# Extension Phenomena for Holomorphic Geometric Structures*

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**Abstract.** The most commonly encountered types of complex analytic $G$-structures and Cartan geometries cannot have singularities of complex codimension 2 or more.

*Key words:* Hartogs extension; Cartan geometry; parabolic geometry; $G$-structure

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## Contents



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# 1 Introduction

This article introduces methods to prove that various holomorphic geometric structures (to be defined below) on complex manifolds cannot have singularities of complex codimension 2 or more; in other words, they extend holomorphically across subsets of complex codimension 2 or more. Singularities can occur on complex hypersurfaces. Simple (but perhaps less important) examples will show that some other types of holomorphic geometric structures *can* have low dimensional singularities.

Specifically, we prove that

1. The underlying holomorphic principal bundle of a Cartan geometry or $G$-structure extends holomorphically across a subset of complex codimension 2 or more just when the Cartan geometry or $G$-structure does (see Theorem 4.18 and Theorem 4.20 on page 19).
2. Holomorphic higher order structures extend across subsets of complex codimension 2 or more just when their underlying first order $G$-structures extend (see Proposition 6.9 on page 33).
3. The following holomorphic geometric structures extend holomorphically across subsets of complex codimension 2 or more:
   (a) contact structures (see Theorem 5.9 on page 22),
   (b) reductive $G$-structures (see Example 5.22 on page 24),
   (c) scalar conservation laws (see Example 5.43 on page 30),
   (d) web geometries on surfaces (see Example 5.26 on page 25),
   (e) reductive Cartan geometries (see Example 8.2 on page 37) and
   (f) parabolic geometries (see Theorem 8.5 on page 37).

We also present a dictionary between holomorphic extension problems for geometric structures and holomorphic extension problems for maps to complex homogeneous spaces.

The most striking example of our extension theorems is that of 2nd order scalar ordinary differential equations (Example 8.9 on page 37), for which we prove that the geometry invariant under point transformations extends across subsets of complex codimension 2 or more, and give examples in which the geometry invariant under fiber-preserving transformations does not.

Every manifold and Lie group in these notes is complex, and every map and bundle is holomorphic. I write Lie groups as $G$, $H$, etc. and their Lie algebras as $\mathfrak{g}$, $\mathfrak{h}$, etc.

## 2 Definitions of Cartan geometries and first order structures

For completeness, I will define the geometric structures of interest.

### 2.1 First order structures

**Definition 2.1.** If $V \to M$ is a holomorphic vector bundle, the *frame bundle* of $V$, also called the *associated principal bundle* and denoted $FV$, is the set of complex linear isomorphisms of fibers of $V$ with some fixed vector space $V_0$. Clearly $FV \to M$ is a holomorphic principal right $\mathrm{GL}(V_0)$-bundle, under the action $r_h u = h^{-1}u$. When we need to be precise about the choice of vector space $V_0$, we will refer to $FV$ as the $V_0$-valued frame bundle.

**Definition 2.2.** Suppose that $V_0$ is a complex vector space and $\rho : G \to \mathrm{GL}(V_0)$ is a holomorphic representation of a complex Lie group $G$. If $V \to M$ is a vector bundle and $FV$ its $V_0$-valued frame bundle, then $G$ acts on $FV$ by having $g \in G$ take $u \in FV \mapsto \rho(g)^{-1}u \in FV$. We write this action as $r_g u = \rho(g)^{-1}u$. A *$G$-structure* on a manifold $M$ is a principal $G$-bundle $E$ together with a $G$-equivariant bundle morphism map $E \to FTM$. See Gardner [19] or Ivey and Landsberg [32] for an introduction to $G$-structures.

If we want to discuss $G$-structures for various groups $G$, we will call them *first order structures*.

**Definition 2.3.** The *standard flat $G$-structure* associated to a representation $\rho : G \to \mathrm{GL}(V_0)$ is the trivial bundle $E = V_0 \times G \to M = V_0$ mapped to $FV_0 = V_0 \times \mathrm{GL}(V_0) \to V_0$ by $(v_0, g) \mapsto (v_0, \rho(g))$.

**Example 2.4.** Suppose that $E \to FTM$ is a $G$-structure for a complex representation $G \to \mathrm{GL}(V_0)$. Let $K$ be the kernel of $G \to \mathrm{GL}(V_0)$, and suppose that $G \to \mathrm{GL}(V_0)$ has closed image. Then $E/K \subset FTM$ is a $G/K$-structure, called the *underlying embedded first order structure*, because it is embedded in $FTM$.

**Definition 2.5.** Fix a complex manifold $M$, a vector space $V_0$ with $\dim V_0 = \dim M$, and take $FTM$ the $V_0$-valued frame bundle. Let $\pi : FTM \to M$ be the bundle mapping. We define a $V_0$-valued 1-form $\sigma$ on $FTM$, called the *soldering form*, by $v \lrcorner \sigma_{(m,u)} = u\left(\pi'(m,u)v\right)$, for any $v \in T_{(m,u)}FTM$.

### 2.2 Cartan geometries

**Definition 2.6.** Let $H \subset G$ be a closed subgroup of a Lie group, with Lie algebras $\mathfrak{h} \subset \mathfrak{g}$. A *Cartan geometry* modelled on $G/H$ (also called a $G/H$-geometry) on a manifold $M$ is a choice of principal right $H$-bundle $E \to M$, and 1-form $\omega \in \Omega^1(E) \otimes \mathfrak{g}$ called the *Cartan connection*, which satisfies the following conditions:

1. Denote the right action of $g \in G$ on $e \in E$ by $r_g e = eg$. The Cartan connection transforms in the adjoint representation:
$$r_g^* \omega = \mathrm{Ad}_g^{-1}\, \omega.$$
2. $\omega_e : T_e E \to \mathfrak{g}$ is a linear isomorphism at each point $e \in E$.
3. For each $A \in \mathfrak{g}$, define a vector field $\vec{A}$ on $E$ by the equation $\vec{A} \lrcorner \omega = A$. Then the vector fields $\vec{A}$ for $A \in \mathfrak{h}$ must generate the right $H$-action:
$$\vec{A} = \left.\frac{d}{dt} r_{e^{tA}}\right|_{t=0}.$$

See Sharpe [54] for an introduction to Cartan geometries.

**Definition 2.7.** The *standard flat $G/H$-geometry* is the Cartan geometry whose bundle is $G \to G/H$ and whose Cartan connection is $g^{-1}\, dg$.

**Example 2.8.** Let $M$ be a complex manifold, and $\pi : E \to M$ a $G/H$-geometry. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of $G$ and $H$. Let $V_0 = \mathfrak{g}/\mathfrak{h}$. Let $FTM$ be the $V_0$-valued frame bundle. Let $\sigma = \omega + \mathfrak{h}$, a semibasic 1-form valued in $V_0$. At each point $e \in E$ the 1-form $\sigma$ determines a linear isomorphism $u = u_e : T_m M \to V_0$ by the equation $u(\pi'(e)v) = v \lrcorner \sigma$. Map $e \in E \mapsto u_e \in FTM$. This map is an $H$-structure. Let $H_1 \subset H$ be the subgroup of $H$ consisting of the elements of $H$ which act trivially on $V_0 = \mathfrak{g}/\mathfrak{h}$. The fibers of the map $E \to FTM$ consist of the $H_1$-orbits in $E$. The map descends to a map $E/H_1 \to FTM$ called the *underlying* $(H/H_1)$*-structure* or *underlying first order structure* of the $G/H$-geometry.

**Definition 2.9.** The *kernel* $K$ of a homogeneous space $G/H$ is the largest normal subgroup of $G$ contained in $H$:

$$K = \bigcap_{g \in G} gHg^{-1}.$$

As $G$-spaces, $G/H = (G/K)/(H/K)$, so strictly speaking, the kernel is not an invariant of the homogeneous space, but of the choice of Lie group $G$ and closed subgroup $H$, but we will ignore this subtlety.

**Lemma 2.10 (Sharpe [54]).** *Suppose that $K$ is the kernel of $G/H$, with Lie algebra $\mathfrak{k}$, and that $E \to M$ is a Cartan geometry modelled on $G/H$, with Cartan connection $\omega$. Let $E' = E/K$ and $\omega' = \omega + \mathfrak{k} \in \Omega^1(E) \otimes (\mathfrak{g}/\mathfrak{k})$. Then $\omega'$ drops to a 1-form on $E'$, and $E' \to M$ is a Cartan geometry, called the* reduction *of $E$.*

**Definition 2.11.** A complex homogeneous space $G/H$ is called *reduced* or *faithful* if $K = 1$, and *almost reduced* or *almost faithful* if $K$ is a discrete subgroup of $H$.

## 2.3 Parabolic geometries

**Definition 2.12.** A Lie subalgebra $\mathfrak{p} \subset \mathfrak{g}$ of a semisimple Lie algebra is called *parabolic* if it contains a Borel subalgebra (see Serre [53]). A connected Lie subgroup of a semisimple Lie group is called *parabolic* if its Lie algebra is parabolic. A homogeneous space $G/P$ (with $P$ parabolic and $G$ connected) is called a *rational homogeneous variety*; see Landsberg [37] for an introduction to rational homogeneous varieties.

**Definition 2.13.** A *parabolic geometry* is a Cartan geometry modelled on a rational homogeneous variety. See Čap [11] for an introduction to parabolic geometries.

## 2.4 Curvature

**Definition 2.14.** The *curvature* of a Cartan geometry is the 2-form $d\omega + \frac{1}{2}[\omega, \omega]$. A Cartan geometry is *flat* if its curvature vanishes.

## 2.5 Development

**Definition 2.15.** Suppose that $E_0 \to M_0$ and $E_1 \to M_1$ are two $G/H$-geometries, with Cartan connections $\omega_0$ and $\omega_1$, and $X$ is manifold, perhaps with boundary and corners. A smooth map $\phi_1 : X \to M_1$ is a *development* of a smooth map $\phi_0 : X \to M_0$ if there exists a smooth isomorphism $\Phi : \phi_0^* E_0 \to \phi_1^* E_1$ of principal $H$-bundles so that $\Phi^* \omega_1 = \omega_0$.

**Definition 2.16.** Suppose that $E_0 \to M_0$ and $E_1 \to M_1$ are $G/H$-geometries. Suppose that $\phi_1 : X \to M_1$ is a development of a smooth map $\phi_0 : X \to M_0$ with isomorphism $\Phi : \phi_0^* E_0 \to \phi_1^* E_1$. By analogy with Cartan's method of the moving frame, we will call $e_0$ and $e_1$ *corresponding frames* of the development if $\Phi(e_0) = e_1$.

**Lemma 2.17.** *Suppose that $C$ is a simply connected Riemann surface, that $E_0 \to M_0$ is a $G/H$-geometry, that $\phi_0 : C \to M_0$, and that $e_0 \in \phi_0^* E_0$. Then $\phi_0$ has a unique development $\phi_1 : C \to G/H$ to the model with a unique isomorphism $\Phi : \phi_0^* E_0 \to \phi_1^* G$ so that $\Phi(e_0) = 1 \in G$.*

**Proof.** (This proof is adapted from McKay [40].) The local existence and uniqueness of a development is clear by applying the Frobenius theorem to the Pfaffian system $g^{-1}\, dg = \omega_0$ on $\phi_0^* E_0 \times G$. (The curvature does not affect the involutivity of this Pfaffian system.) The maximal connected integral manifolds project locally diffeomorphically to $\phi_0^* E_0$, because $\omega_0$ is a coframing on them.

Because $C$ is simply connected, $\phi_0^* E_0 \to C$ is a trivial bundle, with a global section $s_0$. If we can develop, then this global section is identified via the isomorphism $\Phi$ with a global section $s_1 : C \to \phi_1^* G$ so that

$$s_1^* g^{-1}\, dg = s_0^* \omega_0. \tag{2.1}$$

Conversely, if we can solve this equation, then there is a unique isomorphism $\Phi$ for which

$$\Phi(s_0 h) = s_1 h$$

for all $h \in H$, by triviality of the bundles. So it suffices to solve equation (2.1).

Equation (2.1) is an ordinary differential equation of Lie type,

$$g^{-1} dg = s_0^* \omega_0,$$

(see Bryant [8, p. 55] or Sharpe [55, p. 118]) so has a unique global solution with given initial condition $g = g_0$ at $t = t_0$. For example, if $G$ is a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$ for some $n$, then global existence and uniqueness of a development follow from writing the ordinary differential equations of Lie type as a linear ordinary differential equation:

$$dg = g s_0^* \omega_0.$$

More generally, one glues together local solutions by using the group action of $G$ to make two local solutions match up at some point. Then the local solutions match near this point by uniqueness. Any compact simply connected subset of $C$ is covered by finitely many domains of such local solutions, which thereby must patch to a global solution. ■

**Lemma 2.18.** *Suppose that $E_0 \to M_0$ is a flat $G/H$-geometry, that $X$ is a simply connected complex manifold, that $\phi_0 : X \to M_0$ is a holomorphic map, and that $e_0 \in \phi_0^* E_0$. Then there is a unique developing map $\phi_1 : X \to G/H$ with isomorphism $\Phi : \phi_0^* E_0 \to \phi_1^* G$ for which $\Phi(e_0) = 1 \in G$.*

**Proof.** Locally, this is the Frobenius theorem. Just as for curves above, the local developments patch together under $G$-action to extend uniquely to all of $X$. ■

# 3 Extensions of maps

## 3.1 Definitions and examples of extension phenomena

**Definition 3.1.** A subset $S \subset M$ of a complex manifold is of *complex codimension* 2 *or more* if each point $s \in S$ lies in an open set $U_s \subset M$ so that $S \cap U_s$ is contained in a complex analytic subvariety $V_s$ of $U_s$, of complex codimension 2 or more.

**Lemma 3.2 (Hartogs extension lemma [22]).** *If $M$ is a complex manifold $M$ and $S \subset M$ is of complex codimension 2 or more, then every holomorphic function $f : M\backslash S \to \mathbb{C}$ extends to a unique holomorphic function $f : M \to \mathbb{C}$.*

We will paraphrase the Hartogs extension lemma as saying that holomorphic functions extend over subsets of complex codimension 2 or more. Krantz [36] provides an introduction to Hartogs extension phenomena, while Merker and Porten [42] give an elegant new proof of the result in a more general form.

Consider two extension problems for holomorphic functions, holomorphic bundles, or holomorphic geometric structures:

- the *Hartogs extension problem* of extending holomorphically from a domain in a Stein manifold to its envelope of holomorphy
- the *Thullen extension problem* of extending holomorphically across a subset of complex codimension 2 or more.

**Remark 3.3.** The expression *Thullen-type extension* is used in complex analysis to mean extension of a holomorphic function (or extension of a holomorphic section of a vector bundle, or extension of a holomorphic vector bundle, etc.) from $M\backslash S$ to $M$, where $M$ is a complex manifold, and $S \subset M$ has the form $S = H\backslash U$, where $H \subset M$ is a complex hypersurface, and $U \subset M$ is an open set intersecting every codimension 1 component of $H$. In this paper, we will use the term *Thullen extension* only in the narrower sense above, for simplicity.

**Definition 3.4.** A complex space $X$ is a *Hartogs extension target* if every holomorphic map $f : D \to X$ from a domain $D$ in a Stein manifold extends to the envelope of holomorphy of $D$.

We will also need the following two simpler and weaker properties, for Thullen extension problems.

**Definition 3.5.** We will say that a complex space $X$ is a *Thullen extension target*, to mean that every holomorphic map $f : M\backslash S \to X$ extends to a holomorphic map $f : M \to X$, where $M$ is any complex manifold and $S \subset M$ is any subset of complex codimension 2 or more. (Informally, we will also say that holomorphic maps to $X$ extend across subsets of complex codimension 2 or more.) Hartogs extension targets are Thullen extension targets. Similarly, we will say that $X$ is a Thullen extension target for local biholomorphisms to mean that local biholomorphisms to $X$ extend across subsets of complex codimension 2 or more, etc.

**Example 3.6.** $\mathbb{C}$ is a Hartogs extension target.

**Example 3.7.** If $X$ is any complex manifold of dimension at least two, and $x \in X$, then $X\backslash x$ is neither a Hartogs extension target nor a Thullen extension target for local biholomorphisms: $\mathrm{id} : X \setminus x \to X \setminus x$ doesn't extend to a map $X \to X \setminus x$.

**Example 3.8.** The map $z \in \mathbb{C}^{n+1}\backslash 0 \to \mathbb{C}z \in \mathbb{P}^n$ doesn't extend over the puncture at 0.

**Example 3.9.** Let $\mathrm{Bl}_m M$ be the blowup of a complex manifold $M$ at a point $m$. Map $M\backslash m \to \mathrm{Bl}_m M$ by the obvious local biholomorphism. This local biholomorphism clearly doesn't extend across the puncture.

**Example 3.10.** Every closed complex subspace of a Hartogs/Thullen extension target is a Hartogs/Thullen extension target.

**Example 3.11.** If $X$ is the blowup of a complex manifold along an analytic subvariety, then $X$ is *not* a Hartogs or Thullen extension target (Hartogs and Thullen extension targets are *minimal*).

**Example 3.12 (Ivashkovich [28, Proposition 3, p. 195]).** A product is a Hartogs/Thullen extension target just when the factors are.

**Example 3.13.** Affine analytic varieties and Stein manifolds are Hartogs and Thullen extension targets. (Recall that every Stein manifold admits a closed holomorphic embedding into $\mathbb{C}^N$ for some integer $N > 0$.)

**Example 3.14.** Pseudoconvex domains in Hartogs/Thullen extension targets are themselves Hartogs/Thullen extension targets.

**Example 3.15.** Let $S$ be the Hopf surface: $\left(\mathbb{C}^2 \backslash 0\right)/(z \sim 2z)$. The Hopf surface is a compact complex surface. Take the map $f : \mathbb{C}^2\backslash 0 \to S$ taking each point $z$ to its equivalence class $[z] \in S$. This map is a local biholomorphism onto $S$, but doesn't extend to $\mathbb{C}^2$ because distinct points of $S$ have preimages arbitrarily close to 0. Therefore the Hopf surface is not an extension target in either sense. See Wehler [61] for an introduction to Hopf surfaces.

**Lemma 3.16 (Ivashkovich [29, Lemma 6, p. 229]).** *If $X$ and $Y$ are complex manifolds and $X \to Y$ is an unramified covering map, then $X$ is a Hartogs/Thullen extension target just when $Y$ is.*

**Proof.** Consider the Thullen extension problem of extending a map $M \setminus S \to X$. Local extensions will clearly glue together to give a global extension. So we can replace our manifold $M$ with a simply connected open subset of $M$. Take $S \subset M$ any subset of complex codimension 2 or more. Then $M\backslash S$ is also simply connected, so we can replace $X$ by any covering space of $X$. The Hartogs extension problem is more subtle; see Ivashkovich [29]. ∎

**Lemma 3.17.** *If $M$ is a domain in Stein manifold, and $\hat{M}$ is the envelope of holomorphy of $M$, then $\pi_1(M) \to \pi_1\left(\hat{M}\right)$ is surjective.*

**Proof.** Let $\Gamma$ be the image of the morphism $\pi_1(M) \to \pi_1\left(\hat{M}\right)$ given by the inclusion $M \subset \hat{M}$. Let $\tilde{M}$ be the universal covering space of $\hat{M}$, and let $M' = \tilde{M}/\Gamma$, with projection mapping $\pi : M' \to \hat{M}$. Clearly there is a section $M \to M'$ over $M$. By Lemma 3.16, since $M'$ is a covering space of $\hat{M}$, the holomorphic map $M \to M'$ extends uniquely to a holomorphic map $\hat{M} \to M'$. By analytic continuation, this map must be a section of $M' \to \hat{M}$, so $M' = \hat{M}$. ∎

**Lemma 3.18.** *If a local biholomorphism extends across a subset of complex codimension 2 or more to a holomorphic map, then it extends uniquely to a local biholomorphism.*

**Proof.** Express the extended map in some local coordinates as a holomorphic function $w(z)$, $z$, $w$ points of some open subsets of $\mathbb{C}^n$. Then $\det w'(z) \neq 0$ away from $z \in S$. But then $\det w'(z)$ and $1/\det w'(z)$ extend holomorphically across $S$ by Hartogs extension lemma. So $\det w'(z) \neq 0$ throughout the domain of the local coordinates. ∎

**Lemma 3.19.** *If $X$ is a Hartogs/Thullen extension target (for local biholomorphisms), and $F_\alpha : X \to \mathbb{C}$ are some holomorphic functions, then $X \backslash \cup_\alpha (F_\alpha = 0)$ is a Hartogs/Thullen extension target (for local biholomorphisms).*

**Proof.** Suppose that $f : M\backslash S \to X\backslash \cup_\alpha (F_\alpha = 0)$. Then $f$ extends to a map $f : M \to X$. Away from $S$, $1/(F_\alpha f)$ is a holomorphic function, so extends to $M$, and therefore $F_\alpha \neq 0$ on $M$. ∎

**Example 3.20.** If $f : M\backslash S \to G$ and $G \subset \mathrm{GL}(n, \mathbb{C})$ is a closed Lie subgroup, then $f$ and $f^{-1}$ extend holomorphically to matrix-valued functions, so $f$ extends to a holomorphic map to $\mathrm{GL}(n, \mathbb{C})$ and therefore $f$ extends to a map to $G$. So closed complex subgroups of $\mathrm{GL}(n, \mathbb{C})$ are Thullen extension targets. Similarly, they are Hartogs extension targets. So are their covering spaces, by Lemma 3.16, so any Lie group which admits a faithful complex Lie algebra representation is a Hartogs and Thullen extension target.

**Lemma 3.21 (Adachi, Suzuki, Yoshida [1]).** *Complex Lie groups are Hartogs and Thullen extension targets.*

We will first give a short proof that they are Thullen extension targets.

**Proof.** Suppose that $M$ is a complex manifold, $S \subset M$ is a subset of complex codimension 2 or more and $f : M\backslash S \to G$ is a holomorphic map to a complex Lie group. We can assume that $M$ and $G$ are connected. Local extensions will glue together to produce a global extension, so we can assume that $M$ is simply connected, and therefore that $M\backslash S$ is too. The 1-form $f^* g^{-1}\, dg$ extends to a 1-form on $M$: just write it out in local coordinates near $S$ and use Hartogs extension lemma. Now apply the fundamental theorem of calculus for maps to Lie groups: Sharpe [54, p. 124]. ■

To carry solve Hartogs extension problems, we will need to extend meromorphic functions as well.

**Theorem 3.22 (Kajiwara and Sakai [33, p. 75]).** *All meromorphic functions on a domain $M$ in a Stein manifold extend meromorphically to the envelope of holomorphy of $M$.*

**Lemma 3.23.** *Any holomorphic vector field on a domain in a Stein manifold extends holomorphically to the envelope of holomorphy.*

**Proof.** Suppose that $M$ is a domain in a Stein manifold, and that $\hat{M}$ is the envelope of holomorphy of $M$; in particular $\hat{M}$ is Stein, and all holomorphic functions on $M$ extend to $\hat{M}$ (see Hörmander [24, p. 116]).

To extend a vector field $v$ from $M$ to $\hat{M}$, take a holomorphic function $h : \hat{M} \to \mathbb{C}$, and extend the holomorphic function $\mathfrak{L}_v h$ to $\hat{M}$. The Leibnitz identity extends. For each point of a Stein manifold $\hat{M}$, there is a holomorphic map $\hat{M} \to \mathbb{C}^n$ which is a local biholomorphism near that point (by definition; see Hörmander [24, p. 116]). Therefore a vector field is uniquely determined by its action on holomorphic functions. ■

**Lemma 3.24.** *Suppose that $M$ is a domain in a Stein manifold. Let $\hat{M}$ be the envelope of holomorphy of $M$. Then $M$ intersects every complex hypersurface in $\hat{M}$.*

**Proof.** Let $H \subset \hat{M}$ be a complex hypersurface. Henri Cartan's Theorem A (Hörmander [24, Theorem 7.2.8, p. 190]) says that for every coherent sheaf $F$ on any Stein manifold every fiber $F_m$ is generated by global sections. Let $\mathcal{O}(H)$ be the line bundle corresponding to the hypersurface $H$. By Cartan's theorem, for each point $m \in H$, there must be a global section of $\mathcal{O}(H)$ not vanishing at $m$. Take such a section, and let $H'$ be its zero locus.

Henri Cartan's theorem B (Hörmander [24, Theorem 7.4.3, p. 199]) says that the cohomology of any coherent sheaf on any Stein manifold vanishes in positive degrees. By the exponential sheaf sequence, (see Hörmander [24, p. 201]) $H^1\left(\hat{M}, \mathcal{O}^\times\right) = H^2\left(\hat{M}, \mathbb{Z}\right)$. In other words, line bundles on $\hat{M}$ are precisely determined by their coholomogy classes. Therefore a divisor is the divisor of a meromorphic function just when it has trivial first Chern class. In particular, $H' - H$ is the divisor of a meromorphic function, say $f : \hat{M} \to \mathbb{C}$. So $f$ is holomorphic on $\hat{M} \setminus H$, and has poles precisely on $H$. If $H$ does not intersect $M$, then $f$ is holomorphic on $M$, so extends holomorphically to $\hat{M}$, i.e. has no poles, so $H$ is empty. ■

**Corollary 3.25.** *Suppose that $M$ is a domain in a Stein manifold, with envelope of holomorphy $\hat{M}$ and that $H \subset \hat{M}$ is a complex hypersurface. Every function meromorphic on $M$ and holomorphic on $M \setminus H$ extends uniquely to a function meromorphic on $\hat{M}$ and holomorphic on $\hat{M} \setminus H$.*

**Proof.** Suppose that $f$ is meromorphic on $M$ and holomorphic on $M \setminus H$, so the poles of $f$ lie on $H$. By the theorem of Kajiwara and Sakai (Theorem 3.22 on the preceding page), every meromorphic function on a domain $M$ in a Stein manifold extends to a meromorphic function on $\hat{M}$. Suppose that $f$ extends to have poles on some hypersurface $H \cup Z$. We can assume that $Z$ contains no component of $H$, so that $H \cap Z$ is a subset of complex codimension 2 or more. But then every component of $Z$ must be a hypersurface not intersecting $M$, so must be empty. ■

**Lemma 3.26 (Hörmander [24] p. 116).** *Suppose that $M$ is a domain in an $n$-dimensional Stein manifold, with envelope of holomorphy $\hat{M}$. For each point $m \in \hat{M}$, there is a map $f : \hat{M} \to \mathbb{C}^n$ which is a local biholomorphism near $m$.*

**Proposition 3.27.** *Let $M$ be a domain in Stein manifold, and $\hat{M}$ be the envelope of holomorphy of $M$. Every holomorphic tensor extends from $M$ to $\hat{M}$.*

**Remark 3.28.** This proposition generalizes a theorem of Aeppli [2].

**Proof.** For simplicity, suppose that we wish to extend a holomorphic 1-form $\omega$ on $M$ to $\hat{M}$. Pick a point $m_0 \in \hat{M}$. Pick a map $f : \hat{M} \to \mathbb{C}^n$ which is a local biholomorphism near $m_0$, say on some open set $\hat{M} \setminus H_f$ containing $m_0$. Write the 1-form $\omega$ on $M$ as $\omega = g\,df$. Then $g$ is a meromorphic function on $M$, and holomorphic on $M \setminus H_f$. By Corollary 3.25 on the facing page, $g$ extends uniquely to a function meromorphic on $\hat{M}$ and holomorphic on $\hat{M} \setminus H_f$. So we can extend $\omega$ to a neighborhood of $m_0$ by $\omega = g\,df$. Clearly the extension is unique where defined. Since $m_0$ is an arbitrary point of $\hat{M}$, $\omega$ extends holomorphically to all of $\hat{M}$. ■

We now finish the proof of Lemma 3.21 on the preceding page: complex Lie groups are Hartogs and Thullen extension targets.

**Proof.** Suppose that $M$ is a domain in a Stein manifold, with envelope of holomorphy $\hat{M}$, and $f : M \to G$ is a holomorphic map to a complex Lie group. We can assume that $M$ and $G$ are connected. The 1-form $f^* g^{-1}\,dg$ extends to a 1-form on $\hat{M}$. Apply the fundamental theorem of calculus for maps to Lie groups: Sharpe [54, p. 124] to see that some covering space $\pi : \tilde{M}$ of $\hat{M}$ bears a holomorphic map $\tilde{f} : \tilde{M} \to G$ satisfying $\tilde{f}^* g^{-1}\,dg = \omega$ (where $\omega$ here denotes the pullback of $\omega$ on $\hat{M}$ to the covering space $\tilde{M}$). Note that we don't require $\tilde{M}$ to be the universal covering space. Moreover, there is a morphism of groups $h : \pi_1\left(\hat{M}\right) \to G$ so that $\tilde{f}\left(\gamma m\right) = h(\gamma)\,\tilde{f}(m)$ for any $\gamma \in \pi_1\left(\hat{M}\right)$ and $m \in \tilde{M}$. This map $\tilde{f}$ is uniquely determined up to the action of $G$. Pick a point $\tilde{m}_0 \in \tilde{M}$ so that the point $m_0 = \pi\left(\tilde{m}_0\right)$ lies in $M$. Arrange by $G$-action that $\tilde{f}\left(\tilde{m}_0\right) = f\left(m_0\right)$. Then clearly $\tilde{f}\left(\tilde{m}\right) = f\left(m\right)$ whenever $m = \pi\left(\tilde{m}\right) \in M$. In particular, $h = 1$ on $\pi_1(M)$. By Lemma 3.17, $h = 1$ on $\pi_1\left(\tilde{M}\right)$. ■

The three following results of Matsushima and Morimoto provide a larger class of examples of Stein manifolds.

**Lemma 3.29 (Matsushima and Morimoto [38, Theorem 2, p. 146]).** *A complex Lie group $G$ is a Stein manifold just when the identity component of the center of $G$ is a linear algebraic group, i.e. contains no complex tori.*

**Lemma 3.30 (Matsushima and Morimoto [38, Theorem 4, p. 147]).** *If a holomorphic principal bundle has Stein base, and Stein fibers, then it has Stein total space.*

**Lemma 3.31 (Matsushima and Morimoto [38, Theorem 6, p. 153]).** *If a holomorphic principal bundle $G$-bundle $E \to M$, has Stein base $M$, and if $H \subset G$ is a closed connected complex Lie subgroup, and $G/H$ is Stein, then $E/H \to M$ has Stein total space.*

**Lemma 3.32.** *Complex reductive homogeneous spaces (i.e. $G/H$ with $H \subset G$ a closed reductive algebraic subgroup of a complex linear algebraic group $G$) are affine analytic varieties.*

Therefore reductive homogeneous spaces are Hartogs and Thullen extension targets.

**Proof.** The categorical quotient $G//H$ (spectrum of $H$-invariant polynomials) of affine algebraic varieties by reductive algebraic groups are affine algebraic varieties; see Procesi [49, Theorem 2, p. 556]. The quotient will in general only parameterize the closed orbits, and admit a holomorphic submersion $G/H \to G//H$. But the $H$-orbits on $G$ are the translates of $H$, so all orbits are closed, and thus $G/H = G//H$. ■

**Corollary 3.33.** *Suppose that $G$ is a complex linear algebraic group and $H \subset G$ is a closed complex reductive algebraic Lie subgroup. Suppose that $M$ is a domain in a Stein manifold, and that $\hat{M}$ is the envelope of holomorphy of $M$. If $E \to \hat{M}$ is a holomorphic principal $G$-bundle, then every holomorphic section of $E/H|_M \to M$ extends uniquely to a holomorphic section of $E/H \to \hat{M}$.*

**Proof.** Let $H^0 \subset H$ be the connected component of the identity element. Then $E/H^0$ is a Stein manifold, by Lemma 3.31 on the preceding page, so a Hartogs extension target. By Lemma 3.16 on page 7, $E/H$ is a Hartogs extension target. Therefore any section $s$ of $E/H|_M \to M$ extends to a unique map $s : \hat{M} \to E/H$. Consider the bundle map $\pi : E \to \hat{M}$. Over $M$, $s$ satisfies $\pi s = \mathrm{id}$. By analytic continuation, this holds over $\hat{M}$ as well. ■

**Lemma 3.34.** *Suppose that $X$ is a Thullen extension target for local biholomorphisms. The complement of any hypersurface in $X$ is also a Thullen extension target for local biholomorphisms.*

**Proof.** If $f : M \backslash S \to X \backslash H$ is a local biholomorphism, then it extends to a local biholomorphism $f : M \to X$. The hypersurface $f^{-1}(H)$ is either empty or else cannot be contained in $S$, and so intersects $M \backslash S$, so $f$ doesn't take $M \backslash S$ to $X \backslash H$. ■

**Lemma 3.35.** *Suppose that $X$ is a Hartogs extension target for local biholomorphisms. The complement of any complex hypersurface in $X$ is also a Hartogs extension target for local biholomorphisms.*

**Proof.** Suppose that $M$ is a domain in a Stein manifold, with envelope of holomorphy $\hat{M}$. If $f : M \to X \backslash H$ is a local biholomorphism, then it extends to a local biholomorphism $f : \hat{M} \to X$. The complex hypersurface $f^{-1}(H)$ cannot intersect $M$, so is empty by Lemma 3.24 on page 8. ■

**Proposition 3.36.** *Suppose that $M$ is a domain in a Stein manifold with envelope of holomorphy $\hat{M}$. If $V \to \hat{M}$ is a holomorphic vector bundle, then every section of $V$ over $M$ extends to a section of $V$ over $\hat{M}$.*

**Proof.** Pick a section $f$ of $V$ over $M$, and a point $\hat{m} \in \hat{M}$. By Cartan's theorem A, there are global sections $f_1, f_2, \dots, f_p$ of $V$ over $\hat{M}$ whose values at $\hat{m}$ are a basis of $V_{\hat{m}}$. These sections are linearly independent away from some hypersurface $H \subset \hat{M}$. On $\hat{M} \setminus H$, $V$ is trivial. So on $M \setminus H$, we can write $f = \sum_i c_i f_i$ for some holomorphic functions $c_i$ on $M \setminus H$.

Check that the functions $c_i$ are meromorphic on $M$ as follows. In any local trivialization $g_1, g_2, \dots, g_p$ for $V$, write $f_i = \sum_j A_{ij} g_j$ and $f = \sum_j b_j g_j$. Let $A = (A_{ij})$. The entries of $A^{-1}$ are rational functions of the entries of $A$, so we can write $f = \sum_{ij} A^{-1}_{ij} b_i f_i$.

Each $c_i$ extends to a holomorphic function on $\hat{M} \setminus H$, and therefore we can extend $f$ to $f = \sum_i c_i f_i$. This extends $f$ to a neighborhood of the arbitrary point $\hat{m}$. ■

**Corollary 3.37.** *Suppose that $M$ is a domain in a Stein manifold with envelope of holomorphy $\hat{M}$. If $E \to \hat{M}$ is a holomorphic $G$-bundle, then every holomorphic connection on $E|_M$ extends uniquely to a holomorphic connection on $E$.*

**Proof.** Suppose that $\pi : E \to \hat{M}$ is a principal $G$-bundle, where $G$ is a complex Lie group with Lie algebra $\mathfrak{g}$. Let $E_0 = E|_M$. A connection on $E_0 \to M$ is a section of the vector bundle $(T^*E_0 \otimes \mathfrak{g})^G \to M$, i.e. the vector bundle whose local sections are precisely the $G$-invariant local sections of $T^*E_0 \otimes \mathfrak{g} \to E_0$. This vector bundle extends to the holomorphic vector bundle $(T^*E \otimes \mathfrak{g})^G \to \hat{M}$. By Proposition 3.36, every section of $(T^*E_0 \otimes \mathfrak{g})^G \to M$ extends to a holomorphic section of $(T^*E \otimes \mathfrak{g})^G \to \hat{M}$. So we can extend $\omega$ to a $\mathfrak{g}$-valued 1-form on $E$.

For each vector $A \in \mathfrak{g}$, denote by $\vec{A}$ the associated vertical vector field on $E$ giving the infinitesimal $\mathfrak{g}$-action. A connection on $E \to \hat{M}$ is precisely a $G$-equivariant section $\omega$ of $T^*E_0 \otimes \mathfrak{g} \to M$, for which $\vec{A} \lrcorner \omega = A$, for $A \in \mathfrak{g}$. Clearly this identity must extend from $E_0$ to $E$ by analytic continuation. ■

**Lemma 3.38.** *Any holomorphic principal bundle on any Stein manifold admits a holomorphic connection.*

**Proof.** Let $G$ denote a complex Lie group. The obstruction to a holomorphic connection on a holomorphic principal $G$-bundle $\pi : E \to M$ is the Atiyah class

$$a\,(M, E) \in H^1\,(M, T^*M \otimes \mathrm{Ad}(E))\,;$$

see Atiyah [4]. Henri Cartan's theorem B says that all positive degree cohomology groups of coherent sheaves on Stein manifolds vanish; see Hörmander [24, Theorem 7.4.3, p. 199]. Therefore the Atiyah class vanishes, and so there is a holomorphic connection. ■

**Corollary 3.39.** *If $M$ is a domain in a Stein manifold, and $E \to M$ is a holomorphic principal bundle with nonzero Atiyah class, then $E$ does not extend to a holomorphic principal bundle on the envelope of holomorphy of $M$.*

**Conjecture 3.40.** *A holomorphic principal bundle $E \to M$ over a domain $M$ in a Stein manifold extends uniquely to a holomorphic principal bundle over the envelope of holomorphy $\hat{M}$ just when the Atiyah class of $E$ vanishes.*

## 3.2 Meromorphic extension theorems

**Definition 3.41.** If $X$ and $Y$ are complex manifolds, and $X$ is connected, a *meromorphic map* (in the sense of Remmert [50, Definition 15, p. 367]) $f : X \to Y$ is a choice of nonempty compact set $f(x) \subset Y$ for each $x \in X$, so that

1. $f(x)$ is a single point for $x$ in some dense open subset of $X$,
2. the pairs of $(x, y)$ with $y \in f(x)$ form an irreducible analytic variety $\Gamma \subset X \times Y$, called the *graph* of $f$.

Clearly the composition $fg$ of a holomorphic map $f$ with a meromorphic map $g$, given by mapping points $(x, y)$ of the graph to $(x, f(y))$, is also a meromorphic map. It is easy to prove (see Remmert [50], Siu [57], Ivashkovich [30]) that there is a subset of complex codimension 2 or more in $X$ (called the *indeterminacy locus*) over which $f$ is the graph of a holomorphic map.

The Thullen extension problem:

**Theorem 3.42 (Siu [57]).** *Take a complex manifold $M$ and a subset of complex codimension 2 or more $S \subset M$. Every holomorphic map $f : M \backslash S \to X$ to a compact Kähler manifold $X$ extends to a meromorphic map $f : M \to X$.*

The Hartogs extension problem:

**Theorem 3.43 (Ivashkovich [30]).** *Every holomorphic map from a domain in a Stein manifold to a compact Kähler manifold extends to a meromorphic map from the envelope of holomorphy.*

**Lemma 3.44.** *Any meromorphic map $M \to G/H$ to a reductive homogeneous space $G/H$ from a domain $M$ in Stein manifold extends uniquely to a meromorphic map $\hat{M} \to G/H$ to its envelope of holomorphy.*

**Proof.** By Lemma 3.32 on page 10, $G/H$ is an affine algebraic variety. Apply the theorem of Kajiwara and Sakai (Theorem 3.22 on page 8 above) for meromorphic functions. ■

### 3.3 Extension theorems for homogeneous spaces

The Thullen extension problem:

**Lemma 3.45 (Ivashkovich [31]).** *Suppose that $X$ is a complex manifold with locally transitive biholomorphism group. A local biholomorphism to $X$ extends across a subset of complex codimension 2 or more to a local biholomorphism just when it extends across that subset to a meromorphic map.*

**Proof.** Take a local biholomorphism $f : M \backslash S \to X$ with $M$ a complex manifold and $S \subset M$ a subset of complex codimension 2 or more. Suppose that $f$ extends meromorphically to $f : M \to X$ with graph $\Gamma \subset M \times X$. We only need to extend holomorphically across $S$ locally, so we can assume that $M$ is Stein and connected and that $X$ is connected. Therefore $\Gamma$ is connected.

For each holomorphic vector field $v_X$ on $X$, define a holomorphic vector field $v_M$, on $M \backslash S$, by setting $v_M(z) = f'(z)^{-1} v_X(f(z))$ for $z \in M \backslash S$. Applying Hartogs extension lemma to the coefficients of $v_M$ in local coordinates, we can extend $v_M$ uniquely to a holomorphic vector field on $M$.

For each holomorphic vector field $v_X$ on $X$, we also define a holomorphic vector field $v_{M \times X}$ on $M \times X$ by taking $v_{M \times X} = (v_X, v_M)$. Let $\mathfrak{g}$ be the Lie algebra of all holomorphic vector fields on $X$. This Lie algebra might be infinite dimensional, and acts transitively on $X$. The action of $\mathfrak{g}$ on $M \times X$ maps to the action of $\mathfrak{g}$ on $X$, and so the orbits in $M \times X$ must be at least as large in dimension as $X$. Let $\Gamma_0$ be the part of the graph of $f$ which lies above $M \backslash S$. Then $H$ is invariant under this Lie algebra action. Moreover $\Gamma_0$ is dense in $\Gamma$. Therefore $\Gamma$ is invariant under the Lie algebra action. Each orbit inside $\Gamma$ has dimension at least that of $X$, while $\Gamma$ has dimension equal to that of $X$. The singular locus of $\Gamma$ is invariant, so is a union of orbits, each of dimension equal to that of $\Gamma$. So $\Gamma$ has empty singular locus, and is a smooth complex manifold of dimension equal to the dimension of $X$. Since $\Gamma$ is connected, and all orbits on $\Gamma$ are open sets, $\Gamma$ is a single orbit.

The projection map $M \times X$ restricted to $\Gamma$ is a holomorphic surjective map $\Gamma \to M$, injective on a Zariski open set. If we can show that $\Gamma \to M$ is a biholomorphism, then we can invert it to a holomorphic map $M \to \Gamma$, and then map $\Gamma \to X$ by the other projection, and the composition $M \to \Gamma \to X$ holomorphically extends $f$. To prove that $\Gamma \to M$ is a biholomorphism, we only have to show that it is a local biholomorphism, since $\Gamma$ is closed in $M \times X$, so the number of sheets of $\Gamma \to M$ won't change at different points of $M$. Since $\Gamma \to M$ is $\mathfrak{g}$-equivariant, the set of points at which $\Gamma \to M$ fails to be a local biholomorphism must be $\mathfrak{g}$-invariant, so empty or all of $\Gamma$. However, $\Gamma \to M$ is a local biholomorphism on a dense open set, so must be a biholomorphism everywhere. ■

The Hartogs extension problem:

**Lemma 3.46 (Ivashkovich [31]).** *A local biholomorphism from a domain in a Stein manifold to a complex homogeneous space extends to a local biholomorphism from the envelope of holomorphy just when it extends to a meromorphic map from the envelope of holomorphy.*

**Proof.** The proof of Lemma 3.45 on the preceding page also works here. ■

**Theorem 3.47 (Ivashkovich [31]).** *Compact Kähler homogeneous spaces of dimension at least two are Hartogs/Thullen extension targets for local biholomorphisms.*

**Proof.** Suppose that $X = G/H$ is a compact Kähler homogeneous space. Wang [60] proves that such spaces are complex homogeneous spaces, i.e. we can take $G$ to be a complex Lie group and $H$ a closed complex Lie subgroup of $G$. We can even assume that $G$ is the biholomorphism group of $X$. Now apply Theorems 3.42 and 3.43 and Lemma 3.45. ■

**Theorem 3.48.** *Suppose that $X$ is a complex manifold, and that the Lie algebra of all holomorphic vector fields on $X$ acts locally transitively. Furthermore suppose that $X$ is an unbranched covering space of a compact Kähler manifold. Then $X$ is a Hartogs/Thullen extension target for local biholomorphisms.*

**Proof.** Again apply Theorems 3.42 and 3.43 and Lemma 3.45. ■

**Table 1.** The complex homogeneous surfaces.

| | symmetry group | stabilizer subgroup |
|---|---|---|
| $\mathbb{P}^2$ | $\mathbb{P}\,\mathrm{SL}(3,\mathbb{C})$ | $\begin{bmatrix} a^0_0 & a^0_1 & a^0_2 \\ 0 & a^1_1 & a^1_2 \\ 0 & a^2_1 & a^2_2 \end{bmatrix}$ |
| $\mathbb{C}^2$ | affine | $\mathrm{GL}(2,\mathbb{C})$ |
| $\mathbb{C}^2$ | special affine | $\mathrm{SL}(2,\mathbb{C})$ |
| $\mathbb{C}^2$ | translation | 0 |
| $\mathbb{C}^2$ | etc. | etc. |
| $\mathbb{C}^2\backslash 0$ | $\mathrm{GL}(2,\mathbb{C})$ | $\begin{pmatrix} 1 & b \\ 0 & c \end{pmatrix}$ |
| $\mathbb{C}^2\backslash 0$ | $\mathrm{SL}(2,\mathbb{C})$ | $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ |
| $\mathbb{P}^1 \times \mathbb{C}$ | $\mathbb{P}\,\mathrm{SL}(2,\mathbb{C}) \times (\mathbb{C}^\times \rtimes \mathbb{C})$ | $\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \times \mathbb{C}^\times$ |
| $\mathbb{P}^1 \times \mathbb{C}$ | $\mathbb{P}\,\mathrm{SL}(2,\mathbb{C}) \times \mathbb{C}$ | $\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}$ |
| $\mathbb{P}^1 \times \mathbb{P}^1\backslash$ diagonal | $\mathbb{P}\,\mathrm{SL}(2,\mathbb{C})$ | $\begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix}$ |
| $\mathbb{P}^1 \times \mathbb{P}^1$ | $\mathbb{P}\,\mathrm{SL}(2,\mathbb{C}) \times \mathbb{P}\,\mathrm{SL}(2,\mathbb{C})$ | $\begin{bmatrix} a & b \\ 0 & 1/a \end{bmatrix} \times \begin{bmatrix} c & d \\ 0 & 1/c \end{bmatrix}$ |
| $\mathcal{O}(n)\,, n \ge 1$ | $(\mathrm{GL}(2,\mathbb{C})/\mathbb{Z}_n) \rtimes \mathrm{Sym}^n\left(\mathbb{C}^2\right)^*$ | $\left\{ \left( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, p \right) \,\middle|\, p(1,0) = 0 \right\}$ |
| $\mathcal{O}(n)\,, n \ge 1$ | $(\mathrm{SL}(2,\mathbb{C})/(\pm^n)) \rtimes \mathrm{Sym}^n\left(\mathbb{C}^2\right)^*$ | $\left\{ \left( \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix}, p \right) \,\middle|\, p(1,0) = 0 \right\}$ |

Clearly if $X$ is any Hartogs/Thullen target for local biholomorphisms, we can replace $X$ by any complex manifold with a common covering space, cut out any hypersurface, replace with a pseudoconvex open set, etc. repeatedly and still have a Hartogs/Thullen extension target for local biholomorphisms, so we have a large collection of complex manifolds to work with. In this paper, we are only interested in homogeneous Hartogs/Thullen extension targets.

**Example 3.49.** Let us determine the extension properties of all complex homogeneous surfaces. Note that in calling a surface *complex homogeneous* we mean homogeneous under a holomorphic action of a complex Lie group. Up to covering, the complex homogeneous surfaces are presented in Table 1 on the preceding page. Huckleberry [25], Mostow [45] and Olver [48, p. 472] provide an introduction to this classification; Mostow provides a proof. The surface $\mathbb{P}^1 \times \mathbb{P}^1 \setminus$ diagonal is the set of pairs of distinct lines in the plane, acted on by linear maps of the plane. Consider the usual line bundle $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n} \to \mathbb{P}^1$, whose fibers are choices of line in $\mathbb{C}^2$ and homogeneous polynomial of degree $n$ on that line. In other words, $\mathcal{O}(n)$ is the line bundle on $\mathbb{P}^1$ with first Chern class $n$. We will also denote the total space of this line bundle by the symbol $\mathcal{O}(n)$. This total space is a surface acted on by the group of linear substitutions of variables. It is also acted on by the homogeneous polynomials of degree $n$, by adding the homogeneous polynomial to the polynomial on any given line. The group $\mathbb{Z}_n$ is the group of scalings of variables by roots of unity. The group $\pm^n$ is $\pm 1$ if $n$ is even, and 1 if $n$ is odd.

The symmetry groups listed are all of the connected complex Lie groups that act transitively on each given surface, except for the surface $\mathbb{C}^2$. On $\mathbb{C}^2$, there are finite dimensional Lie groups of all positive dimensions greater than one, acting transitively. They are classified, but the classification is a little complicated and irrelevant here; see Olver [48, Cases 1.5–1.9, p. 472]. There are also some disconnected Lie groups acting on these same surfaces and containing the connected ones listed; see Huckleberry [25].

| Homogeneous surface | Hartogs target | Thullen target | local biholomorphism Thullen target |
|---|---|---|---|
| $\mathbb{P}^2$ | × | × | ✓ |
| $\mathbb{C}^2$ | ✓ | ✓ | ✓ |
| $\mathbb{C}^2\setminus 0$ | × | × | × |
| $\mathbb{P}^1 \times \mathbb{C}$ | × | × | ✓ |
| $\mathbb{P}^1 \times \mathbb{P}^1 \setminus$ diagonal | ✓ | ✓ | ✓ |
| $\mathbb{P}^1 \times \mathbb{P}^1$ | × | × | ✓ |
| $\mathcal{O}(n), n \geq 1$ | × | × | × |

Map $\mathbb{C}^2\setminus 0 \to \mathcal{O}(n)$ by taking a point $z \neq 0$ to the homogeneous polynomial $p_z$ of degree $n$ on the span of $z$ which takes the value 1 on $z$. This map is a local biholomorphism and doesn't extend across the puncture.

To see which complex homogeneous surfaces are Hartogs or Thullen extension targets, we start by looking at covering spaces, and which complex surfaces contain rational curves. If a complex surface contains a rational curve, then it isn't a Hartogs or Thullen extension target, as we saw in Example 3.10 on page 6. Keep in mind that we can apply Theorem 3.47 on the preceding page to any compact Kähler complex manifold with the same universal covering space as a given space; for example, $\mathbb{P}^1 \times \mathbb{C}$ is the universal covering space of $\mathbb{P}^1 \times E$ for any elliptic curve $E$, to which we can apply Theorem 3.47. Finally, notice that $\mathbb{P}^1 \times \mathbb{P}^1\setminus$diagonal is an affine variety, being a reductive homogeneous space:

$$\mathbb{P}^1 \times \mathbb{P}^1\setminus\text{diagonal} = \mathbb{P}\,\mathrm{SL}(2,\mathbb{C}) \,/ \left\{ \begin{bmatrix} a & 0 \\ 0 & \frac{1}{a} \end{bmatrix} \right\}.$$

**Example 3.50.** Now that we have some understanding of local biholomorphisms to complex homogeneous spaces, consider immersions to those spaces. Take two elliptic curves $C_0 = \mathbb{C}/\Lambda_0$

and $C_1 = \mathbb{C}/\Lambda_1$. The manifold $M = \mathbb{P}^1 \times C_1 \times C_2$ is a complex homogeneous projective variety. The immersion $\mathbb{C}^2 \backslash 0 \to M$, $(z_1, z_2) \mapsto \left(\frac{z_1}{z_2}, z_1 + \Lambda_1, z_2 + \Lambda_2\right)$, does not extend across the puncture. We can embed $M$ into projective space to see that immersions to projective space don't always extend.

### 3.4 Extensions of integral maps of invariant differential relations

**Definition 3.51.** Suppose that $G/H$ is a complex homogeneous space. An *invariant relation* of first order is a complex submanifold $R \subset (\mathfrak{g}/\mathfrak{h}) \otimes \mathbb{C}^{n*}$, for some integer $n$, invariant under the obvious action of $H \times \mathrm{GL}\,(n, \mathbb{C})$. Given any holomorphic map $f : M \to G/H$ from any $n$-dimensional complex manifold $M$, define $f^{(1)} : FTM \times_M f^*G \to (\mathfrak{g}/\mathfrak{h}) \otimes \mathbb{C}^{n*}$ by $f^{(1)}(u, g) = \left(L_g^{-1}\right)'(f(m))f'(m)u^{-1}$. An *integral map* of $R$ is a holomorphic map $f : M \to G/H$ of an $n$-dimensional complex manifold $M$ for which $f^{(1)}$ is valued in $R$. A relation $R$ will be called a *Thullen extension relation* if every integral map of the relation $f : M \backslash S \to G/H$, with $M$ a complex manifold and $S \subset M$ a subset of complex codimension 2 or more, extends holomorphically across $S$. We will use the term *Hartogs extension relation* analogously.

**Example 3.52.** The set of linear isomorphisms $\mathbb{C}^{n*} \to \mathfrak{g}/\mathfrak{h}$ is a Hartogs and a Thullen extension relation.

We will see that extension relations are very closely related to extension problems for various geometric structures, giving rise to numerous examples. Clearly if $R_0 \subset R_1$ are invariant relations and $R_1$ is a Thullen/Hartogs extension relation, then $R_0$ is too. So naturally we want to focus on finding maximal Hartogs/Thullen extension relations.

## 4 Extensions of bundles and geometric structures

### 4.1 Extending holomorphic bundles

The results of this subsection are well known (see Okonek et al. [47, Chapter II.1.1], Siu [57]), but we provide elementary proofs for completeness.

**Definition 4.1.** Suppose that $M$ is a complex manifold, $S \subset M$ a closed subset, $G$ a complex Lie group, and $E \to M$ a holomorphic principal $G$-bundle. We will say that $E$ extends across $S$ to mean that there is a holomorphic principal $G$-bundle $E' \to M$ and a $G$-equivariant biholomorphism $E \to E'|_{M \backslash S}$.

**Lemma 4.2.** *The extension of a holomorphic principal bundle across a subset of complex codimension* 2 *or more is unique.*

**Proof.** Suppose that $E' \to M$ and $E'' \to M$ are two holomorphic principal $G$-bundles, that $S \subset M$ is a subset of complex codimension 2 or more, and that $F : E'|_{M \backslash S} \to E''|_{M \backslash S}$ is a holomorphic principal $G$-bundle isomorphism. If $F$ extends holomorphically to some open set, then clearly it does so uniquely. We can therefore replace $M$ by an open subset on which $E'$ and $E''$ are both holomorphically trivial. So $F : (M \backslash S) \times G \to (M \backslash S) \times G$ is expressed as $F(z, g) = (z, f(z)g)$, for some map $f : M \backslash S \to G$. By Lemma 3.21 on page 8, complex Lie groups are Thullen extension targets, so $f$ extends to a map $f : M \to G$, and this extends $F$ to an isomorphism $F(z, g) = (z, f(z)g)$. ■

Clearly the same proof as above shows that the Hartogs extension problem for holomorphic principal bundles has at most one solution.

**Definition 4.3.** Suppose that $M$ is a complex manifold, $S \subset M$ is a closed subset, and that $E \to M\backslash S$ is a holomorphic fiber bundle. Say that $E$ is *holomorphically trivial along* $S$ if for each $s \in S$, there is an open set $U \subset M$ containing $s$ on which $E|_{U\backslash S}$ is holomorphically trivial.

**Proposition 4.4.** *Any holomorphic fiber bundle extends across a closed subset if and only if it is holomorphically trivial along that subset.*

**Proof.** If such an extension $E'$ exists, then take $U$ any open set containing $s$ on which $E'$ is trivial.

Conversely, suppose that for every point $s \in S$, we have an open set $U$ containing $s$ on which $E|_{U\backslash S}$ is trivial. Cover $M\backslash S$ by all possible open sets on which $E$ is trivial and pick a trivialization on each. Let $F$ be a typical fiber of $E$. To each pair $V, W$ of open sets from this cover, associate a holomorphic transition map $\phi^V_W : (V \cap W) \times F \to F$. Then for this particular subset $U$, define $\phi^U_V = \phi^{U\backslash S}_V$, and $\phi^V_U = \phi^V_{U\backslash S}$. Clearly $\phi^U_V \phi^V_U = \mathrm{id}$ and $\phi^U_V \phi^V_W \phi^W_U = \mathrm{id}$, so these transition maps determine a unique holomorphic bundle $E' \to M$ with obvious canonical isomorphism $E'|_{M\backslash S} = E$. ■

**Corollary 4.5.** *Any holomorphic principal bundle extends across a closed nowhere dense subset if and only if it has a holomorphic section near each point of that subset.*

**Lemma 4.6.** *Covering spaces extend uniquely across subsets of complex codimension* 2 *or more.*

**Proof.** Suppose that $M$ is a complex manifold and $S \subset M$ is a subset of complex codimension 2 or more. Pick a covering space $Z \to M\backslash S$. Pick any point $s \in S$ and any simply connected open set $U \subset M$ containing $s$. Then $U\backslash S$ is also simply connected, so $Z|_{U\backslash S} \to U\backslash S$ is holomorphically trivial. Apply Lemma 4.2 on the preceding page and Proposition 4.4. ■

**Lemma 4.7.** *Suppose that $\tilde{G} \to G$ is a Lie group morphism with discrete kernel $K$. Any holomorphic principal $\tilde{G}$-bundle $\tilde{E}$ has quotient $E = \tilde{E}/K$. Moreover, $\tilde{E}$ extends across subsets of complex codimension* 2 *or more just when $E$ does.*

**Proof.** Suppose that $M$ is a complex manifold, that $S \subset M$ is a subset of complex codimension 2 or more, and that $\tilde{E} \to M\backslash S$ is a holomorphic principal $\tilde{G}$-bundle. Clearly if $\tilde{E}$ extends over $M$, then $E = \tilde{E}/K$ does too. Suppose that $E$ extends over $M$, so has a local section $\sigma$ defined near some point $s \in S$, say on an open set $U$. The preimage of $\sigma$ in $\tilde{E}$ is a covering space of $U\backslash S$; apply Lemma 4.7. ■

**Corollary 4.8.** *Take any two holomorphic line bundles $L_1$ and $L_2$ with a common tensor power, say $L_1^{\otimes n_1} = L_2^{\otimes n_2}$, for some integers $n_1$ and $n_2$. Then $L_1$ extends across a subset of complex codimension* 2 *or more just when $L_2$ does.*

**Proof.** The associated principal bundles are both covering spaces of the associated principal bundle of $L_1^{\otimes n_1} = L_2^{\otimes n_2}$, so we can apply Lemma 4.7 to each. ■

**Example 4.9.** On $\mathbb{C}^2\backslash 0$, all real rank 2 bundles are smoothly trivial. There is an infinite dimensional space of holomorphic line bundles on $\mathbb{C}^2\backslash 0$ which are not holomorphically trivial; see Serre [52, p. 372] for proof (without explicit examples). So smooth category obstructions are not enough to decide whether holomorphic bundles extend across punctures.

## 4.2 Relative extension problems for holomorphic bundles

**Definition 4.10.** Suppose that $G$ and $G'$ are complex Lie groups and that $\rho : G \to G'$ is a morphism of complex Lie groups. Suppose that $E \to M$ is a holomorphic principal right $G$-bundle. Let $G$ act on $E \times G'$ by having $g \in G$ act on $(e, g')$ to give

$$r_g\left(e, g'\right) = \left(r_g e, g' \rho\left(g\right)\right).$$

We denote the quotient as $E' = E \times_G G'$. Let $G'$ act on $E \times G'$ by

$$R_{h'}\left(e, g'\right) = \left(e, \left(h'\right)^{-1} g'\right)$$

for $e \in E$ and $g', h' \in G'$. This action commutes with the $G$-action, so descends to a right $G'$-action on $E' = E \times_G G'$. Moreover $E' \to M$ is a principal right holomorphic $G'$-bundle.

Consider the map $e \in E \mapsto (e, 1) \in E \times G'$. Compose this map with the obvious quotient map $E \times G' \to E \times_G G' = E'$ to make a $G$-equivariant bundle map $p : E \to E'$. We define $q : E \times G' \to G'$ by $q\left(e, g'\right) = g'$. The map $q$ is $G$-equivariant, so descends to a map $q : E' \to G'/\rho\left(G\right)$. Moreover, the composition $qp$ is the constant map to the $\rho\left(G\right)$ coset. Conversely, the $\rho\left(G\right)$-subbundle $E/\ker\rho = p\left(E\right) \subset E'$ is precisely $E/\ker\rho = q^{-1}\rho\left(G\right)$.

**Lemma 4.11.** *Take a morphism $\rho : G \to G'$ of complex Lie groups. Suppose that $E$ is a principal $G$-bundle. Let $E' = E \times_G G'$ as above. If $E$ extends across a subset then $E'$ extends as $E' = E \times_G G'$. If the kernel of $\rho$ is discrete and the image of $\rho$ is a closed subgroup of $G'$, and if holomorphic maps to $G'/\rho(G)$ extend across subsets of complex codimension 2 or more, then $E$ extends across a subset of complex codimension 2 or more just when $E'$ does.*

**Proof.** Suppose that $M$ is a complex manifold and $S \subset M$ is a subset of complex codimension 2 or more. Suppose that $E' \to M \backslash S$ extends holomorphically to a principal $G'$-bundle $E' \to M$, and that the image of $\rho$ is a closed subgroup of $G'$, and that $G'/\rho\left(G\right)$ is a Thullen extension target. We can replace $M$ with a small neighborhood of a point $s \in S$, in which $E' \to M$ is trivial: $E' = M \times G'$.

Denote a typical point of $E'$ as $\left(z, g'\right)$ with $z \in M$ and $g' \in G'$. Write $q$ as $q\left(z, g'\right) \in G'/\rho\left(G\right)$. Because $G'/\rho\left(G\right)$ is a Thullen extension target, we can extend the map $q$ to $E'$ by extending it to be holomorphic for $z \in M$ for each fixed $g'$. Then $q^{-1}\rho\left(G\right)$ is a holomorphic $G/\ker\rho$-subbundle of $E'$ extending $E/\ker\rho$. By Lemma 4.7 on the facing page, $E$ extends across $S$. ■

## 4.3 Extending bundles via a connection

**Example 4.12.** Affine connections are given in local coordinates by Christoffel symbols, which are holomorphic functions. The Christoffel symbols extend holomorphically across subsets of complex codimension 2 or more by Hartogs lemma. Therefore affine connections holomorphically extend across subsets of complex codimension 2 or more. We have already generalized this result to holomorphic connections on bundles in Lemma 3.37 on page 11.

**Example 4.13.** If $G \subset \mathrm{GL}\left(n, \mathbb{C}\right)$ is a closed complex Lie subgroup, and $E \to M \backslash S$ is a $G$-structure equipped with a torsion-free connection, then the connection extends to a torsion-free connection of $FTM = E \times_G \mathrm{GL}\left(n, \mathbb{C}\right)$ with holonomy in $G$. The connection extends over $S$ by the last example. The holonomy of a loop $\gamma$ passing through $S$ is the limit of the holonomy of any sequence of loops $\gamma_j$ converging to $\gamma$ uniformly. We can choose each $\gamma_j$ to avoid $S$, so the holonomy of $\gamma_j$ lies in $G$. Therefore the parallel transport on $FTM$ preserves a foliation by $G$-subbundles, one leaf $E'$ of which contains and therefore extends $E$ from $E \to M \backslash S$ to $E' \to M$.

**Proposition 4.14 (Buchdahl and Harris [10]).** *A holomorphic principal bundle or vector bundle on a complex manifold extends across a subset of complex codimension 2 or more just when it admits a holomorphic connection near each point of that subset.*

**Remark 4.15.** This solves the Thullen extension problem for holomorphic bundles with holomorphic connections; the Hartogs extension problem is unsolved.

**Proof.** Suppose that $E \to M \backslash S$ is a holomorphic principal bundle, with $M$ a complex manifold and $S \subset M$ a subset of complex codimension 2 or more. We can assume that $M$ is a ball $B \subset \mathbb{C}^n$, and that the holomorphic connection is defined on all of $E$. Pick any point $z_0 \in B \backslash S$. Each complex line through $z_0$ intersects $B$ in a disk. Each complex line also intersects $B \backslash S$ in a disk, except for those lines which intersect points of $S$, which yield disks with finitely many punctures. Parallel transport between any two points of a disk is well defined, because the connection is holomorphic. There could be monodromy around a punctured disk, but since the punctured disk is a limit of unpunctured disks (coming from the nearby complex lines through $z_0$), the monodromy is trivial. Therefore if we pick an initial point of the fiber $E_{z_0}$, we can parallel transport it around all of the disks through $z_0$, obtaining a global holomorphic section of $E$ over $B \backslash S$. By Corollary 4.5 on page 16, the bundle is holomorphically trivial. Extend the connection across $S$ by writing it out in local coordinates and applying Hartog's extension lemma to the Christoffel symbols.

For a vector bundle, consider the associated principal bundle. ■

**Definition 4.16.** Suppose that $M$ is a complex manifold, $S \subset M$ is an analytic subset, $G$ is a complex Lie group, and $E \to M \backslash S$ is a holomorphic principal $G$-bundle. Each open set $U \subset M$ containing $S$ has an Atiyah class

$$a\left(E|_{U \backslash S}\right) \in H^1\left(U \backslash S, T^*\left(U \backslash s\right) \otimes \left(E|_{U \backslash S} \times_G \mathfrak{g}\right)\right),$$

and these Atiyah classes pullback under inclusions of open sets. For each point $s \in S$, define the *Atiyah class* of $E$ at $s$, written $a(E, s)$, to be the inverse limit of Atiyah classes $a\left( E|_{U \backslash S}\right)$ over all open sets $U \subset M$ containing $s$.

$$a\left(E, s\right) \in \varprojlim_{s \in U} H^1\left(U \backslash S, T^*\left(U \backslash s\right) \otimes \left(E|_{U \backslash S} \times_G \mathfrak{g}\right)\right).$$

If $S$ is subset of complex codimension 2 or more then $a(E, s) = 0$ for all $s \in S$ just when $E$ extends across $S$.

## 4.4 Extending geometric structures by extending bundles

**Definition 4.17.** We will say that $G/H$-geometries *extend across subsets of complex codimension* 2 *or more* to mean that if $M$ is a complex manifold, $S \subset M$ a subset of complex codimension 2 or more, and $E \to M \backslash S$ is a holomorphic $G/H$-geometry then $E$ extends to a unique holomorphic $G/H$-geometry on $M$. Equivalently, we will refer to solving the Thullen extension problem for $G/H$-geometries.

Similar terminology will be used for first order structures, etc.

**Theorem 4.18.** *The underlying holomorphic principal bundle of a holomorphic Cartan geometry extends across a subset of complex codimension 2 or more to a holomorphic principal bundle just when the Cartan geometry extends across the subset.*

**Proof.** The result is local, so we can assume that $M$ is a ball $B \subset \mathbb{C}^n$, and that $E \to B \backslash S$ is a Cartan geometry, extending as a principal bundle to some $E' \to B$, which we can assume is holomorphically trivial, $E' = B \times H$. Denote points of $E'$ as $(z,h)$, $z \in B$ and $h \in H$. Take linear coordinates $z^1, z^2, \dots, z^n$ on $\mathbb{C}^n$. From part (3) of the definition of a Cartan geometry (Definition 2.6 on page 3), $\omega$ restricts to $h^{-1}\,dh$ on each fiber $\{z\} \times H$. Therefore $\omega - h^{-1}\,dh$ is a multiple of $dz^1, dz^2, \dots, dz^n$, say $\omega = h^{-1}\,dh + \operatorname{Ad}(h)^{-1}\left(\Gamma_j(z,h)\,dz^j\right)$, with $\Gamma_j(z,h)$ holomorphic for $z$ away from $S$ and valued in $\mathfrak{g}$. From part (1) of the definition of a Cartan geometry, $\Gamma_j(z,h)$ is independent of $h$, say $\Gamma_j(z)$, holomorphic for $z \neq 0$. Therefore $\Gamma_j(z)$ admits a unique extension to a holomophic function on the ball $B$, extending $\omega$ to a holomorphic 1-form on $B \times H = E'$. The properties (1) and (3) of Cartan connections then follow immediately. In terms of a basis $\{e_i\}$ of $\mathfrak{g}$, we can write $\omega = \omega^i e_i$. The tangent bundle of $B \times H$ is holomorphically trivial, so $B \times H$ admits a global holomorphic volume form. Pick any holomorphic volume form $\Omega$ on $B \times H$. The holomorphic function

$$f = \frac{\Omega}{\omega^1 \wedge \omega^2 \wedge \dots \wedge \omega^{\dim G}} : B \backslash S \to \mathbb{C}$$

extends across $S$ by Hartogs lemma, so $\omega$ satisfies property (2) of a Cartan connection. ■

**Theorem 4.19.** *Consider a holomorphic Cartan geometry defined on a domain in Stein manifold. The underlying holomorphic principal bundle of the Cartan geometry extends to a holomorphic principal bundle on the envelope of holomorphy just when the Cartan geometry extends.*

**Proof.** Suppose that $\pi : E \to M$ is a holomorphic Cartan geometry modelled on $G/H$ with Cartan connection $\omega$. Suppose that $M$ is a domain in a Stein manifold with envelope of holomorphy $\hat{M}$. Suppose that $E$ extends to a holomorphic principal $H$-bundle $E' \to \hat{M}$. The Cartan connection $\omega$ is a holomorphic section of the vector bundle $T^*E \otimes \mathfrak{g}$ on $E$. Being $H$-equivariant, $\omega$ is also a section of the vector bundle $(T^*E \otimes \mathfrak{g})^H \to M$, whose local sections are $H$-equivariant local sections of $T^*E \otimes \mathfrak{g}$. This latter vector bundle extends to the holomorphic vector bundle $(T^*E' \otimes \mathfrak{g})^H \to \hat{M}$. By Proposition 3.36 on page 10, the section $\omega$ extends to a section of this vector bundle, which we will also denote as $\omega$. This section is thus an $H$-invariant 1-form valued in $\mathfrak{g}$, the first property of a Cartan connection (see Definition 2.6 on page 3).

Let $Z$ be the set of points $e \in E'$ at which $\omega_e : T_e E' \to \mathfrak{g}$ is *not* a linear isomorphism. Clearly $Z$ is a hypersurface, given by the one equation $\det \omega_e = 0$. Moreover, this hypersurface is $H$-invariant, so projects to a hypersurface in $\hat{M}$. This hypersurface doesn't intersect $M$, so is empty by Lemma 3.24 on page 8. Therefore $\omega$ satisfies the second property of a Cartan connection.

Over $M$, $\omega$ satisfies $\vec{A} \lrcorner \omega = A$, for any $A \in \mathfrak{h}$. By analytic continuation, this must also hold over $\hat{M}$, the third and final property of a Cartan connection. ■

**Theorem 4.20.** *The underlying holomorphic principal bundle of a holomorphic first order structure extends across a subset of complex codimension 2 or more to a holomorphic principal bundle just when the first order structure extends across the subset.*

**Proof.** The problem is local, so we can assume that $M$ is a ball $B \subset \mathbb{C}^n$, and that $E' \to M$ is holomorphically trivial, $E' = B \times G$. Denote points of $E'$ as $(z,g)$, $z \in B$ and $g \in G$. Take linear coordinates $z^1, z^2, \dots, z^n$ on $\mathbb{C}^n$. We have a map $\phi : E \to FT\left(M \backslash S\right), \phi\left(z,g\right) = \left(z, u\left(z,g\right)\right)$, defined for $z \in B \backslash S$, with $u(z,g) \in \operatorname{GL}\left(n,\mathbb{C}\right)$. Moreover, $u(z,g) = g^{-1} u(z,1)$. So we can consider the map $u(z) = u(z,1)$ as a map $u : B \backslash S \to \operatorname{GL}\left(n,\mathbb{C}\right)$. Clearly $u$ extends to a matrix-valued function on $B$. Moreover, so does $u^{-1}$, so clearly $u$ extends to a map $u : B \to \operatorname{GL}\left(n,\mathbb{C}\right)$, and we extend $\phi$ to $E'$ by $\phi(z,g) = \left(z, g^{-1} u(z)\right)$. ■

**Conjecture 4.21.** *The underlying holomorphic principal bundle of a holomorphic first order structure with trivial kernel extends from a domain $M$ in a Stein manifold to a holomorphic principal bundle on the envelope of holomorphy of $M$ just when the first order structure extends as well.*

# 5 Extending first order structures

## 5.1 Inextensible examples

**Example 5.1.** Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be the stabilizer of a nonzero vector $v_0 \in \mathbb{C}^n$. A $G$-structure on a manifold $M$ is precisely a nowhere vanishing vector field on $M$. Suppose that $\phi : E \to FTM$ is a $G$-structure on $M$, with underlying principal right $G$-bundle $\pi : E \to M$. Take any point $e \in E$. Since $G \subset \mathrm{GL}(n, \mathbb{C})$ is a subgroup, and $\phi$ is a $G$-bundle morphism, this point $e$ is identified with a point $u = \phi(e) \in FTM$. This point $u$ is a linear isomorphism $u : T_m M \to \mathbb{C}^n$, where $m = \pi(e)$. Consider the vector $X = u^{-1}(v_0) \in T_m M$. Denote this vector $X(e)$. Clearly since $v_0 \neq 0$ and $u$ is a linear isomorphism, $X(e) \neq 0$.

Under $G$-action, $\phi(eg) = g^{-1}\phi(e)$, so $X(eg) = u^{-1}g^{-1}(v_0) = u^{-1}(v_0) = X(e)$. Therefore $X(e)$ depends only on the point $m \in M$: $X = X(m)$, i.e. $X$ is a vector field on $M$. By local triviality of $\pi : E \to M$, $X$ is a holomorphic vector field, nowhere vanishing.

Conversely, suppose that $X$ is a nowhere vanishing vector field on a manifold $M$. Let $E$ be the set of all pairs $e = (m, u)$ for which $m \in M$ and $u : T_m M \to \mathbb{C}^n$ is a linear isomorphism satisfying $u(X(m)) = v_0$. Clearly $E \subset FTM$ is a $G$-structure. So we have an isomorphism between the category of $G$-structures on $M$ and the category of nowhere vanishing vector fields on $M$.

Vector fields extend across subsets of complex codimension 2 or more, by Hartogs extension lemma applied in local coordinates to the component functions of the vector field. (They even extend to the envelope of holomorphy, if there is one.) Clearly the associated $G$-structure extends across such a subset just when the vector field extends without zeroes. As an example, take an invertible $n \times n$ matrix $A$, and let $X$ denote the vector field $X(z) = Az$ on $\mathbb{C}^n$. Clearly $X$ doesn't vanish except at $z = 0$. Moreover, there is a unique holomorphic extension of $X$ from $\mathbb{C}^n \backslash 0$ to $\mathbb{C}^n$. Let $S = \{0\}$, $M = \mathbb{C}^n$, and let $E$ be the $G$-structure on $M \backslash S$ associated to $X$. Then $E$ does not extend holomorphically as a $G$-structure to $M$. By Theorem 4.20 on the previous page, the holomorphic bundle $E \to \mathbb{C}^n \backslash 0$ does not extend to a holomorphic principal bundle over $\mathbb{C}^n$. So we cannot always solve the Thullen or Hartogs extension problems for first order structures.

**Example 5.2.** Consider the 1-form $\alpha = \sum z^i \, dz^i$ on $\mathbb{C}^n$. On $\mathbb{C}^n \backslash 0$, take the hyperplane field $\alpha = 0$. Let $G$ be the group of linear maps on $\mathbb{C}^n$ preserving a hyperplane, say $\mathbb{C}^{n-1} \subset \mathbb{C}^n$. Let $E$ be the set of pairs $(z, u)$ with $z \in M \backslash S$ and $u \in \mathrm{GL}(n, \mathbb{C})$ for which $u$ takes the hyperplane $(\alpha = 0)$ to the fixed hyperplane $\mathbb{C}^{n-1}$. Clearly $E$ is a $G$-structure on $M \backslash S$.

Let us show that this $G$-structure does not extend across the puncture at 0. If this $G$-structure extends across the puncture, then we can take a local section, say $u(z)$, defined near 0, and define a 1-form $\beta$ by $v \lrcorner \beta = dz^1(u(v))$. This 1-form $\beta$ doesn't vanish at any point, and annihilates tangent vectors precisely on the hyperplanes, as does $\alpha$, so $\beta = h\alpha$ for some nonzero function $h$ away from 0. Extend $\beta$ and $h$ and $1/h$ to 0 by the Hartogs extension lemma. So $\beta$ vanishes at 0, a contradiction. So we cannot always solve the Hartogs or Thullen extension problems for hyperplane fields.

**Example 5.3.** Generalizing the previous example, we can take any closed complex subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$, any nonconstant map $f : \mathbb{P}^1 \to \mathrm{GL}(n, \mathbb{C}) / G$ (assuming there is a nonconstant map, which is a complicated constraint on the choice of $G$). For a point $z \in \mathbb{C}^2 \backslash 0$, write $[z]$ for the complex line through $z$ and 0, mapping $\mathbb{C}^2 \backslash 0 \to \mathbb{P}^1$. We map $F : \left(\mathbb{C}^2 \backslash 0\right) \times \mathbb{C}^{n-2} \to \mathrm{GL}\left(n, \mathbb{C}\right)/G$ by $F(z, w) = f([z])$. Take the bundle $\mathrm{GL}\left(n, \mathbb{C}\right) \to \mathrm{GL}\left(n, \mathbb{C}\right)/G$ and let $E = F^* \mathrm{GL}\left(n, \mathbb{C}\right)$ be the pullback. The $G$-structure can't extend across the puncture at $z = 0$ because the function $f$ on $E$ is defined on $\mathbb{C}^2 \backslash 0$ and is constant along all lines through $z = 0$, with different constants along different lines. Intuitively, this tells us that we cannot solve the Hartogs or Thullen extension problems for $G$-structures for "large" subgroups $G \subset \mathrm{GL}\left(n, \mathbb{C}\right)$ (for example, parabolic subgroups). For such "large" subgroups, we will need to use additional hypotheses on torsion (see Section 5.5 on page 27).

**Example 5.4.** For some choices of group $G$, all $G$-structures extend across subsets of complex codimension 2 or more. For example, an $\mathrm{SO}\left(n, \mathbb{C}\right)$-structure (a.k.a. a holomorphic Riemannian metric) is given in local coordinates by a symmmetric matrix $g = (g_{ij})$ of holomorphic functions with $\det g \neq 0$. The functions will extend holomorphically over such subsets as will the function $1/\det g$, by Hartogs lemma. Therefore any holomorphic Riemannian metric extends holomorphically over subsets of complex codimension 2 or more. Exactly the same trick works for holomorphic symplectic structures. Similar remarks apply to the Hartogs extension problem for these structures.

**Example 5.5.** On $\mathbb{C}^4$, with coordinates $z^0$, $z^1$, $w^0$, $w^1$, let $\omega = dz^0 \wedge dz^1 + dw^0 \wedge dw^1$ and let $L(z, w) = (iz, -iw)$. Then the span of a pair of vectors

$$(z, w), L(z, w)$$

is a Lagrangian 2-plane for $\omega$, unless $z = 0$ or $w = 0$, in which case it is a sub-Lagrangian line. On $\mathbb{C}^4 \backslash \left((z = 0) \cup (w = 0)\right)$, this Lagrangian foliation is holomorphic, and does not extend across $z = 0$ or $w = 0$. Consider the usual map $\pi : z \in \mathbb{C}^4 \backslash 0 \to \mathbb{C}^{\times} z \in \mathbb{P}^3$. Let $\alpha = (z, w) \lrcorner\, \omega$. The hyperplane field $V_{(z,w)} = \ker \alpha$ has the fibers of $\pi$ as Cauchy characteristics, and so descends to a hyperplane field on $\mathbb{P}^3$, which is a holomorphic contact structure. The Lagrangian 2-planes project to Legendre lines, foliating $\mathbb{P}^3$ away from the two lines $[z = 0]$ and $[w = 0]$. This Legendre foliation does not extend across those two lines.

**Example 5.6.** A *plane field* on a manifold $M$ is a vector subbundle $V \subset TM$. An *Engel* 2-plane field $V \subset TM$ on a 4-fold $M$ is a 2-plane field so that near each point $m \in M$, there are local sections $X$ and $Y$ for which $X, Y, [X, Y]$ and $[[X, Y], Y]$ are linearly independent. Every Engel 2-plane field is locally isomorphic to every other; see Ehlers et al. [16], Kazarian, Montgomery and Shapiro [34], Vogel [59], or Zhitomirskiĭ [62]. In particular, all Engel 2-plane fields are locally isomorphic to the plane field

$$dy - p\, dx = 0, \qquad dp - q\, dx = 0.$$

In other words, this is the plane field spanned by the vector fields

$$\frac{\partial}{\partial q}, \qquad \frac{\partial}{\partial x} + p \frac{\partial}{\partial y} + q \frac{\partial}{\partial p}.$$

However, it is easy to check that the plane field

$$dy - pq\, dx = 0, \qquad dp - q^2\, dx = 0,$$

spanned by the vector fields

$$\frac{\partial}{\partial q}, \qquad \frac{\partial}{\partial x} + pq \frac{\partial}{\partial y} + q^2 \frac{\partial}{\partial p}$$

is also an Engel 2-plane field except on the subset $p = q = 0$. On that subset, the 2-plane field fails to be Engel, as the required brackets of any local sections lose their linear independence. Therefore Engel 2-plane fields do *not* always extend holomorphically across complex codimension 2 subsets to Engel 2-plane fields.

**Conjecture 5.7.** *Holomorphic Engel* 2*-plane fields extend holomorphically across subsets of complex codimension* 2 *or more to holomorphic* 2*-plane fields, and across subsets of complex codimension* 3 *or more to holomorphic Engel* 2*-plane fields.*

## 5.2 Contact structures

**Definition 5.8.** A *hyperplane field* on a complex manifold $M$ is a holomorphic line subbundle of the holomorphic cotangent bundle. If $L \subset T^*M$ is a hyperplane field, then locally $L$ is spanned by a nonzero 1-form, say $\alpha$. The hyperplane field is a *contact structure* if $\alpha \wedge (d\alpha)^n \neq 0$, where $M$ has dimension $2n+1$.

A contact structure can be considered a first order structure with torsion condition; see Example 5.2 on page 20.

**Theorem 5.9.** *Holomorphic contact structures extend holomorphically across subsets of complex codimension* 2 *or more.*

To prove this theorem, we will need to prove two minor results.

**Lemma 5.10.** *Suppose that $M$ is a complex manifold, that $S \subset M$ is a subset of complex codimension* 2 *or more, and that $M \backslash S$ bears a contact structure. If, for each point $s \in S$, the contact structure is spanned by a nonzero* 1*-form $\alpha$ defined in an open set of the form $U \backslash S$, where $U$ is an open subset of $M$ containing $s$, then the contact structure extends holomorphically to a unique contact structure on $M$.*

**Proof.** Imagine a holomorphic contact structure on $M^{2n+1} \backslash S$ with a choice of holomorphic contact form in a neighborhood of $s$, say $\alpha$, so that $\alpha \wedge d\alpha^n \neq 0$. Applying Hartogs extension to the coefficients of $\alpha$ in local coordinates near $s$, the contact form $\alpha$ extends uniquely as a holomorphic 1-form across $S \cap U$. Take $\Omega$ a holomorphic volume form defined near $s$, and let $f = \Omega / \alpha \wedge d\alpha^n$. By Hartog's extension lemma, $f$ extends to a holomorphic function across $S \cap U$, so $\alpha$ extends to a contact form. ■

**Corollary 5.11.** *Suppose that $M$ is a complex manifold, that $S \subset M$ is a subset of complex codimension* 2 *or more, and that $M \backslash S$ bears a contact structure $L \subset T^*(M \backslash S)$. Then the line bundle $L$ extends across $S$ as a holomorphic line bundle if and only if the contact structure extends across $S$ as a holomorphic contact structure.*

**Proof.** If the line bundle $L$ extends as a holomorphic line bundle across the puncture, then $L$ is locally trivial, so we can choose a local section, $\alpha$, and apply Lemma 5.10. ■

**Example 5.12.** On the other hand, as we saw in Example 5.2 on page 20, the hyperplane field $z_i \, dz_i = 0$ does not extend across 0 as a hyperplane field. Its associated line bundle is trivial over $\mathbb{C}^n \backslash 0$, having global section $z_i \, dz_i$. Therefore the line bundle extends holomorphically across 0. This hyperplane field is not a contact plane field. Clearly the contact condition simplifies the extension problem. More generally, we should expect nonvanishing torsion of a first order structure to be helpful in extension problems.

Now we return to proving Theorem 5.9.

**Proof.** Suppose that $S \subset M$ is a subset of complex codimension 2 or more in a complex manifold, and that $M \backslash S$ bears a contact structure $L \subset T^*(M \backslash S)$. Consider the inclusion $\iota : L \to T^*(M \backslash S)$ as a linear map. Take the transpose $\alpha = \iota^t : T(M \backslash S) \to L^{\otimes -1}$, which is a 1-form valued in $L^{\otimes -1}$. The expression $\alpha \wedge (d\alpha)^n$ is a section of the bundle $K_{M \backslash S} \otimes L^{\otimes(-n-1)}$, nowhere vanishing, so an isomorphism $L^{\otimes(n+1)} \to K_{M \backslash S}$. Clearly $K_{M \backslash S}$ extends to $K_M$. Therefore $L^{\otimes(n+1)}$ extends across $S$. Apply Corollary 4.8 on page 16 to conclude that $L$ extends holomorphically across $S$. Apply Corollary 5.11 on the preceding page to conclude that the contact structure extends. ∎

**Example 5.13.** Consider the real analytic contact structure

$$\cos z \, dx - \sin z \, dy = 0$$

on $\mathbb{R}^3$, and then compactify $\mathbb{R}^3$ to the 3-sphere. Near the point at infinity, the contact planes wind infinitely often around certain great circles, so the contact structure doesn't extend. Perhaps there is no complex analytic analogue of overtwisted contact structures.

**Proposition 5.14.** *For every holomorphic subbundle $V \subset TM$ of the tangent bundle of a domain $M$ in a Stein manifold, there is a subset of complex codimension 2 or more $S \subset \hat{M}$ of the envelope of holomorphy $\hat{M}$ of $M$, so that $V$ extends uniquely to a holomorphic subbundle of $T\left(\hat{M} \setminus S\right)$.*

**Remark 5.15.** The author would like to thank Sergei Ivashkovich for providing this proof.

**Proof.** Suppose that $V \subset TM$ is a holomorphic subbundle, say of rank $k$, and $M$ is a domain in a Stein manifold, and $\hat{M}$ is the envelope of holomorphy of $M$. There is a holomorphic embedding $\hat{M} \to \mathbb{C}^N$ for some integer $N$; see Hörmander [24, Theorem 5.3.9, p. 135].

Let $f : M \to \operatorname{Gr}(k, N), m \mapsto V_m$. By Theorem 3.43 on page 12, $f$ extends to a meromorphic map $f : \hat{M} \to \operatorname{Gr}(k, N)$. By analytic continuation, the equation $f(m) \subset T_m M$ for $m \in M$ ensures that $f(m) \subset T_m \hat{M}$ for $m \in \hat{M}$. The indeterminacy locus of the map $f$ is of complex codimension 2 or more. ∎

An alternative proof making use of more elementary results, on meromorphic functions rather than maps, could proceed by taking affine coordinates on the Grassmannian and then applying the Kajiwara–Sakai theorem (Theorem 3.22 on page 8).

**Theorem 5.16.** *Holomorphic contact structures extend uniquely from any domain $M$ in Stein manifold to the ennvelope of holomorphy of $M$.*

**Remark 5.17.** The author would like to thank Sergei Ivashkovich for providing this proof.

**Proof.** By Proposition 5.14, the problem reduces to extension in complex codimension 2, so the result follows from Theorem 5.9 on the facing page. ∎

**Example 5.18 (Robert Bryant).** Let $X$ be a complex a 2-torus. Naturally $\mathbb{P}T^*X$ has the obvious contact structure. But $\mathbb{P}T^*X$ also has various holomorphic 2-plane fields. Note that $\mathbb{P}T^*X = X \times \mathbb{P}^1$. In linear coordinates $z, w$ on $X$, and affine coordinate $p$ on $\mathbb{P}^1$, pick a rational function $f(p)$ and consider the 2-plane field $dw - f(p)\, dz = 0$. Note that near points where $f(p) = \infty$, we can write this 2-plane field as

$$\frac{dw}{f(p)} - dz = 0,$$

so the 2-plane field is holomorphic at all points of $X \times \mathbb{P}^1$. The 2-plane field is invariant under translations of $X$.

The critical points of $f$ are precisely the points where the 2-plane fields fails to be a contact structure. These points form a union of hypersurfaces, each one a torus. These tori are homogeneous under the translation action of $X$. So we have many examples of holomorphic 2-plane fields on a homogeneous smooth projective variety, which are contact structures except on some disjoint subvarieties, each of which is homogeneous. The pseudogroup of local isomorphisms of any of these 2-plane fields will active transitively on the dense open set where the 2-plane field is a contact structure, and also on the various smooth hypersurfaces on which the 2-plane field fails to be a contact structure.

### 5.3 Reducing to a homogeneous space extension problem

**Definition 5.19.** Take a complex Lie group $G$ and complex representation $\rho : G \to \mathrm{GL}(V_0)$ with closed image. Suppose that $\phi : E \to FTM$ is a holomorphic $G$-structure, with $FTM$ the $V_0$-valued frame bundle. We define $\check{\phi} : FTM \times_M E \to \mathrm{GL}(V_0)$ by $\check{\phi}(u, e) = u\phi(e)^{-1}$. Under $G$-action, $\check{\phi}(u, r_g e) = \check{\phi}(u, e)g$. Therefore we can quotient by right $G$-action, to produce a map $\check{\phi} : FTM \to \mathrm{GL}(V_0)/\rho(G)$, with $\check{\phi}(u) = u\phi(e)^{-1}\rho(G)$ for some element $u \in FT_mM$ and $e \in E_m$. If we change the choice of $e \in E_m$, say to $eg$, we change $\phi(e)^{-1}$ to $\phi(eg)^{-1} = \phi(e)^{-1}\rho(g)$, not affecting $\check{\phi}(u)$. So $\phi : E \to FTM$ determines $\check{\phi} : FTM \to \mathrm{GL}(V_0)/\rho(G)$, equivariant under right $\mathrm{GL}(V_0)$-action. Moreover, the composition $\check{\phi}\phi$ is the constant map to the $\rho(G)$ coset. The $\rho(G)$-structure $\phi(E) \subset FTM$ (i.e. the underlying embedded first order structure) is precisely $\check{\phi}^{-1}\rho(G)$.

**Theorem 5.20.** *Suppose that $G \subset \mathrm{GL}(n, \mathbb{C})$ is a reductive algebraic group. Suppose that $M$ is a domain in a Stein manifold. Then every holomorphic $G$-structure on $M$ extends uniquely to a holomorphic $G$-structure on the envelope of holomorphy of $M$.*

**Proof.** A $G$-structure on $M$ is equivalent to a section of $FTM/G \subset FT\hat{M}/G$. The total space of $FT\hat{M}/G$ is a Hartogs extension target by Corollary 3.33 on page 10. Therefore if $s : M \to FTM/G$ is a $G$-structure, then $s$ extends uniquely to a holomorphic map $s : \hat{M} \to FT\hat{M}/G$. Let $\pi : FT\hat{M}/G \to \hat{M}$ denote the bundle map. Then $\pi s$ is the identity on $M$, and therefore by analytic continuation is the identity on $\hat{M}$. So the extension is also a holomorphic $G$-structure ■

**Theorem 5.21.** *Suppose that $\rho : G \to \mathrm{GL}(n, \mathbb{C})$ is a representation with closed image and discrete kernel. Then all holomorphic $G$-structures extend across subsets of complex codimension 2 or more if and only if all holomorphic maps to $\mathrm{GL}(n, \mathbb{C})/\rho(G)$ from $n$-folds extend across such subsets.*

**Proof.** We can quotient by the kernel, by Lemma 4.11 on page 17, so assume that $G \subset \mathrm{GL}(n, \mathbb{C})$ is a closed subgroup. Suppose that all holomorphic maps from $n$-folds to $\mathrm{GL}(n, \mathbb{C})/G$ extend across subsets of complex codimension 2 or more. We need only prove the result locally. The local result is clear from Theorem 5.20. Take a $G$-structure $\phi : E \to FT(M \backslash S)$ on $M \backslash S$, with $M$ an $n$-fold, and $S$ a subset of complex codimension 2 or more. In local coordinates $z^1, \dots, z^n$ on $M$ near a point $s \in S$, points of $FTM$ look like $(z, u)$ with $z \in \mathbb{C}^n$ and $u \in \mathrm{GL}(n, \mathbb{C})$. The map $\check{\phi}$ is $\check{\phi}(z, u) = uf(z)$ for some map $f : M \backslash S \to \mathrm{GL}(n, \mathbb{C})/G$. Extend $f$ to a holomorphic map $f : M \to \mathrm{GL}(n, \mathbb{C})/G$. Now let $E'$ be the set of points of the form $(z, u) \in FTM$ so that $uf(z) = G \in \mathrm{GL}(n, \mathbb{C})/G$. This bundle $E' \to M$ extends $E$. By Theorem 4.20 on page 19, the $G$-structure extends holomorphically.

Next suppose that $\mathrm{GL}(n, \mathbb{C})/G$ is not a Thullen extension target for $n$-folds. Pick an $n$-dimensional complex manifold $M$ and a holomorphic map $f : M \backslash S \to \mathrm{GL}(n, \mathbb{C})/G$ which does not extend to $M$. There must be some point $s \in S$ near which $f$ doesn't extend holomorphically. We can replace $M$ by any neighborhood of $s$, so we can assume that $M$ is a ball $B$ in $\mathbb{C}^n$. Think of $\mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})/G$ as a holomorphic principal right $G$-bundle. Let $E \to B \backslash S$ be the pullback bundle $E = f^* \mathrm{GL}(n, \mathbb{C})$. By definition, $E$ is a principal right $G$-subbundle of $(B \backslash S) \times \mathrm{GL}(n, \mathbb{C}) = FT(B \backslash S)$, hence a $G$-structure on $B \backslash S$. Suppose that this $G$-structure $E$ extends holomorphically to a $G$-structure $E'$ on $B$. If need be, we replace $B$ by a smaller ball around $s$ on which the bundle $E'$ is trivial, $E' = B \times G$. Then we can take the section $B \times \{1\}$ of $E'$, and map it to $\mathrm{GL}(n, \mathbb{C})$ and then quotient by $G$ to extend $f$ to a holomorphic map $f : B \to \mathrm{GL}(n, \mathbb{C})/G$. ■

**Example 5.22 (Hwang and Mok [26]).** By Lemma 3.32 on page 10, if $G$ is a reductive algebraic group, then $\mathrm{GL}(n, \mathbb{C})/G$ is an affine variety, so holomorphic $G$-structures extend across subsets of complex codimension 2 or more and extend from domains in Stein manifolds to their envelopes of holomorphy.

**Example 5.23.** Again consider holomorphic Riemannian metrics: $G = \mathrm{SO}(n, \mathbb{C})$. Then

$$\mathrm{GL}(n, \mathbb{C}) / \mathrm{SO}(n, \mathbb{C}) = X \backslash (f = 0),$$

with $X$ the set of complex quadratic forms, and $f$ the determinant. Therefore $\mathrm{SO}(n, \mathbb{C})$-structures (holomorphic Riemannian metrics) extend across subsets of complex codimension 2 or more, and extend from domains in Stein manifolds to their envelopes of holomorphy.

**Example 5.24.** Consider almost symplectic structures, $G = \mathrm{Sp}(2n, \mathbb{C})$. Because $G$ is reductive, almost symplectic structures extend across subsets of complex codimension 2 or more. Alternately, pick a complex volume form $\Omega \in \Lambda^{2n} \left(\mathbb{C}^{2n}\right)^*$, and define $f : \Lambda^2 \left(\mathbb{C}^{2n}\right)^* \to \mathbb{C}$ by $f(\alpha) = \alpha^n / \Omega$. Then $\mathrm{GL}(2n, \mathbb{C}) / \mathrm{Sp}(2n, \mathbb{C}) \backslash (f = 0) \subset \Lambda^2 \left(\mathbb{C}^{2n}\right)^*$ is the set of symplectic forms. Therefore $\mathrm{Sp}(2n, \mathbb{C})$-structures (holomorphic almost symplectic structures) extend across subsets of complex codimension 2 or more, and extend from domains in Stein manifolds to their envelopes of holomorphy.

**Example 5.25.** Given a 3-form $\sigma$ on $\mathbb{C}^7$, define $B_\sigma : \mathbb{C}^7 \otimes \mathbb{C}^7 \to \Lambda^7 \left(\mathbb{C}^7\right)^*$ by $B_\sigma(u, v) = \frac{1}{6} (u \lrcorner \sigma) \wedge (v \lrcorner \sigma) \wedge \sigma$. Pick some nonzero $\Omega \in \Lambda^7 \left(\mathbb{C}^7\right)^*$. Let $f(\sigma) = \det (B_\sigma / \Omega)$. Say that $\sigma$ is *nondegenerate* if $f(\sigma) \neq 0$, i.e. $B_\sigma / \Omega$ is a nondegenerate quadratic form. For instance, if we write $dz^{ij}$ to mean $dz^i \wedge dz^j$, etc., then the 3-form

$$\sigma_0 = dz^{123} + dz^{145} + dz^{167} + dz^{246} - dz^{257} - dz^{347} - dz^{356}$$

is nondegenerate. The degenerate forms clearly form an affine analytic hypersurface ($f \neq 0$) inside the space of 3-forms.

It turns out (see Bryant [7]) that the nondegenerate 3-forms are precisely the orbit of $\sigma_0$ under $\mathrm{GL}(7, \mathbb{C})$-action in $\Lambda^3 \left(\mathbb{C}^7\right)^*$. Moreover the stabilizer of $\sigma_0$ is the exceptional simple Lie group $G_2$. Therefore $\mathrm{GL}(7, \mathbb{C}) / G_2 = \Lambda^3 \left(\mathbb{C}^7\right)^* \backslash (f = 0)$ is a Thullen and Hartogs extension target. So holomorphic $G_2$-structures (even with torsion) extend across subsets of complex codimension 2 or more and extend from domains in Stein manifolds to their envelopes of holomorphy.

**Example 5.26.** A *web* on a surface $M$ is a choice of three nowhere tangent foliations by curves. We will see that webs extend across punctures. At each point $m \in M$, there is a linear isomorphism $T_m M \to \mathbb{C}^2$ taking the tangent lines of the curves to the horizontal axis, vertical axis and diagonal. This linear isomorphism is unique up to rescaling. If we let $E$ be the set of all such linear isomorphisms at all points of $M$, then $E \to M$ is a $\mathbb{C}^\times$-structure. A web is therefore a $\mathbb{C}^\times$-structure. Clearly $\mathrm{GL}(2, \mathbb{C}) / \mathbb{C}^\times = \mathbb{P}\,\mathrm{SL}(2, \mathbb{C})$ is covered by $\mathrm{SL}(2, \mathbb{C})$, an affine variety so a Thullen extension target. Therefore holomorphic webs extend across punctures, and extend from domains in Stein manifolds to their envelopes of holomorphy. This is surprising because foliations by curves in a surface need not extend across punctures.

It turns out that a web is equivalent to a $\mathbb{P}^1 \times \mathbb{P}^1 \backslash$diagonal-geometry; see Example 8.3 on page 37. Webs can also be described as a certain type of $\mathbb{C}^2$-geometry, i.e. as modelled on $G/H = \mathbb{C}^2$ for certain groups $H$ and $G$, which we leave to the reader to work out.

**Example 5.27.** There is a cubic form on $\mathbb{C}^{27}$ whose stabilizer is $E_6$. Since $E_6$ is a reductive algebraic group, holomorphic $E_6$-structures (i.e. symmetric cubic forms in tangent spaces, with stabilizer isomorphic to $E_6$) extend across subsets of complex codimension 2 or more and extend from domains in Stein manifolds to their envelopes of holomorphy.

**Example 5.28.** The space of solutions of certain scalar complex analytic ordinary differential equations bears a canonical $\mathrm{GL}(2,\mathbb{C})$-structure, for a particular embedding $\mathrm{GL}(2,\mathbb{C}) \subset \mathrm{GL}(5,\mathbb{C})$; see Doubrov [14], Dunajski and Tod [15] or Godliński and Nurowski [20]. Because $\mathrm{GL}(2,\mathbb{C})$ is a reductive algebraic group, these structures extend over subsets of complex codimension 2 or more and extend from domains in Stein manifolds to their envelopes of holomorphy.

**Example 5.29.** An *almost product structure* is a $G$-structure where

$$G = \mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(n-k,\mathbb{C}) \subset \mathrm{GL}(n,\mathbb{C}) .$$

Equivalently, an almost product structure is a pair of complementary transverse plane fields. Clearly $G$ is a reductive algebraic group, so almost product structures extend across subsets of complex codimension 2 or more and extend from domains in Stein manifolds to their envelopes of holomorphy.

**Example 5.30 (Ivashkovich [28, Proposition 3, p. 196]).** If $X$ and $Y$ are Thullen extension targets for local biholomorphisms, then so is $X \times Y$.

In fact, a much stronger result is true, which does not apparently follow from our theorems above.

**Theorem 5.31 (Ivashkovich [28, Proposition 4, p. 196]).** *If the base and fiber of a holomorphic fibration are Hartogs extension targets, then so is the total space.*

**Example 5.32.** Consider a Hopf fibration: let $M = \left(\mathbb{C}^3 \setminus 0\right) / (z \sim 2z)$ and map $F : M \to \mathbb{P}^2$, $F(z) = \mathbb{C}z \in \mathbb{P}^2$. The map $F$ is a fibration $E \to M \to \mathbb{P}^2$ with fiber $E = (\mathbb{C} \setminus 0) / (z \sim 2z)$ an elliptic curve. By Theorem 3.47 on page 13, the base is a Hartogs and Thullen target for local biholomorphisms. By Lemma 3.16, the fiber is a Hartogs and Thullen target. The total space admits the obvious map $\mathbb{C}^3 \backslash 0 \to M$ which does not extend to $\mathbb{C}^3$. Therefore the total space is neither a Hartogs nor a Thullen target, even for local biholomorphisms, . Similarly we can construct the obvious bundle $F \times F : M \times M \to \mathbb{P}^2 \times \mathbb{P}^2$ with fiber $E \times E$, and map $\left(\mathbb{C}^3 \backslash 0\right) \times \left(\mathbb{C}^3 \backslash 0\right) \to M \times M$. The map doesn't extend to $\mathbb{C}^6$, i.e. over the complex codimension 3 subset $0 \times \mathbb{C}^3 \cup \mathbb{C}^3 \times 0$. The reader can build many more examples along the same lines in various dimensions. It might be significant that $M$ is not Kähler.

**Conjecture 5.33.** *Consider a holomorphic fibration whose total space is Kähler. Suppose that the base and fiber are Hartogs extension targets for local biholomorphisms. Then so is the total space.*

## 5.4 Relative extension problems for first order structures

**Definition 5.34.** Suppose that $G_0$ and $G_1$ are complex Lie groups, and that $\rho_0 : G_0 \to G_1$ and $\rho_1 : G_1 \to \mathrm{GL}(n,\mathbb{C})$ are Lie group morphisms. We then treat $\mathbb{C}^n$ as both a $G_0$-module and a $G_1$-module, using representations $\rho_1 \rho_0$ and $\rho_1$ respectively. Every $G_0$-structure $\phi : E_0 \to FTM$ induces a $G_1$-bundle $E \times_{G_0} G_1 \to M$. We define $\phi : E \times G_1 \to FTM$ by $\phi(e, g_1) = \rho_1(g_1)^{-1} \phi(e)$. This map clearly descends to $E \times_{G_0} G_1$, giving the *induced $G_1$-structure*.

**Theorem 5.35.** *Suppose that $G_0$ and $G_1$ are complex Lie groups, and that $\rho_0 : G_0 \to G_1$ and $\rho_1 : G_1 \to \mathrm{GL}(n,\mathbb{C})$ are Lie group morphisms, with closed images. If a $G_0$-structure extends across a subset then the induced $G_1$-structure does as well. Suppose that $\rho_0$ has discrete kernel and that holomorphic maps to $G_1/\rho_0(G_0)$ extend across subsets of complex codimension 2 or more. Then any $G_0$-structure extends over a subset of complex codimension 2 or more just when its induced $G_1$-structure extends over the same subset.*

**Proof.** Combine Theorem 4.20 on page 19 and Lemma 4.11 on page 17. ■

**Example 5.36.** Holomorphic spin structures, $G = \operatorname{Spin}(n, \mathbb{C})$, extend across subsets of complex codimension 2 or more, because the induced $\operatorname{SO}(n, \mathbb{C})$-structures do.

## 5.5 Torsion and invariant relations

**Definition 5.37.** Suppose that $G$ is a complex Lie group with Lie algebra $\mathfrak{g}$ and $V_0$ is a complex $G$-module, say with representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V_0)$. Let $\delta : \mathfrak{gl}(V_0) \otimes V_0^* \to V_0 \otimes \Lambda^2 (V_0)^*$ by $\delta(A \otimes \xi)(v, w) = \rho(A)(v)w - \rho(A)(w)v$. We define $\mathfrak{g}^{(1)}$ and $H^{0,2}(\mathfrak{g})$ to be the kernel and cokernel of $\delta$:

$$0 \longrightarrow \mathfrak{g}^{(1)} \longrightarrow \mathfrak{g} \otimes V^* \xrightarrow{\delta} V_0 \otimes \Lambda^2 (V_0)^* \xrightarrow{[\,]} H^{0,2}(\mathfrak{g}) \longrightarrow 0$$

denoting the quotient map to the cokernel as $t \mapsto [t]$.

Suppose that $E \to FTM$ is a $G$-structure. Denote the projection $FTM \to M$ as $\pi : FTM \to M$ . We define the *soldering* 1*-form* $\sigma \in \Omega^1(FTM)$ by $v \lrcorner \sigma_u = u(\pi'(u)v)$. We will also denote the pullback of $\sigma$ to $E$ as $\sigma$.

For each $A \in \mathfrak{g}$, we write the associated infinitesimal generator of the right $G$-action on $E$ as $\vec{A}$. Denote the projection $E \to M$ as $\pi_E : E \to M$. For each open set $U \subset M$, a *pseudoconnection* for $E$ over $U$ is a choice of 1-form $\gamma \in \Omega^1\left(\pi_E^{-1}U\right) \otimes \mathfrak{g}$ so that $\vec{A} \lrcorner \gamma = A$ for all $A \in \mathfrak{g}$. The local existence of a pseudoconnection is obvious, because $E$ is locally trivial. Any two pseudoconnections $\gamma$ and $\gamma'$ defined over the same open set $U \subset M$ differ by $\gamma' - \gamma = Q\sigma$, where $Q : \pi_E^{-1}U \to \mathfrak{g} \otimes V_0^*$. Moreover, any choice of such a function $Q$ yields a new pseudoconnection $\gamma' = \gamma + Q\sigma$.

The *torsion* of a pseudoconnection $\gamma$ is the function $t : \pi_E^{-1}U \to V_0 \otimes \Lambda^2 (V_0)^*$ for which

$$d\sigma + \gamma \wedge \sigma = \frac{1}{2} t\sigma \wedge \sigma.$$

If we change the pseudoconnection to $\gamma' = \gamma + Q\omega$, then we change the torsion to $t' = t + \delta Q$. Therefore the *intrinsic torsion* $[t] : E \to H^{0,2}(\mathfrak{g})$ of the $G$-structure is well defined globally. A *torsion relation* is a $G$-invariant subset of $H^{0,2}(\mathfrak{g})$. An *integral structure* of a torsion relation is a first order structure whose torsion lies in the torsion relation.

For simplicity, assume that $\rho : G \to \operatorname{GL}(V_0)$ has discrete kernel. Consider the map $[\delta] : (\mathfrak{gl}(V_0)/\mathfrak{g}) \otimes V_0^* \to H^{0,2}(\mathfrak{g})$, given on $S \in \mathfrak{gl}(V_0) \otimes V_0^*$ by $[\delta](S) = [\delta S]$. Given a torsion relation $R \subset H^{0,2}(\mathfrak{g})$, let $R'$ be the preimage of $R$ under $[\delta]$, and call $R'$ the *induced invariant relation* for maps to $\operatorname{GL}(V_0)/G$.

**Theorem 5.38.** *Suppose that $G \to \operatorname{GL}(n, \mathbb{C})$ is a representation with discrete kernel and closed image. Let $R$ be a torsion relation for $G$, and $R'$ the induced invariant relation for maps to $\operatorname{GL}(n, \mathbb{C})/G$. Integral maps of $R'$ extend across subsets of complex codimension 2 or more just when integral structures of $R$ extend across subsets of complex codimension 2 or more.*

**Proof.** Suppose that $M$ is a complex manifold, $S \subset M$ a subset of complex codimension 2 or more, and $E \to FT(M\backslash S)$ is a $G$-structure which is an integral structure for $R$. We only need to extend $E$ locally, so we can assume that $M$ is an open subset of $\mathbb{C}^n$, and so $FTM = M \times \operatorname{GL}(n, \mathbb{C})$. By Theorem 5.35 on the facing page, we can assume that $G \subset \operatorname{GL}(n, \mathbb{C})$. Therefore $E$ is a subbundle of the trivial bundle $(M\backslash S) \times \operatorname{GL}(n, \mathbb{C})$, and is therefore determined as the pullback bundle $f^* \operatorname{GL}(n, \mathbb{C})$ of the bundle $\operatorname{GL}(n, \mathbb{C}) \to \operatorname{GL}(n, \mathbb{C})/G$ by a map $f : M\backslash S \to \operatorname{GL}(n, \mathbb{C})/G$. So if we can extend $f$ across $S$, then we can extend $E$ across $S$ to a principal right $G$-bundle $E \to M$, and therefore by Theorem 4.20 on page 19 the $G$-structure extends across $S$. Therefore we need only prove that $f$ extends across $S$.

Take any local section $u$ of $E \to M\backslash S$. This section is then a matrix $u : \text{open} \subset M\backslash S \to \mathrm{GL}(n,\mathbb{C})$. Then clearly $f = uG$. We need to compute $f' : T_m M \to T_{f(m)} (\mathrm{GL}(n,\mathbb{C})/G)$. If we let $\pi : \mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(n,\mathbb{C})/G$ be the bundle map, then $f = uG = \pi u$, so $f'(m) = \pi'(u(m))u'(m)$. Denote left translation by any matrix $g \in \mathrm{GL}(n,\mathbb{C})$ as $L_g$. Let us see why $f$ is an integral map of $R'$. We need to compute $f^{(1)} : FTM \times_M f^* \mathrm{GL}(n,\mathbb{C}) \to (\mathfrak{gl}(\mathfrak{n},\mathbb{C})/\mathfrak{g}) \otimes \mathbb{C}^{n*}$. But $FTM = M \times \mathrm{GL}(n,\mathbb{C})$, so $FTM \times_M f^* G = M \times \mathrm{GL}(n,\mathbb{C}) \times G$, identified by taking $(m,h,g) \in M \times \mathrm{GL}(n,\mathbb{C}) \times G \to (m,h,u(m)g) FTM \times_M f^* G$. So $f^{(1)}(m,h,g) = \left(L^{-1}_{u(m)g}\right)' (f(m)) f'(m) h^{-1}$. In order to test if $f$ is an integral map of $R'$, it suffices to take $g$ and $h$ to be the identity:

$$\begin{aligned}\left(L_u^{-1}\right)' f'(m) &= \left(L_u^{-1}\right)' \pi'(u(m))u'(m) = \left(L_u^{-1}\pi\right)' (u(m))u'(m) = \left(\pi L_u^{-1}\right)' (u(m))u'(m) \\ &= \pi'(m) \left(L_u^{-1}\right)' (u(m))u'(m) = \pi'(m) u^{-1}\, du = u^{-1}\, du + \mathfrak{g}.\end{aligned}$$

So it suffices to show that $u^{-1}\, du + \mathfrak{g} \in \Omega^1(E) \otimes (\mathfrak{gl}(n,\mathbb{C})/\mathfrak{g})$ is a 1-form valued in $R'$.

We can work entirely locally on $M$, so we can assume that $M$ is a domain in $\mathbb{C}^n$ with coordinates $z^1, z^2, \dots, z^n$. Then $\sigma = u^{-1}\, dz$, and so $d\sigma = -\left(u^{-1}\, du\right) \wedge \sigma$. But we also have $d\sigma = -\gamma \wedge \sigma + \frac{1}{2} t\sigma \wedge \sigma$. By Cartan's lemma, we can write $u^{-1}\, du = \gamma + \frac{1}{2} t\sigma + \frac{1}{2} Q\sigma$ with $Q \in \mathbb{C}^n \otimes \mathrm{Sym}^2 (\mathbb{C}^n)^*$. So

$$u^{-1}\, du + \mathfrak{g} = \frac{1}{2} t\sigma + \frac{1}{2} Q\sigma + \mathfrak{g}.$$

Clearly $\delta(t+Q) = t$, so $[\delta(t+Q)] = [t]$. Because $[t] \in R$, we must have (independent of choices made of trivializations and pseudoconnections) $t + Q + \mathfrak{g} \otimes V_0^* \in R'$, so that $u^{-1}\, du + \mathfrak{g}$ is valued in $R'$. Therefore if integral maps of $R'$ extend across subsets of complex codimension 2 or more, then integral structures of $R$ extend as well.

Suppose that $f : M\backslash S \to \mathrm{GL}(n,\mathbb{C})/G$ is an integral map of $R'$. Take the bundle $\mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(n,\mathbb{C})/G$ and pullback to a $G$-bundle $E = f^* \mathrm{GL}(n,\mathbb{C})$. In order to extend $f$ across $S$, it suffices to do so locally, so we can assume that $M$ is an open subset of $\mathbb{C}^n$, and so $FTM$ is holomorphically trivial, $FTM = M \times \mathrm{GL}(n,\mathbb{C})$. We can then map $E \to FTM$ by the identity map, a $G$-structure. The torsion is then clearly in $R$ by the same arguments as above. ∎

The reader can see how to prove a relative version of the previous theorem.

**Conjecture 5.39.** *Suppose that $G \to \mathrm{GL}(n,\mathbb{C})$ is a representation with discrete kernel and closed image. Let $R$ be a torsion relation for $G$, and $R'$ the induced invariant relation for maps to $\mathrm{GL}(n,\mathbb{C})/G$. Integral maps of $R'$ extend from any domain $M$ in any Stein manifold to the envelope of holomorphy of $M$ just when integral structures of $R$ extend from any domain $M$ in any Stein manifold to the envelope of holomorphy of $M$.*

**Example 5.40.** We have seen that holomorphic contact structures extend across subsets of complex codimension 2 or more. We can consider a contact structure as a hyperplane field, i.e. a $G$-structure where $G$ is the subgroup of $\mathrm{GL}(2n+1,\mathbb{C})$ preserving the hyperplane $\mathbb{C}^{2n} \subset \mathbb{C}^{2n+1}$. The torsion module is $H^{0,2}(\mathfrak{g}) = \Lambda^2 \left(\mathbb{C}^{2n}\right)^* \otimes \left(\mathbb{C}^{2n+1}/\mathbb{C}^{2n}\right)$. If $t$ is the torsion of a $G$-structure, then $t^n \in \Lambda^{2n} \left(\mathbb{C}^{2n}\right)^* \otimes \left(\mathbb{C}^{2n+1}/\mathbb{C}^{2n}\right)^{\otimes n}$. A $G$-structure is a contact structure just when $t^n \neq 0$. Let $R \subset H^{0,2}(\mathfrak{g})$ be the open set of $t \in H^{0,2}(\mathfrak{g})$ so that $t^n \neq 0$. Looking at the relevant matrices, it is easy to see that $\mathfrak{gl}(2n+1,\mathbb{C})/\mathfrak{g} = \mathbb{C}^{2n*} \otimes \left(\mathbb{C}^{2n+1}/\mathbb{C}^{2n}\right)$. The induced invariant relation $R' \subset (\mathfrak{gl}(2n+1,\mathbb{C})/\mathfrak{g}) \otimes \mathbb{C}^{2n+1*}$ is the set of tensors $L \in \mathbb{C}^{2n*} \otimes \left(\mathbb{C}^{2n+1}/\mathbb{C}^{2n}\right) \otimes \mathbb{C}^{2n+1*}$ for which the induced tensor $L \in \mathbb{C}^{2n*} \otimes \left(\mathbb{C}^{2n+1}/\mathbb{C}^{2n}\right) \otimes \mathbb{C}^{2n*}$ has symplectic antisymmetrization in the obvious indices.

We conclude that if $M$ is any complex manifold of dimension $2n+1$, and $S \subset M$ any subset of complex codimension 2 or more, then any holomorphic map $f : M\backslash S \to \mathbb{P}^{2n}$ satisfying this invariant relation $R'$ extends across $S$. We can compute the resulting differential relation on $f$ easily. In local coordinates, write

$$f(z) = \begin{bmatrix} 1 \\ f_1(z) \\ \vdots \\ f_{2n}(z) \end{bmatrix}.$$

The $f$ is an integral map for $R'$ just when

$$\Omega = \left( \frac{\partial f_j}{\partial z_k} - \frac{\partial f_k}{\partial z_j} \right) dz^j \wedge dz^k$$

satisfies $\Omega^n \neq 0$. By Theorem 5.38 on page 27, the relation $R'$ is a Thullen extension relation, because contact structures extend holomorphically across subsets of complex codimension 2 or more. Indeed just looking at this relation, it is clearly a Thullen extension relation. It is not so obvious that $R'$ is invariant under projective transformations. This relation $R'$ is the only known Thullen extension relation for maps $M^{2n+1} \to \mathbb{P}^{2n}$.

## 5.6 Harmless reductions of first order structures

A simple trick allows us to effectively reduce first order structures for the sake of solving the Thullen extension problem. Suppose that $G \subset L \subset \mathrm{GL}(n, \mathbb{C})$, each a closed subgroup of the next. Suppose that $L_0 \subset L$ is another closed subgroup and that $G$ acts transitively on $L/L_0$. Then we will say that the pair of subgroups $L_0 \subset L$ are *harmless* for $G$. We will then let $G_0 \subset G$ be the subgroup fixing the identity coset of $L/L_0$, so $G/G_0 = L/L_0$, and $G_0 = G \cap L_0$.

For example, $L = \mathrm{GL}(n, \mathbb{C})$ and $L_0 = \mathrm{SL}(n, \mathbb{C})$ are harmless for any $G \subset \mathrm{GL}(n, \mathbb{C})$ whose identity component is not contained in $\mathrm{SL}(n, \mathbb{C})$.

We can reduce the Thullen extension problem for $G$-structures to the Thullen extension problem for $G_0$-structures.

**Lemma 5.41.** *Suppose that $M$ is a complex manifold and that $S \subset M$ is a subset of complex codimension 2 or more. Suppose that $E \to FT(M \backslash S)$ is a $G$-structure. Take any harmless pair $L_0 \subset L$ for $G$. Then $E \to FT(M \backslash S)$ induces an $L$-structure $E \times_G L$ as above. Suppose that the $L$-structure extends across $S$. Then $M$ is covered by open sets $U_a$ so that $E|_{U_a \backslash S}$ admits a holomorphic reduction to a $G_0$-structure $E_a$. The $G$-structure $E$ extends across $S$ just when every one of these $G_0$-structures $E_a$ extends across $U_a \cap S$.*

**Proof.** Suppose that $E \times_G L \to M \backslash S$ extends to some $L$-structure $E' \to M$. Pick any open set $U \subset M$ on which $E'$ admits an $L_0$-reduction, say $E'_0 \to U$. Then let $E_0 = E \cap E'_0$.

Let us see why $E_0 \to U \backslash S$ is a $G_0$-structure. Take any open set $U_0 \subset U$ on which $E$ is holomorphically trivial, say $E|_{U_0} = U_0 \times G$. Then $E' = U_0 \times L$. If we shrink $U_0$ we can arrange that the $L_0$-structure is also trivial, $E'_0 = U_0 \times L_0$, mapped to $E' = U_0 \times L$ by some bundle map $\phi(m, \ell_0) = r_{\ell_0} \phi(m)$, for any $m \in U_0$ and $\ell_0 \in L_0$, for some holomorphic map $\phi : U_0 \to L$. So then $E_0$ is the set of pairs $(m, \ell_0)$ for which $r_{\ell_0} \phi(m) \in G$. Clearly $E_0$ is acted on freely by $G_0$, since the equation $r_{\ell_0} \phi(m) \in G$ is $G_0$-equivariant. We only need to show that $G_0$ acts transitively on the fibers of $E_0 \to U_0$. Consider two points lying in the same fiber on $E_0 \to U_0$, say $(m, \ell_0)$ and $(m, \ell_1)$. So $r_{\ell_0} \phi(m) \in G$ and $r_{\ell_1} \phi(m) \in G$. Let $\ell = \phi(m)$. We have $\ell \ell_0 \in G$ and $\ell \ell_1 \in G$. So $(\ell \ell_0)^{-1} \ell \ell_1 \in G$. So $\ell_0^{-1} \ell_1 \in G$ and therefore $\ell_0^{-1} \ell_1 \in G_0$. We see that $E_0 \to U_0$ is a holomorphic $G_0$-subbundle of $E$. ∎

**Example 5.42.** If $G \subset \mathrm{GL}(n, \mathbb{C})$ has identity component not contained in $\mathrm{SL}(n, \mathbb{C})$, then any $G$-structure can be locally reduced to a $G_0$-structure, $G_0 = G \cap \mathrm{SL}(n, \mathbb{C})$, and the $G$-structure extends across a subset of complex codimension 2 or more just when all of the $G_0$-structures do; roughly speaking we can pick a local holomorphic volume form.

Once we have achieved a harmless reduction, we can apply Cartan's method of equivalence (see Gardner [19], Ivey and Landsberg [32] or Sternberg [58]) to this $G_0$-structure, to try to reduce it further, if possible.

**Example 5.43.** For this example (but not for any subsequent part of this paper), we will expect the reader to be conversant with Cartan's method of equivalence. Bryant, Griffiths and Hsu [9] constructed out of any scalar conservation law an equivalent first order structure. We will see why holomorphic scalar conservation laws extend across subsets of complex codimension 2 or more.

Their $G$-structure has structure equations (in a slight alteration of their notation)

$$d \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} = - \begin{pmatrix} 2\phi & 0 & 0 \\ 0 & \phi & 0 \\ 0 & \mu & -\phi \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} + \begin{pmatrix} K\omega^2 \wedge \omega^3 \\ \omega^1 \wedge \omega^3 \\ 0 \end{pmatrix}.$$

The structure group $G$ is the group of matrices of the form

$$\begin{pmatrix} g^2 & 0 & 0 \\ 0 & g & 0 \\ 0 & h & g^{-1} \end{pmatrix}$$

for any nonzero real number $g$ and any real number $h$.

Complexify: consider holomorphic scalar conservation laws, so our group $G$ has $g$ and $h$ complex. Picking a local holomorphic volume form, as described in the previous example, we can harmlessly reduce to the group of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & h & \pm 1 \end{pmatrix}.$$

This structure group is still not a reductive algebraic group, but we can apply Cartan's method of equivalence to the reduced structure. The structure equations are identical, but $\phi$ becomes semibasic, i.e.

$$\phi = -t_1\, \omega^1 - t_2\, \omega^2 - t_3\, \omega^3$$

for some holomorphic functions $t_1$, $t_2$, $t_3$ on the total space of each $G_0$-structure. (The minus signs are for convenience in the following calculations.)

Take exterior derivatives of all of the structure equations, to see that $dt_1 = \mu$ on the fibers of the total space of each $G_0$-structure. So $t$ translates under $G_0$-action, and therefore the set $B_1 = (t_1 = 0)$ is a $G_1$-subbundle, where $G_1$ is the group of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}.$$

The structure group is now a reductive algebraic group, so we can extend each $G_1$-structure across the subset of complex codimension 2 or more, and therefore extend the original $G$-structure.

**Example 5.44.** For Engel 2-plane fields on 4-folds, harmless reduction doesn't help solve the Thullen extension problem. The structure group of an Engel 2-plane field (see Ehlers et al. [16]) can only extend to a reductive algebraic group $L$ by taking $L = \mathrm{GL}(4, \mathbb{C})$. Then the only choice of $L_0$ is $\mathrm{SL}(4, \mathbb{C})$. This harmless reduction reduces the Engel structure by picking a volume form. But the symmetry pseudogroup of the standard Engel 2-plane field on the standard 2-jet bundle together with the standard volume form is given by prolongations of maps $X = X(x), Y = Y(x, y) = Y(x, y_0) + y\, X'(x)^{1/3}$. This pseudogroup is still infinite dimensional, so the structure is still of infinite type, and the structure group is not a reductive algebraic group.

**Example 5.45.** The author hopes that harmless reduction might solve the Thullen extension problem for Clelland's $G$-structure associated to a parabolic partial differential equation for one function of $1 + 2$ variables to prove that these equations extend across subsets of complex codimension 2 or more; see Clelland [13]. Note that such a result would only extend the differential equation, and not its solutions.

**Example 5.46.** If $G \subset \mathrm{GL}(n, \mathbb{C})$ preserves a hyperplane $P$, then a holomorphic $G$-structure induces a holomorphic hyperplane field. If that hyperplane field is a holomorphic contact structure, then that contact structure extends across subsets of complex codimension 2 or more. If $G$ does not preserve a 1-form vanishing on the hyperplane $P$, then picking a local choice of contact form is a harmless reduction; this was our method in Theorem 5.9 on page 22.

We will also apply harmless reduction to holomorphic parabolic geometries in Theorem 8.5 on page 37. There is an obvious analogue of harmless reduction for maps to homogeneous spaces, following the ideas of Theorem 5.38 on page 27, which we leave the reader to explore.

**Remark 5.47.** We leave it to the reader to generalize harmless reduction to the Hartogs extension problem for a domain $M$ in a Stein manifold, as long as $M$ bears a holomorphic volume form (in particular for Riemann domains).

# 6 Higher order structures

**Definition 6.1.** Fix a complex manifold $M$, a vector space $V_0$ with $\dim V_0 = \dim M$, and take $FTM$ the $V_0$-valued frame bundle. Let $\pi : FTM \to M$ be the bundle mapping. We define a 1-form $\sigma$ on $FTM$, called the *soldering form*, by $v \lrcorner \sigma_u = u\,(\pi'(u)v)$. Suppose that $E \to FTM$ is a $G$-structure. Let $\mathfrak{g}$ be the Lie algebra of $G$. For any element $A \in \mathfrak{g}$, write $\vec{A}$ for the vector field on $E$ generating the right $H$-action, i.e.

$$\left.\frac{d}{dt} r_{e^{tA}}\right|_{t=0} = \vec{A}.$$

We will always also denote the pullback of $\sigma$ on $E$ as $\sigma$. A *pseudoconnection* 1*-form* at a point $e \in E$ is a 1-form $U \in T^*_e E \otimes \mathfrak{g}$ so that $\vec{A} \lrcorner U = A$.

Pseudoconnection 1-forms exist at each point $e$ of $E$. The set of all pseudoconnection 1-forms at all points of the total space $E$ of a $G$-structure form a principal right $\mathfrak{g} \otimes V_0^*$-bundle over $E$, under the action $r_{A \otimes \xi} U = U - (A\sigma) \wedge (\xi\sigma)$. The reader can consult Gardner [19] or Ivey and Landsberg [32].

**Definition 6.2.** A *torsion function* on a $G$-structure $E \to FTM$ is a holomorphic function $t : E \to \Lambda^2 (V_0)^* \otimes V_0$ so that,

1. $t$ is $G$-equivariant,
2. at each point $e \in E$, there is a pseudoconnection 1-form $U$ so that $d\sigma + U \wedge \sigma = \frac{1}{2} t\sigma \wedge \sigma$.

Not every $G$-structure admits a torsion function in this sense, but the most important examples of $G$-structures do; see Gardner [19] or Ivey and Landsberg [32].

**Definition 6.3.** If $\rho : G \to \mathrm{GL}(V_0)$ is a holomorphic representation of a complex Lie group $G$ with Lie algebra $\mathfrak{g}$, let $\delta : \mathfrak{g} \otimes V_0^* \to V_0 \otimes \Lambda^2(V_0)^*$ be defined by $\delta(A \otimes \xi)(v_1, v_2) = (Av_1)\xi(v_2) - (Av_2)\xi(v_1)$. We define $\mathfrak{g}^{(1)} = \ker\delta$.

**Definition 6.4.** If $E \to FTM$ is a $G$-structure with torsion function $t$, then define the *prolongation* $E^{(1)}$ of $E$ (with respect to $t$) to be the bundle of all pairs $(e, U)$ of points $e \in E$ and pseudoconnection 1-forms $U$ at $e$ with $d\sigma + U \wedge \sigma = \frac{1}{2} t\sigma \wedge \sigma$.

It is easy to check that $E^{(1)} \to E$ is a principal right $\mathfrak{g}^{(1)}$-bundle (a subbundle of the bundle of pseudoconnection 1-forms), with $\mathfrak{g}^{(1)}$ acting as a subgroup of $\mathfrak{g} \otimes V_0^*$.

**Example 6.5.** Suppose that $\pi : E \to M$ is a $G/H$-geometry with Cartan connection $\omega$. Let $\sigma = \omega + \mathfrak{h} \in \Omega^1(E) \otimes (\mathfrak{g}/\mathfrak{h})$. There is a unique function $K : E \to \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}$, the *curvature*, satisfying

$$d\omega + \frac{1}{2}[\omega, \omega] = \frac{1}{2} K\sigma \wedge \sigma.$$

Let $V_0 = \mathfrak{g}/\mathfrak{h}$. Map $E \to FTM$ by $e \in E \mapsto \omega_e + \mathfrak{h} \in FT_{\pi(m)}M$. This map is an $H$-structure. The 1-form $\sigma$ is called the *soldering form*.

Take the the obvious projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$. Take any linear splitting of vector spaces $s : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$. The function $t(a, b) = [s(a), s(b)] + K(a, b) + \mathfrak{h}$ (for $a, b \in \mathfrak{g}/\mathfrak{h} = V_0$) is a torsion function just when $s$ satisfies

$$\mathrm{Ad}(h)[s(A), s(B)] + \mathfrak{h} = [s(\mathrm{Ad}(h)A), s(\mathrm{Ad}(h)B)] + \mathfrak{h}$$

for all $A, B \in \mathfrak{g}/\mathfrak{h}$. (Warning: $t$ is *not* necessarily the same as the object which is usually called the *torsion* of the $G/H$-geometry. Moreover it is not clear that there is always such a splitting $s$ satisfying this complicated condition.) The intrinsic torsion $[t]$ is of course independent of the choice of splitting $s$. The prolongation of the underlying first order structure is $E \times_H \mathfrak{h}^{(1)}$.

**Definition 6.6.** Suppose that $G$ is a complex Lie group with Lie algebra $\mathfrak{g}$ and $V_0$ is a complex $G$-module. Recall the exact sequence

$$0 \longrightarrow \mathfrak{g}^{(1)} \longrightarrow \mathfrak{g} \otimes V^* \xrightarrow{\delta} V_0 \otimes \Lambda^2(V_0)^* \xrightarrow{[]} H^{0,2}(\mathfrak{g}) \longrightarrow 0.$$

Map $\mathfrak{g}^{(1)} \to \mathrm{GL}(V_1)$ by

$$Q \mapsto Q', Q'(v, A) = (v, A + Qv).$$

Suppose that $E \to FTM$ is a $G$-structure with torsion function $t$ and prolongation $E^{(1)}$. Let $\pi^{(1)} : E^{(1)} \to E$ be the bundle map. Let $FE$ be the $V_1$-valued frame bundle of $E$. Map $E^{(1)} \to FE$ by taking $\gamma \mapsto \pi(\gamma)^*\sigma \oplus \gamma$. Then $E^{(1)} \to FE$ is a $\mathfrak{g}^{(1)}$-structure on $E$, called the *prolongation* of $E$.

There is a right action of $Q \in \mathfrak{g}^{(1)}$ given by $r_Q U = U - Q\sigma$ (where $U \in E^{(1)}$). This action satisfies

$$r_Q^*\sigma = \sigma, \qquad r_Q^*\gamma = \gamma - Q\sigma.$$

There is a natural action of $G$ on $E^{(1)}$, commuting with the bundle map $E^{(1)} \to E$, which is (for $g \in G$):

$$r_g U = \mathrm{Ad}_g^{-1}\left(U\left(r_g^{-1}\right)'\right).$$

Form the semidirect product $G \rtimes \mathfrak{g}^{(1)}$ with multiplication

$$(g_1, Q_1)(g_2, Q_2) = (g_1 g_2, Q_1 + g_1 Q_2)$$

where $gQ$ means the element of $\mathrm{Sym}^2(V_0)^* \otimes V_0$ defined by

$$gQ(u, v) = g\left(Q\left(g^{-1}u, g^{-1}v\right)\right).$$

We can write a point of $E^{(1)}$ as $(u, U)$ where $u \in E$ and $U$ is a pseudoconnection at $u$. The two group actions fuse together to an action of the semidirect product:

$$r_{(g,Q)}(u, U) = \left(g^{-1}u, \mathrm{Ad}_g^{-1}\left(U - Qu\pi'\right)\left(r_g^{-1}\right)'\right).$$

This action makes $E^{(1)} \to M$ into a principal right $G \rtimes \mathfrak{g}^{(1)}$-bundle. (It is *not* a $G \rtimes \mathfrak{g}^{(1)}$-structure.) Under the $G$-action

$$r_g^* \sigma = g^{-1}\sigma, \qquad r_g^* \gamma = \mathrm{Ad}_g^{-1} \gamma.$$

Therefore under the $G \rtimes \mathfrak{g}^{(1)}$-action

$$r_{(g,Q)}^* \begin{pmatrix} \sigma \\ \gamma \end{pmatrix} = \begin{pmatrix} g^{-1}\sigma \\ \mathrm{Ad}_g^{-1}(\gamma - Q\sigma) \end{pmatrix}.$$

**Definition 6.7.** A *second order structure* on a manifold $M$ with first order structure $E \to FTM$ and torsion function $t$ and $G$-submodule $G_1 \subset \mathfrak{g}^{(1)}$ is a principal $G_1$-bundle $E_1 \to E$ and $G \rtimes G_1$-equivariant bundle map $E_1 \to E^{(1)}$ over $M$. (In this definition, $E^{(1)}$ is the prolongation of the first order structure using $t$ as torsion function.)

**Definition 6.8.** Suppose that $\phi_1 : E_1 \to E$ is a second order structure over a $G$-structure $\phi : E \to M$. We define $\check{\phi}_1 : E^{(1)} \to \mathfrak{g}^{(1)}/G_1$ by $\check{\phi}_1(\gamma) = Q + G_1$ if $Q \in \mathfrak{g}^{(1)}$ and $r_Q \gamma \in E_1$. Much as before, $\check{\phi}_1^{-1} G_1 = E_1$.

We leave the reader to define higher order structures of all orders by induction.

**Proposition 6.9.** *A holomorphic higher order structure with discrete kernel extends holomorphically across a subset of complex codimension* 2 *or more just when its underlying first order structure extends holomorphically across that subset.*

**Proof.** It is enough (by induction) to prove the result for second order structures. Since the result is local, replace the complex manifold $M$ by a ball $B$. As usual, because $E \to B \backslash S$ extends holomorphically, say to a principal $G$-bundle $E' \to B$ we can replace $B$ by a smaller ball to arrange that $E' \to B$ is holomorphically trivial, $E' = B \times G$, and so $E = (B \backslash S) \times G$. Therefore $E^{(1)} = (B \backslash S) \times G \times \mathfrak{g}^{(1)}$. Over $E$ we have a $G_1$-structure $E_1 \to E$, i.e. $E_1 \to (B \backslash S) \times G$. So we have a map $\check{\phi}_1 : E^{(1)} \to \mathfrak{g}^{(1)}/G_1$, i.e. $\check{\phi}_1 : (B \backslash S) \times G \times \mathfrak{g}^{(1)} \to \mathfrak{g}^{(1)}/G_1$. The quotient $\mathfrak{g}^{(1)}/G_1$ is a quotient of vector spaces, so a vector space, and therefore a Thullen extension target. Therefore for each choice of $g \in G$ and $Q \in \mathfrak{g}^{(1)}$, this map $\check{\phi}_1$ extends to a holomorphic map on $B$, and therefore extends to a holomorphic map on $B \times G \times \mathfrak{g}^{(1)} = E^{(1)}$. We extend $E_1$ holomorphically to be $E_1' = \check{\phi}_1^{-1}(G_1)$. ∎

**Theorem 6.10.** *Let $M$ be a domain in a Stein manifold and $\hat{M}$ the envelope of holomorphy of $M$. A holomorphic higher order structure with trivial kernel extends holomorphically from $M$ to $\hat{M}$ just when its underlying first order structure extends holomorphically from $M$ to $\hat{M}$.*

**Proof.** It is enough (by induction) to prove the result for second order structures. Suppose that our higher order structure is $E_1 \to E \to M$, in the notation of Definition 6.7 on the preceding page. Suppose that $E$ extends to a bundle over $\hat{M}$, which we denote by $E'$. Consider the torsion function $t : E \to \Lambda^2 (V_0)^* \otimes V_0$. The torsion function is a section of the holomorphic vector bundle $E \times_G \left(\Lambda^2 (V_0)^* \otimes V_0\right)$. By Proposition 3.36 on page 10, this section extends to a section of $E' \times_G \left(\Lambda^2 (V_0)^* \otimes V_0\right)$, and therefore the torsion extends to a holomorphic function on $E'$.

By Lemma 3.38 on page 11, there is a connection for $E' \to \hat{M}$, say $\alpha$. If $\alpha$ has torsion, say $t_\alpha$, and $E$ has torsion function $t$, then replace $\alpha$ by $\alpha - \frac{1}{2} t_\alpha \sigma + \frac{1}{2} t\,\sigma$. Every pseudoconnection 1-form $U$ at any point $u \in E$ has the form $U = \alpha + p\,\sigma$, where $\sigma$ is the soldering 1-form and $p \in \mathbb{C}^{n*} \otimes \mathfrak{g}$. The torsion of $U$ is then $t + \delta p$, so we will need $p \in \mathfrak{g}^{(1)}$. In order that $U \in E_1$, we need to pick $p$ to lie in some $G_1$-orbit in $\mathfrak{g}^{(1)}$. This orbit may vary at different points of $E$. We can keep track of this orbit as a function $f : E \to \mathfrak{g}^{(1)}/G_1$. This function is a section of $E \times_G \left(\mathfrak{g}^{(1)}/G_1\right)$ so once again extends uniquely holomorphically to a function $f : E' \to \mathfrak{g}^{(1)}/G_1$. We then produce a bundle $E_1' \to E'$ consisting precisely of the pseudoconnections $U = \alpha + p\,\sigma$ for which $\delta p = 0$ and $p + G_1 = f$. ■

### 6.1 Parabolic geometries and higher order structures

**Example 6.11.** Consider $G = \mathrm{GL}(n, \mathbb{C})$. Then $\mathfrak{g}^{(1)} = V_0 \otimes \mathrm{Sym}^2 (V_0)^*$. There is an obvious $G$-submodule $G_1 = V_0^*$ included by taking $a \in V_0^*$ to the operator $(v_1, v_2) \mapsto a(v_1)\, v_2 + a(v_2)\, v_1$. A *normal projective connection* on a manifold $M$ is a second order structure with first order structure $E = FTM$, torsion function $t = 0$, and $G$-submodule $G_1 \subset \mathfrak{g}^{(1)}$. See Borel [6] or Molzon and Mortensen [44] for more information. Normal projective connections extend across subsets of complex codimension 2 or more, and extend from domains in Stein manifolds to envelopes of holomorphy, because the underlying first order structure is just the frame bundle. In fact, normal projective connections turn out to be equivalent to Cartan connections modelled on projective space (again see Borel [6]), so we will also solve the Thullen extension problem for normal projective connections in Theorem 8.5 on page 37.

**Example 6.12.** A *normal contact projective connection* is a similar sort of higher order structure, to complicated to explain in detail; see Fox [17]. Its underlying first order structure is a contact structure, so that we have solved the Thullen and Hartogs extension problems for normal contact projective connections by combining Theorem 5.9 on page 22 (to extend the contact structure) with Proposition 6.9 on the previous page (to extend the second order structure across subsets of complex codimension 2 or more) or Theorem 6.10 on the preceding page (to extend from a domain in Stein manifold to the envelope of holomorphy). Normal contact projective connections turn out to be equivalent to Cartan connections modelled on the projectivized cotangent bundle of projective space (again see Fox [17]), so we will also solve the Thullen extension problem for normal contact projective connections in Theorem 8.5 on page 37.

## 7 Extending flat Cartan geometries

### 7.1 Extending by development

**Definition 7.1.** Consider a $G/H$-geometry $\pi : E \to M$, and let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie algebras of $H \subset G$. The 1-form $\sigma = \omega + \mathfrak{h} \in \Omega^1(E) \otimes (\mathfrak{g}/\mathfrak{h})$ is called the *soldering form* of the Cartan geometry. The curvature can be written as $d\omega + \frac{1}{2}[\omega, \omega] = \frac{1}{2} K \sigma \wedge \sigma$ for a unique function $K : E \to \mathfrak{g} \otimes \Lambda^2 (\mathfrak{g}/\mathfrak{h})^*$, called the *curvature function*. The curvature function transforms under $H$-action according to the obvious equivariance, so is a section of

$$E \times_H \left(\mathfrak{g} \otimes \Lambda^2 (\mathfrak{g}/\mathfrak{h})^*\right),$$

which we will also call the *curvature*. We will call a Cartan geometry *flat* if its curvature vanishes.

**Lemma 7.2.** *A Cartan geometry is locally isomorphic to its model just when it is flat.*

This lemma is well known (see Sharpe [54, Theorem 5.1, p. 212]) and doesn't require complex analyticity.

**Proof.** Take $E \to M$ the Cartan geometry, with Cartan connection $\omega$. Apply the Frobenius theorem to the Pfaffian system $\omega - g^{-1}\,dg = 0$ on $E \times G$. The integral manifolds of this Pfaffian system are locally graphs of local isomorphisms. ■

**Lemma 7.3.** *Suppose that $E \to M$ is a flat $G/H$-geometry. Let $H$ act on $E \times G$ by the right action $(e, g)h = (eh, gh)$. Take any connected integral manifold $Z$ of the Pfaffian system $\omega - g^{-1}\,dg = 0$ on $E \times G$. Let $ZH$ be the union of all $H$-orbits through points of $Z$. Then $ZH$ is an $H$-equivariant covering space of $E$, and the total space of a Cartan geometry on a covering space of $M$.*

**Proof.** We provide a sketch; see McKay [40] for details. Above each curve in $E$, the Pfaffian system is an ordinary differential equation of Lie type, so has global solution. Therefore each integral manifold $Z$ is a covering space of a path component of $E$. The $H$-orbits are Cauchy characteristics of the Pfaffian system. Therefore the group of path components of $H$ acts permuting integral manifolds. The union $ZH$ of all of these $H$-orbits is therefore a union over path components of $H$, say $ZH = Z\pi_0(H)$, a discrete union of distinct connected integral manifolds, so an integral manifold. Moreover, $ZH$ is acted on by $H$ freely and properly, because $H$ acts freely and properly on $E \times G$. Therefore $E' = ZH \to M' = ZH/H$ is a Cartan geometry with Cartan connection $\omega$. ■

**Proposition 7.4.** *Suppose that $G/H$ is a complex homogeneous space. Then local biholomorphisms to $G/H$ extend across subsets of complex codimension 2 or more if and only if flat holomorphic $G/H$-geometries extend across such subsets.*

**Proof.** Suppose that $G/H$ is a Thullen extension target for local biholomorphisms. We can replace $M$ by any open neighborhood of a point $s \in S$, so we can assume that $M$ is a ball $B$. Then $B \backslash S$ is simply connected. If $Z$ is any integral manifold of the Pfaffian system $\omega - g^{-1}\,dg = 0$ on $E \times G$, then $ZH$ is the graph of a local isomorphism $f : B \backslash S \to G/H$. Because $G/H$ is a Thullen extension target for local biholomorphisms, we can extend the map $f$. The pullback of the bundle $G \to G/H$ holomorphically extends the bundle $E$. By Theorem 4.18 on page 18, the Cartan geometry extends holomorphically.

Suppose that some local biholomorphism $f : M \backslash S \to G/H$ does not extend to $M$, with $S \subset M$ a subset of complex codimension 2 or more. Then the pullback $E = f^*G$ of the standard flat $G/H$-geometry is a $G/H$-geometry on $M \backslash S$. To be precise, $E$ is the set of pairs $(m_0, g_0) \in (M \backslash S) \times G$ so that $f(m_0) = g_0 H \in G/H$. Suppose that this $G/H$-geometry extends to a $G/H$-geometry $E' \to M$. The curvature vanishes on $E$, a dense open set in $E'$, so the $G/H$-geometry on $E'$ is flat. Pick a maximal integral manifold $Z$ of the Pfaffian system $\omega - g^{-1}\,dg = 0$ on $E' \times G$ passing through a point of the fiber $E'_s$. This integral manifold $Z$ is uniquely determined up to left $G$-action on $G$ and right $H$-action on $E'$. After perhaps translating $Z$ by right $H$-action, we can arrange that $Z$ passes through a point of the form $(m_0, g_0, g_0) \in E \times G$. This leaf $Z$ is then locally the graph of the local isomorphism $(m_0, g_0) \in E \mapsto g_0 \in G$ of $G/H$-geometries, and extends this isomorphism to a neighborhood of $E'_s$ in $E'$. Therefore $Z$ is the graph of a local isomorphism of $G/H$-geometries taking an open neighborhood of $s \in M$ to an open set in $G/H$, extending $f$. ■

### 7.2 Examples of inextensible flat Cartan geometries

**Lemma 7.5 (Sharpe [54, Theorem 3.15, p. 188]).** *If $\pi : E \to M$ is any $G/H$-geometry then the Cartan connection of $E$ maps*

$$\begin{tikzcd}
0 \arrow[r] & \ker \pi'(e) \arrow[r] \arrow[d] & T_e E \arrow[r] \arrow[d] & T_m M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{h} \arrow[r] & \mathfrak{g} \arrow[r] & \mathfrak{g}/\mathfrak{h} \arrow[r] & 0
\end{tikzcd}$$

*for any points $m \in M$ and $e \in E_m$; thus*

$$TM = E \times_H (\mathfrak{g}/\mathfrak{h}) .$$

*Under this identification, holomorphic vector fields are identified with $H$-equivariant holomorphic functions $E \to \mathfrak{g}/\mathfrak{h}$.*

**Example 7.6.** Reconsider Example 5.1 on page 20. Take $G \subset \mathrm{GL}(n, \mathbb{C})$ any Lie group acting transitively on $\mathbb{C}^n \backslash 0$, and let $H$ be the stabilizer of some vector $v_0 \in \mathbb{C}^n \backslash 0$. Obviously $G/H = \mathbb{C}^n \backslash 0$. For example, $G = \mathrm{Sp}(2n, \mathbb{C})$ on $\mathbb{C}^{2n}$, or $G = G_2$ on $\mathbb{C}^7$. Then the standard flat $G/H$-geometry, on $G/H = \mathbb{C}^n \backslash 0$ does not extend across the puncture at 0. For any $G/H$-geometry $E \to M$, the constant function $v_0$ on $E$ is identified with a nowhere vanishing vector field by Lemma 7.5. On the model $G/H = \mathbb{C}^n \backslash 0$, this vector field is the Euler vector field $Z(z) = z$. Clearly the Euler vector field cannot extend to $\mathbb{C}^n$ holomorphically without vanishing at the origin. If the $G/H$-structure were to extend holomorphically to the origin, then the associated nowhere vanishing vector field would also extend holomorphically, and remain nowhere vanishing.

## 8 Extending Cartan geometries

**Theorem 8.1.** *Suppose that $H \subset G$ is closed complex Lie subgroup of a complex Lie group. If holomorphic maps to $G/H$ extend across subsets of complex codimension 2 or more, then $G/H$-geometries extend holomorphically across such subsets.*

This solves the Thullen extension problem for various Cartan geometries; the analogous Hartogs extension problem is unsolved.

**Proof.** As usual we can assume that our $G/H$-geometry is on a ball minus a subset of complex codimension 2 or more, say $E \to B \backslash S$. Fix a point $z_0 \in B \backslash S$. Consider the complex affine lines through $z_0$. Away from $z_0$, each point of $B$ lies on a unique disk from this family.

Pick a point $e_0 \in E_{z_0}$ in the fiber above $z_0$, and develop each disk so that the frame $e_0$ is carried to the frame $1 \in G$. The development of each disk is well defined, by Lemma 2.17 on page 5, except possibly for development along the disks through $z_0$ which hit points of $S$.

The disk $D$ through $z_0$ and some point of $S$ has to be punctured at all points of $D \cap S$ before we develop, because the Cartan geometry is not defined on $S$. The development is defined on the universal covering space of the punctured disk. However, by continuity of solutions of ordinary differential equations, the monodromy must be the limit of the monodromies of nearby punctured disks. The nearby disks have no punctures, so the monodromy is trivial: the development is a holomorphic map on each disk.

These developments fit together into a single holomorphic map $\phi_1 : B \backslash S \to G/H$. The development of each disk, say $D$, yields an isomorphism $\Phi_D : E|_D \to \phi_1^* G|_D$. This map clearly extends to a bundle isomorphism $\Phi : E \to \phi_1^* G$ above $B \backslash S$. Since the map $\phi_1$ extends across $S$, so does the bundle $\phi_1^* G$, and therefore so does $E$. ■

**Example 8.2.** If $G/H$ is a reductive homogeneous space, then $G/H$-geometries extend across subsets of complex codimension 2 or more.

**Example 8.3.** Among Cartan geometries on surfaces, this theorem applies to

- $\mathbb{C}^2$-geometries, for any of the various complex Lie groups acting transitively on $\mathbb{C}^2$ and
- $\mathbb{P}^1 \times \mathbb{P}^1 \backslash$diagonal-geometries; see Example 5.26 on page 25.

**Example 8.4.** Beloshapka, Ezhov and Schmalz [5] study a particular class of CR-manifolds, those of dimension 4, CR-dimension 1, CR-codimension 2, with Engel CR-distribution. They associate to each such CR-structure a canonical Cartan geometry modelled on a homogeneous space $G/H$. If we complexify this type of geometry, the resulting complex analytic geometric structures are Cartan geometries modelled on the complexification of $G/H$. This complexification turns out to be an affine space, and therefore these complex analytic geometries extend across subsets of complex codimension 2 or more.

**Theorem 8.5.** *Parabolic geometries extend across subsets of complex codimension* 2 *or more.*

**Proof.** Pick a complex manifold $M$, a subset $S \subset M$ of complex codimension 2 or more, and a parabolic geometry $E \to M \backslash S$ modelled on some rational homogeneous variety $G/P$. We only need to extend $E$ locally, so we can assume that $M$ admits a holomorphic volume form. By McKay [41, Theorem 2, p. 2], because $M \backslash S$ has a holomorphic volume form, $E \to M \backslash S$ admits a holomorphic reduction $E_0 \subset E$ of structure group to a reductive algebraic group $P_0 \subset P$, and $E_0$ has a holomorphic connection. By Proposition 4.14 on page 18, $E_0$ extends to a holomorphic bundle $E_0' \to M$ with holomorphic connection. Then $E$ extends to $E' = E_0' \times_{P_0} P$. ∎

**Remark 8.6.** Hong [23] proved a related result. She assumes the existence of a family of minimal rational curves on a Fano manifold, satisfying some complicated conditions on their tangent lines at a generic point. These conditions turn out to encode a flat parabolic geometry outside a subset of the Fano manifold of complex codimension 2 or more. She demonstrates that the parabolic geometry extends holomorphically along the curves, to become holomorphic at every point of the manifold. It is possible to prove Hong's theorem by using the theorem above, but one would first need results of Ochiai [46] to demonstrate that the geometry encoded by the tangents of the minimal rational curves is actually a parabolic geometry. We will leave this to the reader. Mok [43] made use of this result, and of the result of Hwang and Mok on codimension 2 extension of reductive $G$-structures, in his study of the contact parabolic geometry of $G_2$.

**Example 8.7.** This theorem provides another proof that local biholomorphisms to rational homogeneous varieties extend across subsets of complex codimension 2 or more.

**Example 8.8.** This theorem is an application of the method of harmless reduction (see Section 5.6 on page 29).

**Example 8.9.** There are (at least) two natural geometric structures associated to a 2nd order scalar ordinary differential equation. We contrast their extension theory.

Recall the complex homogeneous surface $\mathcal{O}(n)$ (see Example 3.49 on page 14). It turns out (see Dunajski and Tod [15]) that every complex analytic scalar ordinary differential equation of order $n+1$

$$\frac{d^{n+1}w}{dz^{n+1}} = f\left(z, w, \frac{dw}{dz}, \dots, \frac{d^n w}{dz^n}\right)$$

imposes an $\mathcal{O}(n)$-geometry on the open subset of the $(z, w)$-plane where the equation is defined, invariant under fiber-preserving transformations, i.e. transformations preserving the solutions of the differential equation and the vertical lines $z = \text{const}$. Every $\mathcal{O}(n)$-geometry with vanishing torsion arises in this fashion; again see Dunajski and Tod [15] for a definition of torsion and a proof.

An $\mathcal{O}(n)$-geometry on a surface has as its underlying first order structure a foliation with affine structure on its leaves. This foliation comes from the invariant foliation (indeed line bundle mapping) $\mathcal{O}(n) \to \mathbb{P}^1$. The affine structure of the fibers endows the leaves of any $\mathcal{O}(n)$-geometry with affine structures. We have already seen an example of an $\mathcal{O}(n)$-geometry in Example 3.49 on page 14: map $\mathbb{C}^2 \backslash 0 \to \mathcal{O}(n)$ by taking a point $z \neq 0$ to the homogeneous polynomial $p_z$ of degree $n$ on the span of $z$ which takes the value 1 on $z$. The foliation is the foliation of $\mathbb{C}^2 \backslash 0$ by lines through 0, so doesn't extend across the puncture at 0.

Let $A$ be any $2 \times 2$ invertible complex matrix which has all of its eigenvalues inside the unit disk. Consider the *Hopf surface* $S = \left(\mathbb{C}^2 \backslash 0\right) / (z \sim Az)$, which is a smooth compact complex surface. The $\mathcal{O}(n)$-geometry on $\mathbb{C}^2 \backslash 0$ descends to the Hopf surface $S$. There are infinitely many holomorphic $\mathcal{O}(n)$-geometries on certain Hopf surfaces; the author hopes to classify these in a sequel to this paper.

A different approach: every 2nd order scalar ordinary differential equation

$$\frac{d^2 w}{dz^2} = f\left(z, w, \frac{dw}{dz}\right)$$

gives rise to a contact 3-fold, with coordinates say $z, w, p$, and contact structure $dw - p\,dz = 0$. This 3-fold has two nowhere tangent Legendre fibrations: (1) $dw - p\,dz = dp - f\,dz = 0$ and (2) $dw = dz = 0$. This contact structure and pair of Legendre fibrations is invariant under point transformations of the differential equation. Conversely, we define a *path geometry* to be any pair of nowhere tangent holomorphic Legendre fibrations in a holomorphic contact 3-fold. Path geometries, being parabolic geometries modelled on $\mathbb{P}T\mathbb{P}^2$ (see McKay [39] for proof), extend across subsets of complex codimension 2 or more.

A similar story occurs in 3rd order: a 3rd order scalar ordinary differential equation is equivalent to a parabolic geometry on an appropriate manifold, and this equivalence is invariant under contact transformations; see Sato and Yoshikawa [51] for proof. The induced $\mathcal{O}(2)$-geometry of a 3rd order scalar ODE might not extend across a subset of complex codimension 2 or more, but the contact invariant parabolic geometry does extend across such subsets.

**Remark 8.10.** This striking difference between fiber-preserving and point transformations is probably quite important in understanding 2nd order ODEs, and probably the most important observation in this paper.

**Remark 8.11.** Čap and Schichl [12] proved that any parabolic geometry satisfying a a mild (but complicated) hypothesis is a local product structure, with each local factor bearing a normal projective connection, normal contact projective connection, or a first order structure consisting of various subbundles of the tangent bundle, equipped with a reduction of structure group to a reductive algebraic group. Suppose that we have a holomorphic parabolic geometry on a domain in a Stein manifold, which satisfies the mild hypothesis of Čap and Schichl. The local product structure is a first order structure with reductive structure group, so extends to the envelope of holomorphy (see Example 5.29 on page 26). The various subbundles of the tangent bundle extend, except perhaps on a subset of complex codimension 2 or more (see Example 5.6 on page 21). The normal contact projective connections and normal projective connections will extend by arguments which only slightly generalize those of Examples 6.11 and 6.12 on page 34. Parabolic geometries extend across subsets of complex codimension 2 or more, so we can work modulo such subsets.

In order that extension of the parabolic geometry be parabolic, the various plane fields need to have constant symbol algebra, up to complex codimension 2; see Čap and Schichl [12] for more information. This is not difficult to prove, since the symbol algebra changes type only on an analytic set. The mild hypothesis of Čap and Schichl will therefore be sufficient to solve the Thullen and Hartogs extension problems. We hope to solve both extension problems for all holomorphic parabolic geometries (without even a mild hypothesis) in a sequel to this paper.

# 9 Extension of local isomorphisms

A related question: for which holomorphic geometric structures do local isomorphisms extend holomorphically across subsets of complex codimension 2 or more?

**Example 9.1.** We produce an example of a holomorphic local isomorphism of $G$-structures that fails to extend across a point, even though the $G$-structures are flat, the isomorphism maps to a compact complex homogeneous space, and the homogeneous space $\mathrm{GL}(n, \mathbb{C})/G$ is a Hartogs extension target.

Let $G$ be the set of $n \times n$ matrices of the form $2^k I$ for $k \in \mathbb{Z}$. A $G$-structure is a choice of holomorphic framing of a complex manifold, up to rescaling by factors of 2. The quotient $\mathrm{GL}(2, \mathbb{C})/G$ is covered by $\mathrm{GL}(2, \mathbb{C})$, which is a Thullen and Hartogs extension target, and therefore $\mathrm{GL}(2, \mathbb{C})/G$ is also a Thullen and Hartogs extension target. Therefore $G$-structures extend holomorphically across subsets of complex codimension 2 or more.

Let $S$ be the Hopf manifold $\left(\mathbb{C}^2 \backslash 0\right)/(z \sim 2z)$, and take the map $f : \mathbb{C}^2 \backslash 0 \to S$ taking each point $z$ to its equivalence class $[z] \in S$. All linear automorphisms of $\mathbb{C}^2$ commute with $z \mapsto 2z$, so they descend to biholomorphisms of the Hopf surface. These linear automorphisms act transitively on $\mathbb{C}^2 \backslash 0$, so act transitively on the Hopf surface.

Any local biholomorphism $f : M \to N$ between complex manifolds determines a map $f_1 : FTM \to FTN$ by $Ff(m, u) = u \circ f'(m)^{-1}$ for $m \in M$ and $u \in FT_m M$. Since $FT\left(\mathbb{C}^2 \backslash 0\right) = \left(\mathbb{C}^2 \backslash 0\right) \times \mathrm{GL}(2, \mathbb{C})$ is a trivial bundle, we can compose with the obvious global section of that bundle, to obtain a map $f : \mathbb{C}^2 \backslash 0 \to FTS$, a $G$-structure on $S$. We can map the standard flat $G$-structure on $\mathbb{C}^2 \backslash 0$ to the given $G$-structure on $S$, also called $f$, by

$$f\left(z, 2^k I\right) = f(z) 2^k I.$$

This map is a local isomorphism of $G$-structures, taking the standard flat $G$-structure to the $G$-structure on the Hopf surface, $\mathbb{C}^2 \backslash 0 \to S$. The map doesn't extend holomorphically to $\mathbb{C}^2$. Clearly we can generalize to Hopf manifolds of all dimensions.

# 10 Rigidity

Many of the theorems in this paper have obvious generalizations to families of first order structures, higher order structures and Cartan geometries. We will provide some examples to expose the general pattern.

We will think of a holomorphic submersion $\pi : M^{n+p} \to Z^p$ as a $p$-parameter family of complex manifolds.

**Definition 10.1.** The *vertical bundle* $V = V_\pi$ of such a holomorphic submersion $\pi : M \to Z$ is $V = \ker \pi' \subset TM$. The *vertical frame bundle* is the $\mathrm{GL}(n, \mathbb{C})$-bundle $FV \to M$. Suppose that $\rho : G \to \mathrm{GL}(n, \mathbb{C})$ is a complex representation of a complex Lie group $G$. A *holomorphic family of $G$-structures* on a holomorphic submersion $\pi : M^{n+p} \to Z^p$ is a holomorphic principal right $G$-bundle $E \to M$ and a $G$-equivariant holomorphic bundle map $E \to FV$.

If $p : E \to M$ is a holomorphic principal bundle and $\pi : M \to Z$ is a holomorphic submersion, let $VE \to E$ be the bundle $VE = \ker \pi p \subset TE$. Suppose that $G/H$ is a complex homogeneous space. A *family of Cartan geometries* modelled on $G/H$ on a holomorphic submersion $\pi : M \to Z$ is a holomorphic principal right $H$-bundle $E \to M$ and a section $\omega$ of $VE \otimes \mathfrak{g}$, called the *Cartan connection*, satisfying the following conditions:

1. Denote the right action of $g \in G$ on $e \in E$ by $r_g e = eg$. The Cartan connection transforms in the adjoint representation:
$$r_g^* \omega = \mathrm{Ad}_g^{-1} \omega.$$
2. $\omega_e : V_e E \to \mathfrak{g}$ is a linear isomorphism at each point $e \in E$.
3. For each $A \in \mathfrak{g}$, define a vector field $\vec{A}$ on $E$, tangent to the fibers of $\pi p : E \to Z$, by the equation $\vec{A} \lrcorner \omega = A$. Then the vector fields $\vec{A}$ for $A \in \mathfrak{h}$ must generate the right $H$-action:
$$\vec{A} = \left. \frac{d}{dt} r_{e^{tA}} \right|_{t=0} .$$

If $G/P$ is a rational homogeneous variety, then a family of Cartan geometries modelled on $G/P$ is called a family of *parabolic geometries.*

Recall the foliation of $\mathbb{C}^n \setminus 0$ by radial lines. Clearly foliations, and even holomorphic submersions, do not generally extend to envelopes of holomorphy. Therefore we will pose the Hartogs extension problem only for holomorphic submersions which are already assumed to extend.

**Theorem 10.2.** *Suppose that $G \subset \mathrm{GL}(n, \mathbb{C})$ is a reductive algebraic group. Suppose that $M^{n+p}$ and $Z^p$ are both domains in Stein manifolds, with envelopes of holomorphy $\hat{M}$ and $\hat{Z}$ respectively. Suppose that $\pi : \hat{M} \to \hat{Z}$ is a holomorphic submersion. Then every holomorphic family of $G$-structures on $\pi : M \to Z$ extends uniquely to a holomorphic family of $G$-structures on $\pi : \hat{M} \to \hat{Z}$.*

**Proof.** Let $VM$ be the vertical bundle $\ker \pi'$ of $\pi : M \to Z$ and $V\hat{M}$ be vertical bundle $\ker \pi'$ of $\pi : \hat{M} \to \hat{Z}$. A $G$-structure on $M$ is equivalent to a section of $FVM/G \subset FV\hat{M}/G$. The total space of $FV\hat{M}/G$ is a Hartogs extension target by Corollary 3.33 on page 10. Therefore if $s : M \to FVM/G$ is a $G$-structure, then $s$ extends uniquely to a holomorphic map $s : \hat{M} \to FV\hat{M}/G$. Let $p : FV\hat{M}/G \to \hat{M}$ denote the bundle map. Then $ps$ is the identity on $M$, and therefore by analytic continuation is the identity on $\hat{M}$. So the extension is also a holomorphic family of $G$-structures. ∎

The reader should compare the proof to that of Theorem 5.20 on page 24; it is a small change of notation. We give one more example:

**Theorem 10.3.** *Families of parabolic geometries extend across subsets of complex codimension* 2 *or more.*

First we will need a lemma from representation theory:

**Definition 10.4.** Suppose that $P \subset G$ is a parabolic subgroup of a complex semisimple Lie group. There is a Cartan subgroup of $G$, say $H$, which lies in $P$. Fixing a choice of Cartan subgroup, we induce a choice of root system for $G$. Divide up the roots of $G$ into those whose root spaces lie in $\mathfrak{p}$ and the rest. The former we will call $P$-roots, and their root spaces we will call $P$-root spaces. Among the $P$-roots $\alpha$, there are those for which $-\alpha$ is also a $P$-root; call these $M$-roots. The remaining $P$-roots we call $N$-roots. The roots which are not $P$-roots we call $N^-$-roots.

**Lemma 10.5 (Langlands).** *Suppose that $P \subset G$ is a parabolic subgroup of a complex semisimple Lie group. Then $P$ admits a decomposition $P = MAN$ into connected complex Lie subgroups, where $M$ is a complex semisimple Lie group, $A$ is a abelian linear algebraic Lie group, $MA$ is a maximal reductive algebraic subgroup of $P$, and $N$ is a unipotent subgroup. At most a finite subgroup of $MA$ acts trivially on $\mathfrak{g}/\mathfrak{p}$.*

For proof see Knapp [35, p. 478].

Now we prove Theorem 10.3.

**Proof.** It is clearly that solutions of Hartogs or Thullen extension problems are unique when they exist, so we can treat this problem as a local problem. Pick a complex manifold $M$, a subset $S \subset M$ of complex codimension 2 or more, a holomorphic submersion $\pi : M \to Z$, with vertical bundle $VM \to M$, and a family of parabolic geometries $E \to M \backslash S$ modelled on some rational homogeneous variety $G/P$. In terms of the Langlands decomposition, $P = MAN$, the maximal reductive subgroup of $P$ is $MA$. We only need to extend $E$ locally, so we can assume that $\det V^*M$ admits a nowhere vanishing holomorphic section. By McKay [41, Theorem 2, p. 2], $E \to M\backslash S$ admits a holomorphic reduction $E_0 \subset E$ of structure group to the maximal reductive algebraic group $MA \subset P$. By Lemma 10.5, the group $MA$ acts on $\mathfrak{g}/\mathfrak{p}$ with only some finite subgroup $K \subset MA$ acting trivially. Therefore the underlying family of first order structures is exactly an embedding $E_0/K \to FV\,(M \setminus S)$.

Since the problem is local, we can assume that $M$ is a ball in $\mathbb{C}^{n+p}$. But then $M \setminus S$ has envelope of holomorphy $M$. By Theorem 10.2 on the facing page, the underlying first order structure extends to a bundle mapping $E_0'' \to FVM$ for a holomorphic principal right $MA/K$-bundle $E_0'' \to M$. The covering bundle $E_0 \to E_0'$ extends to a covering bundle $E_0' \to E_0''$, say, by Lemma 4.6 on page 16. Let $E' = E_0' \times_{MA} P$. By Proposition 3.36 on page 10, we can extend the holomorphic section $\omega$ of the bundle $(VE \otimes \mathfrak{g})^H \to M \setminus S$ to a holomorphic section of $(VE' \otimes \mathfrak{g})^H \to M$.

This section is thus an $H$-invariant section of $VE \otimes \mathfrak{g}$, the first property of a Cartan connection for a family of Cartan geometries.

Let $H$ be the set of points $e \in E'$ at which $\omega_e : V_eE' \to \mathfrak{g}$ is *not* a linear isomorphism. Clearly $H$ is a hypersurface, given by the one equation $\det \omega_e = 0$. Moreover, this hypersurface is $P$-invariant, so projects to a hypersurface in $\hat{M}$. This hypersurface doesn't intersect $M$, so is empty by Lemma 3.24 on page 8. Therefore $\omega$ satisfies the second property of a Cartan connection.

Over $M$, $\omega$ satisfies $\vec{A} \lrcorner \omega = A$, for any $A \in \mathfrak{h}$. By analytic continuation, this must also hold over $\hat{M}$, the third and final property of a Cartan connection. ■

**Example 10.6.**

**Proposition 10.7.** *Pick a rational homogeneous variety $G/P$. Pick a holomorphic fibration $\pi : M^{n+p} \to Z^p$ whose fibers are biholomorphic to $G/P$ away from a set of complex codimension 2 or more in $Z$. There is a holomorphic principal $G$-bundle $B \to Z$ so that this fibration is a holomorphic fiber bundle $M = B/P \to Z$.*

**Proof.** Suppose that $\pi : M^{n+p} \to Z^p$ is a holomorphic fibration. Let $S_Z \subset Z$ be a subset of complex codimension 2 or more, and let $S_M = \pi^{-1}S_Z$. Suppose that each fiber $M_z \subset M$ is biholomorphic to $G/P$, as long as $z$ is not a point of $S_Z$. A rational homogeneous variety admits a unique $G/P$-geometry: the standard flat one; see McKay [39, Corollary 7, p. 16]. Therefore $\pi : M \setminus S_M \to Z \setminus S_Z$ admits a unique holomorphic family of $G/P$-geometries. By Theorem 10.3 on the facing page, this family of parabolic geometries extends holomorphically to a family of parabolic geometries on $M$. Therefore each of the fibers $M_s$ for $s$ in $S_Z$ also bears a parabolic geometry. By continuity of the curvature, the parabolic geometry on each fiber $M_s$ is flat. Every holomorphic fibration is a fiber bundle of smooth real manifolds. Since $G/P$ is compact and simply connected, each fiber $M_s$ is also compact and simply connected. Therefore $M_s$ has a holomorphic developing map to $G/P$, a local biholomorphism. Because the fiber $M_s$ is compact, the developing map is a covering map. Because $G/P$ is simply connected, the developing map is a biholomorphism. Therefore all fibers are biholomorphic to $G/P$, and the biholomorphisms preserve the parabolic geometry, so preserve the $G$-action. Local triviality is clear by using the developing map on a local section. ∎

This is a mild form of rigidity for $G/P$, much weaker than results of Hong [23] and of Hwang and Mok [26, 27].

## 11 Conclusion

Holomorphic geometric structures which occur naturally apparently almost always extend across subsets of complex codimension 2 or more, and extend from a domain in a Stein manifold to its envelope of holomorphy. I found the idea behind the Hartog's lemma for Cartan geometries in Gunning [21, p. 126]; the same idea appears for $G$-structures in Hwang and Mok [26]. Similarly, you might hope to extend Cartan geometries or $G$-structures across real or complex analytic submanifolds and analytic subsets using well known extension techniques; see Siu [57] for an introduction to these techniques. For example, restrict holomorphic geometric structures to the boundary of a pseudoconvex domain, to obtain geometric structures on the boundary. It seems natural to ask which geometric structures on the boundary arise in this way.

The *generalized Cartan geometries* of Alekseevsky and Michor [3] have the same definition as Cartan geometries (see Definition 2.6 on page 3), except that they omit condition 2. Examples arise naturally from maps to homogeneous spaces, and also from compactification problems in geometric structures (see Gallo et al. [18] where the authors refer to these geometries as *branched* rather than *generalized*). The Thullen and Hartogs extension problems for maps to complex homogeneous spaces (which can be studied using invariants derived via Cartan's method of the moving frame) are largely untouched, besides the work of Ivashkovich [31]. The obvious meromorphic extension problems for meromorphic geometric structures are open.

### 11.1 Curvature of Hermitian metrics and extension problems

It is easy to generalize many of the theorems of this paper to utilize integral curvature bounds.

**Theorem 11.1 (Shevchishin [56]).** *A holomorphic vector bundle on an $n$-manifold extends across a subset of complex codimension 2 or more just when it admits a Hermitian metric with $L^n$-bounded curvature.*

This result in particular characterizes the nondegenerate plane fields that extend across subsets of complex codimension 2 or more, although the characterization is not easy to apply to examples, since we have to pick a metric for which we can see how to bound the integral.

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