Monopoles and Modifications of Bundles over Elliptic Curves*

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Abstract. Modifications of bundles over complex curves is an operation that allows one to construct a new bundle from a given one. Modifications can change a topological type of bundle. We describe the topological type in terms of the characteristic classes of the bundle. Being applied to the Higgs bundles modifications establish an equivalence between different classical integrable systems. Following Kapustin and Witten we define the modifications in terms of monopole solutions of the Bogomolny equation. We find the Dirac monopole solution in the case $R\times$ (elliptic curve). This solution is a three-dimensional generalization of the Kronecker series. We give two representations for this solution and derive a functional equation for it generalizing the Kronecker results. We use it to define Abelian modifications for bundles of arbitrary rank. We also describe non-Abelian modifications in terms of theta-functions with characteristic.

Key words: integrable systems; field theory; characteristic classes

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1 Introduction

The modifications (or the Hecke transformation) of bundles over complex curves is a correspondence between two bundles E and $\tilde{E}$. It is isomorphism in a complement of some divisor. A modification can change the topological type of the original bundle. From the field-theoretical point of view the modification is provided by a gauge transformation of sections, which is singular at the divisor. In [1] we apply this procedure to the Higgs bundles. The Higgs bundles are the phase spaces of the Hitchin integrable systems [2]. Modifications acts on the phase space as a symplectic transformation. In this special case we call the modification the Symplectic Hecke Correspondence. For the Higgs bundles over elliptic curves with marked points Symplectic Hecke Correspondence leads to a symplectomorphism between different classical integrable systems such as

- Elliptic Calogero–Moser system $\Leftrightarrow$ Elliptic $\mathrm{GL}(N,\mathbb{C})$ Top, [1];
- Calogero–Moser field theory $\Leftrightarrow$ Landau–Lifshitz equation, [1, 4];
- Painlevé VI $\Leftrightarrow$ non-autonomous Zhukovsky–Volterra gyrostat, [3].

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In these examples modifications increase the degree of the underlying bundles on one. In general, modifications act as the Bäcklund transformations of integrable systems. If degree of the bundles (modula rank) is not changed then modifications produce what is called the autoBäcklund transformations. It turned out that the modification in the first example is equivalent to the twist of R-matrices [5, 6] that transforms the dynamical R-matrices of the IRF models of the $\mathrm{GL}(N)$ type [7] to the vertex R-matrices [8] corresponding to the $\mathrm{GL}(N)$ generalization of the XYZ models.

The modifications are parameterized by vectors $\vec{m}$ of the weight lattices P of $\mathrm{SL}(N,\mathbb{C})$. If $\vec{m}$ belongs to the root sublattice $Q \subset P$, then the modified bundle $\tilde{E}$ has the same degree as E. Otherwise, the degree of bundle is changed. The modifications can be described by changing another topological invariant. It is a characteristic class of a bundle. Let the base of E be a Riemann surface Σ_g of genus g. Then the characteristic class of E is an element of $H^2(\Sigma_g,\mathbb{Z}_N) \sim \mathbb{Z}_N$, where $\mathbb{Z}_N \sim P/Q$ is a center of $\mathrm{SL}(N,\mathbb{C})$. Another example of the characteristic classes, is the characteristic class of spin-bundles, that will not considered here, is the Stiefel–Whitney class $H^2(\Sigma_g, Z_2)$.

Here we discuss a field-theoretical interpretation of modifications. It was established in [9] that the modifications are related to the Dirac monopole configurations in a topological version of the $\mathcal{N}=4$ four-dimensional super-symmetric Yang–Mills theory. If "the space-time" of the topological theory has the form $\mathbb{R}^2 \times \Sigma_g$, then the modifications of E over Σ_g are parameterized by the monopoles charges.

To describe the modification it is sufficient to neglect the "time" dependence and consider $\mathbb{R} \times \Sigma_g$. The condition for fields to preserve the supersymmetry amounts to the Bogomolny equation.

The aims of this paper are

- To define modifications and describe their interrelations with the Bogomolny equation following [9]. We consider a special configuration of the space-time $\mathbb{R}^2 \times \Sigma_\tau$, where Σ_τ is an elliptic curve with the modular parameter τ.
- To find solutions of the Bogomolny equation in the case of line bundles over Σ_τ. They are generalizations of the Kronecker series [17]. We give two representations of the solution and prove their equivalence by means of the functional equation generalizing the Kronecker functional equation.
- To describe non-Abelian modifications that are not related directly to solutions of the Bogomolny equation and follows from our previous results.

2 Characteristic classes of holomorphic bundles over complex curves

We describe holomorphic bundles over complex curves Σ_g of genus g and define their characteristic classes.

2.1 Global description

Let $\pi_1(\Sigma_g)$ be a fundamental group of Σ_g. It has $2g$ generators $\{a_\alpha, b_\alpha\}$, corresponding to the fundamental cycles of Σ_g with the relation

$$\prod_{\alpha=1}^{g}[a_\alpha, b_\alpha] = 1, \tag{2.1}$$

where $[a_\alpha, b_\alpha] = a_\alpha b_\alpha a_\alpha^{-1} b_\alpha^{-1}$ is the group commutator.

Let ρ be a representation of π_1 in $\mathbb{C}^N$. Consider a holomorphic adjoint $\mathrm{GL}(N,\mathbb{C})$ bundle E over Σ_g. In fact, E is a $\mathrm{PGL}(N,\mathbb{C}) \sim \mathrm{PSL}(N,\mathbb{C})$ bundle, because the center of $\mathrm{GL}(N,\mathbb{C})$ does not act in the adjoint representation. The bundle E can be defined by holomorphic transition matrices of its sections $s \in \Gamma(E)$ around the fundamental cycles. Let $z \in \Sigma_g$ be a fixed point. Then

$$s(a_\alpha z) = \rho(a_\alpha)s(z), \qquad s(b_\beta z) = \rho(b_\beta)s(z).$$

Due to (2.1) we have

$$\prod_{\alpha=1}^{g} [\rho(a_\alpha), \rho(b_\alpha)] = \mathrm{Id}. \tag{2.2}$$

Let $\mathcal{K}$ be an extension of π_1 by the cyclic group $\mathbb{Z}_N \sim \mathbb{Z}/N\mathbb{Z}$

$$1 \to \mathbb{Z}_N \to \mathcal{K} \to \pi_1(\Sigma_g) \to 1. \tag{2.3}$$

The group $\mathcal{K}$ is defined by the relation

$$\prod_{\alpha=1}^{g} [a_\alpha, b_\alpha] = \omega, \qquad \omega^N = 1.$$

Let $\hat\rho$ be a representation of $\mathcal{K}$ in $\mathrm{GL}(N,\mathbb{C})$. Then using $\hat\rho$ as transition matrices we define a bundle over Σ_g. But now instead of (2.2) we have

$$\prod_{\alpha=1}^{g} [\hat\rho(a_\alpha), \hat\rho(b_\alpha)] = \omega\,\mathrm{Id}. \tag{2.4}$$

Here $\omega\,\mathrm{Id}$ is the generator of the center $\mathcal{Z}(\mathrm{SL}(N,\mathbb{C})) \sim \mathbb{Z}_N$ of $\mathrm{SL}(N,\mathbb{C})$. It means that $\hat\rho$ can serve as transition matrices only for $\mathrm{PSL}(N,\mathbb{C})$ bundles, but not for $\mathrm{SL}(N,\mathbb{C})$ or $\mathrm{GL}(N,\mathbb{C})$ bundles. Note, that the fibers of the $\mathrm{PSL}(N,\mathbb{C})$-bundles are spaces of representations with highest weights from the root lattice Q (A.2) including the adjoint representation with the highest weight $\varpi_1 + \varpi_{N-1}$ (A.3). For the $\mathrm{SL}(N,\mathbb{C})$ representations the highest weights belong to the weight lattice P (A.4). In this way elements from the factor group $P/Q \sim \mathcal{Z}(\mathrm{SL}(N,\mathbb{C}))$ (A.6) define an obstruction to lift $\mathrm{PSL}(N,\mathbb{C})$ bundles to $\mathrm{SL}(N,\mathbb{C})$ bundles.

The obstruction has a cohomological interpretation. Consider the exact sequence following from (A.1)

$$\to H^1(\Sigma_g, \mathrm{SL}(N,\mathbb{C})) \to H^1(\Sigma_g, \mathrm{PSL}(N,\mathbb{C})) \to H^2(\Sigma_g, \mathcal{Z}(\mathrm{SL}(N,\mathbb{C}))) \to \cdots .$$

The groups $H^1(\Sigma_g, \mathrm{SL}(N,\mathbb{C}))$, $H^1(\Sigma_g, \mathrm{PSL}(N,\mathbb{C}))$ are the moduli space of $\mathrm{SL}(N,\mathbb{C})$ and $\mathrm{PSL}(N,\mathbb{C})$ bundles. Then $H^2(\Sigma_g, \mathcal{Z}(\mathrm{SL}(N,\mathbb{C})))$ defines an obstruction to lift $\mathrm{PSL}(N,\mathbb{C})$ bundles to $\mathrm{SL}(N,\mathbb{C})$ bundles. We call $\xi \in H^2(\Sigma_g, \mathbb{Z}_N)$ *the characteristic class* of a $\mathrm{PSL}(N,\mathbb{C})$ bundle. In fact, $H^2(\Sigma_g, \mathbb{Z}_N) \sim \mathbb{Z}_N$ and ω in (2.4) represents $\xi \in H^2(\Sigma_g, \mathbb{Z}_N)$.

This construction can be generalized to any factor-group $G_l = \mathrm{SL}(N,\mathbb{C})/\mathbb{Z}_l$, where l is a nontrivial divisor of N, ($N = pl$, $l \neq 1, N$). Consider an extension $\mathcal{K}_l$ of $\pi_1(\Sigma_g)$ by $\mathbb{Z}_l$ (compare with (2.3))

$$1 \to \mathbb{Z}_l \to \mathcal{K}_l \to \pi_1(\Sigma_g) \to 1.$$

Let E_l be a holomorphic G_l-bundle. The fibers of E_l belong to a irreducible representation of G_l with a highest weight $\nu \in \Gamma(G_l)$ (A.7). Then the transition matrices representing $\mathcal{K}_l$ satisfy the relation

$$\prod_{\alpha=1}^{g} [\hat\rho(a_\alpha), \hat\rho(b_\alpha)] = \omega^p\,\mathrm{Id}, \qquad (\omega^p)^l = 1. \tag{2.5}$$

It follows from the exact sequence

$$1 \to \mathbb{Z}_l \to \mathrm{SL}(N,\mathbb{C}) \to G_l \to 1,$$

that elements from $H^2(\Sigma_g,\mathbb{Z}_l) \sim \mathbb{Z}_l$ are obstructions to lift G_l bundle E_l to a $\mathrm{SL}(N,\mathbb{C})$-bundle. The group $\mathbb{Z}_l$ can be identified with the center of the dual group ${}^LG_l \sim G_p = \mathrm{SL}(N,\mathbb{C})/\mathbb{Z}_p$ (see (A.11) and (A.10)). Thus, the obstructions to lift G_l bundles E_l to a $\mathrm{SL}(N,\mathbb{C})$ bundles are defined by $H^2(\Sigma_g,\mathcal{Z}({}^LG_l))$.

On the other hand, since $\mathbb{Z}_p$ is a center of G_l we have the sequence

$$1 \to \mathbb{Z}_p \to G_l \to \mathrm{PSL}(N,\mathbb{C}) \to 1,$$

where $\mathbb{Z}_p$ is a center of G_l. Then elements from $H^2(\Sigma_g,\mathcal{Z}(G_l))$ are obstructions to lift a $\mathrm{PSL}(N,\mathbb{C})$-bundle to a G_l-bundle. Summarizing we have defined two types of the characteristic classes

$$\begin{aligned}
&H^2(\Sigma_g,\mathcal{Z}(G_l)) \text{ – obstructions to lift a } \mathrm{PSL}(N,\mathbb{C}) \text{ bundle to a } G_l \text{ bundle},\\
&H^2(\Sigma_g,\mathcal{Z}({}^LG_l)) \text{ – obstructions to lift a } G_l \text{ bundle to a } \mathrm{SL}(N,\mathbb{C}) \text{ bundle}.
\end{aligned} \tag{2.6}$$

Though for $\omega \neq 1$ $\mathrm{PSL}(N,\mathbb{C})$ bundles cannot be lifted to $\mathrm{SL}(N,\mathbb{C})$ bundles, they can be lifted to $\mathrm{GL}(N,\mathbb{C})$ bundles. From the exact sequence

$$1 \to \mathcal{O}^* \stackrel{\det}{\to} \mathrm{GL}(N,\mathbb{C}) \to \mathrm{PGL}(N,\mathbb{C}) \to 1$$

we have

$$H^1(\Sigma_g,\mathrm{GL}(N,\mathbb{C})) \to H^1(\Sigma_g,\mathrm{PGL}(N,\mathbb{C})) \to H^2(\Sigma_g,\mathcal{O}^*).$$

The Brauer group $H^2(\Sigma_g,\mathcal{O}^*)$ vanishes and therefore, there is no obstruction to lift $\mathrm{PGL}(N,\mathbb{C}) \sim \mathrm{PSL}(N,\mathbb{C})$ bundles to $\mathrm{GL}(N,\mathbb{C})$ bundles. We will demonstrate it below.

2.2 Holomorphic bundles over elliptic curves

We define an elliptic curve ($g = 1$) as the quotient $\Sigma_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. In this case we can construct explicitly the generic transition matrices for G_l-bundles.

The curve has two fundamental cycles $a : (z \to z+1)$, $b : (z \to z+\tau)$. We define a trivial bundle E over Σ_τ by two commuting matrices

$$s(z+1) = \rho_a s(z), \qquad s(z+\tau) = \rho_b s(z), \qquad [\rho_a,\rho_b] = \mathrm{Id}. \tag{2.7}$$

It is a $\mathrm{PGL}(N,\mathbb{C})$-bundle that can be lifted to $\mathrm{SL}(N,\mathbb{C})$ bundles.

Consider a representation of $\hat\rho$ of $\mathcal{K}$ acting on the sections of E as

$$s(z+1) = \hat\rho_a s(z), \qquad s(z+\tau) = \hat\rho_b s(z).$$

with commutation relation (2.4)

$$[\hat\rho_a,\hat\rho_b] = \omega\,\mathrm{Id}.$$

One can choose

$$\hat\rho_a = \mathcal{Q}, \qquad \hat\rho_b = \Lambda, \qquad \mathcal{Q} = \mathrm{diag}\big(1,\omega,\dots,\omega^{N-1}\big), \qquad \Lambda = \begin{pmatrix} 0 & 1 & \dots & 0\\ 0 & 0 & 1 & 0\\ \vdots & \vdots & \ddots & 1\\ 1 & 0 & \dots & 0 \end{pmatrix}. \tag{2.8}$$

The bundle with these transition functions cannot be lifted to $\mathrm{SL}(N,\mathbb{C})$ bundles. Replace $\hat{\rho}_b$ by

$$\hat{\rho}'_b = \exp\left(-\frac{2\pi i}{N}\left(z+\frac{\tau}{2}\right)\right)\Lambda. \tag{2.9}$$

It is a $\mathrm{GL}(N,\mathbb{C})$ bundle since $[\hat{\rho}_a,\hat{\rho}'_b] = Id$ and $\det\hat{\rho}'_b \neq 1$. It follows from (2.9) that a section of the determinant bundle is the theta-function

$$\vartheta(z,\tau) = q^{\frac{1}{8}}\sum_{n\in\mathbb{Z}}(-1)^n e^{\pi i(n(n+1)\tau+2nz)}, \qquad q = \exp 2\pi i\tau. \tag{2.10}$$

It has a simple pole in the fundamental domain $\mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z})$. Therefore, the bundle has degree one. It is called the theta-bundle.

To consider a general case [10] represent the rank as the product $N = pl$. Define the transition matrix

$$\hat{\rho}_a = \mathcal{Q}, \tag{2.11}$$
$$\hat{\rho}_b = \mathbf{e}(\vec{u}_l)\Lambda^p, \tag{2.12}$$

where

$$\vec{u}_l = \mathrm{diag}(\overbrace{\mathbf{u}_p,\ldots,\mathbf{u}_p}^{l}), \qquad \mathbf{u}_p = (\tilde{u}_1,\ldots,\tilde{u}_p).$$

Since $[\mathcal{Q},\Lambda^p] = \omega^p\,\mathrm{Id}_l$, $\omega^p = \exp\frac{2\pi i}{l}$

$$[\hat{\rho}_a,\hat{\rho}_b] = \omega^p\,\mathrm{Id}_N.$$

Comparing this relation with (2.5) we conclude that (2.11) and (2.12) serve as the transition matrices for a G_l-bundle over Σ_τ. Therefore ω^p represents an element from $H^2(\Sigma_\tau,\mathcal{Z}(^LG_l)) \sim \mathbb{Z}_p$. It is an obstruction (2.6).

As in (2.9), modify the transition matrix

$$\hat{\rho}_b \to \hat{\rho}'_b = \exp\left\{-\frac{2\pi i}{p}\left(z+\frac{\tau}{2}\right)\right\}\hat{\rho}_b.$$

We come to the $\mathrm{GL}(N,\mathbb{C})$-bundle of degree $p \pmod N$.

2.3 Local description

There exists another description of a holomorphic bundles over Σ_g. Let w_0 be a fixed point on Σ_g and D_{w_0} ($D^\times_{w_0}$) be a disc (punctured disc) with a center w_0 with a local coordinate z. A bundle E over Σ_g can be trivialized over D and over $\Sigma_g \setminus w_0$. These two trivializations are related by a $\mathrm{GL}(N,\mathbb{C})$ transformation $g(z)$, holomorphic on $D^\times_{w_0}$. If we consider another trivialization over D then g is multiplied from left by an invertible matrix h on D. Likewise, a trivialization over $\Sigma_g \setminus w_0$ is determined up to the multiplication on the right $g \to gh$, where $h \in \mathrm{GL}(N,\mathbb{C})$ is holomorphic on $\Sigma_g \setminus w_0$. Thus, the set of isomorphism classes of rank N vector bundles is described as a double-coset

$$\mathrm{GL}(N,\mathbb{C})(D_{w_0}) \setminus \mathrm{GL}(N,\mathbb{C})(D^\times_{w_0})/\mathrm{GL}(N,\mathbb{C})(\Sigma_g \setminus w_0),$$

where $\mathrm{GL}(N,\mathbb{C})(U)$ denote the group of $\mathrm{GL}(N,\mathbb{C})$-valued holomorphic functions on U.

Let $\det g(z) = 1$. If $g(ze^{2\pi i}) = g(z)$ then it defines a $\mathrm{SL}(N,\mathbb{C})$-bundle over Σ_g. But if the monodromy is nontrivial

$$g(ze^{2\pi i}) = \omega g(z), \qquad \omega^N = 1, \tag{2.13}$$

then $g(w)$ is a transition matrix for a $\mathrm{PSL}(N,\mathbb{C})$-bundle but not for a $\mathrm{SL}(N,\mathbb{C})$-bundle. This relation is similar to (2.4).

Let us choose a trivialization of E over D by choosing N linear independent holomorphic sections $\vec{s} = (s_1, s_2, \dots, s_N)$. Thereby, the bundle E over D is represented by a sum of N line bundles $\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_N$. The sections over $\Sigma_g \setminus w_0$ are obtained by the action of the transition matrix $\vec{s}\,' = \vec{s}g$.

Let $\vec{m}$ belongs to the root lattice ($\vec{m} = (m_1, m_2, \dots, m_N) \in Q$) (A.2). Transform the restriction of the section $\vec{s}$ on $D^{\times}_{w_0}$ as

$$s_j \to z^{-m_j} s_j, \qquad j = 1, \dots, N. \tag{2.14}$$

Then the transition matrix is transformed by the diagonal matrix

$$g(z) \to \mathrm{diag}\big(z^{-m_1}, z^{-m_2}, \dots, z^{-m_N}\big) g(z). \tag{2.15}$$

It implies the transformation of line bundles over D

$$\mathcal{L}_j \to \mathcal{L}_j \otimes \mathcal{O}(m_j).$$

In this way we come to the new bundle $\tilde{E}$ (*the modified bundle*). It is defined by the new transition matrix (2.15). This transformation of the bundle E to $\tilde{E}$ (or more exactly the map of sheaves of its sections)

$$\Gamma(E) \stackrel{\Xi(\vec{m})}{\longrightarrow} \Gamma(\tilde{E}), \qquad \Xi(\vec{m}) \sim \mathrm{diag}\big(z^{-m_1}, z^{-m_2}, \dots, z^{-m_N}\big),$$

is called *the modification or the Hecke transformation* of type $\vec{m} = (m_1, m_2, \dots, m_N)$. In field-theoretical terms it corresponds to *the t'Hooft operator*, generating by monopoles (see below).

Let us relax the condition $\sum_{j=1}^N m_j = 0$. Then the modification $\Xi(\vec{m})$ changes the topology of E. We come to a nontrivial bundle of degree $\deg(\tilde{E}) = \deg(E) + \sum_{j=1}^N m_j$. In next section we illustrate this fact.

Now assume that $\vec{m}$ belongs to the weight lattice P (A.4). Then the modification $\Xi(\vec{m})$ changes the characteristic class of a $\mathrm{PGL}(N,\mathbb{C})$-bundle E. To prove it let us pass to the basis of the fundamental weights (A.3)

$$\vec{m} = \sum_{j=1}^N m_j e_j = \sum_{k=1}^{N-1} n_k \varpi_k.$$

It follows from (A.3) that m_j and n_k are related as

$$\begin{aligned}
& m_1 = \frac{1}{N}((N-1)n_1 + (N-2)n_2 + \cdots + n_{N-1}), \\
& m_2 = \frac{1}{N}(-n_1 + (N-2)n_2 + \cdots + n_{N-1}), \\
& \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
& m_N = \frac{1}{N}(-n_1 - 2n_2 - \cdots - (N-1)n_{N-1}), \qquad n_k = m_k - m_{k+1}.
\end{aligned}$$

Rewrite the modification in the form of the product of the diagonal matrices

$$\Xi(\vec{n}) \sim \prod_{k=1}^{N-1} \mathrm{diag}\big(z^{-n_k \varpi_k}\big). \tag{2.16}$$

It follows from (A.3) that the monodromy of this matrix around the point $z = 0$ is

$$\exp\left(-\frac{2\pi i}{N}\sum_{k=1}^{N-1} kn_k\right)\mathrm{Id}_N. \tag{2.17}$$

Therefore, the characteristic class of the adjoint bundle is unchanged if

$$\sum_{k=1}^{N-1} kn_k = N\sum_{j=1}^{N-1} m_j = 0, \qquad (\mathrm{mod}\, N).$$

In this case the weight vector $\vec{m}$ belongs to the root lattice Q. Otherwise, we come to the non-trivial monodromy (2.17). It is an obstruction to lift the $\mathrm{PGL}(N, \mathbb{C})$-bundle to a $\mathrm{SL}(N, \mathbb{C})$-bundle. This element can be identified with the monodromy (2.4) and in this way with an element from $H^2(\Sigma, \mathbb{Z}_N)$. As it was mentioned above, the modified bundle $\tilde{E}$ can be lifted to a $\mathrm{GL}(N, \mathbb{C})$ bundle. Let us act on the modified sections (2.14) by the scalar matrix

$$h = z^{\frac{2\pi i}{N}\sum_{k=1}^{N-1} kn_k}\,\mathrm{Id}_N.$$

It is a $\mathrm{GL}(N, \mathbb{C})$ gauge transformation. The monodromy of the new transition matrix is trivial. Therefore, we come to the $\mathrm{GL}(N, \mathbb{C})$ bundle. The bundle is topologically nontrivial – it has degree

$$p = \sum_{k=1}^{N-1} kn_k = N\sum_{j=1}^{N-1} m_j. \tag{2.18}$$

It follows from (2.17) that the characteristic class ξ and the degree p are related as

$$\xi = \exp\frac{2\pi i}{N}p.$$

The set of modifications that changes the degree on p is defined as solutions of (2.18) in integers n_k.

Assume that the bundle E is equipped with a holomorphic connection. On $D^\times_{w_0}$ it takes the form $(\partial_z + A_z)dz$ and can be considered as an element of the affine Lie coalgebra $\widehat{\mathrm{gl}}^*(N, \mathbb{C})(D^\times_{w_0})$ The gauge transformation (2.15) acts on A_w acts as the coadjoint action

$$(A_z)_{jk}dz \to \left(z^{m_k - m_j}(A_z)_{jk}(1 - \delta_{jk}) - m_j z^{-1}\delta_{jk}\right)dz. \tag{2.19}$$

Let $\vec{m} \in P$. Then the first term in the r.h.s. is well defined, since $m_k - m_j$ is integer. The last term represents the shift action (A.14) of the affine group $\bar{W}_a$ (A.13) on the connection. The topology of E is not changed if $\vec{m} \in Q$ and we come to description of the characteristic class as elements from factor group $\bar{W}_a/W_a$ (A.15). We come again to this point in Section 4.

Let $N = pl$ with $l \neq 1, N$ and $G_l = \mathrm{SL}(N, \mathbb{C})/\mathbb{Z}_l$ (A.9). Consider the gauge transformation (2.16) with $\vec{m}$ $(\vec{\varpi}) \in \Gamma({}^L G)$ (A.9). For example, we can take $\vec{\varpi} = (p, 0, \ldots, 0)$. Then the monodromy (2.17) belongs to the group $\mathbb{Z}_l$. It means that the modified bundle $\tilde{E}$ is the G_l-bundle that cannot be lifted to the $\mathrm{SL}(N, \mathbb{C})$-bundle (see (2.6)).

The modification can be performed in an arbitrary number of points w_a, $(a = 1, \ldots, n)$. To this end define the isomorphism classes of vector bundles as the quotient

$$\prod_{a=1}^{n} \mathrm{GL}(N, \mathbb{C})(D_{w_a}) \setminus \prod_{a=1}^{n} \mathrm{GL}(N, \mathbb{C})(D^\times_{w_a})/\mathrm{GL}(N, \mathbb{C})(\Sigma_g \setminus (w_1, \ldots, w_a)).$$

We have n transition matrices $g_a(z_a)$ representing an element of the quotient, where z_a is a local coordinate. Let $\Xi(\vec{m}_a)$ denotes the modification of E at w_a and $\Xi = \prod_{a=1}^{n} \Xi(\vec{m}_a)$. The order of modifications in the product is irrelevant, since they commute. To calculate the monodromy of Ξ we choose the same orientation in all points w_a. The characteristic class of ξ of modified bundle $\tilde{E}$ corresponds to

$$\prod_{a=1}^{n} \exp\left(-\frac{2\pi i}{N}\sum_{k=1}^{N-1} k n_k^a\right).$$

3 Bogomolny equation

Definition. Let $W = \mathbb{R} \times \Sigma_g$. Consider a bundle V over W equipped with the curvature F. Let ϕ be a zero form on W taking value in sections of the adjoint bundle $\phi \in \Omega^0(W, \mathrm{End}\, V)$. It is the so-called Higgs field.

The Bogomolny equation on W takes the form

$$F = *D\phi. \tag{3.1}$$

Here $*$ is the Hodge operator on W with respect to the metric ds^2 on W. In local coordinates $(z, \bar{z})$ on Σ_g and y on the real line $ds^2 = g|dz|^2 + dy^2$, where $g(z, \bar{z})|dz|^2$ is a metric on Σ_g. Then the Hodge operator is defined as

$$\star dy = \tfrac{1}{2} i g dz \wedge d\bar{z}, \qquad \star dz = -i dz \wedge dy, \qquad \star d\bar{z} = i d\bar{z} \wedge dy,$$

and (3.1) becomes

$$\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + [A_z, A_{\bar{z}}] = \frac{ig(z,\bar{z})}{2} \left(\partial_y \phi + [A_y, \phi]\right), \tag{3.2a}$$

$$\partial_y A_z - \partial_z A_y + [A_y, A_z] = i(\partial_z \phi + [A_z, \phi]), \tag{3.2b}$$

$$\partial_y A_{\bar{z}} - \partial_{\bar{z}} A_y + [A_y, A_{\bar{z}}] = -i(\partial_{\bar{z}} \phi + [A_{\bar{z}}, \phi]). \tag{3.2c}$$

In what follows we will consider only $\mathrm{PSL}(N, \mathbb{C})$-bundles.

A monopole solution of this equation is defined in the following way. Let $\tilde{W} = (W \setminus \vec{x}^0 = (y = 0, z = z_0))$. The Bianchi identity $DF = 0$ on $\tilde{W}$ implies that ϕ can be identified with the Green function for the operator $\star D \star D$

$$\star D \star D\phi = M\delta(\vec{x} - \vec{x}^0), \tag{3.3}$$

$$M = \mathrm{diag}(m_1, m_2, \ldots, m_N) \in \mathrm{gl}(N, \mathbb{C}), \qquad \vec{m} = (m_1, m_2, \ldots, m_N) \in P \quad \text{(A.4)}, \tag{3.4}$$

and $(m_1, m_2, \ldots, m_N)$ are the monopole charges. We explain below this choice of M. This equation means that ϕ is singular at $\vec{x}^0$.

Boundary conditions and gauge symmetry. In what follows except Section 3.1 we assume that $\partial_y \phi$ vanishes when $y \to \pm\infty$. It is the Neumann boundary conditions for the Higgs field, while the gauge fields are unspecified. Let $V_\pm$ be restrictions of V to the bundles over Σ_g on the "left end" and "right end" of $W : y \to \pm\infty$. These bundles are flat. It follows from (3.7a), where the gauge $A_y = 0$ is assumed. It was proved in [9] that in absence of the source $M = 0$ in (3.3) the only solutions of (3.1) with these boundary conditions are $F = 0$, $\phi = 0$. Note that these boundary conditions differ from ones chosen in [9].

The Bogomolny equation defines a transformation $V_- \to V_+$. (E and $\tilde{E}$ in our notations in Introduction.) We will see in next sections that in general the characteristic classes of bundles are changed under these transformations. It depends on the monopole charges $\vec{m} = (m_1, m_2, \ldots, m_N)$.

The system (3.2) is invariant with respect to the gauge group $\mathcal{G}$ action:

$$A_z \to hA_zh^{-1} + \partial_z hh^{-1}, \qquad A_{\bar z} \to hA_{\bar z}h^{-1} + \partial_{\bar z} hh^{-1},$$
$$A_y \to hA_yh^{-1} + \partial_y hh^{-1}, \qquad \phi \to h\phi h^{-1}, \tag{3.5}$$

where $h \in \mathcal{G}$ is a smooth map $W \to \mathrm{GL}(N,\mathbb{C})$. To preserve the r.h.s. in (3.3) it should satisfy the condition $[h(\vec{x}^0), M] = 0$.

Assume for simplicity that V is an adjoint bundle. Since the gauge fields for $y = \pm\infty$ are unspecified and only flat we can act on them by boundary values of the gauge group $\mathcal{G}|_{y=\pm\infty} = \mathcal{G}_\pm$. Then $\mathcal{M}_\pm = \{V_\pm\}/\mathcal{G}_\pm$ are the moduli spaces of flat bundles.

Relations to integrable systems. The moduli spaces of flat bundles are phase spaces of non-autonomous Hamiltonian systems related to the isomonodromy problems over Σ_g. The isomonodromy problem takes the form

$$[\partial_z + A_z, \Psi] = 0, \qquad [\partial_{\bar z} + A_{\bar z}, \Psi] = 0. \tag{3.6}$$

Here $\Psi \in \Omega^0(\Sigma_g, \mathrm{Aut}\, V)$ is the Baker–Akhiezer function. These system is compatible for any degree of bundle, because it is defined in the adjoint representation. One example of these systems we have mentioned in Introduction ($V_- \to$ Painlevé VI) and ($V_+ \to$ Zhukovsky–Volterra gyrostat).

It is known, that the moduli space of flat bundles are deformation (the Whitham deformation) of the phase spaces of the Hitchin integrable systems – the moduli spaces of the Higgs bundles. To consider this limit one should replace a holomorphic connection by the κ-connection $\kappa\partial_z + A_z$ introduced by P. Deligne and take a limit $\kappa \to 0$. It is a quasi-classical limit in the linear problem (3.6). Details can be found in [11, 12, 13, 14]. In this way a monopole solution put in a correspondence (symplectic Hecke correspondence) two Hitchin systems (the first and the last examples in Introduction). But Bogomolny equation tells us more. It describes an evolution from one type of system to another.

It is possible to generalize (3.3) and consider multi-monopole sources $\sum_a M_a\delta(\vec{x} - \vec{x}_a^0)$ in the r.h.s. This generalization will correspond to modifications in a few points of Σ_g described at the end of previous section.

It is interesting that in some particular cases this situation was discussed in the frameworks of a supersymmetric Yang–Mills theory [15, 16][1]. It was observed there that a monopole configuration corresponds to a soliton type evolution along y. Therefore, it can be suggested that the system (3.2) is integrable. We did not succeed to prove this fact, but propose a linear problem related to the Bogomolny equation. An associated linear problem allows one in principal to apply the methods of the Inverse Scattering Problem or the Whitham approximation to find solutions [18]. Assume that the metric g on Σ_g is a constant. Then the system (3.2) is the compatibility condition for the linear system

$$\left(\partial_z + A_z + \tfrac{1}{2}\lambda^{-1} g(\partial_y + A_y + i\phi)\right)\psi = 0,$$
$$\left(\partial_{\bar z} + A_{\bar z} + \tfrac{1}{2}\lambda g(\partial_y + A_y - i\phi)\right)\psi = 0,$$

where $\lambda \in \mathbb{C}P^1$ is a spectral parameter. It can be suggested that monopole solution of (3.2) corresponds to a soliton solution of this system. We will not develop here this approach[2].

Gauge fixing. Choose a gauge fixing conditions as: $A_{\bar z} = 0$. Holomorphic functions $h = h(y,z)$ preserve this gauge. Then

$$-\partial_{\bar z} A_z = \frac{ig}{2}\left(\partial_y\phi + [A_y, \phi]\right),$$

[1]We are grateful to A. Gorsky who bring our attention to this point.

[2]The SU(2) case and $W = \mathbb{R}^3$ was analyzed in [19] for different boundary conditions.

$$\partial_y A_z - \partial_z A_y + [A_z, A_y] = i(\partial_z \phi + [A_z, \phi]),$$
$$\partial_{\bar z} A_y = i \partial_{\bar z} \phi.$$

The last equation means that $A_y - i\phi$ is holomorphic. It follows from (3.5) that the gauge transformation of this function is

$$A_y - i\phi \to h(A_y - i\phi)h^{-1} + \partial_y h h^{-1}.$$

Thus, we can keep $A_y = i\phi$ by using holomorphic and y-independent part of the gauge group ($\partial_y h = 0$). Finally, we come to the system

$$\partial_{\bar z} A_z = -\frac{ig}{2}\partial_y \phi, \tag{3.7a}$$
$$\partial_y A_z - 2i\partial_z \phi + 2i[A_z, \phi] = 0, \tag{3.7b}$$
$$A_y = i\phi, \tag{3.7c}$$
$$A_{\bar z} = 0. \tag{3.7d}$$

Two upper equations from (3.7) lead to the Laplace type equation

$$\partial_y^2 \phi + \frac{4}{g}(\partial_z \partial_{\bar z} \phi + \partial_{\bar z}[A_z, \phi]) = 0. \tag{3.8}$$

In scalar case (3.8) is simplified

$$\partial_y^2 \phi + \frac{4}{g}\partial_z \partial_{\bar z} \phi = 0. \tag{3.9}$$

3.1 Rational solution in scalar case

In this subsection we replace Σ_g by $\mathbb{C}$. The coordinates $z, \bar z$ on $\mathbb{C}$ will play the role of local coordinates on Σ_g. Consider (3.9) on $\tilde W = \mathbb{R} \times \mathbb{C} \setminus (0,0,0)$. In this particular case we can choose the boundary conditions in the following form:

$$\phi|_{y=\pm\infty} = 0, \tag{3.10}$$
$$A_z|_{y=\pm\infty} = 0. \tag{3.11}$$

The solution of (3.9) with $g = 1$ satisfying (3.10) has the form:

$$\phi = c\frac{1}{\sqrt{y^2 + z\bar z}}, \tag{3.12}$$

where c is a constant. So in fact we deal here with the Laplace equation on $\mathbb{R} \times \mathbb{C} \setminus (0,0,0)$. It follows from (3.11) and from the equation $\partial_{\bar z} A_z = -\frac{i}{2}\partial_y \phi$ (3.7a) that

$$\begin{aligned} &A_z(z,\bar z, y) = A_z^+(z,\bar z, y), \qquad y > 0 \quad \text{and} \quad y = 0, \quad z \neq 0,\\ &A_z(z,\bar z, y) = A_z^-(z,\bar z, y), \qquad y < 0, \end{aligned} \tag{3.13}$$

where

$$A_z^+(z,\bar z,y) = -ic\left(\frac{1}{z}\frac{y}{\sqrt{y^2+z\bar z}} - \frac{1}{z}\right) + \text{const},$$
$$A_z^-(z,\bar z,y) = -ic\left(\frac{1}{z}\frac{y}{\sqrt{y^2+z\bar z}} + \frac{1}{z}\right) + \text{const},$$

and $A_z(z,\bar{z},y)$ is a connection on the line bundle $\mathcal{L}$ over $\tilde{W}$. The connection has a jump $-2ic\frac{1}{z}$ at $y=0$. To deal with smooth connections we compensate it by a holomorphic gauge transform that locally near $\vec{x}_0$ has the form $h \sim z^m$. Here m should be integer, because h is a smooth function. Notice that all holomorphic line bundles over S^2 are known to be $\mathcal{O}(m)$-bundles, $m \in \mathbb{Z}$. Thus, we have $c = i\frac{m}{2}$, $m \in \mathbb{Z}$. This usually referred as a quantization of the monopole charge. In fact the constant c contains factor 4π (area of a unit sphere) which yields a proper normalization of delta-function and appears in Gauss's law. The gauge transformation h is the modification (2.14), (2.15) for line bundles over $\mathbb{C}P^1$. This is what we mean saying that the described 3-dimensional construction characterizes the modification of the corresponding bundle.

Consider for a moment the general situation $W = \mathbb{R} \times \Sigma_g$ and let $z, \bar{z}$ be local coordinates on Σ_g. Locally near $\vec{x}_0 = (0,0,0)$ connections corresponding to solutions of (3.9) have the form (3.13). Let S^2 be a small sphere surrounding the point $\vec{x}_0$ in W and $\Sigma_{g,\pm}$ be the left and right boundaries of W and $\mathcal{L}_\pm$ are the corresponding restrictions of $\mathcal{L}$. Then (as it is explained in [9] in detail)

$$\int_{\Sigma_{g,+}} F = \int_{\Sigma_{g,-}} F + m,$$

where F is a curvature of the connection A. In other words, the monopole solution with the charge m increases the degree of bundle by m ($\deg \mathcal{L}_+ = \deg \mathcal{L}_- + m$).

3.2 Elliptic solution in scalar case

The Laplace equation (3.9) on Σ_τ has the form

$$\partial_y^2 \phi + 4(\mathcal{I}m(\tau))^2 \partial_z \partial_{\bar{z}} \phi = 0, \tag{3.14}$$

or

$$\partial_y^2 \phi + (2\pi\alpha)^2 \partial_z \partial_{\bar{z}} \phi = 0, \qquad \alpha^{-1} = \frac{2\pi i}{\tau - \bar{\tau}},$$

and $\mathcal{I}m(\tau)$ is the area of parallelogram of periods. We give two representations of the Green function ϕ and prove their equivalence using the same technique as for the Kronecker series described in [17].

A naive elliptic solution of (3.14) on $\tilde{W}$ is obtained by averaging (3.12) over the lattice $\Gamma = \mathbb{Z} \oplus \tau\mathbb{Z} \subset \mathbb{C}$:[3]

$$\phi(z,y) = c \sum_{\gamma \in \Gamma} \frac{1}{\sqrt{(\pi\alpha y)^2 + |z+\gamma|^2}}. \tag{3.15}$$

However the series diverges. That is why we consider its generalization

$$\mathcal{R}(s,x,z,y) = c \sum_{\gamma \in \Gamma} \frac{\chi(\gamma,x)}{((\pi\alpha y)^2 + |z+\gamma|^2)^s}, \qquad \mathcal{R}\left(\tfrac{1}{2},0,z,y\right) = \phi(z,y), \tag{3.16}$$

where

$$\chi(\gamma,x) = e^{\alpha^{-1}(\gamma\bar{x} - \bar{\gamma}x)}$$

[3] We omit here and in what follows the $\bar{z}$ dependence.

is a character $\mathbb{Z}\times\mathbb{Z}\to\mathbb{C}^*$ of the additive group Γ and s, x are complex parameters. The characters are double-periodic

$$\chi(\gamma,x+1)=\chi(\gamma,x),\qquad \chi(\gamma,x+\tau)=\chi(\gamma,x),\qquad \gamma\in\Gamma,$$

while the series $\mathcal{R}(s,x,z,y)$ are quasi-periodic

$$\begin{aligned}&\mathcal{R}(s,x,z+1,y)=e^{\alpha^{-1}(x-\bar{x})}\mathcal{R}(s,x,z,y),\\&\mathcal{R}(s,x,z+\tau,y)=e^{\alpha^{-1}(x\bar{\tau}-\bar{x}\tau)}\mathcal{R}(s,x,z,y).\end{aligned}\tag{3.17}$$

The variable x describes behavior of $\mathcal{R}(s,x,z,y)$ on the lattice Γ. In other words, x parameterizes the moduli space of line bundles on Σ_τ. Note that for $\mathcal{R}e\, s>1$ the series in the r.h.s. of (3.16) converges. The function

$$\mathcal{R}\left(\tfrac{1}{2},x,z,y\right)=c\sum_{\gamma\in\Gamma}\frac{\chi(\gamma,x)}{((\pi\alpha y)^2+|z+\gamma|^2)^{\frac{1}{2}}}=\phi(x,z,y)\tag{3.18}$$

is the formal solution of (3.14) with the quasi-periodicity conditions (3.17).

Another representation of the Green function can be obtained by the Fourier transform. Define the delta-functions

$$\delta(y)=\int_{-\infty}^{+\infty}dpe^{2\pi ipy},\qquad \delta^{(2)}(z,\bar{z})=\sum_{\gamma\in\Gamma}\chi(\gamma,z).$$

Then

$$\sum_{\gamma\in\Gamma}\chi(\gamma+x,z)=\chi(x,z)\sum_{\gamma\in\Gamma}\chi(\gamma,z)=\chi(x,z)\delta^2(z,\bar{z})=\delta^2(z,\bar{z}),$$

Let $\tilde{\phi}$ be the Green function with the quasi-periodicity (3.17)

$$\partial_y^2\tilde{\phi}+(2\pi\alpha)^2\partial_z\partial_{\bar{z}}\tilde{\phi}=c\delta(y)\delta^{(2)}(z,\bar{z}).$$

Expanding it in the Fourier harmonics we find

$$\tilde{\phi}=-\frac{c}{4\pi^2}\sum_{\gamma\in\Gamma}\int_{-\infty}^{+\infty}dp\frac{e^{2\pi ipy}}{p^2+|\gamma+z|^2}\chi(\gamma+x,z).$$

Integrating over p provides factor π and leads to the following expression:

$$\tilde{\phi}(x,z,y)=-\frac{c}{4\pi}\sum_{\gamma\in\Gamma}\frac{1}{|\gamma+x|}e^{-2\pi|\gamma+x||y|}\chi(\gamma+x,z).\tag{3.19}$$

It is worthwhile to note that the solution (3.19) is well defined. Our goal is to find interrelations between (3.19) and (3.18).

Consider a generalization of (3.19)

$$I(s,x,z,y)=2c\pi^s y^{s-\frac{1}{2}}\sum_{\gamma\in\Gamma}\frac{K_{s-\frac{1}{2}}(2\pi|y||\gamma+x|)}{|\gamma+x|^{s-\frac{1}{2}}}\chi(\gamma+x,z).\tag{3.20}$$

Here K_ν is the Bessel–Macdonald function

$$K_\nu(2\pi yz) = \frac{\Gamma(\nu+\frac{1}{2})(z)^\nu}{2(\pi y)^\nu\Gamma(\frac{1}{2})}\int_{-\infty}^{+\infty} dp\frac{e^{2\pi ipy}}{(p^2+z^2)^{\nu+\frac{1}{2}}}.$$

The function $I(s,x,z,y)$ is the Green function for the pseudo-differential operator

$$\left(\partial_y^2 + 4\alpha^2\pi^2\partial_z\partial_{\bar{z}}\right)^s$$

on $\mathbb{R}\times\Sigma_\tau$ with the boundary conditions (3.17). Since

$$K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}e^{-x},$$

we conclude that for $s=1$ I coincides with $\tilde{\phi}$ (3.19) up to constant.

We are going to establish a relation between (3.16) and (3.20), and in this way between (3.15) and (3.19). Let us prove that

$$I(s,x,z,y) = c\sum_{\gamma\in\Gamma}\int_{-\infty}^{\infty} dp\int_0^\infty \frac{dt}{t}t^s e^{-t(p^2+|\gamma+x|^2)+2\pi ipy}\chi(\gamma+x,z). \tag{3.21}$$

In fact, using the integral representation for the Gamma-function

$$\Gamma(s) = \int_0^\infty \frac{dt}{t}t^s e^{-t} \tag{3.22}$$

and taking the integral over t in (3.21) we come to (3.20).

The representation (3.21) is universal and can serve to define $\mathcal{R}$ (3.16)

Lemma 3.1. *The function $\mathcal{R}(s,x,z,y)$ has a representation as the Fourier integral*

$$\mathcal{R}(s,x,z,y) = \frac{1}{\Gamma(s)\chi(\gamma,z)}\int_\infty^\infty dk\, I\left(s,z,x,\frac{k}{\pi\alpha}\right)e^{-2\pi iky}. \tag{3.23}$$

Proof. Substitute in (3.23) $I(s,x,z,y)$ (3.21) and take first integral over k. We come to the condition $p=\pi\alpha y$. Then using the integral representation for the Gamma-function (3.22) we obtain (3.16). ■

Remark 3.1. The series (3.20) is a three-dimensional generalization of the Kronecker series (see [17])

$$K(x,x_0,s) = \sum_\gamma \chi(\gamma,x_0)|x+\gamma|^{-2s}.$$

Using the Poisson summation formula Kronecker proved that

$$\Gamma(s)K(x,x_0,s) = \alpha^{1-2s}\Gamma(1-s)K(x_0,x,1-s)\chi(x,x_0)\,.$$

Our purpose is to generalize this functional equation for the 3-dimensional case $\Sigma_\tau\times\mathbb{R}$. It takes the following form.

Lemma 3.2. *The function $I(s,x,z,y)$ satisfies the functional equation:*

$$I(s,x,z,y) = \chi(x,z)\pi^{-\frac{1}{2}}\alpha^{-2s+1}\int_{-\infty}^{+\infty} dk\, I\left(\frac{3}{2}-s, z, x, \frac{k}{\pi\alpha}\right) e^{-2\pi iky}. \tag{3.24}$$

Proof. Following [17] we subdivide integral (3.21) into two parts

$$I(s,x,z,y) = c\sum_{\gamma\in\Gamma}\int_{-\infty}^{\infty} dp \int_0^T \frac{dt}{t} t^s e^{-t(p^2+|\gamma+x|^2)+2\pi ipy}\chi(\gamma+x,z)$$
$$+ c\sum_{\gamma\in\Gamma}\int_{-\infty}^{\infty} dp \int_T^{\infty} \frac{dt}{t} t^s e^{-t(p^2+|\gamma+x|^2)+2\pi ipy}\chi(\gamma+x,z), \qquad T\in\mathbb{R}, \quad T>0.$$

The second term is a well defined function for all s. Consider the first one. It is well known that for the series

$$\Theta(t,x,x_0) = \sum_{\gamma} e^{-t|x+\gamma|^2}\chi(\gamma,x_0)$$

the following functional equation holds:

$$\Theta(t,x,x_0) = (\alpha t)^{-1}\Theta\big(\alpha^{-2}t^{-1}, x_0, x\big)\chi(x_0,x).$$

The latter follows from the Poisson summation formula which states that the averaging of function over some lattice equals the averaging of its Fourier transform over the dual lattice. In the above case the functional equation appears after the Fourier transform for the Gauss integral. Then

$$\sum_{\gamma\in\Gamma}\int_{-\infty}^{\infty} dp \int_0^T \frac{dt}{t} t^s e^{-t(p^2+|\gamma+x|^2)+2\pi ipy}\chi(\gamma+x,z)$$
$$= \sum_{\gamma\in\Gamma}\int_{-\infty}^{\infty} dp \int_0^T \frac{dt}{t} t^s e^{-tp^2-\alpha^{-2}t^{-1}|\gamma+z|^2+2\pi ipy}\chi(\gamma+z,x)\chi(x,z)(\alpha t)^{-1}$$
$$\overset{\substack{\text{integrating}\\ \text{over } p}}{=} \sum_{\gamma\in\Gamma}\int_0^T \frac{dt}{t} t^s e^{-\pi^2y^2t^{-1}-\alpha^{-2}t^{-1}|\gamma+z|^2}\chi(\gamma+z,x)\chi(x,z)(\alpha t)^{-1}\sqrt{\frac{\pi}{t}}$$
$$\overset{\substack{\text{making}\\ \text{substitution}\\ \alpha^{-2}t^{-1}\to t}}{=} \sum_{\gamma\in\Gamma}\int_{\alpha^{-2}T^{-1}}^{\infty} \frac{dt}{t} t^{\frac{3}{2}-s}\sqrt{\pi}\alpha^{2-2s} e^{-t((\pi\alpha y)^2+|\gamma+z|^2)}\chi(\gamma+z,x)\chi(x,z).$$

Let $T=\alpha^{-1}$. Then

$$I(s,x,z,y) = c\sum_{\gamma\in\Gamma}\int_{\alpha^{-1}}^{\infty} \frac{dt}{t} t^{\frac{3}{2}-s}\sqrt{\pi}\alpha^{2-2s} e^{-t((\pi\alpha y)^2+|\gamma+z|^2)}\chi(\gamma+z,x)\chi(x,z)$$
$$+ c\sum_{\gamma\in\Gamma}\int_{-\infty}^{\infty} dp \int_{\alpha^{-1}}^{\infty} \frac{dt}{t} t^s e^{-t(p^2+|\gamma+x|^2)+2\pi ipy}\chi(\gamma+x,z). \tag{3.25}$$

The proof follows from (3.25). One should only substitute $I(s,x,z,y)$ from (3.25), into (3.24). Formula (3.25) represents I as the sum of two terms. Direct evaluation shows that the first (of two) term from the l.h.s. of (3.24) equals to the second one from the r.h.s. and vice versa. ■

From Lemmas 3.1 and 3.2 we come to the main result of this section

$$\boxed{\mathcal{R}(\tfrac{3}{2} - s, x, z, y) = \tfrac{\sqrt{\pi}\alpha^{2s-1}}{\Gamma(\frac{3}{2}-s)} I(s, x, z, y).}$$

Now put $s = 1$. Then one can see that well-defined series

$$\pi \sum_{\gamma\in\Gamma} \chi(\gamma + x, z) \frac{e^{-2\pi|y||\gamma+x|}}{|\gamma + x|} \tag{3.26}$$

describes the analytic continuation of the divergent series

$$\pi \sum_{\gamma\in\Gamma} \chi(\gamma, x) \frac{1}{\sqrt{(\pi\alpha y)^2 + |\gamma + z|^2}}.$$

We use (3.19) as the Green function. Then

$$A_z(z, \bar{z}, y, x) = -\frac{ic}{4\pi} \frac{1}{\pi^2\alpha^2} \mathrm{sgn}(y) \sum_{\gamma\in\Gamma} \frac{1}{\gamma + x} e^{-2\pi|\gamma+x||y|} \chi(\gamma + x, z), \tag{3.27}$$

$$\mathrm{sgn}(y) = 1 \quad \text{for} \quad y \geq 0, \qquad \mathrm{sgn}(y) = -1 \quad \text{for} \quad y < 0.$$

Notice that the jump of A (while coming through $y = 0,\ z = 0$) is obviously defined by the jump of $\mathrm{sgn}(y)$.

Remark 3.2. Note that (3.27) is a formal solution of the Bogomolny equation. For $x \neq 0$ it is not a connection of a line bundle over Σ_τ due to its monodromies similar to (3.17). We will use this solution in next section to define a genuine connection for higher ranks bundles.

In order to compare elliptic configuration with the rational we take $x = 0$. Then on the line $y = 0$ the connection is proportional to

$$A_z \sim \sum_{\gamma\neq 0} \frac{1}{\gamma} \chi(\gamma, z) = E_1(z) - \alpha^{-1}(z - \bar{z}),$$

where $E_1(z) = \partial \ln \vartheta(z)$ is the so-called first Eisenstein series and $\vartheta(z)$ is the theta-function (2.10). $E_1(z)$ has a simple pole at $z = 0$ with $\mathrm{Res}_{z=0} E_1(z) = 1$ and the connection A_z is double-periodic. In terms of (3.5) the gauge transformation h compensating the jump of the connection is given by an integer power of theta function $\vartheta^m(z)$, $m \in \mathbb{Z}$. Thus

$$\partial \log h = \partial \log \vartheta^m(z) = mE_1(z).$$

4 Arbitrary rank case

Here we describe modification of vector bundles of an arbitrary rank. First, we repeat arguments of [9] and justify the choice M in (3.4). As before, we consider $\mathrm{PSL}(N, \mathbb{C}) = G_{ad}$-bundles.

Near the singular point $\vec{x}^0$ the bundle V is splited in a sum of line bundles. Using the solution (3.12) for a line bundle we take the Higgs field near the singularity in the form

$$\phi = \frac{i}{2\sqrt{y^2 + z\bar{z}}} \mathrm{diag}(m_1, \ldots, m_N).$$

It follows from (3.13) that A_z undergoes a discontinuous jump at $y = 0$

$$A_z^+ - A_z^- = \frac{i}{z} \mathrm{diag}(m_1, m_2, \ldots, m_N). \tag{4.1}$$

To get rid of the singularity of A at $z = 0$, as in the Abelian case, one can perform the singular gauge transform Ξ that behaves near $z = 0$ as (2.16)

$$\Xi = \mathrm{diag}\big(z^{-m_1}, z^{-m_2}, \dots, z^{-m_N}\big).$$

Assume that $\vec{m}$ belongs to the weight lattice $\vec{m} \in P$. It means that Ξ is inverse to the cocharacter γ_{ad} of $\mathrm{PSL}(N,\mathbb{C})$ ($\gamma_{ad} \in t(G_{ad}) = P^\vee \sim P$ (A.5)). As it was explained before, the modified bundle V_+ can not be lifted to a $\mathrm{SL}(N,\mathbb{C})$ bundle. On the other hand, if $\vec{m} = (m_1, \dots, m_N)$ belongs to the root lattice Q (A.5), then $\Xi^{-1} = \bar{\gamma}$ and there is no obstruction to lift V_+ to an $\mathrm{SL}(N,\mathbb{C})$ bundle. Note that (4.1) describes the affine group $\bar{W}_a$ (A.13) action in the former case and the affine group W_a (A.12) action in the latter case. From field-theoretical point of view it is an action of the t'Hooft operator on A_z (see (2.19)).

If $N = pl$, ($l \neq 1, N$) one can consider the intermediate situation and γ_{G_l} (A.8). It means that $\vec{m} \in t({}^L G_l) \sim \Gamma(G_p)$. This embedding provides the modification that allows the $\mathrm{PGL}(N,\mathbb{C})$-bundle to lift to the $G_l = \mathrm{SL}(N,\mathbb{C})/\mathbb{Z}_l$-bundle but not to a $\mathrm{SL}(N,\mathbb{C})$ bundle. In this way the monopole charges are related to the characteristic classes of bundles.

One can use the maps to the Cartan subgroups of the solution $\phi(z)$ for a line bundle over Σ_τ (3.26) with $x = 0$

$$\phi \to \phi \cdot \mathrm{diag}(m_1, m_2, \dots, m_N).$$

Unfortunately, in this case V being restricted on Σ_τ is splitting globally over Σ_τ and defines an unstable bundle, though it allows one to describe its modifications.

There exists a map of $\phi(z, y, x)$ and $A_z(z, y, x)$ with $x \neq 0$ to a non-semisimple elements of $\mathrm{sl}(N,\mathbb{C})$

$$\begin{pmatrix} 0 & k_1\phi(z,y,x_1) & \dots & k_{N-1}\phi(z,y,x_{N-1}) \\ 0 & 0 & \dots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & k_1A_z(z,y,x_1) & \dots & k_{N-1}A_z(z,y,x_{N-1}) \\ 0 & 0 & \dots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix}.$$

Since these matrices commute they are solutions of the matrix equation (3.8). The connection has a jump at $y = 0$. The bundle is characterized by the diagonal monodromy matrices (2.7)

$$\rho_a = \mathrm{diag}(a_1, a_2, \dots, a_N), \qquad \rho_b = \mathrm{diag}(b_1, b_2, \dots, b_N),$$

where

$$a_1 = \prod_{j=1}^{N-1} \sigma_j^{\frac{1}{N}}, \qquad a_2 = a_1\sigma_1^{-1}, \qquad a_N = a_1\sigma_{N-1}^{-1},$$
$$b_1 = \prod_{j=1}^{N-1} \varsigma_j^{\frac{1}{N}}, \qquad b_2 = a_1\varsigma_1^{-1}, \qquad b_N = a_1\varsigma_{N-1}^{-1},$$

$\sigma_j = \exp(\alpha^{-1}(x_j - \bar{x}_j))$, $\varsigma_j = \exp(\alpha^{-1}(x_j\bar{\tau} - \bar{x}_j\tau))$. Note that they are y-independent. Moreover, the singular gauge transform, leading to a continues solution of the Bogomolny equation, belongs to the upper nilpotent subgroup and in this way does not change the topological type of the bundle.

Now we describe non-diagonal modifications Ξ of a $\mathrm{PGL}(N,\mathbb{C})$-bundles over Σ_τ. We do not know solutions of the Bogomolny equation in this case, and only can assert that the modification "kill the jump" of A_z at $y=0$:

$$\Xi^{-1}\partial_z\Xi = A_z^+ - \Xi^{-1}A_z^-\Xi.$$

We use the global description of a bundle E in terms of the transition matrices ρ_a, ρ_b (2.7) using the approach of [1]. Let

$$\rho_a = \mathrm{Id}_N, \qquad \rho_b = \mathbf{e}-u, \qquad (u = \mathrm{diag}(u_1,u_2,\ldots,u_N)), \qquad \mathbf{e}(a) = \exp(2\pi i a). \tag{4.2}$$

The group commutator of these matrices is Id_N. Thereby, E can be lifted to a $\mathrm{SL}(N,\mathbb{C})$-bundle.

Define a modification Ξ of E to the bundle $\tilde{E}$ with the transition matrices (2.8). Then Ξ should intertwine the transition matrices

$$\Xi(z+1,\tau) = \mathcal{Q}\times\Xi(z,\tau), \tag{4.3}$$
$$\Xi(z+\tau,\tau) = \Lambda(z,\tau)\times\Xi(z,\tau)\times\mathrm{diag}(\mathbf{e}(u)). \tag{4.4}$$

The matrix $\Xi(z)$ degenerates at $z=0$ and we assume that it has a simple pole. These conditions fix $\Xi(z)$. It can be expressed in terms of the theta-functions with characteristics

$$\Xi_{kj}(z,u_1,\ldots,u_N;\tau) = \frac{\theta\begin{bmatrix} \frac{k}{N}-\frac{1}{2} \\ \frac{N}{2} \end{bmatrix}(z-Nu_j,N\tau)}{\theta^{\frac{1}{N}}(z,\tau)},$$

where

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau) = \sum_{j\in\mathbb{Z}} \exp 2\pi i\left((j+a)^2\frac{\tau}{2} + (j+a)(z+b)\right).$$

The quasi-periodicity properties (4.3), (4.4) follow from the properties of the theta-functions

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z+1,\tau) = \mathbf{e}(a)\theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau),$$
$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z+a'\tau,\tau) = \mathbf{e}\left(-a'^2\frac{\tau}{2} - a'(z+b)\right)\theta\begin{bmatrix} a+a' \\ b \end{bmatrix}(z,\tau).$$

This modification has the type $(\frac{N-1}{N},-\frac{1}{N},\ldots,-\frac{1}{N})$. The modification that allows to lift $\tilde{E}$ to $\mathrm{GL}(N,\mathbb{C})$- bundle is

$$\Xi_1(z) = h(z)\Xi(z) = \theta\begin{bmatrix} \frac{k}{N}-\frac{1}{2} \\ \frac{N}{2} \end{bmatrix}(z-Nu_j,N\tau),$$

where the gauge transformation h is the diagonal matrix

$$h(z) = \theta^{\frac{1}{N}}(z,\tau)\mathrm{Id}_N.$$

This modification intertwine the boundary conditions (4.2) with

$$\rho_a = \mathcal{Q}, \qquad \rho_b = \tilde{\Lambda}, \qquad \tilde{\Lambda} = e^{-2\pi i(\frac{z}{N}+\frac{\tau}{2N})}\Lambda.$$

The last transformation belongs to $\mathrm{GL}(N,\mathbb{C})$. Moreover, it can be proved that

$$\det\left[\frac{\Xi_1(z,u_1,\ldots,u_N;\tau)}{i\eta(\tau)}\right] = \frac{\vartheta(z)}{i\eta(\tau)}\prod_{1\le k<l\le N}\frac{\vartheta(u_l-u_k)}{i\eta(\tau)},$$

where $\eta(\tau) = q^{\frac{1}{24}} \prod_{n>0}(1-q^n)$ is the Dedekind function ($q = \exp 2\pi i\tau$) and $\vartheta(z)$ is the theta-function (2.10). Since $\vartheta(z)$ has a simple pole in Σ_τ the bundle $\tilde{E}$ is a $\mathrm{GL}(N,\mathbb{C})$-bundle of degree one. This modification provides the Symplectic Hecke correspondence between the elliptic Calogero–Moser system and the Elliptic Top.

Now consider the modification of the trivial bundle E with the transition matrices (4.2) to the $\tilde{E} = E_l$ (2.11), (2.12), where $\mathbf{u}_p = (\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_p)$ is the moduli of the modified bundle. The modification takes the form

$$\Xi_{kj}(z,\tau) = \frac{\theta\begin{bmatrix} \frac{k}{l} - \frac{1}{2} \\ \frac{l}{2} \end{bmatrix}(z - l\tilde{u}_i), l\tau)}{\theta^{\frac{1}{l}}(z,\tau)}, \qquad (j = mp + i,\ m = 0, \ldots, l-1).$$

As it was explained in Section 2 the modified bundle can be lifted to $G_l = \mathrm{SL}(N,\mathbb{C})/\mathbb{Z}_l$-bundle, but not to $\mathrm{SL}(N,\mathbb{C})$-bundle.

A SL($N,\mathbb{C}$) and PSL($N,\mathbb{C}$) [20, 21]

The group $\mathrm{SL}(N,\mathbb{C})$ is an universal covering of $\mathrm{PSL}(N,\mathbb{C})$ with the center $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$

$$\mathrm{Id} \to \mathbb{Z}_N \to \mathrm{SL}(N,\mathbb{C}) \to \mathrm{PSL}(N,\mathbb{C}) \to \mathrm{Id}. \tag{A.1}$$

Therefore $\pi_1(\mathrm{PSL}(N,\mathbb{C})) = \mathbb{Z}_N$. The both groups have the same Lie algebra $\mathfrak{G}$.

Roots and weights. The Cartan subalgebra $\mathfrak{H} \subset \mathfrak{G}$ is a hyperplane in $\mathbb{C}^N$

$$\mathfrak{H} = \left\{ \mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{C}^N \mid \sum_{j=1}^{N} x_j = 0 \right\}.$$

The simple roots $\Pi = \{\alpha_k\}$

$$\alpha_1 = e_1 - e_2, \quad \ldots, \quad \alpha_{N-1} = e_{N-1} - e_N$$

form a basis in the dual space $\mathfrak{H}^*$. Here $\{e_j\}$ $j = 1, \ldots, N$ is a canonical basis in $\mathbb{C}^N$. They generate the set of roots of type A_{N-1}

$$R = \{(e_j - e_k),\ j \neq k\}.$$

The root lattice $Q \subset \mathfrak{H}^*$ takes the form

$$Q = \left\{ \sum m_j e_j \,|\, m_j \in \mathbb{Z},\ \sum m_j = 0 \right\}. \tag{A.2}$$

We identify $\mathfrak{H}^*$ and $\mathfrak{H}$ by means of the standard metric on $\mathbb{C}^N$. Then the coroot system

$$R^\vee = \left\{ \alpha^\vee(R) = \frac{2(\alpha^\vee, \beta)}{(\beta,\beta)} \in \mathbb{Z} \text{ for any } \beta \in R \right\}$$

coincides with R, and the coroot lattice $Q^\vee$ coincides with Q.

The fundamental weights ϖ_k, ($k = 1, \ldots, N-1$) are dual to the basis of simple coroots $\Pi^\vee \sim \Pi$ ($\varpi_k(\alpha_k^\vee) = \delta_{kj}$)

$$\varpi_j = e_1 + \cdots + e_j - \frac{j}{N} \sum_{l=1}^{N} e_l, \tag{A.3}$$

$$\varpi_1 = \left(\frac{N-1}{N}, -\frac{1}{N}, \dots, -\frac{1}{N}\right), \quad \varpi_2 = \left(\frac{N-2}{N}, \frac{N-2}{N}, \dots, -\frac{2}{N}\right), \quad \dots,$$
$$\varpi_{N-1} = \left(\frac{1}{N}, \frac{1}{N}, \dots, \frac{1-N}{N}\right).$$

In the basis of simple roots the fundamental weights are

$$\varpi_k = \frac{1}{N}[(N-k)\alpha_1 + 2(N-k)\alpha_2 + \dots + (k-1)(N-k)\alpha_{k-1}$$
$$+ k(N-k)\alpha_k + k(N-k-1)\alpha_{k+1} + \dots + k\alpha_{N-1}].$$

The fundamental weights generate the weights lattice

$$P \subset \mathfrak{H}^*, \qquad P = \left\{\sum_l n_l\varpi_l \,|\, n_l \in \mathbb{Z}\right\}, \tag{A.4}$$
$$P = \sum_{j=1}^{N} m_j e_j, \qquad m_j \in \frac{1}{N}\mathbb{Z}, \qquad m_j - m_k \in \mathbb{Z}.$$

The weight lattice is generated by Q and the vector

$$\varpi_1 = e_1 - \frac{1}{N}\sum_{j=1}^{N} e_j.$$

The weight lattice P defines representations of $\mathrm{SL}(N,\mathbb{C})$, while Q define representations of $\mathrm{PSL}(N,\mathbb{C})$.

The factor-group $P^\vee/Q^\vee$ ($P^\vee \sim P$) is the center $\mathbb{Z}_N$ of $\mathrm{SL}(N,\mathbb{C})$. On the other hand it can be identified with the cyclic group symmetry $e_j \to e_{j+1} \bmod(N)$ of the extended Dynkin graph $\Pi \cup (\alpha_0 = e_N - e_1)$.

Characters and cocharacters. Let $\bar{\mathcal{T}}$ ($\mathcal{T}_{ad}$) be a Cartan torus in $\mathrm{SL}(N,\mathbb{C})$ ($\mathrm{PSL}(N,\mathbb{C})$). Define the groups of characters[4]

$$\bar{\Gamma} = \{\bar{\chi}(x)\} = \{\bar{\mathcal{T}} \to \mathbb{C}^*\}, \qquad \Gamma_{ad} = \{\chi_{ad}(x)\} = \{\mathcal{T}_{ad} \to \mathbb{C}^*\}.$$

They can be identified with lattice groups in $\mathfrak{H}^*$ as follows. Let ϖ_k be a basic weight and $\phi = (\phi_1, \phi_2, \dots, \phi_N)$, $\phi_k = \frac{1}{2\pi i}\ln x_k$. The functions

$$\exp 2\pi i(\varpi_k\phi), \qquad k = 1, \dots, N-1$$

generate a basis in $\bar{\Gamma}$. Similarly, for $\alpha_k \in \Pi$

$$\exp 2\pi i(\alpha_k, \phi), \qquad k = 1, \dots, N-1$$

is a basis in Γ_{ad}. Thereby, we have

$$\bar{\Gamma} = P, \qquad \Gamma_{ad} = Q.$$

Define the dual groups of cocharacters $t(\bar{G}) = \bar{\Gamma}^*$ and $t(G_{ad}) = \Gamma^*_{ad}$ as the maps

$$t(\bar{G}) = \{\bar{\gamma} = \mathbb{C}^* \to \bar{\mathcal{T}}\}, \qquad t(G_{ad}) = \{\gamma_{ad} = \mathbb{C}^* \to \mathcal{T}_{ad}\}.$$

[4]The holomorphic maps of the tori to $\mathbb{C}^*$ such that $\chi(xy) = \chi(x)\chi(y)$ for $x, y \in \mathcal{T}$.

In another way

$$t(\bar{G}) = \{\phi \in \mathfrak{H} \,|\, \bar{\chi}(e^{2\pi i\phi}) = 1\}, \qquad t(G_{ad}) = \{\phi \in \mathfrak{H} \,|\, \bar{\chi}_{ad}(e^{2\pi i\phi}) = 1\}.$$

These groups are the groups of the coweight and coroot lattices

$$t(\bar{G}) = Q^\vee \sim Q, \qquad t(G_{ad}) = P^\vee \sim P. \tag{A.5}$$

The center of $\bar{\Gamma} = \mathrm{SL}(N,\mathbb{C})$ belongs to $\bar{\mathcal{T}}$ and is identified with the factor-group

$$\begin{aligned}&\mathcal{Z}(\mathrm{SL}(N,\mathbb{C})) = P^\vee/t(\bar{G}) \sim P^\vee/Q^\vee \sim \\ &\qquad \sim \pi_1(\mathrm{PSL}(N,\mathbb{C})) \sim t(G_{ad})/Q^\vee \sim P/Q = \mathbb{Z}_N.\end{aligned} \tag{A.6}$$

Let $N = pl$, $(l \neq 1, N)$ and $\mathbb{Z}_l \subset \mathbb{Z}_N$ be a subgroup of Z_N. Define the factor-group

$$G_l = \mathrm{SL}(N,\mathbb{C})/\mathbb{Z}_l.$$

Then the center $\mathcal{Z}(G_l)$ of G_l is $\mathbb{Z}_p$ and $\pi_1(G_l) = \mathbb{Z}_l$. Consider the groups of characters and cocharacters of G_l

$$\Gamma(G_l) = \{\chi_{G_l} : \mathcal{T}(G_l) \to \mathbb{C}^*\}, \tag{A.7}$$

$$t(G_l) = \{\gamma_{G_l} : \mathbb{C}^* \to \mathcal{T}(G_l)\}, \tag{A.8}$$

$(\Gamma^*(G_l) = t(G_l))$. They are lattices in $\mathfrak{H}^*$ and $\mathfrak{H}$ $Q \subset \Gamma(G_l) \subset P$, $Q^\vee \subset t(G_l) \subset P^\vee$. The lattice $\Gamma(G_l)$ is generated by the root lattice Q and the vector $l\varpi_1$, while the lattice $t(G_l)$ is generated by the root lattice Q and the vector $p\varpi_1$

$$\Gamma(G_l) = l\varpi_1 \cup Q, \qquad t(G_l) = p\varpi_1 \cup Q. \tag{A.9}$$

The group $\Gamma(G_l)$ is the weight lattice of G_l because highest weights of irreducible finite-dimensional representations of G_l belong to $\Gamma(G_l)$.

In terms of lattices the center $\mathcal{Z}(G_l)$ and $\pi_1(G_l)$ take the form

$$\begin{aligned}&\mathcal{Z}(G_l) \sim P^\vee/t(G_l) \sim \Gamma(G_l)/Q \sim \mathbb{Z}_p,\\ &\pi_1(G_l) \sim t(G_l)/Q^\vee \sim P/\Gamma(G_l) \sim \mathbb{Z}_l.\end{aligned}$$

A subgroup ${}^LG_l \subset \mathrm{SL}(N,\mathbb{C})$ is the Langlands dual to G_l if

$$t({}^LG_l) \sim \Gamma(G_l) \qquad (\Gamma({}^LG_l) \sim t(G_l)).$$

It implies that

$$\begin{aligned}&\mathcal{Z}({}^LG_l) \sim \mathbb{Z}_l,\\ &\pi_1({}^LG_l) \sim \mathbb{Z}_p.\end{aligned} \tag{A.10}$$

Therefore the dual group is

$${}^LG_l = G_p. \tag{A.11}$$

In particular, ${}^L\mathrm{SL}(N,\mathbb{C}) = \mathrm{PGL}(N,\mathbb{C})$.

Affine Weil group. The affine Weyl group W_a is a semidirect product $Q^\vee \rtimes W$ of the Weyl group W and the group $Q^\vee$. It acts on $\mathfrak{H}$ as

$$x \to x - \frac{2(\alpha, x)}{(\alpha,\alpha)}\alpha^\vee + k\alpha^\vee, \qquad k \in \mathbb{Z}. \tag{A.12}$$

Consider a semidirect product

$$\bar{W}_a = P^\vee \rtimes W. \tag{A.13}$$

In particular, the shift operator

$$x \to x + \vec{m}, \qquad \vec{m} \in P^\vee \tag{A.14}$$

is an element from $\bar{W}_a$. It follows from this construction that the factor group

$$\bar{W}_a/W_a \sim P^\vee/Q^\vee \sim \mathcal{Z}(\mathrm{SL}(N, \mathbb{C})). \tag{A.15}$$

Let again $N = pl$ and define a subgroup $W_a(G_l)$ of $\bar{W}_a$, generated by shifts from $t(G_l)$

$$W_a(G_l) = t(G_l) \rtimes W.$$

The factor group $W_a(G_l)/W_a$ is isomorphic to $t(G)/Q^\vee$ and in this way to $\mathcal{Z}({}^L G_l)$.

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