma_1(v), \gamma_2(v)\phi(x,y))$.*

(i) If $\gamma = (\gamma_1, \gamma_2)$ satisfies $\gamma_2^2|\gamma_1\gamma_2' - \gamma_1'\gamma_2|^5 = |\gamma_1'\gamma_2'' - \gamma_1''\gamma_2'|(\gamma_1')^2 \neq 0$, then φ defines a 3-dimensional indefinite proper affine hypersphere.

(ii) If $\gamma = (\gamma_1, \gamma_2)$ satisfies $\gamma_2^2|\gamma_2'|^5 = |\gamma_1'\gamma_2'' - \gamma_1''\gamma_2'|(\gamma_1')^2 \neq 0$, then φ defines a 3-dimensional indefinite improper affine hypersphere.

Theorem 11. *Let $\phi : N^2 \to \mathbb{R}^3$ be a one-sheeted hyperboloid and let $\gamma : I \to \mathbb{R}^2$ be a curve, such that $\varphi(v,x,y) = (\gamma_1(v), \gamma_2(v)\phi(x,y))$ defines a 3-dimensional indefinite affine hypersphere. Then $\varphi(I \times N^2)$ admits a pointwise $SO(1,1)$-symmetry.*

6.2 The second case: $\nu \equiv 0$ and $H \neq 0$ on M^3

Theorem 12. *Let M^3 be an indefinite proper affine hypersphere of $\mathbb{R}^4$ which admits a pointwise $SO(1,1)$-symmetry. Let $H = b_4^2 - a_{12}^2 (\neq 0)$ on M^3. Then M^3 is affine equivalent with*

$$\varphi: \ I \times N^2 \to \mathbb{R}^4: \ (v,x,y) \mapsto (\gamma_1(v)x, \gamma_1(v)y, \gamma_1(v)f(x,y) + \gamma_2(v), \gamma_1(v)),$$

where $\psi : N^2 \to \mathbb{R}^3 : (x,y) \mapsto (x,y,f(x,y))$ is a hyperbolic paraboloid with affine normal $(0,0,1)$ and $\gamma : I \to \mathbb{R}^2$ is a curve.

Theorem 13. *Let $\psi : N^2 \to \mathbb{R}^3$ be a hyperbolic paraboloid with affine normal $(0,0,1)$, and let $\gamma : I \to \mathbb{R}^2$ be a curve. Define $\varphi : I \times N^2 \to \mathbb{R}^4$ by $\varphi(v,x,y) = (\gamma_1(v)x, \gamma_1(v)y, \gamma_1(v)f(x,y) + \gamma_2(v), \gamma_1(v))$. If $\gamma = (\gamma_1, \gamma_2)$ satisfies $\gamma_1^2|\gamma_1\gamma_2' - \gamma_1'\gamma_2|^5 = |\gamma_1'\gamma_2'' - \gamma_1''\gamma_2'|(\gamma_1')^2 \neq 0$, then φ defines a 3-dimensional indefinite proper affine hypersphere.*

Theorem 14. *Let $\psi : N^2 \to \mathbb{R}^3$ be a hyperbolic paraboloid with affine normal $(0,0,1)$, and let $\gamma : I \to \mathbb{R}^2$ be a curve, such that $\varphi(v,x,y) = (\gamma_1(v)x, \gamma_1(v)y, \gamma_1(v)f(x,y) + \gamma_2(v), \gamma_1(v))$ defines a 3-dimensional indefinite proper affine hypersphere. Then $\varphi(I \times N^2)$ admits a pointwise $SO(1,1)$-symmetry.*

6.3 The third case: $\nu \equiv 0$ and $H = 0$ on M^3

Theorem 15. *Let M^3 be an indefinite improper affine hypersphere of $\mathbb{R}^4$ which admits a pointwise $SO(1,1)$-symmetry. Let $a_{12}^2 = b_4^2$ on M^3. Then M^3 is affine equivalent with either*

$$\varphi: \ I \times N^2 \to \mathbb{R}^4: \ (v,x,y) \mapsto (vx, vy, vf(x,y) - cv^4, v), \quad (a_{12} = b_4) \qquad \textit{or}$$
$$\varphi: \ I \times N^2 \to \mathbb{R}^4: \ (v,x,y) \mapsto (x, y, f(x,y) + cv^3, v^4), \qquad (-a_{12} = b_4),$$

where $\psi : N^2 \to \mathbb{R}^3 : (v,w) \mapsto (v,w,f(v,w))$ is a hyperbolic paraboloid with affine normal $(0,0,1)$ and $c, t \in \mathbb{R}^+$.

Theorem 16. *Let $\psi : N^2 \to \mathbb{R}^3 : (v,w) \mapsto (v,w,f(v,w))$ be a hyperbolic paraboloid with affine normal $(0,0,1)$. Define $\varphi(t,v,w) = (tv, tw, tf(v,w) - ct^4, t)$ or $\varphi(t,v,w) = (v,w,f(v,w) + ct^3, t^4)$, where $t \in \mathbb{R}^+$, $c \neq 0$. Then φ defines a 3-dimensional indefinite improper affine hypersphere, which admits a pointwise $SO(1,1)$-symmetry.*

A Computations for pointwise SO(2)-, S_3- or $\mathbb{Z}_3$-symmetry

$e[1]$:=$\{1,0,0\}$; $e[2]$:=$\{0,1,0\}$; $e[3]$:=$\{0,0,1\}$;

ONB of SO(1,2): *e*[1]=T, *e*[2]=V, *e*[3]=W

Affine metric

h[y_,z_]:=-y[[1]]z[[1]]+Sum[y[[i]]z[[i]],{i,2,3}];

Difference tensor (r=a1, s=a4, u=a6)

K[y_,z_]:= Sum[y[[i]]z[[j]]k[i,j], {i,1,3}, {j,1,3}];
k[1,1]:={-r,0,0}; k[1,2]:={0,s,0}; k[1,3]:={0,0,r-s};
k[2,1]:={0,s,0}; k[2,2]:={-s,u,0}; k[2,3]:={0,0,-u};
k[3,1]:={0,0,r-s}; k[3,2]:={0,0,-u}; k[3,3]:={-(r-s),-u,0};

Ricci tensor, scalar curvature and Pick invariant

[Kx,Ky]z

LK[x_, y_, z_]:=$K[x,K[y,z]]-K[y,K[x,z]]$;
ListLK:={LK[$e[1],e[2],e[1]$], LK[$e[1],e[2],e[2]$], LK[$e[1],e[2],e[3]$],
LK[$e[1],e[3],e[1]$], LK[$e[1],e[3],e[2]$], LK[$e[1],e[3],e[3]$],
LK[$e[2],e[3],e[1]$], LK[$e[2],e[3],e[2]$], LK[$e[2],e[3],e[3]$]};
FullSimplify[ListLK]

Curvature tensor (of the Levi-Civita connection)

R[x_, y_, z_]:=$H(h[y,z]x-h[x,z]y)-K[x,K[y,z]]+K[y,K[x,z]]$;
ListR:={$R[e[1],e[2],e[1]]$, $R[e[1],e[2],e[2]]$, $R[e[1],e[2],e[3]]$,
$R[e[1],e[3],e[1]]$, $R[e[1],e[3],e[2]]$, $R[e[1],e[3],e[3]]$,
$R[e[2],e[3],e[1]]$, $R[e[2],e[3],e[2]]$, $R[e[2],e[3],e[3]]$};
Simplify[ListR]

Ricci tensor (of the Levi-Civita connection)

ric[x_, y_]:=Simplify[$(-h[R[e[1],x,y],e[1]]+h[R[e[2],x,y],e[2]]+h[R[e[3],x,y],e[3]])$];
Listric:={ric[$e[1],e[1]$], ric[$e[1],e[2]$], ric[$e[1],e[3]$], ric[$e[2],e[2]$], ric[$e[2],e[3]$], ric[$e[3],e[3]$]};
Simplify[Listric]

Scalar curvature (of the Levi-Civita connection)

sc:=1/6($-$ric[$e[1],e[1]$] + ric[$e[2],e[2]$] + ric[$e[3],e[3]$]);
Simplify[sc]

Pick invariant

P:=1/6($-(k[1,1][[1]])$^2 + $(k[2,2][[2]])$^2 + $(k[3,3][[3]])$^2 + 3(($k[1,1][[2]]$)^2 + ($k[1,1][[3]]$)^2 $-$ ($k[2,2][[1]]$)^2 $-$ ($k[3,3][[1]]$)^2 + ($k[2,2][[3]]$)^2 + ($k[3,3][[2]]$)^2) $-$ 6($k[1,2][[3]]$)^2);
Simplify[P]

Lemma 1

$r = 2s$; Simplify[Listric]
Simplify[P]

Lemma 8

$R[e[2], e[3], e[2]]$

Lemma 3

Levi-Civita connection (ONB)

$n[1,1]$:={0, a12, a13}; $n[1,2]$:={a12, 0, −b13}; $n[1,3]$:={a13, b13, 0};
$n[2,1]$:={0, a22, a23}; $n[2,2]$:={a22, 0, −b23}; $n[2,3]$:={a23, b23, 0};
$n[3,1]$:={0, a32, a33}; $n[3,2]$:={a32, 0, −b33}; $n[3,3]$:={a33, b33, 0};
Na[y_, z_]:=Sum[$y[[i]]z[[j]]n[i,j]$, {i, 1, 3}, {j, 1, 3}] + Sum[$y[[1]]$Dt[$z[[i]]$, f1]$e[i]$
+ $y[[2]]$Dt[$z[[i]]$, f2]$e[i]$ + $y[[3]]$Dt[$z[[i]]$, f3]$e[i]$, {i, 1, 3}];

Codazzi for K (affine hypersphere)

codazzi[x_, y_, z_]:=Na[x, $K[y,z]$] − K[Na[x,y], z] − K[y, Na[x,z]] − Na[y, $K[x,z]$] + K[Na[y,x], z] + K[x, Na[y,z]];
eq1:=Simplify[codazzi[$e[2], e[1], e[1]$]]; eq2:=Simplify[codazzi[$e[3], e[1], e[1]$]];
eq3:=Simplify[codazzi[$e[1], e[2], e[2]$]]; eq4:=Simplify[codazzi[$e[3], e[2], e[2]$]];
eq5:=Simplify[codazzi[$e[1], e[3], e[3]$]]; eq6:=Simplify[codazzi[$e[2], e[3], e[3]$]];
eq7:=Simplify[codazzi[$e[1], e[2], e[3]$]]; eq8:=Simplify[codazzi[$e[2], e[3], e[1]$]];
eq9:=Simplify[codazzi[$e[3], e[1], e[2]$]];
eq:={eq1, eq2, eq3, eq4, eq5, eq6, eq7, eq8, eq9}; eq

1. case: u^2≠4 s^2

conclusions from eq1,2,4:
Simplify[eq2[[1]] − 2eq4[[1]]]
Simplify[eq1[[3]] + eq2[[2]]]
a13 = 0; a32 = −a23;
eq
conclusions from eq1,2,6:
Simplify[−2eq6[[1]] + eq1[[1]]]
Simplify[eq1[[2]] − eq2[[3]]]
a12 = 0; a33 = a22;
eq
Clear[a13, a12, a32, a33]

2. case: u=2s≠0

$u = 2s$; eq
conclusions from eq8, eq1:
a32 = a23; a13 = −2a23; eq
conclusions from eq3:
b13 = 0; eq
Simplify[eq1[[2]] − eq2[[3]]]
a12 = −(a33 − a22); eq

Simplify[eq3[[2]] − 1/2eq1[[1]] + 2eq1[[2]]]
a33 = −a22; eq
It follows that T(a4)=0, V(a4)=-4a22 a4, W(a4)=4a23 a4.
Simplify[eq4[[2]] + eq2[[1]]]
Simplify[eq4[[3]] + eq1[[1]]]
b23 = −a23; b33 = −a22;

Lemma 4

Gauss for Levi-Civita connection (affine hypersphere)

gaussLC[x_, y_, z_]:=Na[x, Na[y, z]] − Na[y, Na[x, z]] − Na[Na[x, y] − Na[y, x], z] − Hh[y, z]x + Hh[x, z]y + K[x, K[y, z]] − K[y, K[x, z]];
eq11:=Simplify[gaussLC[e[1], e[2], e[2]]]; eq12:=Simplify[gaussLC[e[1], e[3], e[2]]];
eq13:=Simplify[gaussLC[e[2], e[3], e[2]]]; eq14:=Simplify[gaussLC[e[1], e[2], e[1]]];
eq15:=Simplify[gaussLC[e[1], e[3], e[1]]]; eq16:=Simplify[gaussLC[e[2], e[3], e[1]]];
eq17:=Simplify[gaussLC[e[1], e[2], e[3]]]; eq18:=Simplify[gaussLC[e[1], e[3], e[3]]];
eq19:=Simplify[gaussLC[e[2], e[3], e[3]]];
eqG:={eq11, eq12, eq13, eq14, eq15, eq16, eq17, eq18, eq19};

2. case: u=2s ≠0

eqG
Simplify[eq11[[1]] − eq12[[3]]]
Simplify[eq15[[3]] + eq12[[3]]]
Contradiction to eq13.3
Clear[b33, b23, a33, a12, b13, a32, a13, u]

1. case: u^2≠4 s^2

a13 = 0; a32 = −a23; a12 = 0; a33 = a22; eqG

Acknowledgements

Partially supported by the DFG-Project PI 158/4-5 ‘Geometric Problems and Special PDEs’.

References

[1] Bryant R.L., Second order families of special Lagrangian 3-folds, in Perspectives in Riemannian Geometry, *CRM Proc. Lecture Notes*, Vol. 40, Amer. Math. Soc., Providence, RI, 2006, 63–98, math.DG/0007128.

[2] Calabi E., Complete affine hyperspheres. I, in Convegno di Geometria Differenziale (INDAM, Rome, 1971), *Symposia Mathematica*, Vol. X, Academic Press, London, 1972, 19–38.

[3] Dillen F., Equivalence theorems in affine differential geometry, *Geom. Dedicata* **32** (1989), 81–92.

[4] Dillen F., Katsumi N., Vrancken L., Conjugate connections and Radon’s theorem in affine differential geometry, *Monatsh. Math.* **109** (1990), 221–235.

[5] Dillen F., Vrancken L., Calabi-type composition of affine spheres, *Differential Geom. Appl.* **4** (1994), 303–328.

[6] Hu Z., Li H.Z., Vrancken L, Characterizations of the Calabi product of hyperbolic affine hyperspheres, *Results Math.* **52** (2008), 299–314.

[7] Li A.M., Simon U., Zhao G.S., Global affine differential geometry of hypersurfaces, *de Gruyter Expositions in Mathematics*, Vol. 11, Walter de Gruyter & Co., Berlin, 1993.

[8] Lu Y., Scharlach C., Affine hypersurfaces admitting a pointwise symmetry, *Results Math.* **48** (2005), 275–300, math.DG/0510150.

[9] Nölker S., Isometric immersions of warped products, *Differential Geom. Appl.* **6** (1996), 1–30.

[10] Nomizu K., Sasaki T., Affine differential geometry. Geometry of affine immersions, *Cambridge Tracts in Mathematics*, Vol. 111, Cambridge University Press, Cambridge, 1994.

[11] O'Neill B., Semi-Riemannian geometry. With applications to relativity, *Pure and Applied Mathematics*, Vol. 103, Academic Press, Inc. [Harcourt Brace Jovanovich Publishers], New York, 1983.

[12] Ponge R., Reckziegel H., Twisted products in pseudo-Riemannian geometry, *Geom. Dedicata* **48** (1993), 15–25.

[13] Scharlach C., Indefinite affine hyperspheres admitting a pointwise symmetry. Part 1, *Beiträge Algebra Geom.* **48** (2007), 469–491, math.DG/0510531.

[14] Vrancken L., Special classes of three dimensional affine hyperspheres characterized by properties of their cubic form, in Contemporary Geometry and Related Topics, World Sci. Publ., River Edge, NJ, 2004, 431–459.