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# The Symmetrical $H_q$-Semiclassical Orthogonal Polynomials of Class One

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**Abstract.** We investigate the quadratic decomposition and duality to classify symmetrical $H_q$-semiclassical orthogonal $q$-polynomials of class one where $H_q$ is the Hahn's operator. For any canonical situation, the recurrence coefficients, the $q$-analog of the distributional equation of Pearson type, the moments and integral or discrete representations are given.

*Key words:* quadratic decomposition of symmetrical orthogonal polynomials; semiclassical form; integral representations; $q$-difference operator; $q$-series representations; the $q$-analog of the distributional equation of Pearson type

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## 1 Introduction

Orthogonal polynomials (OP) have been a subject of research in the last hundred and fifty years. The orthogonality considered in our contribution is related to a form (regular linear functional) [8, 24] and not only to a positive measure. By classical orthogonal polynomials sequences (OPS), we refer to Hermite, Laguerre, Bessel and Jacobi polynomials. In the literature, the extension of classical (OPS) can be done from different approaches such that the hypergeometric character [7, 8, 11, 18, 22] and the distributional equation of Pearson type [6, 8, 20, 29, 32]. A natural generalization of the classical character is the semiclassical one introduced by J.A. Shohat in [35]. This theory was developed by P. Maroni and extensively studied by P. Maroni and coworkers in the last decade [1, 24, 26, 28, 32]. Let $\Phi$ monic and $\Psi$ be two polynomials, $\deg\Phi = t \geq 0$, $\deg\Psi = p \geq 1$. We suppose that the pair $(\Phi,\Psi)$ is admissible, i.e., when $p = t-1$, writing $\Psi(x) = a_p x^p + \cdots$, then $a_p \neq n+1$, $n \in \mathbb{N}$. A form $u$ is called semiclassical when it is regular and satisfies the distributional equation of Pearson type

$$D(\Phi u) + \Psi u = 0, \tag{1.1}$$

where the pair$(\Phi,\Psi)$ is admissible and $D$ is the derivative operator. The corresponding monic orthogonal polynomials sequence (MOPS) $\{B_n\}_{n\geq 0}$ is called semiclassical. Moreover, if $u$ is semiclassical satisfying (1.1), the class of $u$, denoted $s$ is defined by

$$s := \min\big(\max(\deg\Phi - 2, \deg\Psi - 1)\big) \geq 0,$$

where the minimum is taken over all pairs $(\Phi,\Psi)$ satisfying (1.1). In particular, when $s = 0$ the classical case is recovered.

Symmetrical semiclassical forms of class one are well described in [1], see also [6]; there are three canonical situations:

1) The generalized Hermite form $\mathcal{H}(\mu)$ ($\mu \neq 0$, $\mu \neq -n-\frac{1}{2}$, $n \geq 0$) satisfying the distributional equation of Pearson type

$$D(x\mathcal{H}(\mu)) + \big(2x^2 - (2\mu+1)\big)\mathcal{H}(\mu) = 0. \tag{1.2}$$

2) The generalized Gegenbauer $\mathcal{G}(\alpha,\beta)$ ($\alpha \neq -n-1$, $\beta \neq -n-1$, $\beta \neq -\frac{1}{2}$, $\alpha+\beta \neq -n-1$, $n \geq 0$) satisfying the distributional equation of Pearson type

$$D\big(x(x^2-1)\mathcal{G}(\alpha,\beta)\big) + \big(-2(\alpha+\beta+2)x^2 + 2(\beta+1)\big)\mathcal{G}(\alpha,\beta) = 0. \tag{1.3}$$

For further properties of the generalized Hermite and the generalized Gegenbauer polynomials see [8, 15, 26].

3) The form $\mathcal{B}[\nu]$ of Bessel kind ($\nu \neq -n-1$, $n \geq 0$) [1, 31] satisfying the distributional equation of Pearson type

$$D\big(x^3\mathcal{B}[\nu]\big) - \big(2(\nu+1)x^2 + \tfrac{1}{2}\big)\mathcal{B}[\nu] = 0. \tag{1.4}$$

For an integral representation of $\mathcal{B}[\nu]$ and some additional features of the associated (MOPS) see [14].

Other families of semiclassical orthogonal polynomials of class greater than one were discovered by solving functional equations of the type $P(x)u = Q(x)v$, where $P$, $Q$ are two polynomials cunningly chosen and $u$, $v$ two linear forms [19, 26, 27, 34]. For other relevant works in the semiclassical case see [5, 23].

In [21], instead of the derivative operator, the $q$-difference one is used to establish the theory and characterizations of $H_q$-semiclassical orthogonal $q$-polynomials. Some examples of $H_q$-semiclassical orthogonal $q$-polynomials are given in [2, 13]. The $H_q$-classical case is exhaustively described in [20, 32]. Moreover, in [30] the symmetrical $D_\omega$-semiclassical orthogonal polynomials of class one are completely described by solving the system of their Laguerre–Freud equations where $D_w$ is the Hahn's operator.

So, the aim of this paper is to present the classification of the symmetrical $H_q$-semiclassical orthogonal $q$-polynomials of class one by investigating the quadratic operator $\sigma$, the $q$-analog of the distributional equation of Pearson type satisfied by the corresponding form and some $H_q$-classical situations (see Tables 1 and 2) in connection with our problem. Among the obtained canonical cases, three are well known: two symmetrical Brenke type (MOPS) [8, 9, 10] and a symmetrical case of the Al-Salam and Verma (MOPS) [2]. Also, $q$-analogues of $\mathcal{H}(\mu)$, $\mathcal{G}(\alpha,\beta)$ and $\mathcal{B}[\nu]$ appear. In [3, 33], the authors have established, up a dilation, a $q$-analogues of $\mathcal{H}(\mu)$ and $\mathcal{B}[\nu]$ using other methods. For any canonical case, we determine the recurrence coefficient, the $q$-analog of the distributional equation of Pearson type, the moments and a discrete measure or an integral representation.

## 2 Preliminary and first results

### 2.1 Preliminary and notations

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its topological dual. We denote by $\langle u, f\rangle$ the effect of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$ the moments of $u$. Moreover, a form (linear functional) u is called symmetric if $(u)_{2n+1} = 0$, $n \geq 0$.

Let us introduce some useful operations in $\mathcal{P}'$. For any form $u$, any polynomial $g$ and any $(a,b,c) \in (\mathbb{C}\setminus\{0\}) \times \mathbb{C}^2$, we let $H_q u$, $gu$, $h_a u$, $\tau_b u$, $(x-c)^{-1}u$ and $\delta_c$, be the forms defined by

**Table 1.** Canonical cases.

| $H_q$-classical linear form |
|---|
| case 1.1 $\qquad \mathcal{U}$ |
| $\widehat{\beta}_n = \{1-(1+q)q^n\}q^{n-1}$, $n \geq 0$,<br>$\widehat{\gamma}_{n+1} = (q^{n+1}-1)q^{3n}$, $n \geq 0$,<br>$H_q(x\mathcal{U}) - (q-1)^{-1}(x+1)\mathcal{U} = 0$,<br>$(\mathcal{U})_n = (-1)^n q^{\frac{1}{2}n(n-1)}$, $n \geq 0$,<br>$\mathcal{U} = \sum_{k=0}^{\infty} (-1)^k \frac{q^{-k^2} s(k)}{(q^{-1};q^{-1})_k} \delta_{-q^k}$, $q > 1$,<br>where $s(k) = \sum_{m=0}^{\infty} \frac{q^{-(\frac{1}{2}m(m+1)+km)}}{(q^{-1};q^{-1})_k} \varepsilon_{m+k}$, $\varepsilon_{2k} = (q-1)^k$, $k \geq 0$ and $\varepsilon_{2k+1} = 0$, $k \geq 0$ |
| case 1.2 $\qquad$ little $q$-Laguerre $\mathcal{L}(a,q)$ $\big(a \neq 0,\ a \neq q^{-n-1},\ n \geq 0\big)$ |
| $\widehat{\beta}_n = \{1+a-a(1+q)q^n\}q^n$, $n \geq 0$,<br>$\widehat{\gamma}_{n+1} = a(1-q^{n+1})(1-aq^{n+1})q^{2n+1}$, $n \geq 0$,<br>$H_q(x\mathcal{L}(a,q)) - (aq)^{-1}(q-1)^{-1}\{x-1+aq\}\mathcal{L}(a,q) = 0$,<br>$(\mathcal{L}(a,q))_n = (aq;q)_n$, $n \geq 0$,<br>$\mathcal{L}(a,q) = (aq;q)_\infty \sum_{k=0}^{\infty} \frac{(aq)^k}{(q;q)_k} \delta_{q^k}$, $0 < q < 1$, $0 < a < q^{-1}$,<br>$\langle \mathcal{L}(a,q), f \rangle = K \int_0^{q^{-1}} x^{\frac{\ln a}{\ln q}} (qx;q)_\infty f(x)dx$, $f \in \mathcal{P}$, $0 < q < 1$, $0 < a < q^{-1}$,<br>where $K^{-1} = q^{-\frac{\ln a}{\ln q}-1} \int_0^1 x^{\frac{\ln a}{\ln q}} (x;q)_\infty dx$,<br>$\mathcal{L}(a,q) = \frac{1}{(a;q^{-1})_\infty} \sum_{k=0}^{\infty} \frac{q^{-\frac{1}{2}k(k-1)}}{(q^{-1};q^{-1})_k} (-a)^k \delta_{q^k}$, $q > 1$, $a < 0$ |
| case 1.3 $\qquad$ Wall $\mathcal{W}(b,q)$ $\big(b \neq 0,\ b \neq q^{-n},\ n \geq 0\big)$ |
| $\widehat{\beta}_n = \{b+q-b(1+q)q^n\}q^n$, $n \geq 0$,<br>$\widehat{\gamma}_{n+1} = b(1-q^{n+1})(1-bq^n)q^{2n+2}$, $n \geq 0$,<br>$H_q(x\mathcal{W}(b,q)) - b^{-1}(q-1)^{-1}(q^{-1}x+b-1)\mathcal{W}(b,q) = 0$,<br>$(\mathcal{W}(b,q))_n = q^n(b;q)_n$, $n \geq 0$,<br>$\langle \mathcal{W}(b,q), f \rangle = \frac{(b;q)_\infty}{2} \sum_{k=0}^{\infty} \frac{b^k}{(q;q)_k} \langle \delta_{q^{1+k}}, f \rangle + \frac{K}{2} \int_0^1 x^{\frac{\ln b}{\ln q}-1} (x;q)_\infty f(x)dx$,<br>where $K^{-1} = \int_0^1 x^{\frac{\ln b}{\ln q}-1} (x;q)_\infty dx$, $f \in \mathcal{P}$, $0 < q < 1$, $0 < b < 1$,<br>$\langle \mathcal{W}(b,q), f \rangle = \frac{1}{(bq^{-1};q^{-1})_\infty} \sum_{k=0}^{\infty} \frac{q^{-\frac{1}{2}k(k+1)}}{(q^{-1};q^{-1})_k} (-b)^k \langle \delta_{q^{1+k}}, f \rangle$, $f \in \mathcal{P}$, $q > 1$, $b \neq q^{\pm k}$, $k \geq 0$ |

duality

$$\langle H_q u, f \rangle := -\langle u, H_q f \rangle, \qquad \langle gu, f \rangle := \langle u, gf \rangle, \qquad \langle h_a u, f \rangle := \langle u, h_a f \rangle, \qquad f \in \mathcal{P},$$
$$\langle \tau_b u, f \rangle := \langle u, \tau_{-b} f \rangle, \qquad \langle (x-c)^{-1} u, f \rangle := \langle u, \theta_c f \rangle, \qquad \langle \delta_c, f \rangle := f(c), \qquad f \in \mathcal{P},$$

where $(H_q f)(x) = \frac{f(qx)-f(x)}{(q-1)x}$, $q \in \widetilde{\mathbb{C}} := \big\{z \in \mathbb{C},\ z \neq 0, z^n \neq 1, n \geq 1\big\}$ [16, 18], $(h_a f)(x) = f(ax)$, $(\tau_{-b} f)(x) = f(x+b)$, $(\theta_c f)(x) = \frac{f(x)-f(c)}{x-c}$ [24] and it's easy to see that [20, 26]

$$H_q(fu) = (h_{q^{-1}} f) H_q u + q^{-1}(H_{q^{-1}} f)u, \qquad f \in \mathcal{P}, \qquad u \in \mathcal{P}', \tag{2.1}$$
$$(x-c)((x-c)^{-1}u) = u, \qquad (x-c)^{-1}((x-c)u) = u - (u)_0 \delta_c.$$

*Continuation of Table 1.*

| case 1.4 | Generalized $q^{-1}$-Laguerre $\mathcal{U}^{(\alpha)}(b,q)$ $\big(b\neq 0,\ b\neq q^{n+1+\alpha},\ n\geq 0\big)$ |
|---|---|
| $\widehat{\beta}_n=\{1-q^{-n-1}+q^{-1}(1-bq^{-n-\alpha})\}q^{2n+\alpha+1},\ n\geq 0,$<br>$\widehat{\gamma}_{n+1}=(1-q^{-n-1})(1-bq^{-n-1-\alpha})q^{4n+2\alpha+3},\ n\geq 0,$<br>$H_q(x\mathcal{U}^{(\alpha)}(b,q))+(q-1)^{-1}q^{-\alpha-1}(x+b-q^{\alpha+1})\mathcal{U}^{(\alpha)}(b,q)=0,$<br>$(\mathcal{U}^{(\alpha)}(b,q))_n=(-b)^n(b^{-1}q^{\alpha+1};q)_n,\ n\geq 0,$<br>$\langle\mathcal{U}^{(\alpha)}(b,q),f\rangle=(b^{-1}q^{\alpha+1};q)_\infty\sum_{k=0}^{\infty}\frac{(b^{-1}q^{\alpha+1})^k}{(q;q)_k}\langle\delta_{-bq^k},f\rangle,$<br>$f\in\mathcal{P},\ 0<q<1,\ b>q^{\alpha+1},\ \alpha\in\mathbb{R},$<br>$\langle\mathcal{U}^{(\alpha)}(b,q),f\rangle=\frac{1}{2(b^{-1}q^{\alpha};q^{-1})_\infty}\sum_{k=0}^{\infty}\frac{q^{-\frac{1}{2}k(k-1)}(-b^{-1}q^{\alpha})^k}{(q^{-1};q^{-1})_k}\langle\delta_{-bq^k},f\rangle$<br>$+\frac{K}{2}\int_0^\infty\frac{x^{\alpha-\frac{\ln b}{\ln q}}}{(-b^{-1}x;q^{-1})_\infty}f(x)dx,\ f\in\mathcal{P},\ q>1,\ q^{\alpha}<b<q^{\alpha+1},\ \alpha\in\mathbb{R},$<br>where $K^{-1}=\int_0^\infty\frac{x^{\alpha-\frac{\ln b}{\ln q}}}{(-b^{-1}x;q^{-1})_\infty}dx$ is given by (2.4) | |
| case 1.5 | Alternative $q$-Charlier $\mathcal{A}(a,q)$ $\big(a\neq 0,\ a\neq -q^{-n},\ n\geq 0\big)$ |
| $\widehat{\beta}_n=\frac{1+aq^{n-1}+aq^n-aq^{2n}}{(1+aq^{2n-1})(1+aq^{2n+1})}q^n,\ n\geq 0,$<br>$\widehat{\gamma}_{n+1}=aq^{3n+1}\frac{(1-q^{n+1})(1+aq^n)}{(1+aq^{2n})(1+aq^{2n+1})^2(1+aq^{2n+2})},\ n\geq 0,$<br>$H_q(x^2\mathcal{A}(a,q))-(aq)^{-1}(q-1)^{-1}\{(1+aq)x-1\}\mathcal{A}(a,q)=0,$<br>$(\mathcal{A}(a,q))_n=\frac{1}{(-aq;q)_n},\ n\geq 0,$<br>$\langle\mathcal{A}(a,q),f\rangle=\frac{1}{2(-aq;q)_\infty}\sum_{k=0}^{\infty}\frac{q^{\frac{1}{2}k(k+1)}a^k}{(q;q)_k}\langle\delta_{q^k},f\rangle+q^{\frac{1}{2}(\frac{\ln a}{\ln q}+\frac{1}{2})^2}\frac{(-a^{-1};q)_\infty}{2\sqrt{2\pi\ln q^{-1}}}$<br>$\times\int_0^\infty x^{\frac{\ln a}{\ln q}-\frac{1}{2}}(qx;q)_\infty\exp\left(-\frac{\ln^2 x}{2\ln q^{-1}}\right)f(x)dx,\ f\in\mathcal{P},\ 0<q<1,\ a>0$ | |

Now, we introduce the operator $\sigma:\mathcal{P}\longrightarrow\mathcal{P}$ defined by $(\sigma f)(x):=f(x^2)$ for all $f\in\mathcal{P}$. Consequently, we define $\sigma u$ by duality [8, 25]

$$\langle\sigma u,f\rangle:=\langle u,\sigma f\rangle,\qquad f\in\mathcal{P},\qquad u\in\mathcal{P}'.$$

We have the well known formula [25]

$$f(x)\sigma u=\sigma\big(f\big(x^2\big)u\big).\tag{2.2}$$

Let $\{B_n\}_{n\geq 0}$ be a sequence of monic polynomials with $\deg B_n=n$, $n\geq 0$, the form $u$ is called *regular* if we can associate with it a sequence of polynomials $\{B_n\}_{n\geq 0}$ such that $\langle u,B_mB_n\rangle=r_n\delta_{n,m}$, $n,m\geq 0$; $r_n\neq 0$, $n\geq 0$. The sequence $\{B_n\}_{n\geq 0}$ is then said orthogonal with respect to $u$. $\{B_n\}_{n\geq 0}$ is an (OPS) and it can be supposed (MOPS). The sequence $\{B_n\}_{n\geq 0}$ fulfills the recurrence relation

$$\begin{aligned}&B_0(x)=1,\qquad B_1(x)=x-\beta_0,\\&B_{n+2}(x)=(x-\beta_{n+1})B_{n+1}(x)-\gamma_{n+1}B_n(x),\qquad \gamma_{n+1}\neq 0,\qquad n\geq 0.\end{aligned}\tag{2.3}$$

When $u$ is regular, $\{B_n\}_{n\geq 0}$ is a symmetrical (MOPS) if and only if $\beta_n=0$, $n\geq 0$.

*Continuation of Table 1.*

| case 1.6 | little $q$-Jacobi $\mathcal{U}(a,b,q)$ $\big(ab \neq 0,\ a \neq q^{-n-1},\ b \neq q^{-n-1},\ ab \neq q^{-n},\ n \geq 0\big)$ |
|---|---|
| $\widehat{\beta}_n = \dfrac{(1+a)(1+abq^{2n+1}) - a(1+b)(1+q)q^n}{(1-abq^{2n})(1-abq^{2n+2})}q^n,\ n \geq 0,$ <br> $\widehat{\gamma}_{n+1} = aq^{2n+1}\dfrac{(1-q^{n+1})(1-aq^{n+1})(1-bq^{n+1})(1-abq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})^2(1-abq^{2n+3})},\ n \geq 0,$ <br> $H_q(x(x-b^{-1}q^{-1})\mathcal{U}(a,b,q)) + (abq^2(q-1))^{-1}\{(1-abq^2)x + aq - 1\}\mathcal{U}(a,b,q) = 0,$ <br> $(\mathcal{U}(a,b,q))_n = \dfrac{(aq;q)_n}{(abq^2;q)_n},\ n \geq 0,$ <br> $\langle \mathcal{U}(a,b,q), f\rangle = \dfrac{(aq;q)_\infty}{2(abq^2;q)_\infty}\displaystyle\sum_{k=0}^{\infty}\dfrac{(bq;q)_k}{(q;q)_k}(aq)^k\langle\delta_{q^k}, f\rangle + \dfrac{K}{2}\int_0^{q^{-1}} x^{\frac{\ln a}{\ln q}}\dfrac{(qx;q)_\infty}{(bqx;q)_\infty}f(x)dx,$ <br> $f \in \mathcal{P},\ 0<q<1,\ 0<a<q^{-1},\ b \in ]-\infty,1]\setminus\{0\}$, where $K^{-1} = \displaystyle\int_0^{q^{-1}} x^{\frac{\ln a}{\ln q}}\dfrac{(qx;q)_\infty}{(bqx;q)_\infty}dx,$ <br> $\langle \mathcal{U}(a,b,q), f\rangle = \dfrac{(a^{-1}q^{-1};q^{-1})_\infty}{2(a^{-1}b^{-1}q^{-2};q^{-1})_\infty}\displaystyle\sum_{k=0}^{\infty}\dfrac{(b^{-1}q^{-1};q^{-1})_k}{(q^{-1};q^{-1})_k}(aq)^{-k}\langle\delta_{b^{-1}q^{-k-1}}, f\rangle$ <br> $\qquad + \dfrac{K}{2}\displaystyle\int_0^{b^{-1}} x^{\frac{\ln a}{\ln q}}\dfrac{(bx;q^{-1})_\infty}{(x;q^{-1})_\infty}f(x)dx,$ <br> $f \in \mathcal{P},\ q>1,\ a>q^{-1},\ b \geq 1$ where $K^{-1} = \displaystyle\int_0^{b^{-1}} x^{\frac{\ln a}{\ln q}}\dfrac{(bx;q^{-1})_\infty}{(x;q^{-1})_\infty}dx$ | |
| case 1.7 | $q$-Charlier-II-$\mathcal{U}(\mu,q)$ $\big(\mu \neq 0,\ \mu \neq q^{-n},\ n \geq 0\big)$ |
| $\widehat{\beta}_n = \dfrac{1-(1+q)q^n + \mu q^{2n}}{(1-\mu q^{2n-1})(1-\mu q^{2n+1})}q^{n-1},\ n \geq 0,$ <br> $\widehat{\gamma}_{n+1} = -q^{3n}\dfrac{(1-q^{n+1})(1-\mu q^n)}{(1-\mu q^{2n})(1-\mu q^{2n+1})^2(1-\mu q^{2n+2})},\ n \geq 0,$ <br> $H_q(x(x-\mu^{-1}q^{-1})\mathcal{U}(\mu,q)) - (\mu q(q-1))^{-1}\{(\mu q-1)x - 1\}\mathcal{U}(\mu,q) = 0,$ <br> $(\mathcal{U}(\mu,q))_n = (-1)^n\dfrac{q^{\frac{1}{2}n(n-1)}}{(\mu q;q)_n},\ n \geq 0,$ <br> $\langle \mathcal{U}(\mu,q), f\rangle = \dfrac{1}{(\mu^{-1}q^{-1};q^{-1})_\infty}\displaystyle\sum_{k=0}^{\infty}\dfrac{q^{-\frac{1}{2}k(k+1)}}{(q^{-1};q^{-1})_k}(-\mu^{-1})^k\langle\delta_{\mu^{-1}q^{-k-1}}, f\rangle,\ f \in \mathcal{P},\ q>1,\ \mu<0$ | |

Lastly, let us recall the following standard expressions [8, 11, 20]

$$(a;q)_0 := 1, \qquad (a;q)_n := \prod_{k=1}^{n}\big(1-aq^{k-1}\big), \qquad n \geq 1,$$

$$(a;q)_\infty := \prod_{k=0}^{\infty}\big(1-aq^k\big), \qquad |q|<1,$$

the $q$-binomial theorem [4, 17]

$$\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}z^k = \frac{(az;q)_\infty}{(z;q)_\infty}, \qquad |z|<1, \qquad |q|<1,$$

$$\int_0^\infty t^{x-1}\frac{(-at;q)_\infty}{(-t;q)_\infty}dt = \begin{cases} \dfrac{\pi}{\sin(\pi x)}\dfrac{(a;q)_\infty}{(aq^{-x};q)_\infty}\dfrac{(q^{1-x};q)_\infty}{(q;q)_\infty}, & x \in \mathbb{R}_+\setminus\mathbb{N},\ |a|<q^x,\ 0<q<1, \\ \dfrac{(-q)^m}{1-q^m}\dfrac{(q^{-1};q^{-1})_m}{(aq^{-1};q^{-1})_m}\ln(q^{-1}), & x = m \in \mathbb{N}^\star,\ |a|<q^m,\ 0<q<1. \end{cases} \tag{2.4}$$

*Continuation of Table* 1.

| case 1.8 | Generalized Stieltjes–Wigert $\mathcal{S}(\omega,q)$ $\big(\omega\neq q^{-n},\ n\geq 0\big)$ |
|---|---|
| $\widehat{\beta}_n=\{(1+q)q^{-n}-q-\omega\}q^{-n-\frac{3}{2}},\ n\geq 0,$<br>$\widehat{\gamma}_{n+1}=(1-q^{n+1})(1-\omega q^{n})q^{-4n-4},\ n\geq 0,$<br>$H_q(x(x+\omega q^{-\frac{3}{2}})\mathcal{S}(\omega,q))-(q-1)^{-1}\{x+(\omega-1)q^{-\frac{3}{2}}\}\mathcal{S}(\omega,q)=0,$<br>$(\mathcal{S}(\omega,q))_n=q^{-\frac{1}{2}n(n+2)}(\omega;q)_n,\ n\geq 0,$<br>$\mathcal{S}(\omega,q)=(\omega^{-1};q^{-1})_\infty\sum_{k=0}^{\infty}\frac{\omega^{-k}}{(q^{-1};q^{-1})_k}\delta_{-\omega q^{-k-\frac{3}{2}}},\ q>1,\ \omega>1,$<br>$\langle\mathcal{S}(\omega,q),f\rangle=K\int_0^\infty\frac{x^{\frac{\ln\omega}{\ln q}-1}}{(-q^{\frac{3}{2}}\omega^{-1}x;q)_\infty}f(x)dx,\ f\in\mathcal{P},\ 0<q<1,\ 0<\omega<1,$<br>where $K^{-1}=\int_0^\infty\frac{x^{\frac{\ln\omega}{\ln q}-1}}{(-q^{\frac{3}{2}}\omega^{-1}x;q)_\infty}dx$ is given by (2.4),<br>$\langle\mathcal{S}(\omega,q),f\rangle=K_\omega\int_{q^{-\frac{1}{2}}\lvert\omega\rvert}^{\infty}(-q^{-\frac{1}{2}}\lvert\omega\rvert x^{-1};q)_\infty\exp\left(-\frac{\ln^2x}{2\ln q^{-1}}\right)f(x)dx,$<br>$f\in\mathcal{P},\ 0<q<1,\ \omega\leq 0,$<br>where $K_\omega^{-1}=\int_{q^{-\frac{1}{2}}\lvert\omega\rvert}^{\infty}(-q^{-\frac{1}{2}}\lvert\omega\rvert x^{-1};q)_\infty\exp\left(-\frac{\ln^2x}{2\ln q^{-1}}\right)dx,$<br>in particular $K_0=\sqrt{\frac{q}{2\pi\ln q^{-1}}}$ | |

**Table 2.** Limiting cases.

| $H_q$-classical linear form | |
|---|---|
| case 2.1 | $q$-analogue of Laguerre $\mathbf{L}(\alpha,q)$ $(\alpha\neq-[n]_q-1,\ n\geq 0)$ |
| $\widehat{\beta}_n=q^n\big\{(1+q^{-1})[n]_q+1+\alpha\big\},\ n\geq 0,$<br>$\widehat{\gamma}_{n+1}=q^{2n}[n+1]_q\big\{[n]_q+1+\alpha\big\},\ n\geq 0,$<br>$H_q(x\mathbf{L}(\alpha,q))+(x-1-\alpha)\mathbf{L}(\alpha,q)=0$ | |
| case 2.2 | $q$-analogue of Bessel $\mathbf{B}(\alpha,q)$ $(\alpha\neq\frac{1}{2}(q-1)^{-1},\ \alpha\neq-\frac{1}{2}[n]_q,\ n\geq 0)$ |
| $\widehat{\beta}_n=-2q^n\frac{2\alpha+(1+q^{-1})[n-1]_q-q^{-1}[2n]_q}{(2\alpha+[2n-2]_q)(2\alpha+[2n]_q)},\ n\geq 0,$<br>$\widehat{\gamma}_{n+1}=-4q^{3n}\frac{[n+1]_q(2\alpha+[n-1]_q)}{(2\alpha+[2n-1]_q)(2\alpha+[2n]_q)^2(2\alpha+[2n+1]_q)},\ n\geq 0,$<br>$H_q(x^2\mathbf{B}(\alpha,q))-2(\alpha x+1)\mathbf{B}(\alpha,q)=0$ | |
| case 2.3 | $q$-analogue of Jacobi $\mathbf{J}(\alpha,\beta,q)$ $(\alpha+\beta\neq\frac{3-2q}{q-1},$<br>$\alpha+\beta\neq-[n]_q-2,\ n\geq 0,\ \beta\neq-[n]_q-1,\ n\geq 0$ et $\alpha+\beta+2-(\beta+1)q^n+[n]_q\neq 0,\ n\geq 0)$ |
| $\widehat{\beta}_n=q^{n-1}\frac{(1+q)(\alpha+\beta+2+[n-1]_q)(\beta+1+[n]_q)-(\beta+1)(\alpha+\beta+2+[2n]_q)}{(\alpha+\beta+2+[2n-2]_q)(\alpha+\beta+2+[2n]_q)},\ n\geq 0,$<br>$\widehat{\gamma}_{n+1}=q^{2n}\frac{[n+1]_q(\alpha+\beta+2+[n-1]_q)([n]_q+\beta+1)(\alpha+\beta+2-(\beta+1)q^n+[n]_q)}{(\alpha+\beta+2+[2n-1]_q)(\alpha+\beta+2+[2n]_q)^2(\alpha+\beta+2+[2n+1]_q)},\ n\geq 0,$<br>$H_q(x(x-1)\mathbf{J}(\alpha,\beta,q))-((\alpha+\beta+2)x-(\beta+1))\mathbf{J}(\alpha,\beta,q)=0$ | |

## 2.2 Some results about the $H_q$-semiclassical character

A form $u$ is called $H_q$-semiclassical when it is regular and there exist two polynomials $\Phi$ and $\Psi$, $\Phi$ monic, $\deg\Phi=t\geq 0$, $\deg\Psi=p\geq 1$ such that

$$H_q(\Phi u)+\Psi u=0, \tag{2.5}$$

the corresponding orthogonal polynomial sequence $\{B_n\}_{n\geq 0}$ is called $H_q$-semiclassical [21].

The $H_q$-semiclassical character is kept by a dilation [21]. In fact, let $\{a^{-n}(h_a B_n)\}_{n\geq 0}$, $a \neq 0$; when $u$ satisfies (2.5), then $h_{a^{-1}}u$ fulfills the $q$-analog of the distributional equation of Pearson type

$$H_q\big(a^{-t}\Phi(ax)h_{a^{-1}}u\big) + a^{1-t}\Psi(ax)h_{a^{-1}}u = 0,$$

and the recurrence coefficients of (2.3) are

$$\frac{\beta_n}{a}, \qquad \frac{\gamma_{n+1}}{a^2}, \qquad n \geq 0.$$

Also, the $H_q$-semiclassical form $u$ is said to be of class $s = \max(p-1, t-2) \geq 0$ if and only if [21]

$$\prod_{c\in\mathcal{Z}_\Phi} \big\{ \big|q(h_q\Psi)(c) + (H_q\Phi)(c)\big| + \big|\langle u, q(\theta_{cq}\Psi) + (\theta_{cq}\circ\theta_c\Phi)\rangle\big| \big\} > 0, \tag{2.6}$$

where $\mathcal{Z}_\Phi$ is the set of zeros of $\Phi$. In particular, when $s = 0$ the form $u$ is usually called *$H_q$-classical* (Al-Salam–Carlitz, big $q$-Laguerre, $q$-Meixner, Wall, ...) [20].

**Lemma 1 ([21]).** *Let $u$ be a symmetrical $H_q$-semiclassical form of class $s$ satisfying* (2.5)*. The following statements holds*

*i) If $s$ is odd then the polynomial $\Phi$ is odd and $\Psi$ is even.*

*ii) If $s$ is even then the polynomial $\Phi$ is even and $\Psi$ is odd.*

In the sequel we are going to use some $H_q$-classical forms [20], resumed in Table 1 (canonical cases: 1.1–1.8) and Table 2 (limiting cases: 2.1–2.3). In fact, when $q \to 1$ in results of Table 2, we recover the classical Laguerre $\mathcal{L}(\alpha)$, Bessel $\mathcal{B}(\alpha)$ and $h_{-\frac{1}{2}} \circ \tau_{-1}\mathcal{J}(\alpha,\beta)$ respectively where $\mathcal{J}(\alpha,\beta)$ is the Jacobi classical form [24].

Moreover in what follows we are going to use the logarithmic function denoted by Log : $\mathbb{C}\setminus\{0\} \longrightarrow \mathbb{C}$ defined by

$$\operatorname{Log} z = \ln|z| + i\operatorname{Arg} z, \qquad z \in \mathbb{C}\setminus\{0\}, \qquad -\pi < \operatorname{Arg} z \leq \pi,$$

Log is the principal branch of log and includes $\ln : \mathbb{R}^+\setminus\{0\} \longrightarrow \mathbb{R}$ as a special case. Consequently, the principal branch of the square root is

$$\sqrt{z} = \sqrt{|z|}\, e^{i\frac{\operatorname{Arg} z}{2}}, \qquad z \in \mathbb{C}\setminus\{0\}, \qquad -\pi < \operatorname{Arg} z \leq \pi.$$

## 2.3 On quadratic decomposition of a symmetrical regular form

Let $u$ be a symmetrical regular form and $\{B_n\}_{n\geq 0}$ be its MOPS satisfying (2.3) with $\beta_n = 0$, $n \geq 0$. It is very well known (see [8, 25]) that

$$B_{2n}(x) = P_n\big(x^2\big), \qquad B_{2n+1}(x) = xR_n\big(x^2\big), \qquad n \geq 0,$$

where $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are the two MOPS related to the regular form $\sigma u$ and $x\sigma u$ respectively. In fact, [8, 25]

$u$ is regular $\Leftrightarrow$ $\sigma u$ and $x\sigma u$ are regular,

$u$ is positive definite $\Leftrightarrow$ $\sigma u$ and $x\sigma u$ are positive definite.

Furthermore, taking

$$
\begin{aligned}
&P_0(x) = 1, \qquad P_1(x) = x - \beta_0^P, \\
&P_{n+2}(x) = \big(x - \beta_{n+1}^P\big) P_{n+1}(x) - \gamma_{n+1}^P P_n(x), \qquad \gamma_{n+1}^P \neq 0, \qquad n \geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&R_0(x) = 1, \qquad R_1(x) = x - \beta_0^R, \\
&R_{n+2}(x) = \big(x - \beta_{n+1}^R\big) R_{n+1}(x) - \gamma_{n+1}^R R_n(x), \qquad \gamma_{n+1}^R \neq 0, \qquad n \geq 0,
\end{aligned}
$$

we get [8, 25]

$$
\begin{aligned}
&\beta_0^P = \gamma_1, \\
&\beta_{n+1}^P = \gamma_{2n+2} + \gamma_{2n+3}\ , \qquad n \geq 0, \\
&\gamma_{n+1}^P = \gamma_{2n+1}\gamma_{2n+2}, \qquad n \geq 0,
\end{aligned}
\tag{2.7}
$$

and

$$
\begin{aligned}
&\beta_n^R = \gamma_{2n+1} + \gamma_{2n+2}, \qquad n \geq 0, \\
&\gamma_{n+1}^R = \gamma_{2n+2}\gamma_{2n+3}, \qquad n \geq 0.
\end{aligned}
\tag{2.8}
$$

Consequently,

$$
\begin{aligned}
&\gamma_1 = \beta_0^P, \qquad \gamma_2 = \frac{\gamma_1^P}{\beta_0^P}, \\
&\gamma_{2n+1} = \beta_0^P \frac{\prod_{k=1}^{n} \gamma_k^R}{\prod_{k=1}^{n} \gamma_k^P}, \qquad \gamma_{2n+2} = \frac{1}{\beta_0^P} \frac{\prod_{k=1}^{n+1} \gamma_k^P}{\prod_{k=1}^{n} \gamma_k^R}, \qquad n \geq 1.
\end{aligned}
\tag{2.9}
$$

**Proposition 1.** *Let $u$ be a symmetrical regular form.*

*(i) The moments of $u$ are*

$$
(u)_{2n} = (\sigma u)_n, \qquad (u)_{2n+1} = 0, \qquad n \geq 0. \tag{2.10}
$$

*(ii) If $\sigma u$ has the discrete representation*

$$
\sigma u = \sum_{k=0}^{\infty} \rho_k \delta_{\tau_k}, \qquad \sum_{k=0}^{\infty} \rho_k = 1, \tag{2.11}
$$

*then a possible discrete measure of $u$ is*

$$
u = \sum_{k=0}^{\infty} \rho_k \frac{\delta_{\sqrt{\tau_k}} + \delta_{-\sqrt{\tau_k}}}{2}. \tag{2.12}
$$

*(iii) If $u$ is positive definite and $\sigma u$ has the integral representation*

$$
\langle \sigma u, f \rangle = \int_0^{\infty} V(x) f(x) dx, \qquad f \in \mathcal{P}, \qquad \int_0^{\infty} V(x) dx = 1, \tag{2.13}
$$

*then, a possible integral representation of $u$ is*

$$
\langle u, f \rangle = \int_{-\infty}^{\infty} |x| V(x^2) f(x) dx, \qquad f \in \mathcal{P}. \tag{2.14}
$$

**Proof.** $(i)$ is a consequence from the definition of the quadratic operator $\sigma$.

For $(ii)$ taking into account (2.10), (2.11) we get

$$(u)_{2n} = (\sigma u)_n = \sum_{k=0}^{\infty} \rho_k (\sqrt{\tau_k})^{2n} = \sum_{k=0}^{\infty} \rho_k \frac{(\sqrt{\tau_k})^{2n} + (-\sqrt{\tau_k})^{2n}}{2}.$$

But

$$(u)_{2n+1} = 0 = \sum_{k=0}^{\infty} \rho_k \frac{(\sqrt{\tau_k})^{2n+1} + (-\sqrt{\tau_k})^{2n+1}}{2}.$$

Hence the desired result (2.12) holds.

For $(iii)$ consider $f \in \mathcal{P}$ and let us split up the polynomial $f$ accordingly to its even and odd parts

$$f(x) = f^{\mathrm{e}}\big(x^2\big) + x f^{\mathrm{o}}\big(x^2\big). \tag{2.15}$$

Therefore since $u$ is a symmetrical form

$$\langle u, f(x)\rangle = \langle u, f^{\mathrm{e}}\big(x^2\big)\rangle = \langle \sigma u, f^{\mathrm{e}}(x)\rangle. \tag{2.16}$$

From (2.15) we get

$$f^{\mathrm{e}}(x) = \frac{f(\sqrt{x}) + f(-\sqrt{x})}{2}, \qquad x \in \mathbb{R}_+. \tag{2.17}$$

By (2.13) and according to (2.16), (2.17) we recover the representation in (2.14). ■

## 3 Symmetrical $H_{\sqrt{q}}$-semiclassical orthogonal polynomials of class one

**Lemma 2.** *We have*

$$\sigma(H_q u) = (q+1) H_{q^2}(\sigma(xu)), \qquad u \in \mathcal{P}'. \tag{3.1}$$

**Proof.** From the definition of $H_q$ we get

$$(H_q(\sigma f))(x) = (q+1)x(\sigma(H_{q^2} f))(x), \qquad f \in \mathcal{P}.$$

Therefore, $\forall\, f \in \mathcal{P}$,

$$\begin{aligned}\langle \sigma(H_q u), f\rangle &= \langle H_q u, \sigma f\rangle = -\langle u, (q+1)x\sigma(H_{q^2} f)\rangle \\ &= -\langle (q+1)\sigma(xu), H_{q^2} f\rangle = \langle (q+1)H_{q^2}(\sigma(xu)), f\rangle.\end{aligned}$$

Thus the desired result. ■

**Lemma 3.** *Let $u$ be a symmetrical $H_{\sqrt{q}}$-semiclassical form of class one. There exist two polynomials $\varphi$ and $\psi$, $\varphi$ monic, with* $\deg \varphi \le 1$ *and* $\deg \psi = 1$, *such that*

$$H_{\sqrt{q}}\big(x\varphi\big(x^2\big)u\big) + \psi\big(x^2\big)u = 0. \tag{3.2}$$

**Proof.** The result is a consequence from the definition of the class and Lemma 1. ■

**Corollary 1.** *Let $u$ be a symmetrical $H_{\sqrt{q}}$-semiclassical form of class one satisfying (3.2); then $\sigma u$ et $x\sigma u$ are $H_q$-classical satisfying respectively the following $q$-analog of the distributional equation of Pearson type*

$$H_q(x\varphi(x)\sigma u) + \frac{1}{\sqrt{q}+1}\psi(x)\sigma u = 0, \tag{3.3}$$

$$H_q(x\varphi(x)(x\sigma u)) + q^{-1}\left(\frac{1}{\sqrt{q}+1}\psi(x) - \varphi(x)\right)(x\sigma u) = 0. \tag{3.4}$$

**Proof.** First, $\sigma u$ and $x\sigma u$ are regular because $u$ is symmetrical and regular. Applying the quadratic operator $\sigma$ to (3.2) and taking into account (3.1) we get

$$(\sqrt{q}+1)H_q\big(\sigma\big(x^2\varphi\big(x^2\big)u\big)\big) + \sigma\big(\psi\big(x^2\big)u\big) = 0.$$

By (2.2) we get (3.3). Now, multiplying both sides of (3.3) by $q^{-1}x$, using the identity in (2.1), this yields to (3.4). ∎

Regarding Table 1 (cases 1.1–1.8), Table 2 (cases 2.1–2.3) and the $q$-analog of the distributional equation of Pearson type (3.3), (3.4), we consider the following situations for the polynomial $\varphi$ in order to get a $H_{\sqrt{q}}$-semiclassical form from a $H_q$-classical

**A**. $\varphi(x) = 1$ (cases $1.1, 1.2, 1.3, 1.4, 2.1$);

**B**. $\varphi(x) = x$ (cases $1.5, 2.2$);

**C**. $\varphi(x) = x - 1$ (case 2.3);

**D**. $\varphi(x) = x - b^{-1}q^{-1}$ (case 1.6);

**E**. $\varphi(x) = x - \mu^{-1}q^{-1}$ (case 1.7);

**F**. $\varphi(x) = x + \omega q^{-\frac{3}{2}}$ (case 1.8).

**A.** In the case $\varphi(x) = 1$ the $q$-analog of the distributional equation of Pearson type (3.3), (3.4) are

$$H_q(x\sigma u) + \frac{1}{\sqrt{q}+1}\psi(x)\sigma u = 0, \tag{3.5}$$

$$H_q(x(x\sigma u)) + q^{-1}\left(\frac{1}{\sqrt{q}+1}\psi(x) - 1\right)(x\sigma u) = 0. \tag{3.6}$$

**A$_1$.** If $\psi(x) = (\sqrt{q}+1)(x-1-\alpha)$ the $q$-analogue of the Laguerre form $\mathbf{L}(\alpha, q)$, $\alpha \neq -[n]_q - 1$, $n \geq 0$ (case 2.1 in Table 2) satisfying

$$H_q(x\mathbf{L}(\alpha, q)) + (x - 1 - \alpha)\mathbf{L}(\alpha, q) = 0.$$

Comparing with (3.5), (3.6) we get

$$\sigma u = \mathbf{L}(\alpha, q), \qquad \alpha \neq -[n]_q - 1, \qquad n \geq 0, \tag{3.7}$$

and

$$x\sigma u = (1+\alpha)\mathbf{L}\big(q^{-1}(\alpha+2) - 1, q\big), \qquad \alpha \neq -[n]_q - 1, \qquad n \geq 0. \tag{3.8}$$

Taking into account the recurrence coefficients (see case 2.1 in Table 2), by virtue of (3.7), (3.8) and (2.7), (2.8) we get for $n \geq 0$

$$\beta_n^P = q^n\big\{\big(1+q^{-1}\big)[n]_q + 1 + \alpha\big\},$$
$$\gamma_{n+1}^P = q^{2n}[n+1]_q\{[n]_q + 1 + \alpha\},$$
$$\beta_n^R = q^{n+1}\big\{\big(1+q^{-1}\big)[n]_q + q^{-1}(2+\alpha)\big\},$$
$$\gamma_{n+1}^R = q^{2n+2}[n+1]_q\big\{[n]_q + q^{-1}(2+\alpha)\big\}.$$

With the relation $[k-1]_q = q^{-1}[k]_q - q^{-1}$, $k \geq 1$ the system (2.9) becomes for $n \geq 0$

$$\gamma_{2n+1} = q^n([n]_q + 1 + \alpha), \qquad \gamma_{2n+2} = q^n[n+1]_q. \tag{3.9}$$

Writing $\alpha = \mu - \frac{1}{2}$, $\mu \neq -[n]_q - \frac{1}{2}$, $n \geq 0$ and denoting the symmetrical form $u$ by $\mathcal{H}(\mu, q)$ we get the following result:

**Proposition 2.** *The symmetrical form $\mathcal{H}(\mu, q)$ satisfies the following properties:*

1) *The recurrence coefficient $\gamma_{n+1}$ satisfies* (3.9).
2) *$\mathcal{H}(\mu, q)$ is regular if and only if $\mu \neq -[n]_q - \frac{1}{2}$, $n \geq 0$.*
3) *$\mathcal{H}(\mu, q)$ is positive definite if and only if $q > 0$, $\mu > -\frac{1}{2}$.*
4) *$\mathcal{H}(\mu, q)$ is a $H_{\sqrt{q}}$-semiclassical form of class one for $\mu \neq \frac{1}{\sqrt{q}(\sqrt{q}+1)} - \frac{1}{2}$, $\mu \neq -[n]_q - \frac{1}{2}$, $n \geq 0$ satisfying the $q$-analog of the distributional equation of Pearson type*

$$H_{\sqrt{q}}(x\mathcal{H}(\mu, q)) + (\sqrt{q}+1)\left(x^2 - \mu - \frac{1}{2}\right)\mathcal{H}(\mu, q) = 0. \tag{3.10}$$

**Proof.** The results in 1), 2) and 3) are straightforward from (3.9). For 4), it is clear that $\mathcal{H}(\mu, q)$ satisfies (3.10); in this case and by virtue of (2.6), we are going to prove that the class of $\mathcal{H}(\mu, q)$ is exactly one for $\mu \neq \frac{1}{\sqrt{q}(\sqrt{q}+1)} - \frac{1}{2}$, $\mu \neq -[n]_q - \frac{1}{2}$, $n \geq 0$. Denoting $\Phi(x) = x$, $\Psi(x) = (\sqrt{q}+1)\big(x^2 - \mu - \frac{1}{2}\big)$, we have accordingly to (2.6), on one hand

$$\sqrt{q}\big(h_{\sqrt{q}}\Psi\big)(0) + \big(H_{\sqrt{q}}\Phi\big)(0) = 1 - \sqrt{q}(\sqrt{q}+1)\left(\mu + \frac{1}{2}\right) \neq 0,$$

and on the other hand by $(\theta_0\Psi)(x) = (\sqrt{q}+1)x$ and $(\theta_0^2\Phi)(x) = 0$,

$$\langle \mathcal{H}(\mu, q), \sqrt{q}\theta_0\Psi + \theta_0^2\Phi \rangle = 0,$$

taking into account that $u$ is a symmetrical form. ■

**Remark 1.** The symmetrical form $\mathcal{H}(\mu, q)$, $\mu \neq \frac{1}{\sqrt{q}(\sqrt{q}+1)} - \frac{1}{2}$, $\mu \neq -[n]_q - \frac{1}{2}$, $n \geq 0$ is the $q$-analogue of the generalized Hermite one [12] (when $q \to 1$ we recover the generalized Hermite form $\mathcal{H}(\mu)$ (see (1.2)) which is a symmetrical semiclassical form of class one for $\mu \neq 0$, $\mu \neq -n - \frac{1}{2}$, $n \geq 0$ [1, 8, 15, 26]).

**A$_2$.** If $\psi(x) = -(\sqrt{q}-1)^{-1}(x+1)$ the form $\mathcal{U}$ that satisfies the $q$-analog of the distributional equation of Pearson type (see case 1.1 in Table 1)

$$H_q(x\mathcal{U}) - (q-1)^{-1}(x+1)\mathcal{U} = 0.$$

Comparing with (3.5), (3.6) we get

$$\sigma u = \mathcal{U}, \tag{3.11}$$

and

$$x\sigma u = -h_q\mathcal{U}. \tag{3.12}$$

Taking into account (3.11), (3.12), (2.7), (2.8) and the case 1.1 in Table 1 we obtain for $n \geq 0$

$$\beta_n^P = \big\{1 - (1+q)q^n\big\}q^{n-1}, \qquad \gamma_{n+1}^P = \big(q^{n+1} - 1\big)q^{3n},$$
$$\beta_n^R = \big\{1 - (1+q)q^n\big\}q^n, \qquad \gamma_{n+1}^R = \big(q^{n+1} - 1\big)q^{3n+2}.$$

Consequently, the system (2.9) becomes for $n \geq 0$

$$\gamma_{2n+1} = -q^{2n}, \qquad \gamma_{2n+2} = \big(1 - q^{n+1}\big)q^n. \tag{3.13}$$

**Proposition 3.** *The symmetrical form $u$ satisfies the following properties:*

1) *The recurrence coefficient $\gamma_{n+1}$ satisfies* (3.13).
2) *$u$ is regular for any $q \in \widetilde{\mathbb{C}}$.*
3) *$u$ is a $H_{\sqrt{q}}$-semiclassical form of class one satisfying*

$$H_{\sqrt{q}}(xu) - (\sqrt{q}-1)^{-1}\big(x^2+1\big)u = 0. \tag{3.14}$$

4) *The moments of $u$ are*

$$(u)_{2n} = (-1)^n q^{\frac{1}{2}n(n-1)}, \qquad (u)_{2n+1} = 0, \qquad n \geq 0.$$

5) *we have the following discrete representation*

$$u = \sum_{k=0}^{\infty} \frac{(-1)^k q^{-k^2} s(k)}{(q^{-1};q^{-1})_k} \frac{\delta_{iq^{\frac{k}{2}}} + \delta_{-iq^{\frac{k}{2}}}}{2}, \qquad q > 1.$$

**Proof.** The results in 1), 2) are obvious from (3.13). For 3), it is clear that $u$ satisfies (3.14). Denoting $\Phi(x) = x$, $\Psi(x) = -(\sqrt{q}-1)^{-1}\big(x^2+1\big)$, we have (2.6)

$$\sqrt{q}\big(h_{\sqrt{q}}\Psi\big)(0) + \big(H_{\sqrt{q}}\Phi\big)(0) = \frac{1}{1-\sqrt{q}} \neq 0, \qquad \langle u, \sqrt{q}\theta_0\Psi + \theta_0^2\Phi\rangle = 0.$$

Therefore, $u$ is of class one. The results in 4) and 5) are consequence from (2.10)–(2.12) and those for $\mathcal{U}$ (case 1.1 in Table 1). ■

**A$_3$.** If $\psi(x) = -(aq)^{-1}(\sqrt{q}-1)^{-1}(x-1+aq)$ the little $q$-Laguerre form $\mathcal{L}(a,q)$, $a \neq 0$, $a \neq q^{-n-1}$, $n \geq 0$ (case 1.2 in Table 1) satisfying

$$H_q(x\mathcal{L}(a,q)) - (aq)^{-1}(q-1)^{-1}(x-1+aq)\mathcal{L}(a,q) = 0.$$

With (3.5), (3.6) we obtain

$$\sigma u = \mathcal{L}(a,q), \qquad a \neq 0, \qquad a \neq q^{-n-1}, \qquad n \geq 0, \tag{3.15}$$

and

$$x\sigma u = (1-aq)\mathcal{L}(aq,q), \qquad a \neq 0, \qquad a \neq q^{-n-1}, \qquad n \geq 0. \tag{3.16}$$

By virtue of the recurrence coefficients of little $q$-Laguerre polynomials in Table 1, case 1.2, the relations in (3.15), (3.16) and (2.7), (2.8) we get for $n \geq 0$

$$\beta_n^P = \big\{1+a-a(1+q)q^n\big\}q^n,$$
$$\gamma_{n+1}^P = a\big(1-q^{n+1}\big)\big(1-aq^{n+1}\big)q^{2n+1},$$
$$\beta_n^R = \big\{1+aq-a(1+q)q^{n+1}\big\}q^n,$$
$$\gamma_{n+1}^R = a\big(1-q^{n+1}\big)\big(1-aq^{n+2}\big)q^{2n+2}.$$

Therefore (2.9) becomes for $n \geq 0$

$$\gamma_{2n+1} = q^n\big(1-aq^{n+1}\big), \qquad \gamma_{2n+2} = aq^{n+1}\big(1-q^{n+1}\big). \tag{3.17}$$

Comparing with [2], $u$ is a symmetrical case of the Al-Salam–Verma form, $u := \mathcal{SV}(a,q)$. From (3.17), it is easy to see that $\mathcal{SV}(a,q)$ is regular if and only if $a \neq 0$, $a \neq q^{-n-1}$, $n \geq 0$. Also, $\mathcal{SV}(a,q)$ is positive definite if and only if $0 < q < 1$, $0 < a < q^{-1}$ or $q > 1$, $a < 0$.

**Proposition 4.** *The form $\mathcal{SV}(a,q)$ is a $H_{\sqrt{q}}$-semiclassical form of class one for $a\neq 0$, $a\neq q^{-\frac{1}{2}}$, $a\neq q^{-n-1}$, $n\geq 0$ satisfying*

$$H_{\sqrt{q}}(x\mathcal{SV}(a,q)) - (aq)^{-1}(\sqrt{q}-1)^{-1}\big(x^2-1+aq\big)\mathcal{SV}(a,q) = 0. \tag{3.18}$$

*The moments are*

$$(\mathcal{SV}(a,q))_{2n} = (aq;q)_n, \qquad (\mathcal{SV}(a,q))_{2n+1} = 0, \qquad n\geq 0, \tag{3.19}$$

*and the orthogonality relation can be represented*

$$\langle \mathcal{SV}(a,q), f\rangle = \frac{(aq;q)_\infty}{2}\sum_{k=0}^{\infty}\frac{(aq)^k}{(q;q)_k}\left\langle \frac{\delta_{q^{\frac{k}{2}}}+\delta_{-q^{\frac{k}{2}}}}{2}, f\right\rangle$$

$$+\frac{K}{2}\int_{-q^{-\frac{1}{2}}}^{q^{-\frac{1}{2}}} |x|^{2\frac{\ln a}{\ln q}+1}(qx^2;q)_\infty f(x)dx, \qquad f\in\mathcal{P}, \qquad 0<q<1, \qquad 0<a<q^{-1}, \tag{3.20}$$

*with*

$$K^{-1} = q^{-\frac{\ln a}{\ln q}-1}\int_0^1 x^{\frac{\ln a}{\ln q}}(x;q)_\infty dx,$$

*and*

$$\mathcal{SV}(a,q) = \frac{1}{(a;q^{-1})_\infty}\sum_{k=0}^{\infty}\frac{q^{-\frac{1}{2}k(k-1)}(-a)^k}{(q^{-1};q^{-1})_k}\frac{\delta_{q^{\frac{k}{2}}}+\delta_{-q^{\frac{k}{2}}}}{2}, \qquad q>1, \qquad a<0. \tag{3.21}$$

**Proof.** It is direct that the form $\mathcal{SV}(a,q)$ satisfies the $q$-analog of the distributional equation of Pearson type (3.18). Denoting $\Phi(x)=x$, $\Psi(x) = -(aq)^{-1}(\sqrt{q}-1)^{-1}\big(x^2-1+aq\big)$, we have (2.6)

$$\sqrt{q}\big(h_{\sqrt{q}}\Psi\big)(0) + \big(H_{\sqrt{q}}\Phi\big)(0) = \frac{a^{-1}q^{-\frac{1}{2}}-1}{\sqrt{q}-1}\neq 0, \qquad \langle \mathcal{SV}(a,q), \sqrt{q}\theta_0\Psi+\theta_0^2\Phi\rangle = 0,$$

from which we get that $\mathcal{SV}(a,q)$ is of class one because $a\neq 0$, $a\neq q^{-\frac{1}{2}}$, $a\neq q^{-n-1}$, $n\geq 0$. The results mentioned in (3.19)–(3.21) are easily obtained from those well known the properties of the little $q$-Laguerre from (case 1.2 in Table 1) and (2.10)–(2.14). ∎

**Remark 2.** The regular form $\mathcal{SV}(q^{-\frac{1}{2}},q)$ is the discrete $\sqrt{q}$-Hermite form which is $H_{\sqrt{q}}$-classical [20].

**A$_4$.** If $\psi(x) = -b^{-1}(\sqrt{q}-1)^{-1}\big(q^{-1}x+b-1\big)$ the Wall form $\mathcal{W}(b,q)$, $b\neq 0$, $b\neq q^{-n}$, $n\geq 0$ (case 1.3 in Table 1) that satisfies

$$H_q(x\mathcal{W}(b,q)) - b^{-1}(q-1)^{-1}(q^{-1}x+b-1)\mathcal{W}(b,q) = 0.$$

In accordance of (3.5), (3.6) we get

$$\sigma u = \mathcal{W}(b,q), \qquad b\neq 0, \qquad b\neq q^{-n}, \qquad n\geq 0,$$

and

$$x\sigma u = q(1-b)\mathcal{W}(bq,q), \qquad b\neq 0, \qquad b\neq q^{-n}, \qquad n\geq 0.$$

We recognize the Brenke type symmetrical regular form $\mathcal{Y}(b,q)$ [8, 9, 10]. In [13] it is proved that $\mathcal{Y}(b,q)$ is $H_{\sqrt{q}}$-semiclassical of class one for $b\neq 0$, $b\neq\sqrt{q}$, $b\neq q^{-n}$, $n\geq 0$ satisfying

$$H_{\sqrt{q}}(x\mathcal{Y}(b,q)) - b^{-1}\big(q^{\frac{1}{2}}-1\big)^{-1}\big\{q^{-1}x^2+b-1\big\}\mathcal{Y}(b,q) = 0. \tag{3.22}$$

Also in that work, moments, discrete and integral representations are established.

**Remark 3.** Likewise, from (3.22) it is easy to see that $h_{\frac{1}{\sqrt{q}}}\mathcal{Y}(\sqrt{q},q)$ is the $H_{\sqrt{q}}$-classical discrete $\sqrt{q}$-Hermite form [20].

**A$_5$.** If $\psi(x) = (\sqrt{q}-1)^{-1}q^{-\alpha-1}\big(x+b-q^{\alpha+1}\big)$ the generalized $q^{-1}$-Laguerre $\mathcal{U}^{(\alpha)}(b,q)$ form, $b\neq 0$, $b\neq q^{n+1+\alpha}$, $n\geq 0$ and its $q$-analog of the distributional equation of Pearson type (case 1.4 in Table 1)

$$H_q(x\mathcal{U}^{(\alpha)}(b,q)) + (q-1)^{-1}q^{-\alpha-1}(x+b-q^{\alpha+1})\mathcal{U}^{(\alpha)}(b,q) = 0.$$

By (3.5), (3.6) we deduce the following relationships

$$\sigma u = \mathcal{U}^{(\alpha)}(b,q), \qquad b\neq 0, \qquad b\neq q^{n+1+\alpha}, \qquad n\geq 0, \tag{3.23}$$

$$x\sigma u = \big(q^{\alpha+1}-b\big)\mathcal{U}^{(\alpha+1)}(b,q), \qquad b\neq 0, \qquad b\neq q^{n+1+\alpha}, \qquad n\geq 0. \tag{3.24}$$

From Table 1, case 1.4, the relations in (3.23), (3.24) and (2.7), (2.8) we get for $n\geq 0$

$$\begin{aligned}
&\beta_n^P = \big\{1-q^{-n-1}+q^{-1}\big(1-bq^{-n-\alpha}\big)\big\}q^{2n+\alpha+1},\\
&\gamma_{n+1}^P = \big(1-q^{-n-1}\big)\big(1-bq^{-n-1-\alpha}\big)q^{4n+2\alpha+3},\\
&\beta_n^R = \big\{1-q^{-n-1}+q^{-1}\big(1-bq^{-n-\alpha-1}\big)\big\}q^{2n+\alpha+2},\\
&\gamma_{n+1}^R = \big(1-q^{-n-1}\big)\big(1-bq^{-n-2-\alpha}\big)q^{4n+2\alpha+5}.
\end{aligned}$$

Thus, for $n\geq 0$

$$\gamma_{2n+1} = \big(1-bq^{-n-1-\alpha}\big)q^{2n+\alpha+1}, \qquad \gamma_{2n+2} = \big(1-q^{-n-1}\big)q^{2n+\alpha+2}.$$

Consequently, the symmetrical form $u := u(\alpha,b,q)$ is regular if and only if $b\neq 0$, $b\neq q^{n+1+\alpha}$, $n\geq 0$. It is positive definite for $\alpha\in\mathbb{R}$, $q>1$, $b<q^{\alpha+1}$.

**Proposition 5.** *The symmetrical form $u$ is a $H_{\sqrt{q}}$-semiclassical form of class one for $b\neq 0$, $b\neq q^{n+1+\alpha}$, $n\geq 0$, $\alpha\in\mathbb{R}$ satisfying*

$$H_{\sqrt{q}}(xu) + q^{-\alpha-1}\big(q^{\frac{1}{2}}-1\big)^{-1}\big\{x^2+b-q^{\alpha+1}\big\}u = 0.$$

*Moreover, we have the following identities*

$$(u)_{2n} = (-b)^n\big(b^{-1}q^{\alpha+1};q\big)_n, \qquad (u)_{2n+1} = 0, \qquad n\geq 0, \tag{3.25}$$

$$\langle u,f\rangle = K\int_{-\infty}^{\infty}\frac{|x|^{2\alpha-2\frac{\ln b}{\ln q}+1}}{(-b^{-1}x^2;q^{-1})_\infty}f(x)dx, \tag{3.26}$$

*for $f\in\mathcal{P}$, $\alpha\in\mathbb{R}$, $q>1$, $0<b<q^{\alpha+1}$, with*

$$K^{-1} = \int_0^\infty \frac{x^{\alpha-\frac{\ln b}{\ln q}}}{(-b^{-1}x;q^{-1})_\infty}dx$$

*is given by (2.4),*

$$u = \frac{1}{(b^{-1}q^\alpha;q^{-1})_\infty}\sum_{k=0}^{\infty}\frac{q^{-\frac{1}{2}k(k-1)}}{(q^{-1};q^{-1})_k}(-b^{-1}q^\alpha)^k\frac{\delta_{\sqrt{-b}q^{\frac{k}{2}}}+\delta_{-\sqrt{-b}q^{\frac{k}{2}}}}{2}, \tag{3.27}$$

*for $\alpha\in\mathbb{R}$, $q>1$, $b<0$, and*

$$u = \big(b^{-1}q^{\alpha+1};q\big)_\infty\sum_{k=0}^{\infty}\frac{(b^{-1}q^{\alpha+1})^k}{(q;q)_k}\frac{\delta_{i\sqrt{b}q^{\frac{k}{2}}}+\delta_{-i\sqrt{b}q^{\frac{k}{2}}}}{2}, \tag{3.28}$$

*for $\alpha\in\mathbb{R}$, $0<q<1$, $b>q^{\alpha+1}$.*

**Proof.** First, let us obtain the class of the form; denoting

$$\Phi(x) = x, \qquad \Psi(x) = (\sqrt{q}-1)^{-1} q^{-\alpha-1}\big(x^2 + b - q^{\alpha+1}\big),$$

we have

$$\sqrt{q}\big(h_{\sqrt{q}}\Psi\big)(0) + \big(H_{\sqrt{q}}\Phi\big)(0) = \frac{bq^{-\alpha-\frac{1}{2}}-1}{\sqrt{q}-1} \neq 0, \qquad \langle u, \sqrt{q}\theta_0\Psi + \theta_0^2\Phi\rangle = 0,$$

for $b \neq 0$, $b \neq q^{n+1+\alpha}$, $n \geq 0$, $\alpha \in \mathbb{R}$. Thus, $u$ is of class one. The identities given in (3.25)–(3.28) are easily obtained from the properties of the generalized $q^{-1}$-Laguerre $\mathcal{U}^{(\alpha)}(b,q)$ form (Table 1, case 1.4) and (2.10)–(2.14). ■

**B.** In the case $\varphi(x) = x$ the $q$-analog of the distributional equation of Pearson type (3.3), (3.4) are

$$H_q\big(x^2\sigma u\big) + \frac{1}{\sqrt{q}+1}\psi(x)\sigma u = 0, \tag{3.29}$$

$$H_q\big(x^2(x\sigma u)\big) + q^{-1}\left\{\frac{1}{\sqrt{q}+1}\psi(x) - x\right\}(x\sigma u) = 0. \tag{3.30}$$

**B$_1$.** If $\psi(x) = -2(\sqrt{q}+1)(\alpha x+1)$ the $q$-analogue of the Bessel form (case 2.2 in Table 2), the form $\mathbf{B}(\alpha,q)$, $\alpha \neq \frac{1}{2}(q-1)^{-1}$, $\alpha \neq -\frac{1}{2}[n]_q$, $n \geq 0$ satisfying

$$H_q\big(x^2\mathbf{B}(\alpha,q)\big) - 2(\alpha x+1)\mathbf{B}(\alpha,q) = 0.$$

Thus, comparing with (3.29), (3.30), we get

$$\sigma u = \mathbf{B}(\alpha,q), \qquad \alpha \neq \frac{1}{2}(q-1)^{-1}, \qquad \alpha \neq -\frac{1}{2}[n]_q, \qquad n \geq 0,$$

and

$$x\sigma u = -\alpha^{-1}h_{q^{-1}}\mathbf{B}(q^{-1}(\alpha+\frac{1}{2}),q),\ \alpha \neq \frac{1}{2}(q-1)^{-1}, \qquad \alpha \neq -\frac{1}{2}[n]_q, \qquad n \geq 0.$$

By the recurrence coefficients in case 2.2 of Table 2, the relations in (3.29), (3.30) and (2.7), (2.8) we get for $n \geq 0$

$$\begin{aligned}
\beta_n^P &= -2q^n\frac{2\alpha + (1+q^{-1})[n-1]_q - q^{-1}[2n]_q}{(2\alpha+[2n-2]_q)(2\alpha+[2n]_q)},\\
\gamma_{n+1}^P &= -4q^{3n}\frac{[n+1]_q(2\alpha+[n-1]_q)}{(2\alpha+[2n-1]_q)(2\alpha+[2n]_q)^2(2\alpha+[2n+1]_q)},\\
\beta_n^R &= -2q^{n-1}\frac{(2\alpha+1)q^{-1} + (1+q^{-1})[n-1]_q - q^{-1}[2n]_q}{((2\alpha+1)q^{-1}+[2n-2]_q)((2\alpha+1)q^{-1}+[2n]_q)},\\
\gamma_{n+1}^R &= -4q^{3n-2}\frac{[n+1]_q((2\alpha+1)q^{-1}+[n-1]_q)}{((2\alpha+1)q^{-1}+[2n-1]_q)((2\alpha+1)q^{-1}+[2n]_q)^2((2\alpha+1)q^{-1}+[2n+1]_q)}.
\end{aligned}$$

By the relation $[k-1]_q = q^{-1}[k]_q - q^{-1}$, $k \geq 1$, (2.9) leads to for $n \geq 0$

$$\begin{aligned}
\gamma_1 &= -\frac{1}{\alpha}, \qquad \gamma_{2n+2} = 2q^{2n}\frac{[n+1]_q}{(2\alpha+[2n]_q)(2\alpha+[2n+1]_q)},\\
\gamma_{2n+3} &= -2q^{n+1}\frac{(2\alpha+[n]_q)}{(2\alpha+[2n+1]_q)(2\alpha+[2n+2]_q)}.
\end{aligned} \tag{3.31}$$

We put $\alpha = \frac{\nu+1}{2}$, $\nu \neq \frac{2-q}{q-1}$, $\nu \neq -[n]_q - 1$, $n \geq 0$ and denote the symmetrical form $u$ by $\mathcal{B}[\nu, q]$. From (3.31) the form $\mathcal{B}[\nu, q]$ is regular if and only if $\nu \neq \frac{2-q}{q-1}$, $\nu \neq -[n]_q - 1$, $n \geq 0$. Also, it is quite straightforward to deduce that the symmetrical form $\mathcal{B}[\nu, q]$ is $H_{\sqrt{q}}$-semiclassical of class one for $\nu \neq \frac{2-q}{q-1}$, $\nu \neq -[n]_q - 1$, $n \geq 0$ satisfying the $q$-analog of the distributional equation of Pearson type

$$H_{\sqrt{q}}\big(x^3\mathcal{B}[\nu, q]\big) - 2(\sqrt{q} + 1)\left(\frac{\nu+1}{2}x^2 + 1\right)\mathcal{B}[\nu, q] = 0.$$

**Remark 4.** The symmetrical form $h_{(2\sqrt{2})^{-1}}\mathcal{B}[\nu, q]$, $\nu \neq \frac{2-q}{q-1}$, $\nu \neq -[n]_q - 1$, $n \geq 0$ is the $q$-analogue of the symmetrical form $\mathcal{B}[\nu]$ [14] (when $q \to 1$ we recover the symmetrical semiclassical $\mathcal{B}[\nu]$, $\nu \neq -n-1$, $n \geq 0$ of class one, see (1.4)). Also, for any parameter $\alpha \neq -n-1$, $n \geq 0$ the symmetrical form $h_{(2\sqrt{1+\sqrt{q}})^{-1}}\mathcal{B}[-\frac{q^{-\alpha-1}-1}{q-1} - 1, q]$ appears in [33].

**B$_2$.** If $\psi(x) = -(aq)^{-1}(\sqrt{q} - 1)^{-1}((1 + aq)x - 1)$ the Alternative $q$-Charlier $\mathcal{A}(a, q)$ form with $a \neq 0$, $a \neq -q^{-n}$, $n \geq 0$ that satisfies (case 1.5 in Table 1)

$$H_q\big(x^2\mathcal{A}(a, q)\big) - (aq)^{-1}(q - 1)^{-1}\big((1 + aq)x - 1\big)\mathcal{A}(a, q) = 0.$$

Thus

$$\sigma u = \mathcal{A}(a, q), \qquad a \neq 0, \qquad a \neq -q^{-n}, \qquad n \geq 0,$$

and

$$x\sigma u = \frac{1}{1 + aq}\mathcal{A}(aq, q), \qquad a \neq 0, \qquad a \neq -q^{-n}, \qquad n \geq 0.$$

The systems (2.7), (2.8) are for $n \geq 0$

$$\beta_n^P = q^n\frac{1 + aq^{n-1} + aq^n - aq^{2n}}{(1 + aq^{2n-1})(1 + aq^{2n+1})},$$
$$\gamma_{n+1}^P = aq^{3n+1}\frac{(1 - q^{n+1})(1 + aq^n)}{(1 + aq^{2n})(1 + aq^{2n+1})^2(1 + aq^{2n+2})},$$
$$\beta_n^R = q^n\frac{1 + aq^n + aq^{n+1} - aq^{2n+1}}{(1 + aq^{2n})(1 + aq^{2n+2})},$$
$$\gamma_{n+1}^R = aq^{3n+2}\frac{(1 - q^{n+1})(1 + aq^{n+1})}{(1 + aq^{2n+1})(1 + aq^{2n+2})^2(1 + aq^{2n+3})},$$

from which we get for $n \geq 0$

$$\gamma_{2n+1} = q^n\frac{1 + aq^n}{(1 + aq^{2n})(1 + aq^{2n+1})}, \qquad \gamma_{2n+2} = aq^{2n+1}\frac{1 - q^{n+1}}{(1 + aq^{2n+1})(1 + aq^{2n+2})}.$$

Consequently, the symmetrical form $u = u(a, q)$ is regular if and only if $a \neq 0$, $a \neq -q^{-n}$, $n \geq 0$. It is positive definite for $0 < q < 1$, $a > 0$. Also, $u$ is $H_{\sqrt{q}}$-semiclassical of class one for $a \neq 0$, $a \neq -q^{-n}$, $n \geq 0$ satisfying the $q$-analog of the distributional equation of Pearson type

$$H_{\sqrt{q}}\big(x^3u\big) - (aq)^{-1}(\sqrt{q} - 1)^{-1}\big((1 + aq)x^2 - 1\big)u = 0.$$

After some straightforward computations, we get the following representations for the moments and the orthogonality

$$(u)_{2n} = \frac{1}{(-aq; q)_n}, \qquad (u)_{2n+1} = 0, \qquad n \geq 0,$$
$$\langle u, f\rangle = q^{\frac{1}{2}(\frac{\ln a}{\ln q}+\frac{1}{2})^2}\frac{(-a^{-1}; q)_\infty}{\sqrt{2\pi\ln q^{-1}}}\int_{-\infty}^{\infty}|x|^{2\frac{\ln a}{\ln q}}(qx^2; q)_\infty\exp\left(-2\frac{\ln^2|x|}{\ln q^{-1}}\right)f(x)dx,$$

for $f \in \mathcal{P}$, $0 < q < 1$, $a > 0$, and

$$u = \frac{1}{(-aq;q)_\infty} \sum_{k=0}^{\infty} \frac{a^k q^{\frac{1}{2}k(k+1)}}{(q;q)_k} \frac{\delta_{-q^{\frac{k}{2}}} + \delta_{q^{\frac{k}{2}}}}{2}, \qquad 0 < q < 1, \qquad a > 0.$$

**C.** In the case $\varphi(x) = x - 1$ the $q$-analogue of Jacobi form (case 2.3 in Table 2), therefore the $q$-analog of the distributional equation of Pearson type (3.3), (3.4) become

$$H_q(x(x-1)\sigma u) - \big((\alpha+\beta+2)x - (\beta+1)\big)\sigma u = 0,$$

and

$$H_q\big(x(x-1)(x\sigma u)\big) - q^{-1}\big((\alpha+\beta+3)x - (\beta+2)\big)(x\sigma u) = 0.$$

Consequently,

$$\sigma u = \mathbf{J}(\alpha,\beta,q), \tag{3.32}$$

$$x\sigma u = \frac{\beta+1}{\alpha+\beta+2}\mathbf{J}\big(q^{-1}(\alpha+1)-1, q^{-1}(\beta+2)-1, q\big) \tag{3.33}$$

with the constraints

$$\begin{aligned} &\alpha+\beta \neq \frac{3-2q}{q-1}, \qquad \alpha+\beta \neq -[n]_q - 2, \qquad \beta \neq -[n]_q - 1, \\ &\alpha+\beta+2-(\beta+1)q^n + [n]_q \neq 0, \qquad n \geq 0. \end{aligned} \tag{3.34}$$

By Table 2 and (3.32), (3.33), the systems (2.7), (2.8) give for $n \geq 0$

$$\begin{aligned} \beta_n^P &= q^{n-1}\frac{(1+q)(\alpha+\beta+2+[n-1]_q)(\beta+1+[n]_q) - (\beta+1)(\alpha+\beta+2+[2n]_q)}{(\alpha+\beta+2+[2n-2]_q)(\alpha+\beta+2+[2n]_q)}, \\ \gamma_{n+1}^P &= q^{2n}\frac{[n+1]_q(\alpha+\beta+2+[n-1]_q)([n]_q+\beta+1)(\alpha+\beta+2-(\beta+1)q^n+[n]_q)}{(\alpha+\beta+2+[2n-1]_q)(\alpha+\beta+2+[2n]_q)^2(\alpha+\beta+2+[2n+1]_q)}, \\ \beta_n^R &= q^{n-1}\frac{(1+q)(\alpha+\beta+2+[n]_q)(\beta+1+[n+1]_q) - (\beta+2)(\alpha+\beta+2+[2n+1]_q)}{(\alpha+\beta+2+[2n-1]_q)(\alpha+\beta+2+[2n+1]_q)}, \\ \gamma_{n+1}^R &= q^{2n+1}\frac{[n+1]_q(\alpha+\beta+2+[n]_q)([n+1]_q+\beta+1)(\alpha+\beta+2-(\beta+2)q^n+[n+1]_q)}{(\alpha+\beta+2+[2n]_q)(\alpha+\beta+2+[2n+1]_q)^2(\alpha+\beta+2+[2n+2]_q)}. \end{aligned}$$

Using the above results and the relations

$$[k-1]_q = q^{-1}[k]_q - q^{-1}, \qquad [k]_q = q^{k-1} + [k-1]_q, \qquad k \geq 1$$

we deduce from (2.9) for $n \geq 0$

$$\begin{aligned} \gamma_{2n+1} &= q^n \frac{(\alpha+\beta+2+[n-1]_q)(\beta+1+[n]_q)}{(\alpha+\beta+2+[2n-1]_q)(\alpha+\beta+2+[2n]_q)}, \\ \gamma_{2n+2} &= q^n[n+1]_q \frac{\alpha+\beta+2-(\beta+1)q^n+[n]_q}{(\alpha+\beta+2+[2n]_q)(\alpha+\beta+2+[2n+1]_q)}. \end{aligned} \tag{3.35}$$

We denote the symmetrical form $u$ by $\mathcal{G}(\alpha,\beta,q)$. From (3.35) the symmetrical form $\mathcal{G}(\alpha,\beta,q)$ is regular if and only if the conditions in (3.34) hold. It is $H_{\sqrt{q}}$-semiclassical of class one for $\alpha+\beta \neq \frac{3-2q}{q-1}$, $\alpha+\beta \neq -[n]_q - 2$, $\beta \neq -[n]_q - 1$, $\alpha+\beta+2-(\beta+1)q^n+[n]_q \neq 0$, $n \geq 0$, $\beta \neq \frac{1}{\sqrt{q}(\sqrt{q}+1)} - 1$ satisfying

$$H_q\big(x(x^2-1)\mathcal{G}(\alpha,\beta,q)\big) - (\sqrt{q}+1)\big((\alpha+\beta+2)x^2 - (\beta+1)\big)\mathcal{G}(\alpha,\beta,q) = 0.$$

**Remark 5.** The symmetrical form $\mathcal{G}(\alpha,\beta,q)$ is the $q$-analogue of the symmetrical generalized Gegenbauer $\mathcal{G}(\alpha,\beta)$ form (see (1.3)) which is semiclassical of class one for $\alpha\neq -n-1$, $\beta\neq -n-1$, $\beta\neq -\frac{1}{2}$, $\alpha+\beta\neq -n-1$, $n\geq 0$ [1, 6].

**D.** In the case $\varphi(x)=x-b^{-1}q^{-1}$ the little $q$-Jacobi $\mathcal{U}(a,b,q)$ form (case 1.6 in Table 1). The $q$-analog of the distributional equation of Pearson type in (3.3), (3.4) become

$$H_q\big(x\big(x-b^{-1}q^{-1}\big)\sigma u\big)+\big(abq^2(q-1)\big)^{-1}\big(\big(1-abq^2\big)x+aq-1\big)\sigma u=0,$$
$$H_q\big(x\big(x-b^{-1}q^{-1}\big)(x\sigma u)\big)+\big(abq^3(q-1)\big)^{-1}\big(\big(1-abq^3\big)x+aq^2-1\big)(x\sigma u)=0.$$

Hence

$$\sigma u=\mathcal{U}(a,b,q), \tag{3.36}$$
$$x\sigma u=\frac{1-aq}{1-abq^2}\mathcal{U}(aq,b,q) \tag{3.37}$$

with the constraints

$$ab\neq 0,\qquad a\neq q^{-n-1},\qquad b\neq q^{-n-1},\qquad ab\neq q^{-n},\qquad n\geq 0. \tag{3.38}$$

By Table 1 and (3.36), (3.37), the systems (2.7), (2.8) lead to for $n\geq 0$

$$\beta_n^P=q^n\frac{(1+a)(1+abq^{2n+1})-a(1+b)(1+q)q^n}{(1-abq^{2n})(1-abq^{2n+2})},$$
$$\gamma_{n+1}^P=aq^{2n+1}\frac{(1-q^{n+1})(1-aq^{n+1})(1-bq^{n+1})(1-abq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})^2(1-abq^{2n+3})},$$
$$\beta_n^R=q^n\frac{(1+aq)(1+abq^{2n+2})-a(1+b)(1+q)q^{n+1}}{(1-abq^{2n+1})(1-abq^{2n+3})},$$
$$\gamma_{n+1}^R=aq^{2n+2}\frac{(1-q^{n+1})(1-aq^{n+2})(1-bq^{n+1})(1-abq^{n+2})}{(1-abq^{2n+2})(1-abq^{2n+3})^2(1-abq^{2n+4})}.$$

Using the above results and (2.9) we get for $n\geq 0$

$$\gamma_{2n+1}=q^n\frac{(1-aq^{n+1})(1-abq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})},\qquad \gamma_{2n+2}=aq^{n+1}\frac{(1-q^{n+1})(1-bq^{n+1})}{(1-abq^{2n+2})(1-abq^{2n+3})}.$$

Therefore, the symmetrical form $u=u(a,b,q)$ is regular if and only if the conditions in (3.38) are satisfied. Further, the form $u$ is positive definite for $0<q<1$, $0<a<q^{-1}$, $b<1$, $b\neq 0$ or $q>1$, $a>q^{-1}$, $b\geq 1$. Moreover, by virtue of (2.6), the form $u$ is $H_{\sqrt{q}}$-semiclassical of class one for $ab\neq 0$, $a\neq q^{-n-1}$, $b\neq q^{-n-1}$, $ab\neq q^{-n}$, $n\geq 0$, $a\neq q^{-\frac{1}{2}}$

$$H_{\sqrt{q}}\big(x\big(x^2-b^{-1}q^{-1}\big)u\big)+\big(abq^2(\sqrt{q}-1)\big)^{-1}\big(\big(1-abq^2\big)x^2+aq-1\big)u=0.$$

Proposition 1 and the well known representations of the little $q$-Jacobi form (Table 1) allow us to establish the following results

$$(u)_{2n}=\frac{(aq;q)_n}{(abq^2;q)_n},\qquad (u)_{2n+1}=0,\qquad n\geq 0.$$

For $f\in\mathcal{P}$, $0<q<1$, $0<a<q^{-1}$, $b<1$, $b\neq 0$,

$$u=\frac{(aq;q)_\infty}{(abq^2;q)_\infty}\sum_{k=0}^{\infty}\frac{(aq)^k(bq;q)_k)}{(q;q)_k}\frac{\delta_{-q^{\frac{k}{2}}}+\delta_{q^{\frac{k}{2}}}}{2},$$

and

$$\langle u, f\rangle = K\int_{-q^{-\frac{1}{2}}}^{q^{-\frac{1}{2}}} |x|^{2\frac{\ln a}{\ln q}+1}\frac{(qx^2;q)_\infty}{(bqx^2;q)_\infty}f(x)dx,$$

with

$$K^{-1} = \int_0^{q^{-1}} x^{\frac{\ln a}{\ln q}}\frac{(qx;q)_\infty}{(bqx;q)_\infty}dx.$$

For $f\in\mathcal{P}$, $q>1$, $a>q^{-1}$, $b\geq 1$

$$u = \frac{(a^{-1}q^{-1};q^{-1})_\infty}{(a^{-1}b^{-1}q^{-2};q^{-1})_\infty}\sum_{k=0}^{\infty}\frac{(aq)^{-k}(b^{-1}q^{-1};q^{-1})_k}{(q^{-1};q^{-1})_k}\frac{\delta_{-\sqrt{b^{-1}}q^{-\frac{k+1}{2}}}+\delta_{\sqrt{b^{-1}}q^{-\frac{k+1}{2}}}}{2},$$

and

$$\langle u, f\rangle = K\int_{-b^{-\frac{1}{2}}}^{b^{-\frac{1}{2}}} |x|^{2\frac{\ln a}{\ln q}+1}\frac{(bx^2;q^{-1})_\infty}{(x^2;q^{-1})_\infty}f(x)dx,$$

with

$$K^{-1} = \int_0^{b^{-1}} x^{\frac{\ln a}{\ln q}}\frac{(bx;q^{-1})_\infty}{(x;q^{-1})_\infty}dx.$$

**E.** In the case $\varphi(x) = x-\mu^{-1}q^{-1}$ the $q$-Charlier-II-form $\mathcal{U}(\mu,q)$ (case 1.7 in Table 1). From the above assumption (3.3), (3.4) are

$$H_q\big(x\big(x-\mu^{-1}q^{-1}\big)\sigma u\big) - (\mu q(q-1))^{-1}\big((\mu q-1)x-1\big)\sigma u = 0,$$
$$H_q\big(x\big(x-\mu^{-1}q^{-1}\big)(x\sigma u)\big) - (\mu q(q-1))^{-1}\big(\big(\mu q^2-1\big)q^{-1}x-1\big)(x\sigma u) = 0.$$

Thus

$$\sigma u = \mathcal{U}(\mu,q), \qquad \mu\neq 0, \qquad \mu\neq q^{-n}, \qquad n\geq 0,$$
$$x\sigma u = \frac{1}{\mu q-1}h_q\mathcal{U}(\mu q,q), \qquad \mu\neq 0, \qquad \mu\neq q^{-n}, \qquad n\geq 0.$$

Consequently, the systems (2.7), (2.8) for $n\geq 0$ are

$$\beta_n^P = q^{n-1}\frac{1-(1+q)q^n+\mu q^{2n}}{(1-\mu q^{2n-1})(1-\mu q^{2n+1})},$$
$$\gamma_{n+1}^P = -q^{3n}\frac{(1-q^{n+1})(1-\mu q^n)}{(1-\mu q^{2n})(1-\mu q^{2n+1})^2(1-\mu q^{2n+2})},$$
$$\beta_n^R = q^n\frac{1-(1+q)q^n+\mu q^{2n+1}}{(1-\mu q^{2n})(1-\mu q^{2n+2})},$$
$$\gamma_{n+1}^R = -q^{3n+2}\frac{(1-q^{n+1})(1-\mu q^{n+1})}{(1-\mu q^{2n+1})(1-\mu q^{2n+2})^2(1-\mu q^{2n+3})}.$$

On account of (2.9) we have for $n\geq 0$

$$\gamma_{2n+1} = -q^{2n}\frac{1-\mu q^n}{(1-\mu q^{2n})(1-\mu q^{2n+1})}, \qquad \gamma_{2n+2} = q^n\frac{1-q^{n+1}}{(1-\mu q^{2n+1})(1-\mu q^{2n+2})}.$$

From the last result, the symmetrical form $u = u(\mu, q)$ is regular if and only if $\mu \neq 0$, $\mu \neq q^{-n}$, $n \geq 0$. Moreover, by virtue of (2.6), it is clear that $u$ is $H_{\sqrt{q}}$-semiclassical of class one for $\mu \neq 0$, $\mu \neq q^{-n}$, $n \geq 0$ satisfying

$$H_{\sqrt{q}}\big(x\big(x^2 - \mu^{-1}q^{-1}\big)u\big) - (\mu q(\sqrt{q}-1))^{-1}\big\{(\mu q - 1)x^2 - 1\big\}u = 0.$$

Furthermore, by the same procedure as in **D** we get

$$(u)_{2n} = \frac{(-1)^n q^{\frac{1}{2}n(n-1)}}{(\mu q; q)_n}, \qquad (u)_{2n+1} = 0, \qquad n \geq 0,$$
$$u = \frac{1}{(\mu^{-1}q^{-1}; q^{-1})_\infty} \sum_{k=0}^{\infty} \frac{(-\mu^{-1})^k q^{-\frac{1}{2}k(k+1)}}{(q^{-1}; q^{-1})_k} \frac{\delta_{-i\sqrt{-\mu^{-1}}q^{-\frac{k+1}{2}}} + \delta_{i\sqrt{-\mu^{-1}}q^{-\frac{k+1}{2}}}}{2},$$

for $q > 1$, $\mu < 0$.

**F.** In the case $\varphi(x) = x + \omega q^{-\frac{3}{2}}$ the generalized Stieltjes–Wigert form $\mathcal{S}(\omega, q)$ (case 1.8 in Table 1). From (3.3), (3.4) it follows

$$H_q\big(x\big(x + \omega q^{-\frac{3}{2}}\big)\sigma u\big) - (q-1)^{-1}\big(x + (\omega - 1)q^{-\frac{3}{2}}\big)\sigma u = 0,$$
$$H_q\big(x\big(x + \omega q^{-\frac{3}{2}}\big)(x\sigma u)\big) - (q-1)^{-1}\big(x + (\omega q - 1)q^{-\frac{5}{2}}\big)(x\sigma u) = 0.$$

Thus

$$\sigma u = \mathcal{S}(\omega, q), \qquad \omega \neq q^{-n}, \qquad n \geq 0,$$
$$x\sigma u = (1-\omega)q^{-\frac{3}{2}} h_{q^{-1}}\mathcal{S}(\omega q, q), \qquad \omega \neq q^{-n}, \qquad n \geq 0.$$

We obtain for $n \geq 0$

$$\beta_n^P = \big\{(1+q)q^{-n} - q - \omega\big\}q^{-n-\frac{3}{2}},$$
$$\gamma_{n+1}^P = \big(1 - q^{n+1}\big)\big(1 - \omega q^n\big)q^{-4n-4},$$
$$\beta_n^R = \big\{(1+q)q^{-n} - q(1+\omega)\big\}q^{-n-\frac{5}{2}},$$
$$\gamma_{n+1}^R = \big(1 - q^{n+1}\big)\big(1 - \omega q^{n+1}\big)q^{-4n-6}.$$

Thus, (2.9) gives for $n \geq 0$

$$\gamma_{2n+1} = q^{-2n-\frac{3}{2}}\big(1 - \omega q^n\big), \qquad \gamma_{2n+2} = q^{-2n-\frac{5}{2}}\big(1 - q^{n+1}\big). \tag{3.39}$$

We recognize the Brenke type symmetrical orthogonal polynomials [8, 9, 10]

$$B_n = T_n(\cdot; \omega, q), \qquad n \geq 0.$$

We denote $u = \mathcal{T}(w, q)$. Taking into consideration (3.39), the symmetrical form $\mathcal{T}(\omega, q)$ is regular if and only if $\omega \neq q^{-n}$, $n \geq 0$, and it is positive definite for $0 < q < 1$, $\omega < 1$. Furthermore, it is easy to deduce that $\mathcal{T}(\omega, q)$ is $H_{\sqrt{q}}$-semiclassical of class one for $\omega \neq \sqrt{q}$, $\omega \neq q^{-n}$, $n \geq 0$ satisfying the $q$-analog of the distributional equation of Pearson type

$$H_{\sqrt{q}}\big(x\big(x^2 + \omega q^{-\frac{3}{2}}\big)\mathcal{T}(\omega, q)\big) - (\sqrt{q}-1)^{-1}\big(x^2 + (\omega - 1)q^{-\frac{3}{2}}\big)\mathcal{T}(\omega, q) = 0.$$

Finally, with Proposition 1 and the properties of the generalized Stieltjes–Wigert $H_q$-classical form (Table 1, case 1.8) we deduce the following results

$$(\mathcal{T}(\omega,q))_{2n} = q^{-\frac{1}{2}n(n+2)}(\omega;q)_n, \qquad (\mathcal{T}(\omega,q))_{2n+1} = 0, n \geq 0,$$
$$\mathcal{T}(\omega,q) = \left(\omega^{-1};q^{-1}\right)_\infty \sum_{k=0}^{\infty} \frac{\omega^{-k}}{(q^{-1};q^{-1})_k} \frac{\delta_{-i\sqrt{\omega}q^{-\frac{k}{2}-\frac{3}{4}}} + \delta_{i\sqrt{\omega}q^{-\frac{k}{2}-\frac{3}{4}}}}{2}, \qquad q>1, \qquad \omega>1,$$
$$\langle \mathcal{T}(\omega,q), f\rangle = K\int_{-\infty}^{\infty} \frac{|x|^{2\frac{\ln\omega}{\ln q}-1}}{(-q^{\frac{3}{2}}\omega^{-1}x^2;q)_\infty} f(x)dx,$$
$$f\in\mathcal{P}, \qquad 0<q<1, \qquad 0<\omega<1,$$

with

$$K^{-1} = \int_0^{\infty} \frac{x^{\frac{\ln\omega}{\ln q}-1}}{(-q^{\frac{3}{2}}\omega^{-1}x;q)_\infty}dx$$

is given by (2.4),

$$\langle \mathcal{T}(0,q), f\rangle = \sqrt{\frac{q}{2\pi\ln q^{-1}}}\int_{-\infty}^{\infty} |x| \exp\left(\frac{-2\ln^2|x|}{\ln q^{-1}}\right) f(x)dx, \qquad 0<q<1.$$

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