On Brane Solutions Related to Non-Singular Kac–Moody Algebras

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Abstract. A multidimensional gravitational model containing scalar fields and antisymmetric forms is considered. The manifold is chosen in the form $M = M_0 \times M_1 \times \cdots \times M_n$, where M_i are Einstein spaces ($i \geq 1$). The sigma-model approach and exact solutions with intersecting composite branes (e.g. solutions with harmonic functions, S-brane and black brane ones) with intersection rules related to non-singular Kac–Moody (KM) algebras (e.g. hyperbolic ones) are reviewed. Some examples of solutions, e.g. corresponding to hyperbolic KM algebras: $H_2(q,q)$, AE_3, $HA_2^{(1)}$, E_{10} and Lorentzian KM algebra P_{10} are presented.

Key words: Kac–Moody algebras; S-branes; black branes; sigma-model; Toda chains

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1 Introduction

Kac–Moody (KM) Lie algebras [1, 2, 3] play a rather important role in different areas of mathematical physics (see [3, 4, 5, 6] and references therein).

We recall that KM Lie algebra is a Lie algebra generated by the relations [3]

$$\begin{aligned}
&[h_i, h_j] = 0, \qquad [e_i, f_j] = \delta_{ij} h_j,\\
&[h_i, e_j] = A_{ij} e_j, \qquad [h_i, f_j] = -A_{ij} f_j,\\
&(\mathrm{ad} e_i)^{1-A_{ij}}(e_j) = 0 \qquad (i \neq j),\\
&(\mathrm{ad} f_i)^{1-A_{ij}}(f_j) = 0 \qquad (i \neq j).
\end{aligned}$$

Here $A = (A_{ij})$ is a generalized Cartan matrix, $i, j = 1, \dots, r$, and r is the rank of the KM algebra. It means that all $A_{ii} = 2$; A_{ij} for $i \neq j$ are non-positive integers and $A_{ij} = 0$ implies $A_{ji} = 0$.

In what follows the matrix A is restricted to be non-degenerate (i.e. $\det A \neq 0$) and symmetrizable i.e. $A = B\mathcal{D}$, where B is a symmetric matrix and $\mathcal{D}$ is an invertible diagonal matrix ($\mathcal{D}$ may be chosen in such way that all its entries $\mathcal{D}_{ii}$ are positive rational numbers [3]). Here we do not consider singular KM algebras with $\det A = 0$, e.g. affine ones. Recall that affine KM algebras are of much interest for conformal field theories, superstring theories etc. [4, 7].

In the case when A is positive definite (the Euclidean case) we get ordinary finite dimensional Lie algebras [3, 4]. For non-Euclidean signatures of A all KM algebras are infinite-dimensional. Among these the Lorentzian KM algebras with pseudo-Euclidean signatures $(-, +, \dots, +)$ for the Cartan matrix A are of current interest, since they contain a subclass of the so-called hyperbolic KM algebras widely used in modern mathematical physics. Hyperbolic KM algebras are by definition Lorentzian Kac–Moody algebras with the property that removing any node from their Dynkin diagram leaves one with a Dynkin diagram of the affine or finite type. The hyperbolic KM algebras can be completely classified [8, 9] and have rank $2 \leq r \leq 10$. For $r \geq 3$ there is a finite number of hyperbolic algebras. For rank 10, there are four algebras, known as E_{10}, BE_{10}, CE_{10} and DE_{10}. Hyperbolic KM algebras appeared in ordinary gravity [10] ($\mathcal{F}_3 = AE_3 = H_3$), supergravity: [11, 12] (E_{10}), [13] ($\mathcal{F}_3$), strings [14] etc.

The growth of interest in hyperbolic algebras in theoretical and mathematical physics appeared in 2001 after the publication of Damour and Henneaux [15] devoted to a description of chaotic (BKL-type [16]) behaviour near the singularity in string inspired low energy models (e.g. supergravitational ones) [17] (see also [18]). It should be noted that these results were based on a billiard approach in multidimensional cosmology with different matter sources (for $D = 4$ suggested by Chitre [19]) elaborated in our papers [20, 21, 22, 23, 24] (for a review see also [25, 26]).

The description of oscillating behaviour near the singularity in $D = 11$ supergravity [27] (which is related to M-theory [28, 29]) in terms of motion of a point-like particle in a 9-dimensional billiard (of finite volume) corresponding to the Weyl chamber of the hyperbolic KM algebra E_{10} inspired another description of $D = 11$ supergravity in [30]: a formal "small tension" expansion of $D = 11$ supergravity near a space-like singularity was shown to be equivalent (at least up to 30th order in height) to a null geodesic motion in the infinite dimensional coset space $\mathcal{E}_{10}/K(\mathcal{E}_{10})$ (here $K(\mathcal{E}_{10})$ is the maximal compact subgroup of the hyperbolic Kac–Moody group $\mathcal{E}_{10}(R)$).

Recall that E_{10} KM algebra is an over-extension of the finite dimensional Lie algebra E_8, i.e. $E_{10} = E_8^{++}$. But there is another extension of E_8 – the so-called the very extended Kac–Moody algebra of the E_8 algebra – called $E_{11} = E_8^{+++}$. (To get an understanding of very extended algebras and some of their properties see [31] and references therein). It has been proposed by P. West that the Lorentzian (non-hyperbolic) KM algebra E_{11} is responsible for a hidden algebraic structure characterizing 11-dimensional supergravity [32]. The same very extended algebra occurs in IIA [32] and IIB supergravities [33]. Moreover, it was conjectured that an analogous hidden structure is realized in the effective action of the bosonic string (with the KM algebra $k_{27} = D_{24}^{+++}$) [32] and also for pure D dimensional gravity (with the KM algebra A_{D-3}^{+++} [34]). It has been suggested in [35] that all the so-called maximally oxidised theories (see also [6]), possess the very extension G^{+++} of the simple Lie algebra G. It was shown in [36] that the BPS solutions of the oxidised theory of a simply laced group $\mathcal{G}$ form representations of a subgroup of the Weyl transformations of the algebra G^{+++}.

In this paper we briefly review another possibility for utilizing non-singular (e.g. hyperbolic) KM algebras suggested in three of our papers [37, 38, 39]. This possibility (implicitly assumed also in [40, 41, 42, 43, 44, 45, 46]) is related to certain classes of exact solutions describing intersecting composite branes in a multidimensional gravitational model containing scalar fields and antisymmetric forms defined on (warped) product manifolds $M = M_0 \times M_1 \times \cdots \times M_n$, where Einstein factor spaces M_i ($i \geq 1$) are Ricci-flat (at least) for $i \geq 2$. From a pure mathematical point of view these solutions may be obtained from sigma-models or Toda chains corresponding to certain non-singular KM algebras. The information about the (hidden) KM algebra is encoded in intersection rules which relate the dimensions of brane intersections with non-diagonal components of the generalized Cartan matrix A [47]. We deal here with generalized Cartan matrices of the form

$$A_{ss'} \equiv \frac{2(U^s, U^{s'})}{(U^{s'}, U^{s'})}, \qquad s, s' \in S, \tag{1.1}$$

with $(U^s, U^s) \neq 0$, for all $s \in S$ (S is a finite set). Here U^s are the so-called brane (co-)vectors. They are linear functions on $\mathbb{R}^N$, where $N = n + l$ and l is the number of scalar fields. The indefinite scalar product $(\cdot,\cdot)$ [48] is defined on $(\mathbb{R}^N)^*$ and has the signature $(-1,+1,\ldots,+1)$ when all scalar fields have positive kinetic terms, i.e. there are no phantoms (or ghosts). The matrix A is symmetrizable. U^s-vectors may be put in one-to-one correspondence with simple roots α_s of the generalized KM algebra after a suitable normalizing. For indecomposable A (when the KM algebra is simple) the matrices $((U^s,U^{s'}))$ and $((\alpha_s|\alpha_{s'}))$ are proportional to each other. Here $(\cdot|\cdot)$ is a non-degenerate bilinear symmetric form on a root space obeying $(\alpha_s|\alpha_s)>0$ for all simple roots α_s [3].

We note that the minisuperspace bilinear form $(\cdot,\cdot)$ coming from multidimensional gravitational model [48] "does not know" about the definition of $(\cdot|\cdot)$ in [3] and hence there exist physical examples where all (U^s,U^s) are negative. Some examples of this are given below in Section 5. For $D=11$ supergravity and ten dimensional IIA, IIB supergravities all $(U^s,U^s)=2$ [47, 49] and corresponding KM algebras are simply laced. It was shown in our papers [22, 23, 24] that the inequality $(U^s,U^s)>0$ is a necessary condition for the formation of the billiard wall (in one approaches the singularity) by the s-th matter source (e.g. a fluid component or a brane).

The scalar products for brane vectors U^s were found in [48] (for the electric case see also [50, 51, 52])

$$\left(U^s,U^{s'}\right)=d_{ss'}+\frac{d_sd_{s'}}{2-D}+\chi_s\chi_{s'}\langle\lambda_s,\lambda_{s'}\rangle, \tag{1.2}$$

where d_s and $d_{s'}$ are dimensions of the brane worldvolumes corresponding to branes s and s' respectively, $d_{ss'}$ is the dimension of intersection of the brane worldvolumes, D is the total space-time dimension, $\chi_s=+1,-1$ for electric or magnetic brane respectively, and $\langle\lambda_s,\lambda_{s'}\rangle$ is the non-degenerate scalar product of the l-dimensional dilatonic coupling vectors λ_s and $\lambda_{s'}$ corresponding to branes s and s'.

Relations (1.1), (1.2) define the brane intersection rules [47]

$$d_{ss'}=d^o_{ss'}+\frac{1}{2}K_{s'}A_{ss'},$$

$s\neq s'$, where $K_s=(U^s,U^s)$ and

$$d^o_{ss'}=\frac{d_sd_{s'}}{D-2}-\chi_s\chi_{s'}\langle\lambda_s,\lambda_{s'}\rangle \tag{1.3}$$

is the dimension of the so-called orthogonal (or $(A_1\oplus A_1)$-) intersection of branes following from the orthogonality condition [48]

$$\left(U^s,U^{s'}\right)=0, \tag{1.4}$$

$s\neq s'$. The orthogonality relations (1.4) for brane intersections in the non-composite electric case were suggested in [50, 51] and for the composite electric case in [52].

Relations (1.2) and (1.3) were derived in [48] for rather general assumptions: the branes were composite, the number of scalar fields l was arbitrary as well as the signature of the bilinear form $\langle\cdot,\cdot\rangle$ (or, equivalently, the signature of the kinetic term for scalar fields), (Einstein) factor spaces M_i had arbitrary dimensions d_i and signatures. The intersection rules (1.3) appeared earlier (in different notations) in [53, 54, 55] when all $d_i=1$ $(i>0)$ and $\langle\cdot,\cdot\rangle$ was positive definite (in [53, 54] $l=1$ and total space-time had a pseudo-Euclidean signature). The intersection rules (1.3) were also used in [56, 57, 58, 47] in the context of intersecting black brane solutions.

It was proved in [59] that the target space of the sigma model describing composite electromagnetic brane configurations on the product of several Ricci-flat spaces is a homogeneous (coset) space G/H. It is locally symmetric (i.e. the Riemann tensor is covariantly constant: $\nabla\,\mathrm{Riem}=0$) if and only if

$$\left(U^s-U^{s'}\right)\left(U^s,U^{s'}\right)=0$$

for all s and s', i.e. when any two brane vectors U^s and $U^{s'}$, $s \neq s'$, are either coinciding $U^s = U^{s'}$ or orthogonal $(U^s, U^{s'}) = 0$ (for two electric branes and $l = 1$ see also [60]).

Now relations for brane vectors U^s (1.1) and (1.2) (with U^s being identified with roots α_s) are widely used in the G^{+++}-approach [36, 6]. The orthogonality condition (1.4) describing the intersection of branes [50, 51, 52, 48] was rediscovered in [49] (for some particular intersecting configurations of M-theory it was done in [61]). It was found in the context of G^{+++}-algebras that the intersection rules for extremal branes are encoded in orthogonality conditions between the various roots from which the branes arise, i.e. $(\alpha_s|\alpha_{s'}) = 0$, $s \neq s'$, where α_s should be real positive roots ("real" means that $(\alpha_s|\alpha_s) > 0$). In [49] another condition on brane, root vectors was found: $\alpha_s + \alpha_{s'}$ should not be a root, $s \neq s'$. The last condition is trivial for M-theory and for IIA and IIB supergravities, but for low energy effective actions of heterotic strings it forbids certain intersections that does not take place due to non-zero contributions of Chern–Simons terms.

It should be noted that the orthogonality relations for brane intersections (1.4) which appeared in 1996–97, were not well understood by the superstring community at that time. The standard intersection rules (1.3) gave back the well-known zero binding energy configurations preserving some supersymmetries. These brane configurations were originally derived from supersymmetry and duality arguments (see for example [62, 63, 64] and reference therein) or by using a no-force condition (suggested for M-branes in [65]).

2 The model

2.1 The action

We consider the model governed by action

$$S = \frac{1}{2\kappa^2} \int_M d^D z \sqrt{|g|} \Big\{ R[g] - 2\Lambda - h_{\alpha\beta} g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta - \sum_{a \in \mathbf{\Delta}} \frac{\theta_a}{n_a!} \exp[2\lambda_a(\varphi)](F^a)^2_g \Big\} + S_{GH}, \tag{2.1}$$

where $g = g_{MN} dz^M \otimes dz^N$ is the metric on the manifold M, $\dim M = D$, $\varphi = (\varphi^\alpha) \in \mathbb{R}^l$ is a vector of dilatonic scalar fields, $(h_{\alpha\beta})$ is a non-degenerate symmetric $l \times l$ matrix ($l \in \mathbb{N}$), $\theta_a \neq 0$,

$$F^a = dA^a = \frac{1}{n_a!} F^a_{M_1 \ldots M_{n_a}} dz^{M_1} \wedge \cdots \wedge dz^{M_{n_a}}$$

is an n_a-form ($n_a \geq 2$) on a D-dimensional manifold M, Λ is a cosmological constant and λ_a is a 1-form on $\mathbb{R}^l$: $\lambda_a(\varphi) = \lambda_{a\alpha}\varphi^\alpha$, $a \in \mathbf{\Delta}$, $\alpha = 1, \ldots, l$. In (2.1) we denote $|g| = |\det(g_{MN})|$, $(F^a)^2_g = F^a_{M_1 \ldots M_{n_a}} F^a_{N_1 \ldots N_{n_a}} g^{M_1 N_1} \cdots g^{M_{n_a} N_{n_a}}$, $a \in \mathbf{\Delta}$, where $\mathbf{\Delta}$ is some finite set (for example, of positive integers), and $S_{\rm GH}$ is the standard Gibbons–Hawking boundary term [66]. In models with one time all $\theta_a = 1$ when the signature of the metric is $(-1, +1, \ldots, +1)$. κ^2 is the multidimensional gravitational constant.

2.2 Ansatz for composite branes

Let us consider the manifold

$$M = M_0 \times M_1 \times \cdots \times M_n, \tag{2.2}$$

with the metric

$$g = e^{2\gamma(x)} \hat{g}^0 + \sum_{i=1}^{n} e^{2\phi^i(x)} \hat{g}^i, \tag{2.3}$$

where $g^0 = g^0_{\mu\nu}(x) dx^\mu \otimes dx^\nu$ is an arbitrary metric with any signature on the manifold M_0 and $g^i = g^i_{m_i n_i}(y_i) dy_i^{m_i} \otimes dy_i^{n_i}$ is a metric on M_i satisfying the equation

$$R_{m_i n_i}[g^i] = \xi_i g^i_{m_i n_i}, \tag{2.4}$$

$m_i, n_i = 1, \dots, d_i$; $\xi_i = \text{const}$, $i = 1, \dots, n$. Here $\hat{g}^i = p_i^* g^i$ is the pullback of the metric g^i to the manifold M by the canonical projection: $p_i : M \to M_i$, $i = 0, \dots, n$. Thus, (M_i, g^i) are Einstein spaces, $i = 1, \dots, n$. The functions $\gamma, \phi^i : M_0 \to \mathbb{R}$ are smooth. We denote $d_\nu = \dim M_\nu$; $\nu = 0, \dots, n$; $D = \sum_{\nu=0}^n d_\nu$. We put any manifold M_ν, $\nu = 0, \dots, n$, to be oriented and connected. Then the volume d_i-form

$$\tau_i \equiv \sqrt{|g^i(y_i)|}\, dy_i^1 \wedge \dots \wedge dy_i^{d_i},$$

and signature parameter

$$\varepsilon(i) \equiv \text{sign}(\det(g^i_{m_i n_i})) = \pm 1$$

are correctly defined for all $i = 1, \dots, n$.

Let $\Omega = \Omega(n)$ be a set of all non-empty subsets of $\{1, \dots, n\}$. The number of elements in Ω is $|\Omega| = 2^n - 1$. For any $I = \{i_1, \dots, i_k\} \in \Omega$, $i_1 < \dots < i_k$, we denote

$$\begin{aligned}
&\tau(I) \equiv \hat{\tau}_{i_1} \wedge \dots \wedge \hat{\tau}_{i_k}, \\
&\varepsilon(I) \equiv \varepsilon(i_1) \cdots \varepsilon(i_k), \\
&M_I \equiv M_{i_1} \times \dots \times M_{i_k}, \\
&d(I) \equiv \sum_{i \in I} d_i.
\end{aligned}$$

Here $\hat{\tau}_i = p_i^* \hat{\tau}_i$ is the pullback of the form τ_i to the manifold M by the canonical projection: $p_i : M \to M_i$, $i = 1, \dots, n$. We also put $\tau(\varnothing) = \varepsilon(\varnothing) = 1$ and $d(\varnothing) = 0$.

For fields of forms we consider the following composite electromagnetic ansatz

$$F^a = \sum_{I \in \Omega_{a,e}} \mathcal{F}^{(a,e,I)} + \sum_{J \in \Omega_{a,m}} \mathcal{F}^{(a,m,J)}, \tag{2.5}$$

where

$$\mathcal{F}^{(a,e,I)} = d\Phi^{(a,e,I)} \wedge \tau(I), \tag{2.6}$$

$$\mathcal{F}^{(a,m,J)} = e^{-2\lambda_a(\varphi)} * (d\Phi^{(a,m,J)} \wedge \tau(J)) \tag{2.7}$$

are elementary forms of electric and magnetic types respectively, $a \in \mathbf{\Delta}$, $I \in \Omega_{a,e}$, $J \in \Omega_{a,m}$ and $\Omega_{a,v} \subset \Omega$, $v = e, m$. In (2.7) $* = *[g]$ is the Hodge operator on (M, g).

For scalar functions we put

$$\varphi^\alpha = \varphi^\alpha(x), \qquad \Phi^s = \Phi^s(x), \qquad s \in S. \tag{2.8}$$

Thus, φ^α and Φ^s are functions on M_0.

Here and below

$$S = S_e \sqcup S_m, \qquad S_v = \sqcup_{a \in \mathbf{\Delta}} \{a\} \times \{v\} \times \Omega_{a,v}, \qquad v = e, m. \tag{2.9}$$

Here and in what follows $\sqcup$ means the union of non-intersecting sets. The set S consists of elements $s = (a_s, v_s, I_s)$, where $a_s \in \mathbf{\Delta}$ is color index, $v_s = e, m$ is electro-magnetic index and set $I_s \in \Omega_{a_s,v_s}$ describes the location of brane.

Due to (2.6) and (2.7)

$$d(I) = n_a - 1, \qquad d(J) = D - n_a - 1, \tag{2.10}$$

for $I \in \Omega_{a,e}$ and $J \in \Omega_{a,m}$ (i.e. in the electric and magnetic case, respectively).

2.3 The sigma model

Let $d_0 \neq 2$ and

$$\gamma = \gamma_0(\phi) \equiv \frac{1}{2 - d_0} \sum_{j=1}^{n} d_j \phi^j,$$

i.e. the generalized harmonic gauge (frame) is used.

Here we put two restrictions on sets of branes that guarantee the block-diagonal form of the energy-momentum tensor and the existence of the sigma-model representation (without additional constraints):

$$(\mathbf{R1}) \quad d(I \cap J) \leq d(I) - 2, \tag{2.11}$$

for any $I, J \in \Omega_{a,v}$, $a \in \mathbf{\Delta}$, $v = e, m$ (here $d(I) = d(J)$) and

$$(\mathbf{R2}) \quad d(I \cap J) \neq 0 \quad \text{for} \quad d_0 = 1, \qquad d(I \cap J) \neq 1 \quad \text{for} \quad d_0 = 3. \tag{2.12}$$

It was proved in [48] that equations of motion for the model (2.1) and the Bianchi identities:

$$d\mathcal{F}^s = 0, \qquad s \in S_m,$$

for fields from (2.3), (2.5)–(2.8), when Restrictions (2.11) and (2.12) are imposed, are equivalent to the equations of motion for the σ-model governed by the action

$$S_{\sigma 0} = \frac{1}{2\kappa_0^2} \int d^{d_0}x \sqrt{|g^0|} \Big\{ R[g^0] - \hat{G}_{AB} g^{0\mu\nu} \partial_\mu \sigma^A \partial_\nu \sigma^B \\ - \sum_{s \in S} \varepsilon_s \exp(-2U_A^s \sigma^A) g^{0\mu\nu} \partial_\mu \Phi^s \partial_\nu \Phi^s - 2V \Big\}, \tag{2.13}$$

where $(\sigma^A) = (\phi^i, \varphi^\alpha)$, $k_0 \neq 0$, the index set S is defined in (2.9),

$$V = V(\phi) = \Lambda e^{2\gamma_0(\phi)} - \frac{1}{2} \sum_{i=1}^{n} \xi_i d_i e^{-2\phi^i + 2\gamma_0(\phi)}$$

is the potential,

$$(\hat{G}_{AB}) = \begin{pmatrix} G_{ij} & 0 \\ 0 & h_{\alpha\beta} \end{pmatrix} \tag{2.14}$$

is the target space metric with

$$G_{ij} = d_i \delta_{ij} + \frac{d_i d_j}{d_0 - 2}$$

and co-vectors

$$U_A^s = U_A^s \sigma^A = \sum_{i \in I_s} d_i \phi^i - \chi_s \lambda_{a_s}(\varphi), \qquad (U_A^s) = (d_i \delta_{iI_s}, -\chi_s \lambda_{a_s \alpha}), \qquad s = (a_s, v_s, I_s). \quad (2.15)$$

Here $\chi_s = +1$ for $v_s = e$ and $\chi_s = -1$ for $v_s = m$;

$$\delta_{iI} = \sum_{j \in I} \delta_{ij}$$

is an indicator of i belonging to I: $\delta_{iI} = 1$ for $i \in I$ and $\delta_{iI} = 0$ otherwise; and

$$\varepsilon_s = (-\varepsilon[g])^{(1-\chi_s)/2} \varepsilon(I_s) \theta_{a_s}, \qquad s \in S, \qquad \varepsilon[g] \equiv \operatorname{sign} \det(g_{MN}). \qquad (2.16)$$

More explicitly (2.16) reads

$$\varepsilon_s = \varepsilon(I_s)\theta_{a_s} \quad \text{for} \quad v_s = e, \qquad \varepsilon_s = -\varepsilon[g]\varepsilon(I_s)\theta_{a_s} \quad \text{for} \quad v_s = m.$$

For finite internal space volumes V_i (e.g. compact M_i) and electric p-branes (i.e. all $\Omega_{a,m} = \varnothing$) the action (2.13) coincides with the action (2.1) when $\kappa^2 = \kappa_0^2 \prod_{i=1}^n V_i$.

3 Solutions governed by harmonic functions

3.1 Solutions with block-orthogonal set of U^s and Ricci-flat factor-spaces

Here we consider a special class of solutions to equations of motion governed by several harmonic functions when all factor spaces are Ricci-flat and the cosmological constant is zero, i.e. $\xi_i = \Lambda = 0$, $i = 1, \dots, n$. In certain situations these solutions describe extremal black branes charged by fields of forms.

The solutions crucially depend upon scalar products of U^s-vectors $(U^s, U^{s'})$; $s, s' \in S$, where

$$(U, U') = \hat{G}^{AB} U_A U'_B, \qquad (3.1)$$

for $U = (U_A), U' = (U'_A) \in \mathbb{R}^N$, $N = n + l$ and

$$(\hat{G}^{AB}) = \begin{pmatrix} G^{ij} & 0 \\ 0 & h^{\alpha\beta} \end{pmatrix}$$

is the inverse matrix to the matrix (2.14). Here as in [67] we have

$$G^{ij} = \frac{\delta^{ij}}{d_i} + \frac{1}{2-D}, \qquad i, j = 1, \dots, n.$$

The scalar products (3.1) for vectors U^s were calculated in [48] and are given by

$$(U^s, U^{s'}) = d(I_s \cap I_{s'}) + \frac{d(I_s) d(I_{s'})}{2-D} + \chi_s \chi_{s'} \lambda_{a_s \alpha} \lambda_{a_{s'} \beta} h^{\alpha\beta}, \qquad (3.2)$$

where $(h^{\alpha\beta}) = (h_{\alpha\beta})^{-1}$, and $s = (a_s, v_s, I_s)$, $s' = (a_{s'}, v_{s'}, I_{s'})$ belong to S. This relation is a very important one since it encodes brane data (e.g. intersections) via the scalar products of U-vectors.

Let

$$S = S_1 \sqcup \dots \sqcup S_k, \qquad S_i \neq \varnothing, \qquad i = 1, \dots, k, \qquad (3.3)$$

and

$$(U^s, U^{s'}) = 0 \tag{3.4}$$

for all $s \in S_i$, $s' \in S_j$, $i \neq j$; $i, j = 1, \dots, k$. Relation (3.3) means that the set S is a union of k non-intersecting (non-empty) subsets $S_1, \dots, S_k$. According to (3.4) the set of vectors $(U^s, s \in S)$ has a block-orthogonal structure with respect to the scalar product (3.1), i.e. it splits into k mutually orthogonal blocks $(U^s, s \in S_i)$, $i = 1, \dots, k$.

Here we consider exact solutions in the model (2.1), when vectors $(U^s, s \in S)$ obey the block-orthogonal decomposition (3.3), (3.4) with scalar products defined in (3.2) [37]. These solutions were obtained from the corresponding solutions to the σ-model equations of motion [37].

Proposition 3.1. *Let (M_0, g^0) be Ricci-flat: $R_{\mu\nu}[g^0] = 0$. Then the field configuration*

$$g^0, \qquad \sigma^A = \sum_{s \in S} \varepsilon_s U^{sA} \nu_s^2 \ln H_s, \qquad \Phi^s = \frac{\nu_s}{H_s}, \qquad s \in S,$$

satisfies the field equations corresponding to the action (2.13) *with $V = 0$ if the real numbers ν_s obey the relations*

$$\sum_{s' \in S} (U^s, U^{s'}) \varepsilon_{s'} \nu_{s'}^2 = -1, \qquad s \in S, \tag{3.5}$$

the functions $H_s > 0$ are harmonic, i.e. $\Delta[g^0] H_s = 0$, $s \in S$, and H_s are coinciding inside blocks: $H_s = H_{s'}$ for $s, s' \in S_i$, $i = 1, \dots, k$.

Using the sigma-model solution from Proposition 3.1 and the relations for contra-variant components [48]:

$$U^{si} = \delta_{iI_s} - \frac{d(I_s)}{D-2}, \qquad U^{s\alpha} = -\chi_s \lambda_{a_s}^{\alpha}, \qquad s = (a_s, v_s, I_s),$$

we get [37]:

$$g = \left(\prod_{s \in S} H_s^{2d(I_s)\varepsilon_s \nu_s^2} \right)^{1/(2-D)} \left\{ \hat{g}^0 + \sum_{i=1}^{n} \left(\prod_{s \in S} H_s^{2\varepsilon_s \nu_s^2 \delta_{iI_s}} \right) \hat{g}^i \right\}, \tag{3.6}$$

$$\varphi^\alpha = -\sum_{s \in S} \lambda_{a_s}^{\alpha} \chi_s \varepsilon_s \nu_s^2 \ln H_s, \tag{3.7}$$

$$F^a = \sum_{s \in S} \mathcal{F}^s \delta_{a_s}^a, \tag{3.8}$$

where $i = 1, \dots, n$, $\alpha = 1, \dots, l$, $a \in \mathbf{\Delta}$ and

$$\mathcal{F}^s = \nu_s dH_s^{-1} \wedge \tau(I_s) \quad \text{for} \quad v_s = e, \tag{3.9}$$

$$\mathcal{F}^s = \nu_s (*_0 dH_s) \wedge \tau(\bar{I}_s) \quad \text{for} \quad v_s = m, \tag{3.10}$$

H_s are harmonic functions on (M_0, g^0) which coincide inside blocks (i.e. $H_s = H_{s'}$ for $s, s' \in S_i$, $i = 1, \dots, k$) and the relations (3.5) on the parameters ν_s are imposed. Here the matrix $((U^s, U^{s'}))$ and parameters ε_s, $s \in S$, are defined in (3.2) and (2.16), respectively; $\lambda_a^\alpha = h^{\alpha\beta} \lambda_{a\beta}$, $*_0 = *[g^0]$ is the Hodge operator on (M_0, g^0) and

$$\bar{I} = \{1, \dots, n\} \setminus I$$

is the dual set. (In (3.10) we redefined the sign of ν_s-parameter.)

3.2 Solutions related to non-singular KM algebras

Now we study the solutions (3.6)–(3.10) in more detail and show that some of them may be related to non-singular KM algebras. We put

$$K_s \equiv (U^s, U^s) \neq 0$$

for all $s \in S$ and introduce the matrix $A = (A_{ss'})$:

$$A_{ss'} \equiv \frac{2(U^s, U^{s'})}{(U^{s'}, U^{s'})}, \qquad s, s' \in S. \tag{3.11}$$

Here some ordering in S is assumed.

Using this definition and (3.2) we obtain the intersection rules [47]

$$d(I_s \cap I_{s'}) = \Delta(s, s') + \frac{1}{2} K_{s'} A_{ss'}, \tag{3.12}$$

where $s \neq s'$, and

$$\Delta(s, s') = \frac{d(I_s) d(I_{s'})}{D-2} - \chi_s \chi_{s'} \lambda_{a_s \alpha} \lambda_{a_{s'} \beta} h^{\alpha\beta} \tag{3.13}$$

defines the so-called "orthogonal" intersection rules [48] (see also [53, 54, 55] for $d_i = 1$).

In $D = 11$ and $D = 10$ (IIA and IIB) supergravity models all $K_s = 2$ and hence the intersection rules (3.12) in this case have a simpler form [47]:

$$d(I_s \cap I_{s'}) = \Delta(s, s') + A_{ss'}, \tag{3.14}$$

where $s \neq s'$, implying $A_{ss'} = A_{s's}$. The corresponding KM algebra is simply-laced in this case.

For $\det A \neq 0$ relation (3.5) may be rewritten in the equivalent form

$$-\varepsilon_s \nu_s^2 (U^s, U^s) = 2 \sum_{s' \in S} A^{ss'} \equiv b_s, \tag{3.15}$$

where $s \in S$ and $(A^{ss'}) = A^{-1}$. Thus, equation (3.5) may be resolved in terms of ν_s for certain $\varepsilon_s = \pm 1$, $s \in S$. We note that due to (3.4) the matrix A has a block-diagonal structure and, hence, for any i-th block the set of parameters $(\nu_s, s \in S_i)$ depends upon the matrix inverse to the matrix $(A_{ss'}; s, s' \in S_i)$.

Now we consider one-block case when the brane intersections are related to some non-singular KM algebras.

Finite-dimensional Lie algebras [38]. Let A be a Cartan matrix of a simple finite-dimensional Lie algebra. In this case $A_{ss'} \in \{0, -1, -2, -3\}$, $s \neq s'$. The elements of inverse matrix A^{-1} are positive (see Chapter 7 in [4]) and hence we get from (3.15) the same signature relation as in the orthogonal case [48]:

$$\varepsilon_s \big(U^s, U^s\big) < 0, \qquad s \in S.$$

When all $(U^s, U^s) > 0$ we get $\varepsilon_s < 0$, $s \in S$.

Moreover, all b_s are natural numbers:

$$b_s = n_s \in \mathbb{N}, \qquad s \in S. \tag{3.16}$$

The integers n_s coincide with the components of the twice dual Weyl vector in the basis of simple co-roots (see Chather 3.1.7 in [4]).

Hyperbolic KM algebras. Let A be a generalized Cartan matrix corresponding to a simple hyperbolic KM algebra.

For the hyperbolic algebras the following relations are satisfied

$$\varepsilon_s(U^s, U^s) > 0, \tag{3.17}$$

since all b_s are negative in the hyperbolic case [31]:

$$b_s < 0, \qquad \text{where} \quad s \in S. \tag{3.18}$$

For $(U^s, U^s) > 0$ we get $\varepsilon_s > 0$, $s \in S$. If $\theta_{a_s} > 0$ for all $s \in S$, then

$$\varepsilon(I_s) = 1 \quad \text{for} \quad v_s = e, \qquad \varepsilon(I_s) = -\varepsilon[g] \quad \text{for} \quad v_s = m.$$

For a metric g of pseudo-Euclidean signature all $\varepsilon(I_s) = 1$ and, hence, all branes are Euclidean or should contain even number of time directions: $2, 4, \dots$. For $\varepsilon[g] = 1$ only magnetic branes may be pseudo-Euclidean.

Remark 3.1. The inequalities (3.18) guarantee the existence of a principal (real) $so(1,2)$ subalgebra for any hyperbolic Kac–Moody algebra [68, 31]. Similarly the inequalities (3.16) imply the existence of a principal $so(3)$ subalgebra for any finite dimensional (semi-)simple Lie algebra. It was shown in [31] that this property is not just restricted to hyperbolic algebras, but holds for a wider class of Lorentzian algebras obeying the inequalities $b_s \leq 0$ for all s.

Example 3.1. $\mathcal{F}_3 = AE_3$ algebra [39]. Now we consider an example of the solution corresponding to the hyperbolic KM algebra $\mathcal{F}_3$ with the Cartan matrix

$$A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \tag{3.19}$$

$\mathcal{F}_3$ contains $\mathbf{A_1^{(1)}}$ affine subalgebra (it corresponds to the Geroch group) and $\mathbf{A_2}$ subalgebra. There exists an example of the solution with the A-matrix (3.19) for 11-dimensional model governed by the action

$$S = \int d^{11}z \sqrt{|g|} \left\{ R[g] - \frac{1}{4!}\left(F^4\right)^2 - \frac{1}{4!}\left(F^{4*}\right)^2 \right\},$$

where $\operatorname{rank} F^4 = \operatorname{rank} F^{4*} = 4$. Here $\mathbf{\Delta} = \{4, 4*\}$. We consider a configuration with two magnetic 5-branes corresponding to the form F^4 and one electric 2-brane corresponding to the form F^{4*}. We denote $S = \{s_1, s_2, s_3\}$, $a_{s_1} = a_{s_3} = 4$, $a_{s_2} = 4*$ and $v_{s_1} = v_{s_3} = m$, $v_{s_2} = e$, where $d(I_{s_1}) = d(I_{s_3}) = 6$ and $d(I_{s_2}) = 3$.

The intersection rules (3.12) read

$$d(I_{s_1} \cap I_{s_2}) = 0, \qquad d(I_{s_2} \cap I_{s_3}) = 1, \qquad d(I_{s_1} \cap I_{s_3}) = 4.$$

For the manifold (2.2) we put $n = 5$ and $d_1 = 2$, $d_2 = 4$, $d_3 = d_4 = 1$, $d_5 = 2$. The corresponding brane sets are the following: $I_{s_1} = \{1, 2\}$, $I_{s_2} = \{4, 5\}$, $I_{s_3} = \{2, 3, 4\}$.

The solution reads

$$g = H^{-12}\left\{-dt \otimes dt + H^9 \hat{g}^1 + H^{13}\hat{g}^2 + H^4 \hat{g}^3 + H^{14}\hat{g}^4 + H^{10}\hat{g}^5\right\}, \tag{3.20}$$

$$F^4 = \frac{dH}{dt}\left\{\nu_{s_1}\hat{\tau}_3 \wedge \hat{\tau}_4 \wedge \hat{\tau}_5 + \nu_{s_3}\hat{\tau}_1 \wedge \hat{\tau}_5\right\},$$

$$F^{4*} = \frac{dH}{dt}\frac{\nu_{s_2}}{H^2} dt \wedge \hat{\tau}_4 \wedge \hat{\tau}_5,$$

where $\nu_{s_1}^2 = \frac{9}{2}$, $\nu_{s_2}^2 = 5$ and $\nu_{s_3}^2 = 2$ (see (3.15)).

All metrics g^i are Ricci-flat ($i = 1, \ldots, 5$) and have Euclidean signatures (this agrees with relations (3.17) and (2.16)), and $H = ht + h_0 > 0$, where h, h_0 are constants. The metric (3.20) may be also rewritten using the synchronous time variable t_s

$$g = -dt_s \otimes dt_s + f^{3/5}\hat{g}^1 + f^{-1/5}\hat{g}^2 + f^{8/5}\hat{g}^3 + f^{-2/5}\hat{g}^4 + f^{2/5}\hat{g}^5,$$

where $f = 5ht_s = H^{-5} > 0$, $h > 0$ and $t_s > 0$. The metric describes the power-law "inflation" in $D = 11$. It is singular for $t_s \to +0$.

In the next example we consider a chain of the so-called B_D-models ($D \geq 11$) suggested in [47]. For $D = 11$ the B_D-model coincides with the truncated (i.e. without Chern–Simons term) bosonic sector of $D = 11$ supergravity [27] which is related to M-theory. For $D = 12$ it coincides with truncated 12-dimensional model from [69] which may be related to F-theory [70].

***B_D*-models.** The B_D-model has the action [47]

$$S_D = \int d^D z \sqrt{|g|}\left\{R[g] + g^{MN}\partial_M \vec{\varphi}\partial_N \vec{\varphi} - \sum_{a=4}^{D-7} \frac{1}{a!}\exp[2\vec{\lambda}_a \vec{\varphi}](F^a)^2\right\}, \tag{3.21}$$

where $\vec{\varphi} = (\varphi^1, \ldots, \varphi^l) \in \mathbb{R}^l$, $\vec{\lambda}_a = (\lambda_{a1}, \ldots, \lambda_{al}) \in \mathbb{R}^l$, $l = D - 11$, $\mathrm{rank} F^a = a$, $a = 4, \ldots, D - 7$. Here vectors $\vec{\lambda}_a$ satisfy the relations

$$\vec{\lambda}_a \vec{\lambda}_b = N(a,b) - \frac{(a-1)(b-1)}{D-2} = \Lambda_{ab}, \qquad N(a,b) = \min(a,b) - 3, \tag{3.22}$$

$a, b = 4, \ldots, D-7$ and $\vec{\lambda}_{D-7} = -2\vec{\lambda}_4$. For $D > 11$ vectors $\vec{\lambda}_4, \ldots, \vec{\lambda}_{D-8}$ are linearly independent. (It may be verified that matrix (Λ_{ab}) is positive definite and hence the set of vectors obeying (3.22) does exist.)

The model (3.21) contains l scalar fields with a negative kinetic term (i.e. $h_{\alpha\beta} = -\delta_{\alpha\beta}$ in (2.1)) coupled to several forms (the number of forms is $(l+1)$) .

For the brane worldvolumes we have the following dimensions (see (2.10))

$$\begin{aligned} &d(I) = 3, \ldots, D-8, \qquad && I \in \Omega_{a,e}, \\ &d(I) = D-5, \ldots, 6, \qquad && I \in \Omega_{a,m}. \end{aligned}$$

Thus, there are $(l+1)$ electric and $(l+1)$ magnetic p-branes, $p = d(I) - 1$. In B_D-model all $K_s = 2$.

Example 3.2. $H_2(q_1, q_2)$ algebra [37]. Let

$$A = \begin{pmatrix} 2 & -q_1 \\ -q_2 & 2 \end{pmatrix}, \qquad q_1 q_2 > 4,$$

$q_1, q_2 \in \mathbb{N}$. This is the Cartan matrix for the hyperbolic KM algebra $H_2(q_1, q_2)$ [3]. From (3.15) we get

$$\varepsilon_s \nu_s^2 (U^s, U^s)(q_1 q_2 - 4) = 2q_s + 4,$$

$s \in \{1, 2\} = S$. An example of the $H_2(q,q)$-solution for B_D-model with two electric p-branes ($p = d_1, d_2$), corresponding to F^a and F^b fields and intersecting on time manifold, is the following [37]:

$$g = H^{-2/(q-2)}\hat{g}^0 - H^{2/(q-2)} dt \otimes dt + \hat{g}^1 + \hat{g}^2,$$

$$F^a = \nu_1 dH^{-1} \wedge dt \wedge \hat{\tau}_1,$$
$$F^b = \nu_2 dH^{-1} \wedge dt \wedge \hat{\tau}_2,$$
$$\vec{\varphi} = \big(\vec{\lambda}_a + \vec{\lambda}_b\big)(q-2)^{-1} \ln H,$$

where $d_0 = 3$, $d_1 = a - 2$, $a = q + 4$, $a < b$, $d_2 = b - 2$, $D = a + b$, and $\nu_1^2 = \nu_2^2 = (q-2)^{-1}$. The signature restrictions are : $\varepsilon(1) = \varepsilon(2) = -1$. Thus, the space-time (M, g) should contain at least three time directions. The minimal D is 15. For $D = 15$ we get $a = 7$, $b = 8$, $d_1 = 5$, $d_2 = 6$ and $q = 3$. (Here we have eliminated a typo in a sign of scalar fields that was originally in [37].)

Remark 3.2. We note that affine KM algebras (with $\det A = 0$) do not appear in the solutions (3.6)–(3.10). Indeed, any affine Cartan matrix satisfy the relations

$$\sum_{s \in S} a_s A_{ss'} = 0$$

with $a_s > 0$ called Coxeter labels [4], $s \in S$. This relation makes impossible the existence of the solution to equation (3.5) (see (3.11)).

Generalized Majumdar–Papapetrou solutions. Now we return to a "multi-block" solution (3.6)–(3.10). Let $M_0 = \mathbb{R}^{d_0}$, $d_0 > 2$, $g^0 = \delta_{\mu\nu} dx^\mu \otimes dx^\nu$, $d_1 = 1$ and $g^1 = -dt \otimes dt$. For

$$H_s = 1 + \sum_{b \in X_s} \frac{q_{sb}}{|x - b|^{d_0 - 2}}, \tag{3.23}$$

where X_s is finite non-empty subset $X_s \subset M_0$, $s \in S$, all $q_{sb} > 0$, and $X_s = X_{s'}$, $q_{sb} = q_{s'b}$ for $b \in X_s = X_{s'}$, $s, s' \in S_j$, $j = 1, \dots, k$. The harmonic functions (3.23) are defined in domain $M_0 \setminus X$, $X = \bigcup_{s \in S} X_s$, and generate the solutions (3.6)–(3.10).

Denote $S(b) \equiv \{s \in S | \quad b \in X_s\}$, $b \in X$. (In the one-block case, when $k = 1$, all $X_s = X$ and $S(b) = S$.) We have a horizon at point b w.r.t. time t, when $x \to b \in X$, if and only if

$$\xi_1(b) \equiv \sum_{s \in S(b)} (-\varepsilon_s) \nu_s^2 \delta_{1 I_s} - \frac{1}{d_0 - 2} \geq 0.$$

This relation follows just from the requirement of the infinite propagation time of light to $b \in X$.

Majumdar–Papapetrou solution. Recall that the well-known 4-dimensional Majumdar–Papapetrou (MP) solution [71] corresponding to the Lie algebra $\mathbf{A_1}$ in our notations reads

$$g = H^2 \hat{g}^0 - H^{-2} dt \otimes dt,$$
$$F = \nu dH^{-1} \wedge dt,$$

where $\nu^2 = 2$, $g^0 = \sum_{i=1}^3 dx^i \otimes dx^i$ and H is a harmonic function. We have one electric 0-brane (point) "attached" to the time manifold; $d(I_s) = 1$, $\varepsilon_s = -1$ and $(U^s, U^s) = 1/2$. In this case (e.g. for the extremal Reissner–Nordström black hole) we get $\xi_1(b) = 1$, $b \in X$. Points b are the points of (regular) horizon.

For certain examples of finite-dimensional semisimple Lie algebras (e.g. for $\mathbf{A_1} \oplus \dots \oplus \mathbf{A_1}$, $\mathbf{A_2}$ etc.) the poles b in H_s correspond to (regular) horizons and the solution under consideration describes a collection of k blocks of extremal charged black branes (in equilibrium) [37].

Hyperbolic KM algebras. Let us consider a generalized one-block ($k = 1$) MP solution corresponding to a hyperbolic KM algebra such that $(U^s, U^s) > 0$ for all $s \in S$. In this case all $\varepsilon_s > 0$, $s \in S$, and hence $\xi_1(b) < 0$. Thus, any point $b \in X$ is not a point of the horizon. (It may be checked using the analysis carried out in [37] that any non-exceptional point b is a singular one). As a consequence, the generalized MP solution corresponding to any hyperbolic KM algebra does not describe a collection of extremal charged black branes (in equilibrium) when all $(U^s, U^s) > 0$.

3.3 Toda-like solutions

3.3.1 Toda-like Lagrangian

Action (2.13) may be also written in the form

$$S_{\sigma 0} = \frac{1}{2\kappa_0^2} \int d^{d_0}x \sqrt{|g^0|}\{R[g^0] - \mathcal{G}_{\hat{A}\hat{B}}(X) g^{0\mu\nu} \partial_\mu X^{\hat{A}} \partial_\nu X^{\hat{B}} - 2V\}, \tag{3.24}$$

where $X = (X^{\hat{A}}) = (\phi^i, \varphi^\alpha, \Phi^s) \in \ R^N$, and the minisupermetric $\mathcal{G} = \mathcal{G}_{\hat{A}\hat{B}}(X) dX^{\hat{A}} \otimes dX^{\hat{B}}$ on the minisuperspace $\mathcal{M} = \mathbf{R}^N$, $N = n + l + |S|$ ($|S|$ is the number of elements in S) is defined by the relation

$$(\mathcal{G}_{\hat{A}\hat{B}}(X)) = \begin{pmatrix} G_{ij} & 0 & 0 \\ 0 & h_{\alpha\beta} & 0 \\ 0 & 0 & \varepsilon_s \exp(-2U^s(\sigma))\delta_{ss'} \end{pmatrix}. \tag{3.25}$$

Here we consider exact solutions to field equations corresponding to the action (3.24)

$$R_{\mu\nu}[g^0] = \mathcal{G}_{\hat{A}\hat{B}}(X) \partial_\mu X^{\hat{A}} \partial_\nu X^{\hat{B}} + \frac{2V}{d_0 - 2} g^0_{\mu\nu}, \tag{3.26}$$

$$\frac{1}{\sqrt{|g^0|}} \partial_\mu [\sqrt{|g^0|} \mathcal{G}_{\hat{C}\hat{B}}(X) g^{0\mu\nu} \partial_\nu X^{\hat{B}}] - \frac{1}{2} \mathcal{G}_{\hat{A}\hat{B},\hat{C}}(X) g^{0,\mu\nu} \partial_\mu X^{\hat{A}} \partial_\nu X^{\hat{B}} = V_{,\hat{C}}, \tag{3.27}$$

where $s \in S$. Here $V_{,\hat{C}} = \partial V / \partial X^{\hat{C}}$.

We put

$$X^{\hat{A}}(x) = F^{\hat{A}}(H(x)),$$

where $F : (u_-, u_+) \to \mathbb{R}^N$ is a smooth function, $H : M_0 \to \mathbb{R}$ is a harmonic function on M_0 (i.e. $\Delta[g^0]H = 0$), satisfying $u_- < H(x) < u_+$ for all $x \in M_0$. We take all factor spaces as Ricci-flat and the cosmological constant is set to zero, i.e. the relations $\xi_i = 0$ and $\Lambda = 0$ are satisfied.

In this case the potential is zero : $V = 0$. It may be verified that the field equations (3.26) and (3.27) are satisfied identically if $F = F(u)$ obeys the Lagrange equations for the Lagrangian

$$L = \frac{1}{2} \mathcal{G}_{\hat{A}\hat{B}}(F) \dot{F}^{\hat{A}} \dot{F}^{\hat{B}} \tag{3.28}$$

with the zero-energy constraint

$$E = \frac{1}{2} \mathcal{G}_{\hat{A}\hat{B}}(F) \dot{F}^{\hat{A}} \dot{F}^{\hat{B}} = 0. \tag{3.29}$$

This means that $F : (u_-, u_+) \to \mathbb{R}^N$ is a null-geodesic map for the minisupermetric $\mathcal{G}$. Thus, we are led to the Lagrange system (3.28) with the minisupermetric $\mathcal{G}$ defined in (3.25).

The problem of integrability will be simplified if we integrate the Lagrange equations corresponding to Φ^s (i.e. the Maxwell-type equations for $s \in S_e$ and Bianchi identities for $s \in S_m$):

$$\frac{d}{du}\left(\exp(-2U^s(\sigma))\dot{\Phi}^s\right) = 0 \Longleftrightarrow \dot{\Phi}^s = Q_s \exp(2U^s(\sigma)), \tag{3.30}$$

where Q_s are constants, and $s \in S$. Here $(F^{\hat{A}}) = (\sigma^A, \Phi^s)$. We put $Q_s \neq 0$ for all $s \in S$.

For fixed $Q = (Q_s, s \in S)$ the Lagrange equations for the Lagrangian (3.28) corresponding to $(\sigma^A) = (\phi^i, \varphi^\alpha)$, when equations (3.30) are substituted, are equivalent to the Lagrange equations for the Lagrangian

$$L_Q = \frac{1}{2} \hat{G}_{AB} \dot{\sigma}^A \dot{\sigma}^B - V_Q, \tag{3.31}$$

where

$$V_Q = \frac{1}{2} \sum_{s \in S} \varepsilon_s Q_s^2 \exp[2U^s(\sigma)], \tag{3.32}$$

the matrix $(\hat{G}_{AB})$ is defined in (2.14). The zero-energy constraint (3.29) reads

$$E_Q = \frac{1}{2} \hat{G}_{AB} \dot{\sigma}^A \dot{\sigma}^B + V_Q = 0. \tag{3.33}$$

3.3.2 The solutions

Here, as above we are interested in exact solutions for a special case when $K_s = (U^s, U^s) \neq 0$, for all $s \in S$, and the generalized Cartan matrix (3.11) is non-degenerate. It follows from the non-degeneracy of the matrix (3.11) that vectors U^s, $s \in S$, are linearly independent. Hence, the number of vectors U^s should not exceed the dimension of $\mathbb{R}^{n+l}$, i.e. $|S| \leq n + l$.

The exact solutions were obtained in [40] and are

$$\begin{aligned} g = & \left(\prod_{s \in S} f_s^{2d(I_s)h_s/(D-2)} \right) \Bigg\{ \exp(2c^0 H + 2\bar{c}^0) \hat{g}^0 \\ & + \sum_{i=1}^{n} \left(\prod_{s \in S} f_s^{-2h_s \delta_{iI_s}} \right) \exp(2c^i H + 2\bar{c}^i) \hat{g}^i \Bigg\}, \end{aligned} \tag{3.34}$$

$$\exp(\varphi^\alpha) = \left(\prod_{s \in S} f_s^{h_s \chi_s \lambda^\alpha_{a_s}} \right) \exp(c^\alpha H + \bar{c}^\alpha), \qquad \alpha = 1, \dots, l, \tag{3.35}$$

and $F^a = \sum_{s \in S} \mathcal{F}^s \delta^a_{a_s}$ with

$$\mathcal{F}^s = Q_s \left(\prod_{s' \in S} f_{s'}^{-A_{ss'}} \right) dH \wedge \tau(I_s), \qquad s \in S_e,$$

$$\mathcal{F}^s = Q_s (*_0 dH) \wedge \tau(\bar{I}_s), \qquad s \in S_m,$$

where $*_0 = *[g^0]$ is the Hodge operator on (M_0, g^0). Here

$$f_s = f_s(H) = \exp(-q^s(H)), \tag{3.36}$$

where $q^s(u)$ is a solution to the Toda-like equations

$$\ddot{q}^s = -B_s \exp\left(\sum_{s' \in S} A_{ss'} q^{s'} \right) \tag{3.37}$$

with $B_s = K_s \varepsilon_s Q_s^2$, $s \in S$, and $H = H(x)$ $(x \in M_0)$ is a harmonic function on (M_0, g^0). Vectors $c = (c^A)$ and $\bar{c} = (\bar{c}^A)$ satisfy the linear constraints

$$U^s(c) = \sum_{i \in I_s} d_i c^i - \chi_s \lambda_{a_s \alpha} c^\alpha = 0, \qquad U^s(\bar{c}) = 0, \qquad s \in S, \tag{3.38}$$

and

$$c^0 = \frac{1}{2 - d_0} \sum_{j=1}^{n} d_j c^j, \qquad \bar{c}^0 = \frac{1}{2 - d_0} \sum_{j=1}^{n} d_j \bar{c}^j. \tag{3.39}$$

The zero-energy constraint reads

$$2E_T + h_{\alpha\beta}c^\alpha c^\beta + \sum_{i=1}^{n} d_i(c^i)^2 + \frac{1}{d_0-2}\left(\sum_{i=1}^{n} d_i c^i\right)^2 = 0, \tag{3.40}$$

where

$$E_T = \frac{1}{4}\sum_{s,s'\in S} h_s A_{ss'}\dot{q}^s\dot{q}^{s'} + \sum_{s\in S} A_s \exp\left(\sum_{s'\in S} A_{ss'}q^{s'}\right) \tag{3.41}$$

is an integration constant (energy) for the solutions from (3.37) and $A_s = \frac{1}{2}\varepsilon_s Q_s^2$.

We note that equations (3.37) correspond to the Lagrangian

$$L_T = \frac{1}{4}\sum_{s,s'\in S} h_s A_{ss'}\dot{q}^s\dot{q}^{s'} - \sum_{s\in S} A_s \exp\left(\sum_{s'\in S} A_{ss'}q^{s'}\right),$$

where $h_s = K_s^{-1}$.

Thus, the solution is given by relations (3.34)–(3.36) with the functions q^s being defined in (3.37) and with relations on the parameters of solutions c^A, $\bar{c}^A$ $(A = i, \alpha, 0)$, imposed by (3.38), (3.39), (3.40).

4 Cosmological-type solutions

Now we consider the case $d_0 = 1$, $M_0 = \mathbb{R}$, i.e. we are interested in applications to the sector with dependence on a single variable. We consider the manifold

$$M = (u_-, u_+) \times M_1 \times \cdots \times M_n$$

with a metric

$$g = we^{2\gamma(u)}du \otimes du + \sum_{i=1}^{n} e^{2\phi^i(u)}\hat{g}^i,$$

where $w = \pm 1$, u is a distinguished coordinate which, by convention, will be called "time"; (M_i, g^i) are oriented and connected Einstein spaces (see (2.4)), $i = 1, \dots, n$. The functions γ, ϕ^i: $(u_-, u_+) \to \mathbb{R}$ are smooth.

Here we adopt the brane ansatz from Section 2 putting $g^0 = wdu \otimes du$.

4.1 Lagrange dynamics

It follows from Subsection 2.3 that the equations of motion and the Bianchi identities for the field configuration under consideration (with the restrictions from Subsection 2.3 imposed) are equivalent to equations of motion for 1-dimensional σ-model with the action

$$S_\sigma = \frac{\mu}{2}\int du\mathcal{N}\left\{G_{ij}\dot{\phi}^i\dot{\phi}^j + h_{\alpha\beta}\dot{\varphi}^\alpha\dot{\varphi}^\beta + \sum_{s\in S}\varepsilon_s \exp[-2U^s(\phi,\varphi)](\dot{\Phi}^s)^2 - 2\mathcal{N}^{-2}V_w(\phi)\right\}, \tag{4.1}$$

where $\dot{x} \equiv dx/du$,

$$V_w = -wV = -w\Lambda e^{2\gamma_0(\phi)} + \frac{w}{2}\sum_{i=1}^{n}\xi_i d_i e^{-2\phi^i+2\gamma_0(\phi)} \tag{4.2}$$

is the potential with $\gamma_0(\phi) \equiv \sum_{i=1}^{n} d_i\phi^i$, and $\mathcal{N} = \exp(\gamma_0 - \gamma) > 0$ is the modified lapse function, $U^s = U^s(\phi,\varphi)$ are defined in (2.15), ε_s are defined in (2.16) for $s = (a_s, v_s, I_s) \in S$, and $G_{ij} = d_i\delta_{ij} - d_i d_j$ are components of "pure cosmological" minisupermetric, $i, j = 1, \dots, n$ [67].

In the electric case ($\mathcal{F}^{(a,m,I)} = 0$) for finite internal space volumes V_i the action (4.1) coincides with the action (2.1) if $\mu = -w/\kappa_0^2$, $\kappa^2 = \kappa_0^2 V_1 \cdots V_n$.

Action (4.1) may be also written in the form

$$S_\sigma = \frac{\mu}{2}\int du \mathcal{N}\left\{\mathcal{G}_{\hat{A}\hat{B}}(X)\dot{X}^{\hat{A}}\dot{X}^{\hat{B}} - 2\mathcal{N}^{-2}V_w\right\}, \tag{4.3}$$

where $X = (X^{\hat{A}}) = (\phi^i, \varphi^\alpha, \Phi^s) \in \mathbb{R}^N$, $N = n + l + |S|$, and minisupermetric $\mathcal{G}$ is defined in (3.25).

Scalar products. The minisuperspace metric (3.25) may be also written in the form $\mathcal{G} = \hat{G} + \sum_{s\in S} \varepsilon_s e^{-2U^s(\sigma)} d\Phi^s \otimes d\Phi^s$, where $\sigma = (\sigma^A) = (\phi^i, \varphi^\alpha)$,

$$\hat{G} = \hat{G}_{AB} d\sigma^A \otimes d\sigma^B = G_{ij} d\phi^i \otimes d\phi^j + h_{\alpha\beta} d\varphi^\alpha \otimes d\varphi^\beta,$$

is the truncated minisupermetric and $U^s(\sigma) = U^s_A\sigma^A$ is defined in (2.15). The potential (4.2) now reads

$$V_w = (-w\Lambda)e^{2U^\Lambda(\sigma)} + \sum_{j=1}^{n} \frac{w}{2}\xi_j d_j e^{2U^j(\sigma)},$$

where

$$U^j(\sigma) = U^j_A\sigma^A = -\phi^j + \gamma_0(\phi), \qquad (U^j_A) = (-\delta^j_i + d_i, 0), \tag{4.4}$$
$$U^\Lambda(\sigma) = U^\Lambda_A\sigma^A = \gamma_0(\phi), \qquad (U^\Lambda_A) = (d_i, 0).$$

The integrability of the Lagrange system (4.3) crucially depends upon the scalar products of co-vectors U^Λ, U^j, U^s (see (3.1)). These products are defined by (3.2) and the following relations [48]

$$\left(U^i, U^j\right) = \frac{\delta_{ij}}{d_j} - 1, \tag{4.5}$$
$$\left(U^i, U^\Lambda\right) = -1, \qquad \left(U^\Lambda, U^\Lambda\right) = -\frac{D-1}{D-2},$$
$$\left(U^s, U^i\right) = -\delta_{iI_s}, \qquad \left(U^s, U^\Lambda\right) = \frac{d(I_s)}{2-D}, \tag{4.6}$$

where $s = (a_s, v_s, I_s) \in S$; $i, j = 1, \dots, n$.

Toda-like representation. We put $\gamma = \gamma_0(\phi)$, i.e. the harmonic time gauge is considered. Integrating the Lagrange equations corresponding to Φ^s (see (3.30)) we are led to the Lagrangian from (3.31) and the zero-energy constraint (3.33) with the modified potential

$$V_Q = V_w + \frac{1}{2}\sum_{s\in S} \varepsilon_s Q_s^2 \exp[2U^s(\sigma)], \tag{4.7}$$

where V_w is defined in (4.2).

4.2 Solutions with $\Lambda = 0$

Here we consider solutions with $\Lambda = 0$.

4.2.1 Solutions with Ricci-flat factor-spaces

Let all spaces be Ricci-flat, i.e. $\xi_1 = \cdots = \xi_n = 0$.

Since $H(u) = u$ is a harmonic function on (M_0, g^0) with $g^0 = w du \otimes du$ we get for the metric and scalar fields from (3.34), (3.35) [40]

$$g = \left(\prod_{s \in S} f_s^{2d(I_s)h_s/(D-2)} \right) \Bigg\{ \exp(2c^0 u + 2\bar{c}^0) w du \otimes du + \sum_{i=1}^{n} \left(\prod_{s \in S} f_s^{-2h_s \delta_{iI_s}} \right) \exp(2c^i u + 2\bar{c}^i) \hat{g}^i \Bigg\}, \tag{4.8}$$

$$\exp(\varphi^\alpha) = \left(\prod_{s \in S} f_s^{h_s \chi_s \lambda^\alpha_{a_s}} \right) \exp(c^\alpha u + \bar{c}^\alpha), \qquad \alpha = 1, \ldots, l, \tag{4.9}$$

and $F^a = \sum_{s \in S} \delta^a_{a_s} \mathcal{F}^s$ with

$$\mathcal{F}^s = Q_s \left(\prod_{s' \in S} f_{s'}^{-A_{ss'}} \right) du \wedge \tau(I_s), \qquad s \in S_e, \tag{4.10}$$

$$\mathcal{F}^s = Q_s \tau(\bar{I}_s), \qquad s \in S_m, \tag{4.11}$$

$Q_s \neq 0$, $s \in S$. Here $f_s = f_s(u) = \exp(-q^s(u))$ and $q^s(u)$ obey Toda-like equations (3.37).

Relations (3.39) and (3.40) take the form

$$c^0 = \sum_{j=1}^{n} d_j c^j, \qquad \bar{c}^0 = \sum_{j=1}^{n} d_j \bar{c}^j,$$

$$2E_T + h_{\alpha\beta} c^\alpha c^\beta + \sum_{i=1}^{n} d_i (c^i)^2 - \left(\sum_{i=1}^{n} d_i c^i \right)^2 = 0,$$

with E_T from (3.41) and all other relations (e.g. constraints (3.38)) are unchanged.

This solution in the special case of an $\mathbf{A_m}$ Toda chain, was obtained earlier in [72] (see also [73]). Some special configurations were considered earlier in [74, 75, 76].

Currently, the cosmological solutions with branes are considered often in a context of S-brane terminology [77]. S-branes were originally space-like analogues of D-branes, see also [78, 79, 80, 81, 45, 82, 83] and references therein.

4.2.2 Solutions with one curved factor-space

The cosmological solution with Ricci-flat spaces may be also modified to the following case: $\xi_1 \neq 0$, $\xi_2 = \cdots = \xi_n = 0$, i.e. one space is curved and others are Ricci-flat and $1 \notin I_s$, $s \in S$, i.e. all "brane" submanifolds do not contain M_1.

The potential (3.32) is modified for $\xi_1 \neq 0$ as follows (see (4.7))

$$V_Q = \frac{1}{2} \sum_{s \in S} \varepsilon_s Q_s^2 \exp[2U^s(\sigma)] + \frac{1}{2} w \xi_1 d_1 \exp[2U^1(\sigma)],$$

where $U^1(\sigma)$ is defined in (4.4) ($d_1 > 1$).

For the scalar products we get from (4.5) and (4.6)

$$(U^1, U^1) = \frac{1}{d_1} - 1 < 0, \qquad (U^1, U^s) = 0 \tag{4.12}$$

for all $s \in S$.

The solution in the case under consideration may be obtained by a little modification of the solution from the previous section (using (4.12), relations $U^{1i} = -\delta^i_1/d_1$, $U^{1\alpha} = 0$) [40]

$$g = \left(\prod_{s \in S} [f_s(u)]^{2d(I_s)h_s/(D-2)}\right) \left\{ [f_1(u)]^{2d_1/(1-d_1)} \exp(2c^1 u + 2\bar{c}^1) \right.$$
$$\left. \times [w du \otimes du + f_1^2(u) \hat{g}^1] + \sum_{i=2}^{n} \left(\prod_{s \in S} [f_s(u)]^{-2h_s \delta_{iI_s}}\right) \exp(2c^i u + 2\bar{c}^i) \hat{g}^i \right\}, \tag{4.13}$$

$$\exp(\varphi^\alpha) = \left(\prod_{s \in S} f_s^{h_s \chi_s \lambda^\alpha_{a_s}}\right) \exp(c^\alpha u + \bar{c}^\alpha), \tag{4.14}$$

$$F^a = \sum_{s \in S} \delta^a_{a_s} \mathcal{F}^s, \tag{4.15}$$

with forms $\mathcal{F}^s$ defined in (4.10) and (4.11).

Here $f_s = f_s(u) = \exp(-q^s(u))$ where $q^s(u)$ obey Toda-like equations (3.37) and

$$f_1(u) = \begin{cases} R \sinh(\sqrt{C_1}(u - u_1)), & C_1 > 0,\ \xi_1 w > 0; \\ R \sin(\sqrt{|C_1|}(u - u_1)), & C_1 < 0,\ \xi_1 w > 0; \\ R \cosh(\sqrt{C_1}(u - u_1)), & C_1 > 0,\ \xi_1 w < 0; \\ |\xi_1(d_1 - 1)|^{1/2}, & C_1 = 0,\ \xi_1 w > 0, \end{cases}$$

u_1, C_1 are constants and $R = |\xi_1(d_1 - 1)/C_1|^{1/2}$.

The vectors $c = (c^A)$ and $\bar{c} = (\bar{c}^A)$ satisfy the linear constraints

$$U^r(c) = U^r(\bar{c}) = 0, \qquad r = s, 1,$$

(for $r = s$ see (3.38)) and the zero-energy constraint

$$C_1 \frac{d_1}{d_1 - 1} = 2E_T + h_{\alpha\beta} c^\alpha c^\beta + \sum_{i=2}^{n} d_i (c^i)^2 + \frac{1}{d_1 - 1} \left(\sum_{i=2}^{n} d_i c^i\right)^2.$$

4.2.3 Special solutions for block-orthogonal set of vectors U^s

Let us consider block-orthogonal case: (3.3), (3.4). In this case we get

$$f_s = (\bar{f}_s)^{b_s}$$

where $b_s = 2 \sum_{s' \in S} A^{ss'}$, $(A^{ss'}) = (A_{ss'})^{-1}$ and

$$\bar{f}_s(u) = \begin{cases} R_s \sinh(\sqrt{C_s}(u - u_s)), & C_s > 0,\ \eta_s \varepsilon_s < 0; \\ R_s \sin(\sqrt{|C_s|}(u - u_s)), & C_s < 0,\ \eta_s \varepsilon_s < 0; \\ R_s \cosh(\sqrt{C_s}(u - u_s)), & C_s > 0,\ \eta_s \varepsilon_s > 0; \\ \frac{|Q^s|}{|\nu_s|}(u - u_s), & C_s = 0,\ \eta_s \varepsilon_s < 0, \end{cases}$$

where $R_s = |Q_s|/(|\nu_s||C_s|^{1/2})$,

$$\eta_s \nu_s^2 = b_s h_s,$$

$\eta_s = \pm 1$, C_s, u_s are constants, $s \in S$. The constants C_s, u_s are coinciding inside the blocks: $u_s = u_{s'}$, $C_s = C_{s'}$, $s, s' \in S_i$, $i = 1, \dots, k$. The ratios $\varepsilon_s Q_s^2/(b_s h_s)$ also coincide inside the blocks, or, equivalently,

$$\frac{\varepsilon_s Q_s^2}{b_s h_s} = \frac{\varepsilon_{s'} Q_{s'}^2}{b_{s'} h_{s'}}, \qquad s, s' \in S_i, \quad i = 1, \dots, k. \tag{4.16}$$

For energy integration constant (3.41) we get

$$E_T = \frac{1}{2} \sum_{s \in S} C_s b_s h_s. \tag{4.17}$$

The solution (4.13)–(4.15) with a block-orthogonal set of U^s-vectors was obtained in [85, 86] (for non-composite case see also earlier paper by K.A. Bronnikov [84]). The generalized KM algebra corresponding to the generalized Cartan matrix A in this case is semisimple. In the special orthogonal (or $A_1 \oplus \cdots \oplus A_1$) case when: $|S_1| = \cdots = |S_k| = 1$, the solution was obtained in [47].

Thus, here we presented a large class of exact solutions for invertible generalized Cartan matrices (e.g. corresponding to hyperbolic KM algebras). These solutions are governed by Toda-type equations. They are integrable in quadratures for finite-dimensional semisimple Lie algebras [87, 88, 89, 90, 91] in agreement with Adler–van Moerbeke criterion [91] (see also [92]).

The problem of integrability of Toda-chains related to Lorentzian (e.g. hyperbolic) KM algebras is much more complicated than in the Euclidean case. This is supported by the result from [93] (based on calculation of the Kovalevskaya exponents) where it was shown that the known cases of algebraic integrability for Euclidean Toda chains have no direct analogues in the case of spaces with pseudo-Euclidean metrics because the full-parameter expansions of the general solution contain complex powers of the independent variable. A similar result, using the Painlevé property, was obtained earlier for 2-dimensional Toda chains related to hyperbolic KM algebras [94].

Remark 4.1. It was shown in [95] that all supergravity billiards corresponding to sigma-models on any U/H non compact-symmetric space and obtained by compactifying supergravity to $D = 3$ are fully integrable. As far as we know this result could not be reformulated in terms of integrability of Toda-chains corresponding to certain Lorentzian (e.g. hyperbolic) KM algebras.

4.3 Examples of S-brane solutions

Example 4.1. S-brane solution governed by E_{10} Toda chain

Let us consider the B_{16}-model in 16-dimensional pseudo-Euclidean space of signature $(-, +, \dots, +)$ with six forms $F^4, \dots, F^9$ and five scalar fields $\varphi^1, \dots, \varphi^5$, see (3.21). Recall that for two branes corresponding to F^a and F^b forms the orthogonal (or $(A_1 + A_1)$-) intersection rules read [47, 46]:

$$(a-1)_e \cap_o (b-1)_e = N(a,b) = \min(a,b) - 3,$$
$$(a-1)_e \cap_o (D-b-1)_m = a - 1 - N(a,b),$$

where $d_v \cap_o d'_{v'}$ denotes the dimension of orthogonal intersection for two branes with the dimensions of their worldvolumes being d and d'. $d_v \cap_o d'_{v'}$ coincides with the symbol $\Delta(s, s')$ from (3.13)[1]. The subscripts $v, v' = e, m$ here indicate whether the brane is electric (e) or magnetic (m) one. In what follows we will be interested in the following orthogonal intersections: $4_e \cap_o 4_e = 2$, $4_e \cap_o 5_e = 2$, $4_e \cap_o 11_m = 3$, $5_e \cap_o 11_m = 4$.

Here we deal with 10 (S-)branes: eight electric branes s_1, s_2, s_3, s_4, s_5, s_6, s_8, s_9 corresponding to 5-form F^5, one electric brane s_7 corresponding to 6-form F^6 and one magnetic brane s_{10} corresponding to 4-form F^4. The brane sets are as follows: $I_1 = \{3, 4, 10, 12\}$, $I_2 = \{1, 6, 7, 12\}$, $I_3 = \{8, 9, 10, 12\}$, $I_4 = \{1, 2, 3, 12\}$, $I_5 = \{5, 6, 10, 12\}$, $I_6 = \{1, 4, 8, 12\}$, $I_7 = \{2, 7, 10, 12, 13\}$, $I_8 = \{3, 6, 8, 12\}$, $I_9 = \{1, 10, 11, 12\}$, $I_{10} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$.

It may be verified that these sets do obey E_{10} intersection rules following from the relations (3.14) (with $I_{s_i} = I_i$) and the Dynkin diagram from Fig. 1.

[1] Here as in [46] our notations differ from those adopted in string theory. For example for intersection of M2- and M5-branes we write $3 \cap_o 6 = 2$ instead of $2 \cap 5 = 1$.

$$\begin{array}{ccccccccc} & & & & & & 10 & & \\ & & & & & & | & & \\ 1 & - 2 & - 3 & - 4 & - 5 & - 6 & - 7 & - 8 & - 9 \end{array}$$

Fig. 1. *Dynkin diagram for E_{10} hyperbolic KM algebra*

Now we present a cosmological S-brane solution from Subsection 4.2.1 for the configuration of ten branes under consideration. In what follows the relations $\varepsilon_s = +1$ and $h_s = 1/2$, $s \in S$, are used.

The metric (4.8) reads:

$$\begin{aligned} g = {} & \left[\left(\prod_{s \neq 7,10} f_s \right)^4 f_7^5 f_{10}^{11} \right]^{1/14} \Biggl\{ -e^{2c^0 t + 2\bar{c}^0} dt \otimes dt + (f_2 f_4 f_6 f_9 f_{10})^{-1} e^{2c^1 t + 2\bar{c}^1} dx^1 \otimes dx^1 \\ & + (f_4 f_7 f_{10})^{-1} e^{2c^2 t + 2\bar{c}^2} dx^2 \otimes dx^2 + (f_1 f_4 f_8 f_{10})^{-1} e^{2c^3 t + 2\bar{c}^3} dx^3 \otimes dx^3 \\ & + (f_1 f_6 f_{10})^{-1} e^{2c^4 t + 2\bar{c}^4} dx^4 \otimes dx^4 + (f_5 f_{10})^{-1} e^{2c^5 t + 2\bar{c}^5} dx^5 \otimes dx^5 \\ & + (f_2 f_5 f_8 f_{10})^{-1} e^{2c^6 t + 2\bar{c}^6} dx^6 \otimes dx^6 + (f_2 f_7 f_{10})^{-1} e^{2c^7 t + 2\bar{c}^7} dx^7 \otimes dx^7 \\ & + (f_3 f_6 f_8 f_{10})^{-1} e^{2c^8 t + 2\bar{c}^8} dx^8 \otimes dx^8 + (f_3 f_{10})^{-1} e^{2c^9 t + 2\bar{c}^9} dx^9 \otimes dx^9 \\ & + (f_1 f_3 f_5 f_7 f_9 f_{10})^{-1} e^{2c^{10} t + 2\bar{c}^{10}} dx^{10} \otimes dx^{10} + (f_9 f_{10})^{-1} e^{2c^{11} t + 2\bar{c}^{11}} dx^{11} \otimes dx^{11} \\ & + \left(\prod_{s=1}^{9} f_s \right)^{-1} e^{2c^{12} t + 2\bar{c}^{12}} dx^{12} \otimes dx^{12} + f_7^{-1} e^{2c^{13} t + 2\bar{c}^{13}} dx^{13} \otimes dx^{13} \\ & + e^{2c^{14} t + 2\bar{c}^{14}} dx^{14} \otimes dx^{14} + e^{2c^{15} t + 2\bar{c}^{15}} dx^{15} \otimes dx^{15} \Biggr\}. \end{aligned}$$

For scalar fields (4.9) we get

$$\varphi^\alpha = \frac{1}{2} \left[-\lambda_{5\alpha} \left(\sum_{s \neq 7,10} \ln f_s \right) - \lambda_{6\alpha} \ln f_7 + \lambda_{4\alpha} \ln f_{10} \right] + c_\varphi^\alpha t + \bar{c}_\varphi^\alpha, \qquad \alpha = 1, \dots, 5.$$

Here we used the relations $\lambda_a^\alpha = -\lambda_{a\alpha}$.

The form fields (see (4.10) and (4.11)) are as follows

$$\begin{aligned} F^4 = {} & Q_{10} dx^{12} \wedge dx^{13} \wedge dx^{14} \wedge dx^{15}, \\ F^5 = {} & Q_1 f_1^{-2} f_2 dt \wedge dx^3 \wedge dx^4 \wedge dx^{10} \wedge dx^{12} + Q_2 f_1 f_2^{-2} f_3 dt \wedge dx^1 \wedge dx^6 \wedge dx^7 \wedge dx^{12} \\ & + Q_3 f_2 f_3^{-2} f_4 dt \wedge dx^8 \wedge dx^9 \wedge dx^{10} \wedge dx^{12} + Q_4 f_3 f_4^{-2} f_5 dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^{12} \\ & + Q_5 f_4 f_5^{-2} f_6 dt \wedge dx^5 \wedge dx^6 \wedge dx^{10} \wedge dx^{12} + Q_6 f_5 f_6^{-2} f_7 dt \wedge dx^1 \wedge dx^4 \wedge dx^8 \wedge dx^{12} \\ & + Q_8 f_7 f_8^{-2} f_9 dt \wedge dx^3 \wedge dx^6 \wedge dx^8 \wedge dx^{12} + Q_9 f_8 f_9^{-2} dt \wedge dx^1 \wedge dx^{10} \wedge dx^{11} \wedge dx^{12}, \\ F^6 = {} & Q_7 f_6 f_7^{-2} f_8 f_{10} dt \wedge dx^2 \wedge dx^7 \wedge dx^{10} \wedge dx^{12} \wedge dx^{13}, \end{aligned} \tag{4.18}$$

where $Q_s \neq 0$, $s = 1, \dots, 10$. Here

$$c^0 = \sum_{j=1}^{15} c^j, \qquad \bar{c}^0 = \sum_{j=1}^{15} \bar{c}^j,$$

$f_s = \exp(-q^s(t))$ and $q^s(t)$ obey Toda-type equations

$$\ddot{q}^s = -2Q_s^2 \exp\left(\sum_{s'=1}^{10} A_{ss'} q^{s'} \right), \qquad s = 1, \dots, 10,$$

where $(A_{ss'})$ is the Cartan matrix for the KM algebra E_{10} (with the Dynkin diagram from Fig. 1) and the energy integration constant

$$E_T = \frac{1}{8}\sum_{s,s'=1}^{10} A_{ss'}\dot{q}^s\dot{q}^{s'} + \frac{1}{2}\sum_{s=1}^{10} Q_s^2 \exp\left(\sum_{s'=1}^{10} A_{ss'}q^{s'}\right),$$

obeys the constraint

$$2E_T - \sum_{\alpha=1}^{5}(c_\varphi^\alpha)^2 + \sum_{i=1}^{15}(c^i)^2 - \left(\sum_{i=1}^{15} c^i\right)^2 = 0.$$

The brane constraints (3.38) are in our case

$$U^1(c) = c^3 + c^4 + c^{10} + c^{12} - \sum_{\alpha=1}^{5}\lambda_{5\alpha}c_\varphi^\alpha = 0, \qquad U^1(\bar{c}) = 0,$$

$$U^2(c) = c^1 + c^6 + c^7 + c^{12} - \sum_{\alpha=1}^{5}\lambda_{5\alpha}c_\varphi^\alpha = 0, \qquad U^2(\bar{c}) = 0,$$

$$U^3(c) = c^8 + c^9 + c^{10} + c^{12} - \sum_{\alpha=1}^{5}\lambda_{5\alpha}c_\varphi^\alpha = 0, \qquad U^3(\bar{c}) = 0,$$

$$U^4(c) = c^1 + c^2 + c^3 + c^{12} - \sum_{\alpha=1}^{5}\lambda_{5\alpha}c_\varphi^\alpha = 0, \qquad U^4(\bar{c}) = 0,$$

$$U^5(c) = c^5 + c^6 + c^{10} + c^{12} - \sum_{\alpha=1}^{5}\lambda_{5\alpha}c_\varphi^\alpha = 0, \qquad U^5(\bar{c}) = 0,$$

$$U^6(c) = c^1 + c^4 + c^8 + c^{12} - \sum_{\alpha=1}^{5}\lambda_{5\alpha}c_\varphi^\alpha = 0, \qquad U^6(\bar{c}) = 0,$$

$$U^7(c) = c^2 + c^7 + c^{10} + c^{12} + c^{13} - \sum_{\alpha=1}^{5}\lambda_{6\alpha}c_\varphi^\alpha = 0, \qquad U^7(\bar{c}) = 0,$$

$$U^8(c) = c^3 + c^6 + c^8 + c^{12} - \sum_{\alpha=1}^{5}\lambda_{5\alpha}c_\varphi^\alpha = 0, \qquad U^8(\bar{c}) = 0,$$

$$U^9(c) = c^1 + c^{10} + c^{11} + c^{12} - \sum_{\alpha=1}^{5}\lambda_{5\alpha}c_\varphi^\alpha = 0, \qquad U^9(\bar{c}) = 0,$$

$$U^{10}(c) = \sum_{i=1}^{11} c^i + \sum_{\alpha=1}^{5}\lambda_{4\alpha}c_\varphi^\alpha = 0, \qquad U^{10}(\bar{c}) = 0.$$

Remark 4.2. For a special choice of integration constants $c^i = 0$ and $c_\varphi^\alpha = 0$, we get a solution governed by E_{10} Toda chain with the energy constraint $E_T = 0$. According to the result from [23] we obtain a never ending asymptotical oscillating behavior of scale factors which is described by the motion of a point-like particle in a billiard $B \subset H^9$. This billiard has a finite volume since E_{10} is hyperbolic.

Special 1-block solution. Now we consider a special 1-block solution (see Subsection 4.2.3). This solution is valid when a special set of charges is considered (see (4.16)):

$$Q_s^2 = Q^2|b_s|,$$

where $Q \neq 0$ and [38]

$$b_s = 2\sum_{s'=1}^{10} A^{ss'} = -60, -122, -186, -252, -320, -390, -462, -306, -152, -230,$$

$s = 1, \dots, 10$. Recall that $(A^{ss'}) = (A_{ss'})^{-1}$.

In this case $f_s = (\bar{f})^{b_s}$, where

$$\bar{f}(t) = \begin{cases} |Q|\sqrt{2/C}\sinh(\sqrt{C}(t-t_0)), & C > 0, \\ |Q|\sqrt{2/|C|}\sin(\sqrt{|C|}(t-t_0)), & C < 0, \\ |Q|\sqrt{2}(t-t_0), & C = 0 \end{cases}$$

and t_0 is a constant.

From (4.17) we get

$$E_T = -620C,$$

where relation $\sum_{s=1}^{10} b_s = -2480$ was used.

For the special solution under consideration the electric monomials in (4.18) have a simpler form

$$\mathcal{F}^s = Q_s \bar{f}^{-2} dt \wedge \tau(I_s),$$

where $s = 1, 2, \dots, 9$.

Solution with one harmonic function. Let $C = 0$ and all $c^i = \bar{c}^i = 0$, $c_\varphi^\alpha = \bar{c}_\varphi^\alpha = 0$. In this case $H = \bar{f}(t) = |Q|\sqrt{2}(t-t_0) > 0$ is a harmonic function on the 1-dimensional manifold $((t_0, +\infty), -dt \otimes dt)$ and our solution coincides with the 1-block solution (3.6)–(3.10) (if $\mathrm{sign}\nu_s = -\mathrm{sign}Q_s$ for all s).

Example 4.2. S-brane solution governed by $HA_2^{(1)}$ Toda chain. Now we consider the B_{11}-model in 11-dimensional pseudo-Euclidean space of signature $(-, +, \dots, +)$ with 4-form F^4.

Here we deal with four electric branes ($SM2$-branes) s_1, s_2, s_3, s_4 corresponding to the 4-form F^4. The brane sets are the following ones: $I_1 = \{1, 2, 3\}$, $I_2 = \{4, 5, 6\}$, $I_3 = \{7, 8, 9\}$, $I_4 = \{1, 4, 10\}$.

It may be verified that these sets obey the intersection rules corresponding to the hyperbolic KM algebra $HA_2^{(1)}$ with the following Cartan matrix

$$A = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \tag{4.19}$$

(see (3.14) with $I_{s_i} = I_i$).

Now we give a cosmological S-brane solution from Subsection 4.2.1 for the configuration of four branes under consideration. In what follows the relations $\varepsilon_s = +1$ and $h_s = 1/2$, $s \in S$, are used.

The metric (4.8) reads:

$$g = (f_1 f_2 f_3 f_4)^{1/3}\Big\{ -e^{2c^0 t + 2\bar{c}^0} dt \otimes dt + (f_1 f_4)^{-1} e^{2c^1 t + 2\bar{c}^1} dx^1 \otimes dx^1 + f_1^{-1} e^{2c^2 t + 2\bar{c}^2} dx^2 \otimes dx^2$$
$$+ f_1^{-1} e^{2c^3 t + 2\bar{c}^3} dx^3 \otimes dx^3 + (f_2 f_4)^{-1} e^{2c^4 t + 2\bar{c}^4} dx^4 \otimes dx^4 + f_2^{-1} e^{2c^5 t + 2\bar{c}^5} dx^5 \otimes dx^5$$

$$+ f_2^{-1} e^{2c^6 t + 2\bar{c}^6} dx^6 \otimes dx^6 + f_3^{-1} e^{2c^7 t + 2\bar{c}^7} dx^7 \otimes dx^7 + f_3^{-1} e^{2c^8 t + 2\bar{c}^8} dx^8 \otimes dx^8$$
$$\left. + f_3^{-1} e^{2c^9 t + 2\bar{c}^9} dx^9 \otimes dx^9 + f_4^{-1} e^{2c^{10} t + 2\bar{c}^{10}} dx^{10} \otimes dx^{10} \right\}.$$

The form field (see (4.10)) is as follows

$$F^4 = Q_1 f_1^{-2} f_2 f_3 dt \wedge dx^1 \wedge dx^2 \wedge dx^3 + Q_2 f_1 f_2^{-2} f_3 dt \wedge dx^4 \wedge dx^5 \wedge dx^6$$
$$+ Q_3 f_1 f_2 f_3^{-2} dt \wedge dx^7 \wedge dx^8 \wedge dx^9 + Q_4 f_3 f_4^{-2} dt \wedge dx^1 \wedge dx^4 \wedge dx^{10},$$

where $Q_s \neq 0$, $s = 1, \dots, 4$. Here

$$c^0 = \sum_{j=1}^{10} c^j, \qquad \bar{c}^0 = \sum_{j=1}^{10} \bar{c}^j,$$

$f_s = \exp(-q^s(t))$ and $q^s(t)$ obey the Toda-type equations

$$\ddot{q}^s = -2Q_s^2 \exp\left(\sum_{s'=1}^{4} A_{ss'} q^{s'} \right),$$

$s = 1, \dots, 4$, where $(A_{ss'})$ is the Cartan matrix (4.19) for the KM algebra $HA_2^{(1)}$ and the energy integration constant

$$E_T = \frac{1}{8} \sum_{s,s'=1}^{4} A_{ss'} \dot{q}^s \dot{q}^{s'} + \frac{1}{2} \sum_{s=1}^{4} Q_s^2 \exp\left(\sum_{s'=1}^{4} A_{ss'} q^{s'} \right),$$

obeys the constraint

$$2E_T + \sum_{i=1}^{10} (c^i)^2 - \left(\sum_{i=1}^{10} c^i \right)^2 = 0.$$

The brane constraints (3.38) read in this case as follows

$$U^1(c) = c^1 + c^2 + c^3 = 0, \qquad U^1(\bar{c}) = 0,$$
$$U^2(c) = c^4 + c^5 + c^6 = 0, \qquad U^2(\bar{c}) = 0,$$
$$U^3(c) = c^7 + c^8 + c^9 = 0, \qquad U^3(\bar{c}) = 0,$$
$$U^4(c) = c^1 + c^4 + c^{10} = 0, \qquad U^4(\bar{c}) = 0.$$

Since $F^4 \wedge F^4 = 0$ this solution also obeys equations of motion of 11-dimensional supergravity.

Special 1-block solution. Now we consider a special 1-block solution (see subsection 4.2.3). This solution is valid when a special set of charges is considered (see (4.16)):

$$Q_s^2 = Q^2 |b_s|,$$

where $Q \neq 0$ and

$$b_s = 2 \sum_{s'=1}^{4} A^{ss'} = -12, -12, -14, -6.$$

In this case $f_s = (\bar{f})^{b_s}$, where $\bar{f}$ is the same as in (4.1).

For the energy integration constant we have

$$E_T = -11C,$$

(see (4.17)).

Example 4.3. S-brane solution governed by P_{10} Toda chain with $E_T = 0$. Now we consider the B_{11}-model in 11-dimensional pseudo-Euclidean space of signature $(-,+,\ldots,+)$ with 4-form F^4.

Here we deal with ten electric branes ($SM2$-branes) $s_1,\ldots,s_{10}$ corresponding to the 4-form F^4. The brane sets are taken from [96, 6] as: $I_1 = \{1,4,7\}$, $I_2 = \{8,9,10\}$, $I_3 = \{2,5,7\}$, $I_4 = \{4,6,10\}$, $I_5 = \{2,3,9\}$, $I_6 = \{1,2,8\}$, $I_7 = \{1,3,10\}$, $I_8 = \{4,5,8\}$, $I_9 = \{3,6,7\}$, $I_{10} = \{5,6,9\}$.

These sets obey the intersection rules corresponding to the Lorentzian KM algebra P_{10} (we call it Petersen algebra) with the following Cartan matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 2 \end{pmatrix}. \tag{4.20}$$

The Dynkin diagram for this Cartan matrix could be represented by the Petersen graph ("a star inside a pentagon"). P_{10} is the Lorentzian KM algebra. It is a subalgebra of E_{10} [96, 6].

Let us present an S-brane solution for the configuration of 10 electric branes under consideration. The metric (4.8) reads:

$$\begin{aligned} g = \left(\prod_{s=1}^{10} f_s\right)^{1/3} \Big\{ & -dt \otimes dt + (f_1 f_6 f_7)^{-1} dx^1 \otimes dx^1 + (f_3 f_5 f_6)^{-1} dx^2 \otimes dx^2 \\ & + (f_5 f_7 f_9)^{-1} dx^3 \otimes dx^3 + (f_1 f_4 f_8)^{-1} dx^4 \otimes dx^4 + (f_3 f_8 f_{10})^{-1} dx^5 \otimes dx^5 \\ & + (f_4 f_9 f_{10})^{-1} dx^6 \otimes dx^6 + (f_1 f_3 f_9)^{-1} dx^7 \otimes dx^7 + (f_2 f_6 f_8)^{-1} dx^8 \otimes dx^8 \\ & + (f_2 f_5 f_{10})^{-1} dx^9 \otimes dx^9 + (f_2 f_4 f_7)^{-1} dx^{10} \otimes dx^{10} \Big\}. \end{aligned}$$

The form field (see (4.10)) is the following

$$\begin{aligned} F^4 = {} & Q_1 f_1^{-2} f_2 f_5 f_{10} dt \wedge dx^1 \wedge dx^4 \wedge dx^7 + Q_2 f_1 f_2^{-2} f_3 f_9 dt \wedge dx^8 \wedge dx^9 \wedge dx^{10} \\ & + Q_3 f_2 f_3^{-2} f_4 f_7 dt \wedge dx^2 \wedge dx^5 \wedge dx^7 + Q_4 f_3 f_4^{-2} f_5 f_6 dt \wedge dx^4 \wedge dx^6 \wedge dx^{10} \\ & + Q_5 f_1 f_4 f_5^{-2} f_8 dt \wedge dx^2 \wedge dx^3 \wedge dx^9 + Q_6 f_4 f_6^{-2} f_9 f_{10} dt \wedge dx^1 \wedge dx^2 \wedge dx^8 \\ & + Q_7 f_3 f_7^{-2} f_8 f_{10} dt \wedge dx^1 \wedge dx^3 \wedge dx^{10} + Q_8 f_5 f_7 f_8^{-2} f_9 dt \wedge dx^4 \wedge dx^5 \wedge dx^8 \\ & + Q_9 f_2 f_6 f_8 f_9^{-2} dt \wedge dx^3 \wedge dx^6 \wedge dx^7 + Q_{10} f_1 f_6 f_7 f_{10}^{-2} dt \wedge dx^5 \wedge dx^6 \wedge dx^9, \end{aligned}$$

where $Q_s \neq 0$, $s = 1,\ldots,10$. Here $f_s = \exp(-q^s(t))$ and $q^s(t)$ obey the Toda-type equations

$$\ddot{q}^s = -2Q_s^2 \exp\left(\sum_{s'=1}^{10} A_{ss'} q^{s'}\right),$$

where $(A_{ss'})$ is the Cartan matrix (4.20) for the KM algebra P_{10} and the energy constraint

$$E_T = \frac{1}{8} \sum_{s,s'=1}^{10} A_{ss'} \dot{q}^s \dot{q}^{s'} + \frac{1}{2} \sum_{s=1}^{10} Q_s^2 \exp\left(\sum_{s'=1}^{10} A_{ss'} q^{s'}\right) = 0$$

is obeyed. Here we used the fact that the two sets of linear equations $U^s(c) = 0$, $U^s(\bar{c}) = 0$, $s = 1, \dots, 10$, have trivial solutions: $c = 0$, $\bar{c} = 0$, due to the linear independence of vectors U^s.

Since $F^4 \wedge F^4 = 0$, this solution also obeys the equations of motion of 11-dimensional supergravity.

Remark 4.3. As pointed out in [96] we do not obtain a never ending asymptotic oscillating behavior of the scale factors in this case since the Lorentzian KM algebra P_{10} is not hyperbolic and the corresponding billiard $B \subset H^9$ has an infinite volume.

Special 1-block solution. Now we consider a special 1-block solution. The calculations give us the following relations

$$b_s = 2 \sum_{s'=1}^{10} A^{ss'} = -2, \qquad s = 1, \dots, 10,$$

and hence the special solution is valid (see (4.16)), when all charges are equal

$$Q_s^2 = Q^2,$$

where $Q \neq 0$. In this case all $f_s = \bar{f}^{-2}$, where

$$\bar{f}(t) = |Q|(t - t_0),$$

and t_0 is constant. The metric (4.3) may be rewritten using the synchronous time variable t_s:

$$g = -dt_s \otimes dt_s + A t_s^{2/7} \sum_{i=1}^{10} dx^i \otimes dx^i,$$

where $A > 0$ and $t_s > 0$. This metric coincides with the power-law, inflationary solution in the model with a one-component perfect fluid when the following equation of state is adopted: $p = \frac{2}{5}\rho$, where p is pressure and ρ is the density of fluid [97, 98].

5 Black brane solutions

In this section we consider the spherically symmetric case of the metric (4.13), i.e. we put $w = 1$, $M_1 = S^{d_1}$, $g^1 = d\Omega^2_{d_1}$, where $d\Omega^2_{d_1}$ is the canonical metric on a unit sphere S^{d_1}, $d_1 \geq 2$. In this case $\xi^1 = d_1 - 1$. We put $M_2 = \mathbb{R}$, $g^2 = -dt \otimes dt$, i.e. M_2 is a time manifold.

Let $C_1 \geq 0$. We consider solutions defined on some interval $[u_0, +\infty)$ with a horizon at $u = +\infty$.

When the matrix $(h_{\alpha\beta})$ is positive definite and

$$2 \in I_s, \qquad \forall\, s \in S,$$

i.e. all branes have a common time direction t, the horizon condition singles out the unique solution with $C_1 > 0$ and linear asymptotics at infinity

$$q^s = -\beta^s u + \bar{\beta}^s + o(1),$$

$u \to +\infty$, where $\beta^s, \bar{\beta}^s$ are constants, $s \in S$ [42, 43].

In this case

$$c^A/\bar{\mu} = -\delta^A_2 + h_1 U^{1A} + \sum_{s \in S} h_s b_s U^{sA}, \qquad \beta^s/\bar{\mu} = 2 \sum_{s' \in S} A^{ss'} \equiv b_s, \tag{5.1}$$

where $s \in S$, $A = (i, \alpha)$, $\bar{\mu} = \sqrt{C_1}$, the matrix $(A^{ss'})$ is inverse of the generalized Cartan matrix $(A_{ss'})$ and $h_1 = (U^1, U^1)^{-1} = d_1/(1 - d_1)$.

Let us introduce a new radial variable $R = R(u)$ through the relations

$$\exp(-2\bar{\mu}u) = 1 - \frac{2\mu}{R^{\bar{d}}}, \qquad \mu = \bar{\mu}/\bar{d} > 0,$$

where $u > 0$, $R^{\bar{d}} > 2\mu$, $\bar{d} = d_1 - 1$. We put $\bar{c}^A = 0$ and $q^s(0) = 0$, $A = (i, \alpha)$, $s \in S$. These relations guarantee the asymptotic flatness (for $R \to +\infty$) of the $(2 + d_1)$-dimensional section of the metric.

Let us denote $H_s = f_s e^{-\beta^s u}$, $s \in S$. Then, solutions (4.13)–(4.15) may be written as follows [41, 42, 43]

$$\begin{aligned} g = & \left(\prod_{s \in S} H_s^{2h_s d(I_s)/(D-2)}\right) \Biggl\{ \left(1 - \frac{2\mu}{R^{\bar{d}}}\right)^{-1} dR \otimes dR + R^2 d\Omega^2_{d_1} \\ & - \left(\prod_{s \in S} H_s^{-2h_s}\right)\left(1 - \frac{2\mu}{R^{\bar{d}}}\right) dt \otimes dt + \sum_{i=3}^{n}\left(\prod_{s\in S} H_s^{-2h_s \delta_{iI_s}}\right)\hat{g}^i \Biggr\}, \end{aligned} \tag{5.2}$$

$$\exp(\varphi^\alpha) = \prod_{s \in S} H_s^{h_s \chi_s \lambda^\alpha_{a_s}}, \tag{5.3}$$

where $F^a = \sum_{s \in S} \delta^a_{a_s} \mathcal{F}^s$, and

$$\mathcal{F}^s = -\frac{Q_s}{R^{d_1}}\left(\prod_{s' \in S} H_{s'}^{-A_{ss'}}\right) dR \wedge \tau(I_s), \qquad s \in S_e, \tag{5.4}$$

$$\mathcal{F}^s = Q_s \tau(\bar{I}_s), \qquad s \in S_m. \tag{5.5}$$

Here $Q_s \neq 0$, $h_s = K_s^{-1}$, $s \in S$, and the generalized Cartan matrix $(A_{ss'})$ is non-degenerate.

Functions $H_s > 0$ obey the equations

$$\frac{d}{dz}\left(\frac{(1 - 2\mu z)}{H_s}\frac{d}{dz}H_s\right) = \bar{B}_s \prod_{s' \in S} H_{s'}^{-A_{ss'}}, \tag{5.6}$$

$$H_s((2\mu)^{-1} - 0) = H_{s0} \in (0, +\infty), \tag{5.7}$$

$$H_s(+0) = 1, \qquad s \in S, \tag{5.8}$$

where $H_s(z) > 0$, $\mu > 0$, $z = R^{-\bar{d}} \in (0, (2\mu)^{-1})$ and $\bar{B}_s = \varepsilon_s K_s Q_s^2/\bar{d}^2 \neq 0$.

There exist solutions to equations (5.6)–(5.7) of polynomial type. The simplest example occurs in orthogonal case [58, 47] (for $d_i = 1$ see also [56, 57]): $(U^s, U^{s'}) = 0$, for $s \neq s'$, $s, s' \in S$. In this case $(A_{ss'}) = \mathrm{diag}(2, \ldots, 2)$ is a Cartan matrix for the semisimple Lie algebra $\mathbf{A_1} \oplus \cdots \oplus \mathbf{A_1}$ and

$$H_s(z) = 1 + P_s z \tag{5.9}$$

with $P_s \neq 0$, satisfying

$$P_s(P_s + 2\mu) = -\bar{B}_s, \qquad s \in S.$$

(For earlier supergravity solutions see [99, 100] and references therein).

In [84, 86, 101] this solution was generalized to a block-orthogonal case (3.3), (3.4). In this case (5.9) is modified as follows

$$H_s(z) = (1 + P_s z)^{b_s}, \tag{5.10}$$

where b_s are defined in (5.1) and parameters P_s coincide inside blocks, i.e. $P_s = P_{s'}$ for $s, s' \in S_i$, $i = 1, \dots, k$. The parameters $P_s \neq 0$ satisfy the relations [86, 101, 46]

$$P_s(P_s + 2\mu) = -\bar{B}_s/b_s, \qquad s \in S, \tag{5.11}$$

and the parameters $\bar{B}_s/b_s$ coincide inside blocks, i.e. $\bar{B}_s/b_s = \bar{B}_{s'}/b_{s'}$ for $s, s' \in S_i$, $i = 1, \dots, k$.

Finite-dimensional Lie algebras. Let $(A_{ss'})$ be a Cartan matrix for a finite-dimensional semisimple Lie algebra $\mathcal{G}$. In this case all powers b_s defined in (5.1) are natural numbers which coincide with the components of twice the dual Weyl vector in the basis of simple co-roots [4] and hence, all functions H_s are polynomials, $s \in S$.

Conjecture 5.1. *Let $(A_{ss'})$ be a Cartan matrix for a semisimple finite-dimensional Lie algebra $\mathcal{G}$. Then the solutions to equations (5.6)–(5.8) (if exist) have a polynomial structure:*

$$H_s(z) = 1 + \sum_{k=1}^{n_s} P_s^{(k)} z^k,$$

where $P_s^{(k)}$ are constants, $k = 1, \dots, n_s$; $n_s = b_s = 2\sum_{s' \in S} A^{ss'} \in \mathbb{N}$ and $P_s^{(n_s)} \neq 0$, $s \in S$.

In the extremal case ($\mu = +0$) an analogue of this conjecture was suggested previously in [76]. Conjecture 5.1 was verified for the $\mathbf{A_m}$ and $\mathbf{C_{m+1}}$ Lie algebras in [42, 43]. Explicit expressions for polynomials corresponding to Lie algebras C_2 and A_3 were obtained in [102] and [103] respectively.

Hyperbolic KM algebras. Let $(A_{ss'})$ be a Cartan matrix for an infinite-dimensional hyperbolic KM algebra $\mathcal{G}$. In this case all powers in (5.1) are negative numbers and hence, we have no chance to get a polynomial structure for H_s. Here we are led to an open problem of seeking solutions to the set of "master" equations (5.6) with boundary conditions (5.7) and (5.8). These solutions define special solutions to Toda-chain equations corresponding to the hyperbolic KM algebra $\mathcal{G}$.

Example 5.1. Black hole solutions for $\boldsymbol{A_1 \oplus A_1}$, $\boldsymbol{A_2}$ and $\boldsymbol{H_2(q,q)}$ KM algebras. Let us consider the 4-dimensional model governed by the action

$$S = \int_M d^4 z \sqrt{|g|} \left\{ R[g] - \varepsilon g^{MN} \partial_M \varphi \partial_N \varphi - \frac{1}{2} e^{2\lambda\varphi} (F^1)^2 - \frac{1}{2} e^{-2\lambda\varphi} (F^2)^2 \right\}.$$

Here F^1 and F^2 are 2-forms, φ is scalar field and $\varepsilon = \pm 1$.

We consider a black brane solution defined on $\mathbb{R}_* \times S^2 \times \mathbb{R}$ with two electric branes s_1 and s_2 corresponding to forms F^1 and F^2, respectively, with the sets $I_1 = I_2 = \{2\}$. Here $\mathbb{R}_*$ is subset of $\mathbb{R}$, $M_1 = S^2$, $g^1 = d\Omega_2^2$, is the canonical metric on S^2, $M_2 = \mathbb{R}$, $g^2 = -dt \otimes dt$ and $\varepsilon_1 = \varepsilon_2 = -1$.

The scalar products of U-vectors are (we identify $U^i = U^{s_i}$):

$$(U^1, U^1) = (U^2, U^2) = \frac{1}{2} + \varepsilon\lambda^2 \neq 0, \qquad (U^1, U^2) = \frac{1}{2} - \varepsilon\lambda^2.$$

The matrix A from (3.11) is a generalized non-degenerate Cartan matrix if and only if

$$\frac{2(U^1, U^2)}{(U^2, U^2)} = -q,$$

or, equivalently,

$$\varepsilon\lambda^2 = \frac{2 + q}{2(2 - q)},$$

where $q = 0, 1, 3, 4, \ldots$. This takes place when

$$\varepsilon = +1, \qquad q = 0, 1, \qquad \varepsilon = -1, \qquad q = 3, 4, 5, \ldots$$

and

$$\lambda^2 = \frac{2+q}{2|2-q|}.$$

The first branch ($\varepsilon = +1$) corresponds to finite dimensional Lie algebras $A_1 \oplus A_1$ ($q = 0$), A_2 ($q = 1$) and the second one ($\varepsilon = -1$) corresponds to hyperbolic KM algebras $H_2(q, q)$, $q = 3, 4, \ldots$. In the hyperbolic case the scalar field φ is a phantom (ghost).

The black brane solution reads (see (5.2)–(5.4))

$$g = (H_1 H_2)^h \left\{ \left(1 - \frac{2\mu}{R}\right)^{-1} dR \otimes dR + R^2 d\Omega_2^2 - (H_1 H_2)^{-2h} \left(1 - \frac{2\mu}{R}\right) dt \otimes dt \right\}, \tag{5.12}$$

$$\exp(\varphi) = (H_1/H_2)^{\varepsilon \lambda h}, \tag{5.13}$$

$$F^s = \frac{Q_s}{R^2} H_s^{-2} (H_{\bar{s}})^q dt \wedge dR, \qquad s = 1, 2. \tag{5.14}$$

Here $h = (2-q)/2$ and $\bar{s} = 2, 1$ for $s = 1, 2$ respectively.

The moduli functions $H_s > 0$ obey the equations (see (5.6))

$$\frac{d}{dz} \left(\frac{(1 - 2\mu z)}{H_s} \frac{d}{dz} H_s \right) = \frac{2Q_s^2}{q-2} H_s^{-2} (H_{\bar{s}})^q, \tag{5.15}$$

with the boundary conditions $H_s((2\mu)^{-1} - 0) = H_{s0} \in (0, +\infty)$, $H_s(+0) = 1$, $s = 1, 2$, imposed. Here $\mu > 0$, $z = 1/R \in (0, (2\mu)^{-1})$. For $q = 0, 1$ the solutions to equations (5.15) with the boundary conditions imposed were given in [41, 42, 43]. They are polynomials of degrees 1 and 2 for $q = 0$ and $q = 1$, respectively. For $q = 3, 4, \ldots$ the exact solutions to equations (5.15) are not known yet.

Special solution with $Q_1^2 = Q_2^2$. Now we consider the special one-block solution with the functions H_s obeying (5.10) and (5.11). Since $b_s = 2/(2-q)$ and $\bar{B}_s = 2Q_s^2/(q-2)$ it takes place when $Q_1^2 = Q_2^2 = Q^2 > 0$. The moduli functions read

$$H_s = H^{2/(2-q)}, \qquad H = 1 + Pz,$$

where $z = 1/R$ and $q \neq 2$. These functions obey $H_s(z) > 0$ for $z \in [0, (2\mu)^{-1}]$ if $P > -2\mu$ ($\mu > 0$). Due to this inequality and the relation $P(P + 2\mu) = Q^2$ (following from (5.11)) we get

$$P = -\mu + \sqrt{\mu^2 + Q^2} > 0.$$

In this special case the solution (5.12)–(5.14) has the following form:

$$g = H^2 \left\{ \left(1 - \frac{2\mu}{R}\right)^{-1} dR \otimes dR + R^2 d\Omega_2^2 - H^{-4} \left(1 - \frac{2\mu}{R}\right) dt \otimes dt \right\},$$

$$\varphi = 0, \tag{5.16}$$

$$F^s = \frac{Q_s}{H^2 R^2} dt \wedge dR, \qquad s = 1, 2.$$

Remarkably, this special solution does not depend upon q. The metric (5.16) coincides with the metric of the Reissner–Nordström solution (when the Maxwell 2-form is $F = \sqrt{2} Q (HR)^{-2} dt \wedge dR$).

In the extremal case $\mu \to +0$ we are lead to the special case of a Majumdar–Papapetrou type solution

$$g = H^2 \hat{g}^0 - H^{-2} dt \otimes dt,$$
$$\varphi = 0,$$
$$F^s = \nu_s dH^{-1} \wedge dt,$$

where $g^0 = \sum_{i=1}^{3} dx^i \otimes dx^i$, H is a harmonic function on $M_0 = \mathbb{R}^3$ and $\nu_s^2 = 1$, $s = 1, 2$. Here $\nu_s = -Q_s/Q$.

6 Conclusions

Here we reviewed several families of exact solutions in multidimensional gravity with a set of scalar fields and fields of forms related to non-singular (e.g. hyperbolic) KM algebras.

The solutions describe composite electromagnetic branes defined on warped products of Ricci-flat, or sometimes Einstein, spaces of arbitrary dimensions and signatures. The metrics are block-diagonal and all scale factors, scalar fields and fields of forms depend on points of some manifold M_0. The solutions include those depending upon harmonic functions, S-branes and spherically-symmetric solutions (e.g. black-branes). Our approach is based on the sigma-model representation obtained in [48] under the rather general assumption on intersections of composite branes (when stress-energy tensor has a diagonal structure).

We were dealing with rather general intersection rules [47] governed by invertible generalized Cartan matrix corresponding to the certain generalized KM Lie algebra $\mathcal{G}$. For $\mathcal{G} = \mathbf{A_1} \oplus \cdots \oplus \mathbf{A_1}$ (r terms) we get the well-known standard (e.g. supersymmetry preserving) intersection rules [53, 54, 55, 48].

We have also considered a class of special "block-orthogonal" solutions corresponding to semisimple KM algebras and governed by several harmonic functions. Certain examples of 1-block solutions (e.g. corresponding to KM algebras $H_2(q,q)$, AE_3) were considered.

In the one-block case a generalization of the solutions to those governed by several functions of one harmonic function H and obeying Toda-type equations was presented.

For finite-dimensional (semi-simple) Lie algebras we are led to integrable Lagrange systems while the Toda chains corresponding to infinite-dimensional (non-singular) KM algebras are not well studied yet. Some examples of S-brane solutions corresponding to Lorentzian KM algebras $HA_2^{(1)} = A_2^{++}$, E_{10} and P_{10} were presented.

We have also considered general classes of cosmological-type solutions (e.g. S-brane and spherically symmetric solutions) governed by Toda-type equations, containing black brane configurations as a special case. The "master" equations for moduli functions have polynomial solutions in the finite-dimensional case (according to our conjecture [41, 42, 43]), while in the infinite-dimensional case we have only a special family of the so-called block-orthogonal solutions corresponding to semi-simple non-singular KM algebras. Examples of 4-dimensional dilatonic black hole solutions corresponding to KM algebras $A_1 \oplus A_1$, A_2 and $H_2(q,q)$ ($q > 2$) were given.

We note that the problem of integrability of Toda chain equations corresponding to (non-singular) KM algebras arises also in the context of fluxbrane solutions [44] that have also a polynomial structure of moduli functions for finite-dimensional Lie algebras (see also [104]). (For similar S-brane solutions governed by polynomial functions and its applications in connection with cosmological problems see [105, 106, 107].)

Here we have considered only the case of non-degenerate matrix A. It is an open problem to find general classes of solutions with branes for the degenerate case when $\det A = 0$ (e.g. corresponding to affine KM algebras). Some special solutions of such type with maximal set of composite electric S-branes (e.g. when A is not obviously a generalized Cartan matrix) were found in [108, 109] and generalized in [110, 111] for arbitrary (anti-)self-dual parallel charge density form of dimension $2m$ defined on Ricci-flat Riemannian sub-manifold of dimension $4m$. In these examples the restrictions on brane intersections (2.11) and (2.12) were replaced by more general condition on the stress-energy tensor: $T_N^M = 0$, $M \neq N$.

It should be noted here that the solutions related to Lorentzian (e.g. hyperbolic) KM algebras considered in Examples 4.1–5.1 are the new ones. (In Example 4.3 the brane configuration from [96, 6]) was used.)

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