Cartan Connections and Lie Algebroids*

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Abstract. This paper is a study of the relationship between two constructions associated with Cartan geometries, both of which involve Lie algebroids: the Cartan algebroid, due to [Blaom A.D., *Trans. Amer. Math. Soc.* **358** (2006), 3651–3671], and tractor calculus [Čap A., Gover A.R., *Trans. Amer. Math. Soc.* **354** (2001), 1511–1548].

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1 Introduction

In 2006 Anthony D. Blaom published a very interesting paper [1] in which he proposed a wide-ranging generalization of Cartan's idea of a geometry as a Klein geometry deformed by curvature. The essential ingredient of Blaom's conception is a Lie algebroid with a linear connection (covariant derivative) that is compatible with the algebroid bracket in a certain sense. Blaom calls such an algebroid a Cartan algebroid, and – rather unfortunately in the circumstances – he calls a connection with the requisite property a Cartan connection. A Cartan geometry in the original sense (perhaps one should add 'as reformulated by Sharpe [4]') of a principal bundle with a Cartan connection form – what Blaom would call a classical Cartan connection – gives rise to a Cartan algebroid, as one would expect. But there are many more examples of the latter than of the former; for instance, a Cartan algebroid arising from a Cartan geometry must necessarily be transitive, but Blaom gives examples of Cartan algebroids which are not transitive. However, in this paper I shall concentrate on the transitive case.

In the introduction to his paper Blaom speculates, in an aside, that there may be a relationship between his work and so-called tractor calculi. The concept of a tractor calculus is based on the methods developed in the 1920s and 1930s by T.Y. Thomas and J.H.C. Whitehead for the study of conformal and projective (differential) geometries. These methods involve the construction in each case of a vector bundle associated with the geometry and a covariant derivative on its sections. Each of these geometries is a Cartan geometry of a special class known as parabolic. The tractor calculus programme is concerned with defining similar structures for all parabolic Cartan geometries. It turns out that the relevant vector bundles also support Lie algebroid brackets. An encyclopaedic account is to be found in a paper of A. Čap and A.R. Gover [3].

One aim of this note is to spell out the exact relationship between Blaom's theory of the Cartan algebroid of a (classical) Cartan geometry and Čap and Gover's tractor calculus. Of course there can be a direct relationship, so far as Blaom's theory is concerned, only when attention is restricted to Cartan geometries. However, it turns out not to be necessary to

*This paper is a contribution to the Special Issue "Élie Cartan and Differential Geometry". The full collection is available at http://www.emis.de/journals/SIGMA/Cartan.html restrict one's attention further to parabolic geometries: quite a lot of tractor calculus works for arbitrary Cartan geometries. Both theories involve linear connections on vector bundles, but though one can use the same vector bundle in each case, the two connections are different. The best way of describing the relationship between the two connections, in my view, is to regard them as secondary constructs, which may be defined in terms of a single object canonically associated with a Cartan geometry. This is an algebroid connection: a collection of operators with covariant-derivative-like properties, but which involve, as it were, differentiation of a section of the algebroid along a section of the algebroid. Both of the papers [1, 3] discuss this algebroid connection, but tend to regard it as dependent on the relevant linear connection; I have reversed the dependency.

A second natural question to ask about Blaom's theory is how to specify which Cartan algebroids derive from Cartan geometries. Those that do must be transitive, but there are further conditions to be satisfied, in part because algebroids derived from Cartan geometries have more structure than just that of an algebroid with Cartan connection (in the sense of Blaom). This extra structure comes to the fore in tractor calculus, so it is no real surprise that one can use the idea of a tractor connection to prove that when suitable extra structure is specified the Cartan algebroid does derive from a Cartan geometry.

In Section 2 of this note I describe a vector bundle associated with a Cartan geometry, called the adjoint tractor bundle, on which the events under discussion take place. In Section 3 I define the Lie algebroid structure on the adjoint tractor bundle, and in Section 4 I specify the two linear connections, the tractor connection of Čap and Gover and Blaom's Cartan connection. In Section 5 I give an alternative derivation of the Lie algebroid structure using the Atiyah algebroid. In Section 6 I prove a theorem which gives sufficient conditions for a Cartan algebroid to come from a Cartan geometry. In the final section I apply some of the ideas introduced earlier to a discussion of Cartan space forms, which give rise to Cartan algebroids which are symmetric in Blaom's terminology.

Most of the concepts and arguments in this note are drawn from one or other of the two main references [1, 3]. This seems pretty-well inevitable given that the purpose of the exercise is to explain how the two fit together. Indeed, what is new herein – and there are several new and I believe illuminating results to be found below – arises from the juxtaposition of the two apparently rather different approaches of these papers, with the occasional support of an appeal to [4].

There are several different types of bracket operations to be considered, which I want to distinguish between notationally. I shall reserve $[\cdot,\cdot]$ for the bracket of vector fields. I shall denote the bracket of any of the finite-dimensional Lie algebras that occur by $\{\cdot,\cdot\}$, and use the same notation for certain natural extensions of such brackets. And I shall use $[\![\cdot,\cdot]\!]$ to denote a Lie algebroid bracket.

Finally in this introduction I briefly review the basic facts aboout Cartan geometries and Lie algebroids that I need.

1.1 Cartan geometries

Recall that a Klein geometry is a homogeneous space G/H of a Lie group G, where H is a closed subgroup of G. I denote by $\mathfrak{g}$ the Lie algebra of G and by $\mathfrak{h}$ the subalgebra of $\mathfrak{g}$ which is the Lie algebra of H. A Cartan geometry on a manifold M, corresponding to a Klein geometry for which G/H has the same dimension, is a principal right H-bundle $\pi : P \to M$ together with a $\mathfrak{g}$-valued 1-form ω on P satisfying the following conditions:

1) the map $\omega_p : T_pP \to \mathfrak{g}$ is an isomorphism for each $p \in P$;
2) $R_h^*\omega = \mathrm{ad}(h^{-1})\omega$ for each $h \in H$, where $h \mapsto R_h$ is the right action; and
3) $\omega(\tilde{\xi}) = \xi$ for each $\xi \in \mathfrak{h}$, where $\tilde{\xi}$ is the fundamental vector field on P corresponding to ξ.

The form ω is the Cartan connection form. From the second condition it follows that for $\xi \in \mathfrak{h}$, $\mathcal{L}_{\tilde{\xi}}\omega + \{\xi, \omega\} = 0$; if H is connected then the differential version is equivalent to the original condition, and I shall assume that H is in fact connected.

The curvature Ω of a Cartan connection is the $\mathfrak{g}$-valued 2-form on P given by

$$\Omega(X, Y) = d\omega(X, Y) + \{\omega(X), \omega(Y)\}.$$

The value of the curvature when one of its arguments is vertical is zero:

$$\Omega(\tilde{\xi}, Y) = \tilde{\xi}(\omega(Y)) - \omega([\tilde{\xi}, Y]) + \{\xi, \omega(Y)\} = (\mathcal{L}_{\tilde{\xi}}\omega)(Y) + \{\xi, \omega(Y)\} = 0.$$

It is clear that in the above definition G has no role other than to provide the Lie algebra $\mathfrak{g}$. Following Sharpe [4] we may replace a Klein geometry as the model of a Cartan geometry by the pair $(\mathfrak{g}, \mathfrak{h})$ consisting of a Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{h}$ (what Sharpe calls an infinitesimal Klein geometry), together with a Lie group H whose Lie algebra is $\mathfrak{h}$, and a representation of H on $\mathfrak{g}$ which restricts to the adjoint representation of H on $\mathfrak{h}$. The representation of H on $\mathfrak{g}$ will be called the adjoint representation and denoted by ad. A Cartan geometry with these data is said to be modelled on $(\mathfrak{g}, \mathfrak{h})$ with group H.

1.2 Lie algebroids

A Lie algebroid is a vector bundle $\mathcal{A} \to M$ equipped with a linear bundle map $\Pi : \mathcal{A} \to TM$ over the identity on M, the anchor, and an $\mathbb{R}$-bilinear skew bracket $[\![\cdot, \cdot]\!]$ on $\Gamma(\mathcal{A})$, the $C^\infty(M)$-module of sections of $\mathcal{A} \to M$, such that

1) $[\![\sigma, f\tau]\!] = f[\![\sigma, \tau]\!] + \Pi(\sigma)(f)\tau$ for $\sigma, \tau \in \Gamma(\mathcal{A})$ and $f \in C^\infty(M)$;
2) $[\![\cdot, \cdot]\!]$ satisfies the Jacobi identity;
3) as a map $\Gamma(\mathcal{A}) \to \Gamma(TM)$, Π is a homomorpism: $\Pi([\![\sigma, \tau]\!]) = [\Pi(\sigma), \Pi(\tau)]$.

(The third of these is in fact a consequence of the other two, but is sufficiently important to be worth mentioning explicitly.)

A Lie algebroid is transitive if its anchor is surjective. The subset of $\mathcal{A}$ consisting of elements mapped to zero by Π is then a vector subbundle called the kernel of the algebroid, $\ker(\mathcal{A})$. The restriction of the algebroid bracket to $\Gamma(\ker(\mathcal{A}))$ is a Lie-algebraic bracket, that is, it is $C^\infty(M)$-bilinear (as well as being skew and satisfying the Jacobi identity). Indeed, a vector bundle with a Lie-algebraic bracket on its sections is a Lie algebroid whose anchor is the zero map. The restriction of the Lie-algebraic bracket to any fibre of $\ker(\mathcal{A}) \to M$ defines a Lie algebra structure on it. It can be shown that for a transitive algebroid $\mathcal{A}$ over a connected base M the Lie algebras on fibres of $\ker(\mathcal{A})$ over different points of M are isomorphic, so that $\ker(\mathcal{A})$ is a Lie algebra bundle.

For a Lie algebroid $\mathcal{A}$, an $\mathcal{A}$-connection D on a vector bundle $E \to M$ is an assignment to each $\sigma \in \Gamma(\mathcal{A})$ of an operator $D_\sigma : \Gamma(E) \to \Gamma(E)$ with the following connection-like properties:

1) D is $\mathbb{R}$-linear in both arguments, and $C^\infty(M)$-linear in the first;
2) $D_\sigma(f\epsilon) = fD_\sigma\epsilon + \Pi(\sigma)(f)\epsilon$ for $f \in C^\infty(M)$, $\epsilon \in \Gamma(E)$.

Since $D_\sigma\epsilon$ is $C^\infty(M)$-linear in σ, for $a \in \mathcal{A}$, $D_\sigma\epsilon(a)$ depends so far as σ is concerned only on $\sigma(a)$, just as is the case for an ordinary connection.

If D is an $\mathcal{A}$-connection on $\mathcal{A}$ then D^* defined by

$$D^*_\sigma\tau = D_\tau\sigma + [\![\sigma, \tau]\!]$$

is also an $\mathcal{A}$-connection on $\mathcal{A}$; it is said to be dual to D. The dual of the dual is the thing one first thought of. An $\mathcal{A}$-connection on $\mathcal{A}$ has torsion T given by

$$T(\sigma,\tau) = D_\sigma\tau - D^*_\sigma\tau = D_\sigma\tau - D_\tau\sigma - [\![\sigma,\tau]\!].$$

Any $\mathcal{A}$-connection has curvature C given by

$$C(\sigma,\tau)\epsilon = D_\sigma D_\tau\epsilon - D_\tau D_\sigma\epsilon - D_{[\![\sigma,\tau]\!]}\epsilon.$$

An $\mathcal{A}$-connection whose curvature vanishes is a homomorphism from sections of $\mathcal{A}$ with the Lie algebroid bracket to operators on $\Gamma(E)$ with the commutator bracket:

$$D_{[\![\sigma,\tau]\!]} = D_\sigma \circ D_\tau - D_\tau \circ D_\sigma.$$

When this holds D is called a representation of $\mathcal{A}$ on E. By the homomorphism property of Π, for any $\sigma \in \Gamma(\mathcal{A})$ the map $s \mapsto [\![\sigma, s]\!]$ maps $\Gamma(\ker(\mathcal{A}))$ to itself; this evidently defines a representation of $\mathcal{A}$ on $\ker(\mathcal{A})$, which is called the canonical representation. (But $\tau \mapsto [\![\sigma,\tau]\!]$ is not an $\mathcal{A}$-connection, so there is no inner representation of $\mathcal{A}$ on itself. Self-representations that restrict to the canonical representation, or in other words extend it, are therefore of some special interest.)

2 The adjoint tractor bundle

Suppose given a Cartan geometry modelled on $(\mathfrak{g},\mathfrak{h})$ with group H. I denote by $\overline{\mathfrak{g}} \to M$ the bundle associated with P by the adjoint action of H on $\mathfrak{g}$, and call it (following [3]) the adjoint tractor bundle of the Cartan geometry. The adjoint tractor bundle is to be distinguished from the adjoint bundle of the principal bundle P, which is the bundle associated with P by the adjoint action of H on $\mathfrak{h}$. The adjoint bundle, which will be denoted by $\overline{\mathfrak{h}}$, is a subbundle of $\overline{\mathfrak{g}}$.

Each of these bundles is a Lie algebra bundle, with bracket derived from the bracket of the corresponding Lie algebra $\mathfrak{g}$ or $\mathfrak{h}$. The bracket on the fibres of $\overline{\mathfrak{g}}$ will therefore be denoted by $\{\cdot,\cdot\}$, and likewise for the fibres of $\overline{\mathfrak{h}}$ (which is a sub-Lie-algebra bundle of $\overline{\mathfrak{g}}$, after all).

Consider next the sections of $\overline{\mathfrak{g}} \to M$. Any element of $\Gamma(\overline{\mathfrak{g}})$ can be regarded as a $\mathfrak{g}$-valued function on P which is H-equivariant; that is to say, a $\mathfrak{g}$-valued function σ on P such that $\sigma(ph) = \mathrm{ad}(h^{-1})\sigma(p)$. I shall use the same symbol (generally ρ, σ or τ) for both the section and the corresponding function; sections of $\overline{\mathfrak{h}}$, or sections of $\overline{\mathfrak{g}}$ which take their values in the subbundle $\overline{\mathfrak{h}}$, will be denoted by s, t and so on. The Lie algebra bracket of fibres extends to a Lie-algebraic bracket of sections, $(\sigma,\tau) \mapsto \{\sigma,\tau\}$, $\sigma,\tau \in \Gamma(\overline{\mathfrak{g}})$. Alternatively, thinking of sections as $\mathfrak{g}$-valued H-equivariant functions, note that the Lie algebra bracket of two such functions is also H-equivariant.

It is a consequence of the equivariance condition for $\sigma \in \Gamma(\overline{\mathfrak{g}})$ that $\tilde{\xi}(\sigma) + \{\xi,\sigma\} = 0$ for all $\xi \in \mathfrak{h}$; when H is connected this is equivalent to the original condition.

We have at our disposal a Cartan connection form ω on P. For any vector field X on P, $\omega(X)$ is a $\mathfrak{g}$-valued function on P; and if X is H-invariant, then $\omega(X)$ is H-equivariant and so defines a section of $\overline{\mathfrak{g}}$. Conversely, by the isomorphism condition, with any $\mathfrak{g}$-valued function σ on P we can associate a vector field X on P such that $\omega(X) = \sigma$; and if σ is a section of $\overline{\mathfrak{g}}$ then X is H-invariant. Thus a Cartan connection determines a 1-1 correspondence between sections of $\overline{\mathfrak{g}}$ and H-invariant vector fields on P. I shall denote the H-invariant vector field corresponding to a section σ by X_σ: thus $\omega(X_\sigma) = \sigma$.

If $s \in \Gamma(\overline{\mathfrak{h}})$, so that considered as a function s takes its values in $\mathfrak{h}$, then X_s is vertical; and conversely. Thus under the correspondence determined by ω the sections of $\overline{\mathfrak{h}}$ correspond to H-invariant vertical vector fields on P. In fact this part of the construction is independent of the Cartan connection: the sections of the adjoint bundle of any principal H-bundle P are in a natural 1-1 correspondence with the invariant vertical vector fields on P. The correspondence may be defined as follows. Let V be a vertical H-invariant vector field on P. Since V is vertical, for each $p \in P$ there is an element $s(p) \in \mathfrak{h}$ such that $V_p = \widetilde{s(p)}|_p$. Then in order that V be H-invariant s must be H-equivariant. Of course the Cartan connection construction reproduces this correspondence when s is a section of $\overline{\mathfrak{h}}$ because it satisfies the condition $\omega(\tilde{\xi}) = \xi$ for $\xi \in \mathfrak{h}$.

Recall that the curvature Ω of a Cartan connection is given by

$$\Omega(X, Y) = d\omega(X, Y) + \{\omega(X), \omega(Y)\}.$$

If X and Y are H-invariant then the $\mathfrak{g}$-valued function $\Omega(X, Y)$ is easily seen to be H-equivariant, and so defines a section of $\overline{\mathfrak{g}}$. We may therefore consider the curvature as a 2-form on M whose value is a section of $\overline{\mathfrak{g}}$; in this guise it is denoted by κ. We have

$$d\omega(X_\sigma, X_\tau) = \kappa(\pi_* X_\sigma, \pi_* X_\tau) - \{\sigma, \tau\}.$$

3 The Lie algebroid structure

Since the bracket of H-invariant vector fields is H-invariant, we can use the vector field bracket on P to define a bracket operation $[\![\cdot, \cdot]\!]$ on $\Gamma(\overline{\mathfrak{g}})$ by

$$[X_\sigma, X_\tau] = X_{[\![\sigma,\tau]\!]}, \qquad \text{or} \qquad [\![\sigma, \tau]\!] = \omega([X_\sigma, X_\tau]).$$

Moreover, for any H-invariant vector field X on P, $\pi_* X$ is a well-defined vector field on M. It is then clear that $\overline{\mathfrak{g}}$, equipped with the bracket $(\sigma, \tau) \mapsto [\![\sigma, \tau]\!]$ and the anchor map $\Pi : \sigma \mapsto \pi_* X_\sigma$, is a Lie algebroid.

The kernel of Π is $\overline{\mathfrak{h}}$. It can be shown directly that if $s, t \in \Gamma(\overline{\mathfrak{h}})$ then $[X_s, X_t] = X_{-\{s,t\}}$, so that the restriction of the Lie algebroid bracket to $\Gamma(\overline{\mathfrak{h}})$ is the negative of the Lie-algebraic bracket: $[\![s, t]\!] = -\{s, t\}$.

For any $\sigma, \tau \in \Gamma(\overline{\mathfrak{g}})$, $X_\sigma(\tau)$ is a $\mathfrak{g}$-valued function which is easily seen to satisfy the H-equivariance condition. The map D from $\Gamma(\overline{\mathfrak{g}})$ to operators on $\Gamma(\overline{\mathfrak{g}})$ given by $D_\sigma \tau = X_\sigma(\tau)$ is a $\overline{\mathfrak{g}}$-connection on $\overline{\mathfrak{g}}$. It is clear that D defines a derivation of the Lie-algebraic bracket:

$$D_\rho\{\sigma, \tau\} = \{D_\rho \sigma, \tau\} + \{\sigma, D_\rho \tau\}.$$

The curvature of D vanishes:

$$D_\rho D_\sigma \tau - D_\sigma D_\rho \tau - D_{[\![\rho,\sigma]\!]}\tau = X_\rho(X_\sigma(\tau)) - X_\sigma(X_\rho(\tau)) - [X_\rho, X_\sigma](\tau) = 0.$$

That is to say, D is a representation of the Lie algebroid $\overline{\mathfrak{g}}$ on itself. I shall call it the fundamental self-representation associated with the Cartan connection; it is called a fundamental D-operator by Čap and Gover [3].

On the other hand, D has torsion given by

$$\begin{aligned} D_\rho \sigma - D_\sigma \rho - [\![\rho, \sigma]\!] &= X_\rho(\omega(X_\sigma)) - X_\sigma(\omega(X_\rho)) - \omega([X_\rho, X_\sigma]) = d\omega(X_\rho, X_\sigma) \\ &= \kappa(\Pi(\rho), \Pi(\sigma)) - \{\rho, \sigma\}. \end{aligned}$$

Recall that $\tilde{\xi}(\sigma) + \{\xi, \sigma\} = 0$. It follows that if s takes its values in $\mathfrak{h}$, so that X_s is vertical, $D_s \sigma = -\{s, \sigma\}$. Then from the torsion formula, since $\Pi(s) = 0$, we have $D_\sigma s = [\![\sigma, s]\!]$; in particular, $D_\sigma(\Gamma(\overline{\mathfrak{h}})) \subset (\Gamma(\overline{\mathfrak{h}}))$. Thus the representation D, restricted to $\Gamma(\overline{\mathfrak{h}})$, reduces to the canonical representation of $\overline{\mathfrak{g}}$ on $\overline{\mathfrak{h}}$ coming from the Lie algebroid structure. One says that D is a representation extending the canonical one.

The key properties of the adjoint tractor bundle $\overline{\mathfrak{g}}$ are these.

1. It is a Lie algebra bundle, whose fibres are modelled on $\mathfrak{g}$.
2. It is at the same time a Lie algebroid.
3. As a Lie algebroid it is transitive, with kernel $\overline{\mathfrak{h}}$ which is a sub-Lie-algebra bundle of $\overline{\mathfrak{g}}$ (relative to the Lie algebra bracket of fibres) modelled on the subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.
4. The restriction of the Lie algebroid bracket to $\Gamma(\overline{\mathfrak{h}})$ (which is automatically a Lie-algebraic bracket) is the negative of the restriction of the Lie-algebraic bracket of $\overline{\mathfrak{g}}$.
5. It is equipped with a representation D of itself on itself, each of whose operators D_σ maps $\Gamma(\overline{\mathfrak{h}})$ to itself, and D extends the canonical representation of $\overline{\mathfrak{g}}$ on $\overline{\mathfrak{h}}$ to $\overline{\mathfrak{g}}$.
6. Each operator D_σ of the representation acts as a derivation of the Lie-algebraic bracket, and for $s \in \Gamma(\overline{\mathfrak{h}})$, D_s coincides with the negative of inner derivation by s.

4 Covariant derivatives

In the previous section it was shown that if $s \in \Gamma(\overline{\mathfrak{h}})$ then both $D_s\sigma + \{s,\sigma\} = 0$ and $D_\sigma s - [\![\sigma, s]\!] = 0$. To put things another way, for $\rho \in \Gamma(\overline{\mathfrak{g}})$, both $D_\rho\sigma + \{\rho,\sigma\}$ and $D_\sigma\rho - [\![\sigma,\rho]\!]$ depend only on $\Pi(\rho)$. We may therefore define, for any vector field U on M, operators ∇^{T}_U and ∇^{C}_U on $\Gamma(\overline{\mathfrak{g}})$ by

$$\left.\begin{aligned} \nabla^{\mathrm{T}}_U\sigma &= D_\rho\sigma + \{\rho,\sigma\} \\ \nabla^{\mathrm{C}}_U\sigma &= D_\sigma\rho - [\![\sigma,\rho]\!] \end{aligned}\right\} \quad \text{for any } \rho \text{ such that } \Pi(\rho) = U.$$

It is easy to see, from the properties of D, that ∇^{T} and ∇^{C} are covariant derivatives; that is, by this process we define two connections (TM-connections) on $\overline{\mathfrak{g}}$. The first of these is the tractor connection. The second is Blaom's Cartan connection; to be exact, in his terminology it is the Cartan connection on the Lie algebroid $\overline{\mathfrak{g}}$ corresponding to the classical Cartan connection ω.

The tractor connection can be specified directly in terms of ω (that is, without reference to D) as follows. For any $u \in T_xM$ and $\hat{u} \in T_pP$, with $\pi(p) = x$, such that $\pi_*\hat{u} = u$

$$\nabla^{\mathrm{T}}_u\sigma = \hat{u}(\sigma) + \{\omega(\hat{u}),\sigma\}.$$

It is clear that for any $\sigma \in \Gamma(\overline{\mathfrak{g}})$ the $\mathfrak{g}$-valued 1-form $d\sigma + \{\omega,\sigma\}$ is H-equivariant and vanishes on vertical vectors, so this formulation makes sense. This is essentially how the tractor connection is defined in [3].

Notice that

$$\nabla^{\mathrm{T}}_u\{\sigma,\tau\} = \{\nabla^{\mathrm{T}}_u\sigma,\tau\} + \{\sigma,\nabla^{\mathrm{T}}_u\tau\} :$$

that is, ∇^{T} is a derivation of the Lie-algebraic bracket. The curvature R^{T} of ∇^{T} is given by

$$R^{\mathrm{T}}(U,V)\sigma = \{\kappa(U,V),\sigma\}.$$

The torsion formula leads to

$$\nabla^{\mathrm{T}}_{\Pi(\sigma)}\tau - \nabla^{\mathrm{T}}_{\Pi(\tau)}\sigma - [\![\sigma,\tau]\!] = \{\sigma,\tau\} + \kappa(\Pi(\sigma),\Pi(\tau)).$$

This could be read as a formula for the Lie algebroid bracket in terms of the tractor connection and its curvature; it is so interpreted in [3].

Blaom's Cartan connection can also be defined directly in terms of ω, as follows. We have

$$D_\sigma\rho - [\![\sigma,\rho]\!] = X_\sigma(\rho) - \omega([X_\sigma, X_\rho]) = (\mathcal{L}_{X_\sigma}\omega)(X_\rho).$$

This suggests that we set

$$\nabla^{\mathrm{C}}_u\sigma = (\mathcal{L}_{X_\sigma}\omega)(\hat{u})$$

where again $u \in T_{\pi(p)}M$ and $\hat{u} \in T_pP$ with $\pi_*\hat{u} = u$. It is not difficult to see that for any H-invariant vector field X on P, $\mathcal{L}_X\omega$ is H-equivariant and vanishes on vertical vectors, so again this makes sense.

This formulation reveals the following interesting fact about ∇^{C}. An H-invariant vector field X on P such that $\mathcal{L}_X\omega = 0$ is an infinitesimal automorphism of the Cartan connection form ω (see [2]). It follows that the infinitesimal automorphisms of ω correspond precisely to the sections of $\overline{\mathfrak{g}} \to M$ which are parallel with respect to ∇^{C}. (In fact ∇^{C} makes a brief incidental appearance in [2], which is in part a study of infinitesimal automorphisms of Cartan connection forms.)

Notice also that $\nabla^{\mathrm{C}}_U\sigma = D^*_\rho\sigma$ where D^* is the $\overline{\mathfrak{g}}$-connection dual to D. Indeed, one could almost sum up the difference between ∇^{T} and ∇^{C} by saying that the first embodies D and the second D^*.

The fact that D is a representation translates into the following formula for ∇^{C}:

$$\nabla^{\mathrm{C}}_U(\llbracket\sigma,\tau\rrbracket) = \llbracket\nabla^{\mathrm{C}}_U\sigma,\tau\rrbracket + \llbracket\sigma,\nabla^{\mathrm{C}}_U\tau\rrbracket + \nabla^{\mathrm{C}}_{\Pi(D_\tau\rho)}\sigma - \nabla^{\mathrm{C}}_{\Pi(D_\sigma\rho)}\tau$$

where $U = \Pi(\rho)$. In [1] Blaom says that a connection ∇ on a Lie algebroid $\mathcal{A}$ is compatible with the Lie algebroid bracket if

$$\nabla_U(\llbracket\sigma,\tau\rrbracket) = \llbracket\nabla_U\sigma,\tau\rrbracket + \llbracket\sigma,\nabla_U\tau\rrbracket + \nabla_{\nabla^*_\tau U}\sigma - \nabla_{\nabla^*_\sigma U}\tau.$$

Here ∇^* is the operator defined by $\nabla^*_\sigma U = \Pi(\nabla_U\sigma) + [\Pi(\sigma), U]$; it is an $\mathcal{A}$-connection on TM which is in a sense dual to ∇. Evidently $\nabla^{\mathrm{C}*}_\sigma\Pi(\rho) = \Pi(D_\sigma\rho)$. So the fact that D is a representation implies that ∇^{C} is compatible with $\llbracket\cdot,\cdot\rrbracket$: ∇^{C} is indeed a Cartan connection in the sense of Blaom.

It is straightforward to calculate the torsion T^* and curvature R^* of the dual D^* of an $\mathcal{A}$-connection on $\mathcal{A}$ in terms of T and R, those of the original $\mathcal{A}$-connection D; the formulas are given in [1]. One finds that $T^* = -T$, and in the case of interest, where D is a representation,

$$R^*(\rho,\sigma)\tau = D_\tau(T(\rho,\sigma)) - T(D_\tau\rho,\sigma) - T(\rho,D_\tau\sigma) = (D_\tau T)(\rho,\sigma).$$

Given the relation between T and the curvature κ, and the fact that D defines a derivation of the Lie-algebraic bracket, one can in effect replace T by κ on the right-hand side. These results can be used to derive further properties of ∇^{C} if desired.

The difference tensor relating the two connections is the curvature:

$$\nabla^{\mathrm{T}}_{\Pi(\rho)}\sigma - \nabla^{\mathrm{C}}_{\Pi(\rho)}\sigma = \kappa(\Pi(\rho),\Pi(\sigma)).$$

5 The Atiyah algebroid and the Cartan connection

Any principal bundle has canonically associated with it a Lie algebroid, its Atiyah algebroid, which in the case of a principal H-bundle $P \to M$ is defined as follows. The action of H on P lifts in a natural way to an action on the tangent bundle TP. As a manifold the Atiyah algebroid of P is TP/H, the quotient of TP under this action. It is a vector bundle over $P/H = M$, each of whose fibres is obtained by identifying the tangent spaces T_pP at points p on the same fibre of $P \to M$. The sections of $TP/H \to M$ may be identified with the H-invariant vector fields on P; the Lie algebroid bracket is the ordinary bracket of such vector fields; the anchor maps an H-invariant vector field on P to its projection onto M. The sections of the kernel of the Atiyah algebroid can therefore be identified with the H-invariant vector fields on P which are vertical (with respect to $\pi : P \to M$). As was pointed out before, any vertical H-invariant vector field V on P takes the form $V_p = \widetilde{s(p)}|_p$ for some H-equivariant $\mathfrak{h}$-valued function s on P. So the kernel of the Atiyah algebroid can be identified with the adjoint bundle of P, that is, $\overline{\mathfrak{h}}$, the vector bundle associated with P by the adjoint action of H on $\mathfrak{h}$.

It is clear from the discussion in Section 3 that as a Lie algebroid $\overline{\mathfrak{g}}$ is isomorphic to the Atiyah algebroid, where the isomorphism is defined by the Cartan connection form ω. In fact ω defines an isomorphism, over the identity on P, of TP with $P \times \mathfrak{g}$, which is equivariant with respect to the actions of H on the two spaces, and therefore passes to the quotients under these actions. As it happens, Blaom in [1] uses the Atiyah algebroid rather than the adjoint tractor bundle in his discussion of the association of a Lie algebroid with a Cartan geometry.

It might indeed be considered preferable to describe the constructions in Sections 2 and 3 in a different way which takes advantage of these observations. Suppose given the data for a model geometry as specified in the Section 1.1, namely an infinitesimal Klein geometry $(\mathfrak{g}, \mathfrak{h})$, a Lie group H realising $\mathfrak{h}$, and the adjoint representation of H on $\mathfrak{g}$. Take a principal H-bundle $P \to M$ where $\dim M = \dim \mathfrak{g} - \dim \mathfrak{h}$. In the absence of a Cartan connection form one can then construct two vector bundles over M: the adjoint tractor bundle $\overline{\mathfrak{g}}$ (the bundle associated with P by the adjoint action of H on $\mathfrak{g}$), and the Atiyah bundle TP/H. These bundles have the same fibre dimension $\dim \mathfrak{g} = \dim M + \dim \mathfrak{h}$. The first is a Lie algebra bundle, the second a Lie algebroid. It is worth emphasising that, given the model data, each of these vector bundles has the stated additional structure regardless of the existence of a Cartan connection form. A Cartan connection form, however, defines a vector bundle isomorphism between the two, which acts as the identity on $\overline{\mathfrak{h}}$ (which can be regarded as a subbundle of each bundle). This isomorphism can be used to carry the Lie algebroid structure from the Atiyah algebroid over to the adjoint tractor bundle – and equally the Lie algebra bundle structure in the opposite direction. In fact the existence of a Cartan connection is equivalent to the existence of a vector bundle isomorphism, over the identity, between these two bundles, which acts as the identity on the common subbundle $\overline{\mathfrak{h}}$.

Theorem 1. *With model data as above, let $P \to M$ be a principal H-bundle where* $\dim M = \dim \mathfrak{g} - \dim \mathfrak{h}$*. Consider two vector bundles over M, with the same fibre dimension: the adjoint tractor bundle $\overline{\mathfrak{g}}$ and the Atiyah bundle TP/H. The Cartan geometries on P modelled on* $(\mathfrak{g}, \mathfrak{h})$ *with group H are in* 1-1 *correspondence with the vector bundle isomorphisms $TP/H \to \overline{\mathfrak{g}}$, over the identity, which reduce to the identity on $\overline{\mathfrak{h}}$.*

Proof. It remains to show that an isomorphism of the bundles defines a Cartan connection on P. Let $\hat{\omega} : TP/H \to \overline{\mathfrak{g}}$ be a vector bundle isomorphism over the identity. We can identify the sections of $TP/H \to M$ with the H-invariant vector fields on P, and the sections of $\overline{\mathfrak{g}} \to M$ with the H-equivariant $\mathfrak{g}$-valued functions on P. Then $\hat{\omega}$ associates with every H-invariant vector field X on P an H-equivariant $\mathfrak{g}$-valued function $\omega(X)$ on P, such that $\omega(X)$ depends $C^\infty(M)$-linearly on its argument. It follows that $\omega(X)(p)$ depends only on the value of X at p, and does so linearly; thus for each p we have a linear map $\omega_p : T_pP \to \mathfrak{g}$, which is an isomorphism. In other words, ω is a $\mathfrak{g}$-valued 1-form on P. The condition that $R_h^*\omega = \operatorname{ad}(h^{-1})\omega$ for each $h \in H$ follows from the fact that $\omega(X)$ is H-equivariant whenever X is H-invariant. The assumed property of $\hat{\omega}$ that it reduces to the identity on $\overline{\mathfrak{h}}$ is equivalent to the condition that $\omega(\tilde{\xi}) = \xi$ for $\xi \in \mathfrak{h}$. ■

These observations suggest an alternative way of defining a Cartan connection form, in effect as an isomorphism of vector bundles. To suggest that one might define a Cartan geometry in this way may seem to be a somewhat idiosyncratic procedure. It is done to emphasise the fact that the approaches of Blaom on the one hand, and Čap and Gover on the other, concentrate on different aspects of the full story.

6 Cartan algebroids and Cartan geometries

I listed earlier the properties of the adjoint tractor bundle associated with a Cartan geometry. I now show that for a certain class of pairs of Lie algebras $(\mathfrak{g}, \mathfrak{h})$ these properties are definitive of Cartan geometries.

Theorem 2. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$ such that*

1) *all derivations of $\mathfrak{g}$ are inner;*
2) *the centre of $\mathfrak{g}$ is $\{0\}$;*
3) *the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$ is $\mathfrak{h}$.*

Let H be the subgroup of the Lie group of inner automorphisms of $\mathfrak{g}$ which leaves $\mathfrak{h}$ invariant; it is a Lie group with Lie algebra $\mathfrak{h}$. Let $\overline{\mathfrak{g}} \to M$ be a transitive Lie algebroid equipped with a self-representation D which extends the canonical representation, such that

1) *$\overline{\mathfrak{g}}$ is at the same time a Lie algebra bundle modelled on $\mathfrak{g}$;*
2) *$\overline{\mathfrak{h}}$, the kernel of $\overline{\mathfrak{g}}$, is a sub-Lie-algebra bundle of $\overline{\mathfrak{g}}$ modelled on the subalgebra $\mathfrak{h}$ of $\mathfrak{g}$;*
3) *the restriction of the Lie algebroid bracket to $\Gamma(\overline{\mathfrak{h}})$ is the negative of the restriction of the Lie-algebraic bracket;*
4) *each operator D_σ of the representation acts as a derivation of the Lie-algebraic bracket, and for $s \in \Gamma(\overline{\mathfrak{h}})$, D_s coincides with the negative of inner derivation by s.*

Then there is a principal H-bundle $\pi : P \to M$ and a Cartan connection form ω on P such that $\overline{\mathfrak{g}}$ is the adjoint tractor bundle of P, the Lie algebroid bracket of $\overline{\mathfrak{g}}$ coincides with that determined by ω, and D is the fundamental self-representation associated with ω.

The proof of this theorem is closely modelled on the proofs of Proposition 2.3 and Theorem 2.7 of [3].

Proof. Let P be the set of all pairs (x, p) where $x \in M$ and p is a Lie algebra isomorphism $\mathfrak{g} \to \overline{\mathfrak{g}}_x$ such that $p(\mathfrak{h}) = \overline{\mathfrak{h}}_x$. Then P can be given a manifold structure such that $(x, p) \mapsto x$ is a smooth submersion π, and $\pi : P \to M$ admits local smooth sections, so it is a fibre bundle over M. For each $h \in H$, $(x, p) \mapsto (x, p \circ \operatorname{ad} h)$, where ad denotes the action of H on $\mathfrak{g}$, defines a right action of H on P which makes P into a principal H-bundle. The map $P \times \mathfrak{g} \to \overline{\mathfrak{g}}$ by $(x, p, \xi) \mapsto p(\xi) \in \overline{\mathfrak{g}}_x$ induces an identification of the adjoint tractor bundle of P with $\overline{\mathfrak{g}}$.

The sections of $\overline{\mathfrak{g}} \to M$ may be identified with the H-equivariant $\mathfrak{g}$-valued functions on P, as before. It follows from the properties of D that $D_\rho\sigma + \{\rho, \sigma\}$ depends only on $\Pi(\rho)$. We may therefore define a covariant derivative operator ∇^{T} on $\Gamma(\overline{\mathfrak{g}})$ by

$$\nabla^{\mathrm{T}}_U\sigma = D_\rho\sigma + \{\rho, \sigma\} \quad \text{for any } \rho \text{ such that } \Pi(\rho) = U.$$

Then ∇^{T}_U is a derivation of the Lie-algebraic bracket on $\Gamma(\overline{\mathfrak{g}})$. Now for any $v \in T_pP$ and any $\sigma \in \Gamma(\overline{\mathfrak{g}})$, $\nabla^{\mathrm{T}}_{\pi_* v}\sigma - v(\sigma)$ is an element of $\mathfrak{g}$ which depends $\mathbb{R}$-linearly on v and on $\sigma(p)$. So we may define an $\mathbb{R}$-linear map $\Phi_p : T_pP \to \operatorname{lin}(\mathfrak{g}, \mathfrak{g})$ (the space of $\mathbb{R}$-linear maps from $\mathfrak{g}$ to $\mathfrak{g}$) by $\Phi_p(v)(\xi) = \nabla^{\mathrm{T}}_{\pi_* v}\sigma - v(\sigma)$ for any $\sigma \in \Gamma(\overline{\mathfrak{g}})$ such that $\sigma(p) = \xi$. Moreover, for each $v \in T_pP$, $\Phi_p(v)$ is a derivation of $\mathfrak{g}$, and by assumption (1) about $(\mathfrak{g}, \mathfrak{h})$ we may write $\Phi_p(v)(\xi) = \{\omega_p(v), \xi\}$. Then $\omega_p : T_pP \to \mathfrak{g}$ is an $\mathbb{R}$-linear map, as a consequence of assumption (2) about $(\mathfrak{g}, \mathfrak{h})$.

Take $v = \tilde{\xi}_p$ for any $\xi \in \mathfrak{h}$. Then $\pi_* v = 0$, and $v(\sigma) = \tilde{\xi}(\sigma)(p) = -\{\xi, \sigma(p)\}$. Thus $\{\omega_p(\tilde{\xi}), \sigma(p)\} = \{\xi, \sigma(p)\}$, so that $\omega_p(\tilde{\xi}) = \xi$, again by assumption (2).

For any H-invariant vector field X on P and any $\sigma \in \Gamma(\overline{\mathfrak{g}})$, $X(\sigma) \in \Gamma(\overline{\mathfrak{g}})$ also. Thus $\nabla^{\mathrm{T}}_{\pi_* X}\sigma - X(\sigma) = \{\omega(X), \sigma\}$ where $\omega(X) \in \Gamma(\overline{\mathfrak{g}})$. So ω is a smooth $\mathfrak{g}$-valued 1-form on P such that $\omega(X)$ is H-equivariant whenever X is H-invariant, whence $R_h^*\omega = \mathrm{ad}(h^{-1})\omega$ for all $h \in H$ (again, appealing implicitly to assumption (2)).

Suppose that $\omega_p(v) = 0$ for some $v \in T_pP$. Let $\rho \in \Gamma(\overline{\mathfrak{g}})$ be such that $\Pi(\rho)(x) = \pi_* v$, $x = \pi(p)$. Then

$$\nabla^{\mathrm{T}}_{\pi_* v}\sigma - v(\sigma) = D_\rho\sigma(p) + \{\rho(p), \sigma(p)\} - v(\sigma) = 0$$

for all $\sigma \in \Gamma(\overline{\mathfrak{g}})$. In particular, this holds for $\sigma = s \in \Gamma(\overline{\mathfrak{h}})$. Now by assumption $D_\rho s(p) \in \mathfrak{h}$, and evidently $v(s) \in \mathfrak{h}$; thus $\{\rho(p), s(p)\} \in \mathfrak{h}$, and so $\{\rho(p), \mathfrak{h}\} \subset \mathfrak{h}$. It follows from assumption (3) that $\rho(p) \in \mathfrak{h}$. But then $\pi_* v = \Pi(\rho)(x) = 0$, so v is vertical; and we know that ω_p is an isomorphism of the vertical subspace of T_pP with $\mathfrak{h}$. So $v = 0$. It follows that for each $p \in P$, $\omega_p : T_pP \to \mathfrak{g}$ is an injective map between vector spaces of the same dimension and is therefore an isomorphism.

Thus ω is a Cartan connection form. For each $\sigma \in \Gamma(\overline{\mathfrak{g}})$ there is a unique H-invariant vector field X_σ on P such that $\omega(X_\sigma) = \sigma$. Then

$$\nabla^{\mathrm{T}}_{\pi_* X_\sigma}\tau = X_\sigma(\tau) + \{\sigma, \tau\} = \nabla^{\mathrm{T}}_{\Pi(\sigma)}\tau = D_\sigma\tau + \{\sigma, \tau\},$$

so $D_\sigma\tau = X_\sigma(\tau)$. Since D is a representation we have

$$D_{[\![\rho,\sigma]\!]}\tau = X_{[\![\rho,\sigma]\!]}(\tau) = [X_\rho, X_\sigma](\tau).$$

Thus

$$[\![\rho, \sigma]\!] = \omega(X_{[\![\rho,\sigma]\!]}) = \omega([X_\rho, X_\sigma]);$$

but the latter expression is just the Lie algebroid bracket of ρ and σ defined by the Cartan connection form ω. ■

Any Lie algebroid equipped with a Cartan connection, that is, a connection which is compatible with the Lie algebroid bracket in the sense given in Section 4, is a Cartan algebroid, in Blaom's terminology; Cartan algebroids are to be regarded as generalizations, at an infinitesimal level, of Cartan geometries. Each Cartan geometry gives rise to a transitive Cartan algebroid, by the construction described in Section 4, but conditions for a Cartan algebroid to come from a Cartan geometry are not given by Blaom. Clearly, one of those conditions must be that the algebroid is transitive. Now a Cartan connection ∇^{C} on any Lie algebroid $\mathcal{A}$ determines an $\mathcal{A}$-connection D: we have merely to reverse the definition of ∇^{C} given earlier, and set $D_\sigma\rho = \nabla^{\mathrm{C}}_{\Pi(\rho)}\sigma + [\![\sigma, \rho]\!]$. Then D is a self-representation such that $D_\sigma s = [\![\sigma, s]\!]$ when $\Pi(s) = 0$. Thus for a transitive algebroid the existence of a Cartan connection is equivalent to the existence of a self-representation extending the canonical representation, as Blaom points out. That is to say, a transitive Cartan algebroid is simply a transitive Lie algebroid with a self-representation extending the canonical representation. The theorem above therefore provides an answer to the question of which Cartan algebroids correspond to Cartan geometries.

7 Symmetric Cartan algebroids

There is one situation in which it is possible to derive a Lie-algebraic structure from a Cartan algebroid: this occurs when the latter is what Blaom calls (locally) symmetric. A Cartan algebroid is symmetric when its Cartan connection ∇^{C} is flat. Such an algebroid is, locally at least, isomorphic to an action algebroid, that is, to an algebroid generated by an infinitesimal action of a Lie algebra on a manifold. The term 'symmetric' comes from this latter property, when the algebra is conceived as an algebra of infinitesimal symmetries.

When the algebroid is transitive one may work with the self-representation D rather than the Cartan connection, and in view of remarks made in Section 4 the condition for symmetry is that the torsion T of D should satisfy $DT = 0$.

Theorem 3. *Let $\mathcal{A} \to M$ be a transitive Cartan algebroid, with self-representation D, which is locally symmetric. Set $\{\sigma, \tau\} = -T(\sigma, \tau)$ where T is the torsion of D: then $\{\cdot, \cdot\}$ is a Lie-algebraic bracket on $\Gamma(A)$ such that*

1) *$\mathcal{A}$ is a Lie algebra bundle, whose Lie-algebraic bracket is $\{\cdot, \cdot\}$;*
2) *the restriction of the Lie algebroid bracket to $\Gamma(\ker(\mathcal{A}))$ is the negative of the restriction of $\{\cdot, \cdot\}$;*
3) *each operator D_σ of the self-representation acts as a derivation of $\{\cdot, \cdot\}$, and for $s \in \Gamma(\ker(\mathcal{A}))$, D_s coincides with the negative of inner derivation by s.*

Proof. Evidently $-T(\sigma, \tau)$ is $C^\infty(M)$-bilinear and skew. Any $\mathcal{A}$-connection D satisfies a Bianchi identity, which when its curvature vanishes is just a differential condition on its torsion, namely

$$\bigoplus_{\rho,\sigma,\tau} (D_\rho T(\sigma, \tau) + T(T(\rho, \sigma), \tau)) = 0,$$

where $\oplus$ stands for the cyclic sum. When $DT = 0$ therefore, $-T$, thought of as defining a bracket, satisfies the Jacobi identity, and so $\{\cdot, \cdot\}$ is a Lie-algebraic bracket on $\Gamma(A)$.

Consider (following [1]) those sections of $\mathcal{A}$ which are parallel with respect to the Cartan connection ∇^{C}. By the properties of flat connections the set of such sections, say $\mathcal{A}_0$, forms a finite-dimensional real vector space whose dimension is the fibre dimension of $\mathcal{A}$. Elements of $\mathcal{A}_0$ are sections σ for which $D_\sigma \tau = [\![\sigma, \tau]\!]$ for every τ (even those $\tau \in \Gamma(\ker(\mathcal{A}))$, since by assumption D extends the canonical representation). It follows from the fact that D is a representation that $\mathcal{A}_0$ is closed under the algebroid bracket. The restriction to $\mathcal{A}_0$ of $-[\![\cdot, \cdot]\!]$ gives it the structure of a Lie algebra. Moreover, each fibre of $\mathcal{A} \to M$ acquires the same Lie algebra structure: the bracket of $a, b \in \mathcal{A}_x$ is $-[\![\alpha, \beta]\!](x)$ where α, β are the unique elements of $\mathcal{A}_0$ such that $\alpha(x) = a$, $\beta(x) = b$. Thus $\mathcal{A}$ is a Lie algebra bundle modelled on the Lie algebra $\mathcal{A}_0$. For any $\alpha, \beta \in \mathcal{A}_0$,

$$T(\alpha, \beta) = D_\alpha \beta - D_\beta \alpha - [\![\alpha, \beta]\!] = [\![\alpha, \beta]\!] - [\![\beta, \alpha]\!] - [\![\alpha, \beta]\!] = [\![\alpha, \beta]\!],$$

so that the restriction of the algebroid bracket to $\mathcal{A}_0$ coincides with the restriction of T. Take a (local) basis $\{\alpha_i\}$ of $\mathcal{A}_0$ (over $\mathbb{R}$). By dimension, $\mathcal{A}_0$ spans $\Gamma(\mathcal{A})$ over $C^\infty(M)$; that is, every $\sigma \in \Gamma(\mathcal{A})$ may be uniquely written in the form $\sigma^i \alpha_i$ (summation over i intended) with $\sigma^i \in C^\infty(M)$. Then

$$T(\sigma, \tau) = \sigma^i \tau^j T(\alpha_i, \alpha_j) = \sigma^i \tau^j [\![\alpha_i, \alpha_j]\!].$$

Thus the Lie-algebraic bracket on $\mathcal{A}$ arising from the fact that it is a Lie algebra bundle is $\{\cdot, \cdot\}$.

Now take $s, t \in \Gamma(\ker(\mathcal{A}))$. Then since $\Pi(s) = \Pi(t) = 0$,

$$[\![s, t]\!] = s^i t^j [\![\alpha_i, \alpha_j]\!] = T(s, t).$$

Thus on $\Gamma(\ker(\mathcal{A}))$ the algebroid bracket coincides with T, and therefore with $-\{\cdot, \cdot\}$.

The condition $DT = 0$ becomes

$$D_\rho\{\sigma, \tau\} = \{D_\rho \sigma, \tau\} + \{\sigma, D_\rho \tau\}.$$

Since D extends the canonical representation of $\mathcal{A}$ on $\ker(\mathcal{A})$, for $s \in \Gamma(\ker(\mathcal{A}))$, $D_\sigma s = [\![\sigma, s]\!]$. Thus

$$T(s, \sigma) = D_s \sigma - D_\sigma s - [\![s, \sigma]\!] = D_s \sigma.$$

That is to say, $D_s \sigma = -\{s, \sigma\}$ for all $s \in \Gamma(\ker(\mathcal{A}))$. ∎

In view of the correspondence mentioned earlier between sections of $\mathcal{A}$ which are parallel with respect to ∇^{C} and infinitesimal automorphisms of the Cartan connection form ω, we see that a transitive Cartan algebroid which is locally symmetric in Blaom's sense is one in which the Lie algebra of infinitesimal automorphisms of ω (which is $\mathcal{A}_0$) has maximal dimension. Since $\mathcal{A}_0$ is closed under the Lie algebroid bracket we may write $[\![\alpha_i, \alpha_j]\!] = C_{ij}^k \alpha_k$; the C_{ij}^k are the structure constants of $\mathcal{A}_0$ relative to the basis $\{\alpha_i\}$. The anchor map, which restricts to a homomorphism of $\mathcal{A}_0$ into $\Gamma(TM)$, defines an infinitesimal action of $\mathcal{A}_0$ on M. We have

$$[\![\sigma, \tau]\!] = \left(\Pi(\sigma)(\tau^k) - \Pi(\tau)(\sigma^k) + \sigma^i \tau^j C_{ij}^k\right) \alpha_k, \qquad \sigma = \sigma^k \alpha_k, \qquad \tau = \tau^k \alpha_k.$$

This realises $\mathcal{A}$ as an action algebroid corresponding to the Lie algebra $\mathcal{A}_0$.

Recall that if one starts with a Cartan geometry one finds that the torsion of the fundamental self-representation is given by

$$T(\sigma, \tau) = \kappa(\Pi(\sigma), \Pi(\tau)) - \{\sigma, \tau\}.$$

A flat Cartan geometry, that is, one for which $\kappa = 0$, is of course symmetric, and the Lie-algebraic bracket constructed by the procedure in the theorem coincides with the original one. But DT can be zero without κ being zero, and then the new Lie-algebraic bracket is different from the original one. In such a case we find ourselves dealing with what Sharpe in [4] calls Cartan space forms and mutation.

The curvature Ω of a Cartan geometry is constant, according to Sharpe, if $\Omega(X, Y)$ (a $\mathfrak{g}$-valued function on the principal H-bundle P) is constant for every pair of vector fields X, Y on P such that $\omega(X)$ and $\omega(Y)$ are constant. A Cartan geometry whose curvature is constant is a Cartan space form.

Until now I have for the most part restricted my attention to vector fields on P which are H-invariant, and vector fields X for which $\omega(X)$ is constant are not H-invariant. However, it is not difficult to show that Ω is constant in Sharpe's sense if and only $DT = 0$. Let $\{e_i\}$ be a basis for $\mathfrak{g}$, $\{E_i\}$ the vector fields on P such that $\omega(E_i) = e_i$. Then $\Omega(E_i, E_j) = \Omega_{ij}^k e_k$, and in a space of constant curvature the coefficients Ω_{ij}^k are constants. Let C_{ij}^k be the structure constants of $\mathfrak{g}$ relative to the basis $\{e_i\}$. Then $C_{ij}^k - \Omega_{ij}^k$ are the structure constants for a new Lie algebra stucture on (the vector space) $\mathfrak{g}$, making it into a new Lie algebra $\mathfrak{g}'$. This process of changing the Lie algebra bracket is called mutation. Note that since Ω vanishes if one of its arguments is vertical, mutation does not change $\mathfrak{h}$.

Starting with a Cartan space form modelled on $(\mathfrak{g}, \mathfrak{h})$ with group H, by carrying out the procedure in Theorem 3 one obtains, for the same Cartan algebroid, the Lie-algebraic bracket appropriate to a Cartan geometry modelled on $(\mathfrak{g}', \mathfrak{h})$ with group H, where $\mathfrak{g}'$ is the mutation of $\mathfrak{g}$ determined by the (constant) curvature Ω. With respect to the new Lie-algebraic bracket we have $T(\sigma, \tau) = -\{\sigma, \tau\}$, so the mutant geometry has curvature zero. This illustrates an observation of Sharpe's, that all Cartan space forms are mutants of flat geometries. Blaom puts the same idea in a different way: his theory, he says, '*has no models*: all symmetric structures are created equal and the curvature [of ∇^{C}] ... merely measures deviation from *some* symmetric structure' (his italics). The tractor connection ∇^{T}, on the other hand, does differentiate between different Cartan space forms.

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