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Quantum Probability, Renormalization and Infinite-Dimensional $*$-Lie Algebras*

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Abstract. The present paper reviews some intriguing connections which link together a new renormalization technique, the theory of $*$-representations of infinite dimensional $*$-Lie algebras, quantum probability, white noise and stochastic calculus and the theory of classical and quantum infinitely divisible processes.

Key words: quantum probability; quantum white noise; infinitely divisible process; quantum decomposition; Meixner classes; renormalization; infinite dimensional Lie algebra; central extension of a Lie algebra

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1 Introduction

The investigations on the stochastic limit of quantum theory in [24] led to the development of quantum white noise calculus as a natural generalization of classical and quantum stochastic calculus. This was initially developed at a pragmatic level, just to the extent needed to solve the concrete physical problems which stimulated the birth of the theory [22, 23, 24, 29]. The first systematic exposition of the theory is contained in the paper [25] and its full development in [31].

This new approach naturally suggested the idea to generalize stochastic calculus by extending it to higher powers of (classical and quantum) white noise. In this sense we speak of *nonlinear white noise calculus*.

This attempt led to unexpected connections between mathematical objects and results emerged in different fields of mathematics and at different times, such as white noise, the representation theory of certain famous Lie algebras, the renormalization problem in physics, the theory of independent increment stationary (Lévy) processes and in particular the Meixner classes, ... The present paper gives an overview of the path which led to these connections.

Our emphasis will be on latest developments and open problems which are related to the renormalized powers of white noise of degree ≥ 3 and the associated Lie algebras. We also briefly review the main results in the case of powers < 3 and, since the main results are scattered in several papers, spanning a rather long time period, we include some bibliographical references which allow the interested reader to reconstruct this development.

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The content of the paper is the following. In Section 2 we state the Lie algebra renormalization problem taking as a model the Lie algebra of differential operators with polynomial coefficients.

In Section 3 we recall the basic notions on $*$-representations of $*$-Lie algebras and their connections with quantum probability. Section 4 describes the standard Fock representation (i.e. for first order fields). Sections 5 and 6 recall some notions on current algebras and their connections with (boson) independent increment processes. The transition from first to second quantization (in the usual framework of Heisenberg algebras) can be considered, from the algebraic point of view, as a transition from a $*$-Lie algebra to its current algebra over $\mathbb{R}$ (or $\mathbb{R}^d$) and, from the probabilistic point of view, as a transition from a class of infinitely divisible random variables to the associated independent increment process. These sections generalize this point of view to Lie algebras more general than the Heisenberg one. Section 7 illustrates the role of renormalization in the quadratic case and shows how, after renormalization, the above mentioned connection between $*$-Lie algebras and independent increment processes, can be preserved, leading to interesting new connections. We also quickly describe results obtained in this direction (giving references to existing surveys for more detailed information). Starting from Section 8 we begin to discuss the case of higher (degree ≥ 3) powers of white noise and we illustrate the ideas that eventually led to the identification of the RHPWN and the w_∞-$*$-Lie algebras (more precisely their closures) (Section 10). We illustrate the content of the no-go theorems and of the various attempts made to overcome the obstructions posed by them. The final sections outline some connections between renormalization and central extensions and some recent results obtained in this direction.

2 Renormalization and differential operators with polynomial coefficients

Since the standard white noises, i.e. the distribution derivatives of Brownian motions, are the prototypes of free quantum fields, the program to find a meaningful way to define higher powers of white noise is related to an old open problem in physics: the renormalization problem. This problem consists in the fact that these higher powers are strongly singular objects and there is no unique way to attribute a meaning to them. That is why one speaks of *renormalized higher powers of white noise* (RHPWN).

The renormalization problem has an old history and we refer to [36] for a review and bibliographical indications. In the past 50 years the meaning of the term *renormalization* has evolved so as to include a multitude of different procedures. A common feature of all these generalizations is that, in the transition from a discrete system to a continuous one, certain expressions become meaningless and one tries to give a mathematical formulation of the continuous theory which *keeps as many properties as possible from the discrete approximation.* One of the main difficulties of the problem consists in its precise formulation, in fact one does not know a priori which properties of the discrete approximation will be preserved in the continuous theory.

The discovery of such properties is one of the main problems of the theory.

The present paper discusses recent progresses in the specification of this problem to a basic mathematical object: the $*$-algebra of differential operators with polynomial coefficients (also called the *full oscillator algebra*).

More precisely, in the present paper, when speaking of the *renormalization problem*, we mean the following:

(*i*) to construct a *continuous analogue* of the $*$-algebra of differential operators with polynomial coefficients acting on the space $\mathcal{C}^\infty(\mathbb{R}^n;\mathbb{C})$ of complex valued smooth functions in $n \in \mathbb{N}$ real variables (here *continuous* means that the space $\mathbb{R}^n \equiv \{\text{functions } \{1,\dots,n\} \to \mathbb{R}\}$ is replaced by the space $\{\text{functions } \mathbb{R} \to \mathbb{R}\}$);

(*ii*) to construct a $*$-representation of this algebra as operators on a Hilbert space $\mathcal{H}$ (all spaces considered in this paper will be complex separable and all associative algebras will have an identity, unless otherwise stated);

(*iii*) the ideal goal would be to have a unitary representation, i.e. one in which the skew symmetric elements of this $*$-algebra can be exponentiated, leading to strongly continuous 1-parameter unitary groups.

In the physical interpretation, $\mathcal{H}$ would be the state space of a physical system with infinitely many degrees of freedom (typically a field, an infinite volume gas, ...) and the 1-parameter unitary groups correspond to time evolutions.

The $*$-algebra of differential operators in n variables with polynomial coefficients can be thought of as a realization of the universal enveloping algebra of the n-dimensional Heisenberg algebra and its Lie algebra structure is uniquely determined by the Heisenberg algebra. In the continuous case the renormalization problem arises from this interplay between the structure of Lie algebra and that of associative algebra. The developments we are going to describe were originated by the idea, introduced in the final part of the paper [25], of *first renormalizing the Lie algebra structure (i.e. the commutation relations), thus obtaining a new $*$-Lie algebra, then proving existence of nontrivial Hilbert space representations.*

In the remaining of this section we give a precise formulation of problem (i) above and explain where the difficulty is. In Section 6 we explain the connection with probability, in particular white noise and other independent increment processes.

2.1 1-dimensional case

The *position operator* acts on $\mathcal{C}^\infty(\mathbb{R};\mathbb{C})$ as multiplication by the independent variable

$$(qf)(x) := xf(x), \qquad x \in \mathbb{R}, \qquad f \in \mathcal{C}^\infty(\mathbb{R};\mathbb{C}).$$

The usual derivation ∂_x also acts on $\mathcal{C}^\infty(\mathbb{R};\mathbb{C})$ and the two operators satisfy the commutation relation

$$[q, \partial_x] = -1,$$

where 1 denotes the identity operator on $\mathcal{C}^\infty(\mathbb{R};\mathbb{C})$. Defining the *momentum operator* by

$$p := \frac{1}{i}\,\partial_x, \qquad (pf)(x) := \frac{1}{i}\left(\frac{df}{dx}\right)(x)$$

one obtains the Heisenberg commutation relations

$$[q, p] = i, \qquad [q, q] = [p, p] = 0,$$

which give a structure of Lie algebra to the vector space generated by the operators q, p, 1. This is the *Heisenberg algebra* $\mathrm{Heis}(\mathbb{R})$. The associative algebra $\mathcal{A}(\mathbb{R})$, algebraically generated by the operators p, q, 1, is the $*$-algebra of differential operators with polynomial coefficients in one real variable and coincides with the vector space

$$\sum_{n\in\mathbb{N}} P_n(q)p^n,$$

where the $P_n(X)$ are polynomials of arbitrary degree in the indeterminate X and almost all the $P_n(X)$ are zero.

There is a unique complex involution $*$ on this algebra such that

$$q^* = q, \qquad p^* = p, \qquad 1^* = 1. \tag{2.1}$$

The $*$-Lie algebra structure on $\mathcal{A}(\mathbb{R})$, induced by the commutator, is uniquely determined by the same structure on the Heisenberg algebra and gives the commutation relations

$$[p^n, q^k] = \sum_{h=1}^{n} (-i)^h \binom{n}{h} k^{(h)} q^{k-h} p^{n-h}, \tag{2.2}$$

where $k^{(h)}$ is the Pochammer symbol:

$$x^{(0)} = 1, \qquad x^{(y)} = x(x-1)\cdots(x-y+1), \qquad x^{(y)} = 0 \quad \text{if} \quad y > x. \tag{2.3}$$

2.2 Discrete case

Let us fix $N \in \mathbb{N}$ and replace $\mathbb{R}$ in Section 2.1 by the function space

$$x \equiv (x_1, \dots, x_N) \in \{\text{functions } \{1, \dots, N\} \to \mathbb{R}\} =: \mathcal{F}_{\{1,\dots,N\}}(\mathbb{R}) \equiv \mathbb{R}^N.$$

Thus $\mathcal{C}^\infty(\mathbb{R}; \mathbb{C})$ is replaced by $\mathcal{C}^\infty(\mathbb{R}^N; \mathbb{C})$. The notation $\mathcal{F}_{\{1,\dots,N\}}(\mathbb{R})$ is more appropriate than $\mathbb{R}^N$ because it emphasizes its algebra structure (for the pointwise operations), which is required in its interpretation as a test function space (see the end of this section). For the values of a function $x \in \mathcal{F}_{\{1,\dots,N\}}(\mathbb{R})$ we will use indifferently the notations x_s or $x(s)$.

The position and momentum operators are then

$$(q_s f)(x) := x_s f(x), \qquad s \in \{1, \dots, N\},$$
$$p_s := \frac{1}{i}\frac{\partial}{\partial x_s}, \qquad (p_s f)(x) = \frac{1}{i}\left(\frac{\partial f}{\partial x_s}\right)(x), \qquad s \in \{1, \dots, N\}, \qquad f \in \mathcal{C}^\infty(\mathbb{R}^N; \mathbb{C}),$$

which give the Heisenberg commutation relations

$$[q_s, p_t] = i\delta_{s,t} \cdot 1, \qquad [q_s, q_t] = [p_s, p_t] = 0, \qquad s, t \in \{1, \dots, N\}, \tag{2.4}$$

where $\delta_{s,t}$ is the Kronecker delta. The involution is defined as in (2.1), i.e.

$$q_s^* = q_s, \qquad p_s^* = p_s, \qquad 1^* = 1, \qquad s \in \{1, \dots, N\}. \tag{2.5}$$

The vector space generated by the operators q_s, p_t, 1 ($s, t \in \{1, \dots, N\}$) has therefore a structure of $*$-Lie algebra, it is called the N*-dimensional Heisenberg algebra* and is denoted

$$\mathrm{Heis}(\mathbb{R}^N) = \mathrm{Heis}(\mathcal{F}_{\{1,\dots,N\}}(\mathbb{R})).$$

The associative $*$-algebra $\mathcal{A}(\mathbb{R}^N) = \mathcal{A}(\mathcal{F}_{\{1,\dots,N\}}(\mathbb{R}))$, algebraically generated by the operators p_s, q_s, 1 ($s, t \in \{1, \dots, N\}$), is the $*$-algebra of differential operators with polynomial coefficients in N real variables and coincides with the vector space

$$\sum_{n \in \mathbb{N}^N} P_n(q) p^n, \qquad p^0 := q^0 := 1, \tag{2.6}$$

where the $P_n(X) = P_n(X_1, \dots, X_N)$ are polynomials of arbitrary degree in the N commuting indeterminates $X_1, \dots, X_N$, almost all the $P_n(X)$ are zero and, for $n = (n_1, \dots, n_N) \in \mathbb{N}^N$, by definition

$$p^n := p_1^{n_1} p_2^{n_2} \cdots p_N^{n_N}.$$

The operation of writing the product of two such operators in the form (2.6) can be called *the normally ordered form* of such an operator with respect to the generators p_s, q_s, 1 ($s,t \in \{1,\dots,N\}$).

Also in this case the $*$-Lie algebra structure on $\mathcal{A}(\mathcal{F}_{\{1,\dots,N\}}(\mathbb{R}))$, induced by the commutator, is uniquely determined by the same structure on the Heisenberg algebra and gives the commutation relations

$$[p_s^n, q_t^k] = \sum_{h=1}^{n} (-i)^h \binom{n}{h} k^{(h)} \delta_{s,t}^h q_t^{k-h} p_s^{n-h}. \tag{2.7}$$

Notice that, with respect to formula (2.2), the new ingredient is the factor $\delta_{s,t}^h$, i.e. the h-th power of the Kronecker delta. Since

$$\delta_{s,t}^h = \delta_{s,t} \tag{2.8}$$

the power is useless, but we kept it to keep track of the number of commutators performed and to make easier the comparison with the continuous case (see Section 2.3 below). Considering $\mathcal{F}_{\{1,\dots,N\}}(\mathbb{R})$ as a test function space and defining the smeared operators

$$p^n q^k(f) := \sum_{t\in\{1,\dots,N\}} p_t^n q_t^k f(t), \qquad f \in \mathcal{F}_{\{1,\dots,N\}}(\mathbb{R})$$

and the scalar product

$$\langle f, g\rangle := \sum_{t\in\{1,\dots,N\}} f(t)g(t)$$

the commutation relations (2.4) and (2.7) become respectively

$$[q(f), p(g)] = i\langle f, g\rangle, \qquad [q(f), q(g)] = [p(f), p(g)] = 0,$$
$$[p^n(f), q^k(g)] = \sum_{h=1}^{n} (-i)^h \binom{n}{h} k^{(h)} q^{k-h} p^{n-h}(fg).$$

Notice that the algebra structure on the test function space is required only when $n + k \geq 3$. The above construction can be extended to the case of an arbitrary discrete set I: all the above formulae continue to hold with the set $\{1,\dots,N\}$ replaced by a generic finite subset of I which might depend on the test function.

2.3 Continuous case

In this section the discrete space $\{1,\dots,N\}$ is replaced by $\mathbb{R}$ and the discrete test function algebra $\mathcal{F}_{\{1,\dots,N\}}(\mathbb{R})$ by an algebra $\mathcal{F}_{\mathbb{R}}(\mathbb{R})$ of smooth functions from $\mathbb{R}$ into itself (for the pointwise operations).

As an analogue of the space $\mathcal{C}^\infty(\mathbb{R}^N;\mathbb{C})$ one can take the space $\mathcal{C}^\infty(\mathcal{F}_{\mathbb{R}}(\mathbb{R});\mathbb{C})$ of all smooth cylindrical functions on $\mathcal{F}_{\mathbb{R}}(\mathbb{R})$ (i.e. functions $f : \mathcal{F}_{\mathbb{R}}(\mathbb{R}) \to \mathbb{C}$ for which there exist $N \in \mathbb{N}$, $\{s_1,\dots,s_N\} \subset \mathbb{R}$, and a smooth function $f_{s_1,\dots,s_N} : \mathbb{R}^N \to \mathbb{C}$, such that $f(x) = f_{s_1,\dots,s_N}(x_{s_1},\dots, x_{s_N})$, $\forall\, x \in \mathcal{F}_{\mathbb{R}}(\mathbb{R})$). This space is sufficient for algebraic manipulations but it is too narrow to include the simplest functionals of interest for the applications in physics or in probability theory: this is where white noise and stochastic calculus play a role. The position operators can be defined as before, i.e.

$$(q_s f)(x) = x_s f(x), \qquad x \in \mathcal{F}_{\mathbb{R}}(\mathbb{R}), \qquad f \in \mathcal{C}^\infty(\mathcal{F}_{\mathbb{R}}(\mathbb{R});\mathbb{C}).$$

The continuous analogue of the partial derivatives $\frac{\partial}{\partial x_s}$, hence of the momentum operators p_s, can be defined by fixing a subspace $\mathcal{F}^0_{\mathbb{R}}(\mathbb{R})$ of $\mathcal{F}_{\mathbb{R}}(\mathbb{R})$ and considering functions $f \in \mathcal{C}^\infty(\mathcal{F}_{\mathbb{R}}(\mathbb{R}); \mathbb{C})$ whose Gateaux derivative in the direction S

$$D_S f(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(f(x+\varepsilon S) - f(x)) = \langle f'(x), S\rangle$$

exists for any test function $S \in \mathcal{F}^0_{\mathbb{R}}(\mathbb{R})$ and is a continuous linear functional on $\mathcal{F}^0_{\mathbb{R}}(\mathbb{R})$ (in some topology whose specification is not relevant for our goals). Denoting $\langle \mathcal{F}^0_{\mathbb{R}}(\mathbb{R})', \mathcal{F}^0_{\mathbb{R}}(\mathbb{R})\rangle$ the duality specified by this topology, the elements of $\mathcal{F}^0_{\mathbb{R}}(\mathbb{R})'$ can be interpreted as distributions on $\mathbb{R}$ and symbolically written in the form

$$\langle f'(x), S\rangle = \int f'(x)(s)S(s)ds, \qquad S \in \mathcal{F}^0_{\mathbb{R}}(\mathbb{R}).$$

The distribution

$$f'(x)(s) =: \frac{\partial f}{\partial x_s}(x) \tag{2.9}$$

is called the *Hida derivative* of f at x with respect to x_s. Intuitively one can think of it as the Gateaux derivative along the δ-function at s, $\delta_s(t) := \delta(s-t)$:

$$\frac{\partial f}{\partial x_s} = D_{\delta_s} f.$$

There is a large literature on the theory of Hida distributions and we refer to [45] for more information. The momentum operators are then defined by

$$p_s = \frac{1}{i}\frac{\partial}{\partial x_s} = \frac{1}{i} D_{\delta_s}$$

and one can prove that the following generalization of the Heisenberg commutation relations holds:

$$[q_s, p_t] = i\delta(s-t)\cdot 1, \qquad [q_s, q_t] = [p_s, p_t] = 0, \qquad s \in \{1, \ldots, N\}, \tag{2.10}$$

where now $\delta(s-t)$ is Dirac's delta and all the identities are meant in the usual sense of operator valued distributions, i.e. one fixes a space of test functions, defines the smeared operators

$$q(f) := \int_{\mathbb{R}} f(t) q_t dt$$

and interprets any distribution identity as a shorthand notation for the identity obtained by multiplying both sides by one test function for each free variable and integrating over all variables.

The involution is defined as in (2.5) with the only difference that now $s \in \mathbb{R}$ and the vector space generated by the operator valued distributions q_s, p_t, 1 $(s, t \in \mathbb{R})$ has therefore a structure of $*$-Lie algebra. This algebra plays a crucial role in quantum field theory and is called the *current algebra of* Heis($\mathbb{R}$) *over* $\mathbb{R}$ (see Section 5) or simply the Boson algebra over $\mathbb{R}$. In the following, when no confusion can arise, we will use the term *Boson algebra* also for the discrete Heisenberg algebra (2.4).

Remark 2.1. One can combine the discrete and continuous case by considering current algebras of Heis($\mathbb{R}$) over

$$\{1, \ldots, N\} \times \{1, \ldots, d\} \times \mathbb{R} \equiv \{1, \ldots, N\} \times \mathbb{R}^d,$$

so that the commutation relations (2.10) become ($j, k \in \{1, \dots, N\}$; $x, y \in \mathbb{R}^d$)

$$[q_j(x), p_k(y)] = \delta_{j,k} i\delta(x-y) \cdot 1, \qquad [q_j(x), q_k(y)] = [p_j(x), p_k(y)] = 0.$$

This corresponds to considering N-dimensional vector fields on $\mathbb{R}^d$ rather than scalar fields on $\mathbb{R}$. The value of the dimension d plays a crucial role in many problems, but not in those discussed in the present paper. Therefore we restrict our discussion to the case $d = 1$. It is however important to keep in mind that all the constructions and statements in the present paper remain true, with minor modifications, when $\mathbb{R}$ is replaced by $\mathbb{R}^d$.

Up to now the discussion of the continuous case has been exactly parallel to the discrete case. Moreover some unitary representations of the continuous analogue of the Heisenberg algebra are known (in fact very few: essentially only Gaussian – quasi-free in the terminology used in physics, see Section 7 below).

However the attempt to build the continuous analogue of the algebra of higher order differential operators with polynomial coefficients leads to some principle difficulties. For example the naive way to define the *second Hida derivative* of f at x with respect to x_s (i.e. $\frac{\partial^2 f}{\partial x_s^2}(x)$) would be to differentiate the "function" $x \mapsto f'(x)(s)$ for fixed $s \in \mathbb{R}$, but even in the simplest examples, one can see that the identity (2.9) defines a distribution so that this map is meaningless.

One might try to forget the concrete realization in terms of multiplication operators and derivatives and to generalize to the continuous case the $*$-Lie algebra structure of $\mathcal{A}(\mathcal{F}_{\{1,\dots,N\}}(\mathbb{R}))$. This can be done for some subalgebras. For example, if the subspace $\mathcal{F}^0_{\mathbb{R}}(\mathbb{R})$ is an algebra for the pointwise operations, then one can extend the $*$-Lie algebra structure of the Boson algebra over $\mathbb{R}$ to first order differential operators (vector fields) by introducing functions of the position operator, which are well defined for any test function $v \in \mathcal{F}^0_{\mathbb{R}}(\mathbb{R})$ by

$$(v(q_s)f)(x) := v(x_s)f(x), \qquad x \in \mathcal{F}_{\mathbb{R}}(\mathbb{R}), \qquad f : \mathcal{F}_{\mathbb{R}}(\mathbb{R}) \to \mathbb{C}$$

and using the commutation relation

$$[v(q_s), p_t] = i\delta(s-t)v'(q_s) \cdot 1 \tag{2.11}$$

which leads to

$$[u(q_{s_1})p_{t_1}, v(q_{s_2})p_{t_2}] = iu(q_{s_1})v'(q_{s_2})\delta(t_1 - s_2)p_{t_2} - iv(q_{s_2})u'(q_{s_1})\delta(s_1 - t_2)p_{t_1}.$$

In terms of test functions and with the notations:

$$u(q; a) := \int_{\mathbb{R}} a_s u(q_s) ds, \qquad p(b) := \int b(s) p_s ds, \qquad a, b \in \mathcal{F}^0_{\mathbb{R}}(\mathbb{R})$$

the above commutator becomes, with $a, b, c, d \in \mathcal{F}^0_{\mathbb{R}}(\mathbb{R})$:

$$[u(q; a)p(b), v(q; c)p(d)] = iv'(q; bc)u(q; a)p(d) - iu'(q; ad)v(q; c)p(b). \tag{2.12}$$

Another interesting class of subalgebras is obtained by considering the vector space generated by arbitrary (smooth) functions of q and first order polynomials in p. The test function form of (2.11) is then

$$[u(q; a), p(b)] = iu'(q; ab) \tag{2.13}$$

which shows that, for any $n \in \mathbb{N}$, the vector space generated by the family $\{u(q; a), p(b)\}$ where u is a complex polynomial of degree $\le n$ and a, b are arbitrary test functions, is a nilpotent $*$-Lie algebra. We will see in Section 14 that the simplest nonlinear case (i.e. $n = 2$) corresponds to the current algebra on the unique nontrivial central extension of the one dimensional Heisenberg algebra.

The right hand sides of (2.12) and (2.13) are well defined so at least we can speak of the $*$-Lie algebra of vector fields in continuously many variables, even if we do not know if some $*$- or unitary representations of this algebra can be built. In the case of the algebra corresponding to (2.12), one can build unitary representations but the interpretation of these representations is still under investigation.

The situation is different with the continuous analogue of the higher order commutation relations (2.7) (i.e. when p_t enters with a power ≥ 2). Here some difficulties arise even at the Lie algebra level.

In fact the continuous analogue of these relations leads to

$$[p_s^n, q_t^k] = \sum_{h=1}^{n} (-i)^h \binom{n}{h} k^{(h)} \delta(s-t)^h q_t^{k-h} p_s^{n-h}, \tag{2.14}$$

which is meaningless because it involves powers of the delta function.

Any rule to give a meaning to these powers in such a way that the brackets, defined by the right hand side of (2.14) induce a $*$-Lie algebra structure, will be called a *renormalization rule*.

There are many inequivalent ways to achieve this goal. Any Lie algebra obtained with this procedure will be called a *renormalized higher power of white noise (RHPWN) $*$-Lie algebra*.

One might argue that, since in the discrete case the identity (2.8) holds, a natural continuous analogue of the commutation relations (2.7) should be

$$[p_s^n, q_t^k] = \sum_{h=1}^{n} (-i)^h \binom{n}{h} k^{(h)} \delta(s-t) q_t^{k-h} p_s^{n-h}.$$

We will see in Section 7 that this naive approach corresponds, up to a multiplicative constant, to Ivanov's renormalization or to consider the current algebra, over $\mathbb{R}$, of the universal enveloping algebra of $\mathrm{Heis}(\mathbb{R})$.

One of the new features of the renormalization problem, brought to light by the present investigation, is that some subtle algebraic obstructions (no-go theorems) hamper this idea at least as far as the Fock representation is concerned (a discussion of this delicate point is in Section 8.2).

A first nontrivial positive result in this programme is that, by separating first and second powers, one can overcome these obstructions in the case $n = 2$ and the results are quite encouraging (see Section 7).

However such a separation becomes impossible for $n \geq 3$. In fact, in Section 12 we will provide strong evidence in support of the thesis that any attempt to force this separation at the level of a Fock type representation, brings back either to the first or to the second order case.

3 $*$-representations of $*$-Lie algebras: connections with quantum probability

Suppose that one fixes a renormalization and defines a RHPWN $*$-Lie algebra in the sense specified above. Then, according to the programme formulated in Section 2, the next step is to build $*$-representations of it. Since different Lie algebra structures will arise from different renormalization procedures, we recall in this section some notions concerning $*$-representations of general $*$-Lie algebras and their connections with quantum probability.

Definition 3.1. A $*$-representation of a $*$-Lie algebra $\mathcal{G}$ is a triple

$$\{\mathcal{H}, \mathcal{H}_0, \pi\},$$

where $\mathcal{H}$ is a Hilbert space, $\mathcal{H}_0$ is a dense sub-Hilbert space of $\mathcal{H}$, π is a representation of $\mathcal{G}$ into the linear operators from $\mathcal{H}_0$ into itself (this implies in particular that the brackets are well defined on $\mathcal{H}_0$), and the elements of $\pi(\mathcal{G})$ are adjointable linear operators from $\mathcal{H}_0$ into itself satisfying

$$\pi(l)^* = \pi(l^*), \qquad \forall\, l \in \mathcal{G}.$$

If moreover the (one-mode) π-field operators

$$F_X(z) := \frac{1}{i}(z\pi(X)^* - \overline{z}\pi(X))$$

are essentially self-adjoint, the $*$-representation π is called unitary.

A vector $\Phi \in \mathcal{H}$ is called cyclic for the representation π if:

(*i*) $\forall\, n \in \mathbb{N}$ the vector

$$\pi(l)^n \Phi \in \mathcal{H} \tag{3.1}$$

is well defined (this is always the case if $\Phi \in \mathcal{H}_0$);

(*ii*) denoting $\mathcal{H}_0(\Phi)$ the algebraic linear span of the vectors (3.1) the triple $\{\mathcal{H}, \mathcal{H}_0(\Phi), \pi\}$ is a $*$-representation of $\mathcal{G}$.

Remark 3.1. If $\{\mathcal{H}, \mathcal{H}_0(\Phi), \pi\}$ is a $*$-representation of $\mathcal{G}$ with cyclic vector Φ, one can always assume that $\mathcal{H}_0 = \mathcal{H}_0(\Phi)$. In this case we omit $\mathcal{H}_0$ from the notations and speak only of *the cyclic $*$-representation* $\{\mathcal{H}, \Phi, \pi\}$.

Any cyclic $*$-representation of $\mathcal{G}$ induces a state φ (positive, normalized linear functional) on the universal enveloping $*$-algebra $U(\mathcal{G})$ of $\mathcal{G}$, namely

$$\varphi(a) := \langle \Phi, \pi_U(a)\Phi \rangle, \qquad a \in U(\mathcal{G}),$$

where π_U is the $*$-representation of $U(\mathcal{G})$ induced by π. Conversely, given a state φ on $U(\mathcal{G})$, the GNS construction gives a cyclic $*$-representation of $U(\mathcal{G})$ hence of $\mathcal{G}$. Thus the problem to construct (non trivial) cyclic $*$-representations of $\mathcal{G}$ (we will only be interested in this type of representations) is equivalent to that of constructing (nontrivial) states on $U(\mathcal{G})$ hence of $\mathcal{G}$. This creates a deep connection with quantum probability. To clarify these connections let us recall (without comments, see [9] for more informations) the following three basic notions of quantum probability:

Definition 3.2.

(*i*) An *algebraic probability space* is a pair $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is an (associative) $*$-algebra and φ a state on $\mathcal{A}$.

(*ii*) An *operator process* in the algebraic probability space $(\mathcal{A}, \varphi)$ is a self-adjoint family G of algebraic generators of $\mathcal{A}$ (typically a set of generators of $\mathcal{A}$).

(*iii*) For any $n \in \mathbb{N}$ and any map $g : k \in \{1, \ldots, n\} \to g_k \in G$ the complex number

$$\varphi(g_1 g_2 \cdots g_n)$$

is called a mixed moment of the process G of order n.

In the above terminology one can say that constructing a $*$-representation of a $*$-Lie algebra $\mathcal{G}$ is equivalent to constructing an algebraic probability space $\{U(\mathcal{G}), \varphi\}$ based on the universal enveloping $*$-algebra $U(\mathcal{G})$ of $\mathcal{G}$ or equivalently, by the Poincaré–Birkhoff–Witt theorem, an *operator process* in $\{U(\mathcal{G}), \varphi\}$ given by any self-adjoint family G of algebraic generators of $\mathcal{G}$.

In the following section we show that when $\mathcal{G}$ is the Boson algebra and φ the Fock state, the resulting algebraic probability space is that of the standard quantum white noise and its restriction to appropriate maximal Abelian (Cartan) subalgebras, gives the standard classical white noise.

4 The Fock representation of the Boson algebra and white noise

In the present section we discuss $*$-representations of the Boson algebra introduced in Section 2.3. All the explicitly known representations of this algebra can be constructed from a single one: the Fock representation.

To define this representation it is convenient to replace the generators q_s, p_t, 1 of the Boson algebra by a new set of generators b_s^+ (creator), b_t (annihilator), 1 (central element, often omitted from notations) defined by

$$b_t^+ = \frac{1}{\sqrt{2}}(q_t - ip_t), \qquad b_t = \frac{1}{\sqrt{2}}(q_t + ip_t). \tag{4.1}$$

The involution (2.5) and the commutation relations (2.10) then imply the relations

$$\begin{aligned} &(b_s^\dagger)^* = b_s, \\ &[b_t, b_s^\dagger] = \delta(t-s), \qquad [b_t^\dagger, b_s^\dagger] = [b_t, b_s] = 0, \end{aligned} \tag{4.2}$$

where $\delta(t-s) := \delta_{s,t}$ is Kronecker's delta in the discrete case and Dirac's delta in the continuous case.

The operator valued distribution form of the universal enveloping algebra $\mathcal{A}$, of the Boson algebra, is the algebraic linear span of the expressions of the form

$$b_{t_n}^{\varepsilon_n} \cdots b_{t_2}^{\varepsilon_2} b_{t_1}^{\varepsilon_1},$$

where $n \in \mathbb{N}$, $t_1, \ldots, t_n \in \mathbb{R}$, $\varepsilon_j \in \{+, -\}$ and $b_{t_n}^+ = b_t^\dagger$, $b_{t_n}^- = b_t$.

This has a natural structure of $*$-algebra induced by (4.2). On this algebra there is a particularly simple state characterized by the following theorem. We outline a proof of this theorem because it illustrates in a simple case the path we have followed to construct analogues of that state in much more complex situations, namely:

(*i*) to formulate an analogue of the Fock condition (4.3);

(*ii*) to use the commutation relations to associate a distribution kernel to any linear functional φ, satisfying the analogue of the Fock condition, in such a way that φ is positive if and only if this kernel is positive definite;

(*iii*) to prove that this kernel is effectively positive definite.

Theorem 4.1. *On the $*$-algebra $\mathcal{A}$, defined above, there exists a unique state φ satisfying*

$$\varphi(b_t^\dagger x) = \varphi(x b_t) = 0, \qquad \forall\, x \in \mathcal{A}. \tag{4.3}$$

Proof. From the commutation relations we know that $\mathcal{A}$ is the algebraic linear span of expressions of the form

$$b_{s_1}^\dagger b_{s_2}^\dagger \cdots b_{s_m}^\dagger b_{t_n} \cdots b_{t_2} b_{t_1} \tag{4.4}$$

(normally ordered products), interpreted as the central element 1 if both $m = n = 0$. Therefore, if a state φ satisfying (4.3) exists, then it is uniquely defined by the properties that $\varphi(1) = 1$ and $\varphi(x) = 0$ for any x of the form (4.4) with either m or $n \neq 0$. It remains to prove that the linear functional defined by these properties is positive.

To this goal notice that the commutation relations imply that $\mathcal{A}$ is also the algebraic linear span of expressions of the form

$$b_{s_1} b_{s_2} \cdots b_{s_m} b^\dagger_{t_n} \cdots b^\dagger_{t_2} b^\dagger_{t_1}$$

(anti-normally ordered products), interpreted as before. A linear functional φ on $\mathcal{A}$ is positive if and only if the distribution kernel

$$K\left(s_1, s_2, \ldots, s_m; t_1, t_2, \ldots, t_2, t_n\right) := \varphi\left(b_{s_1} b_{s_2} \cdots b_{s_m} b^\dagger_{t_n} \cdots b^\dagger_{t_2} b^\dagger_{t_1}\right)$$

is positive definite. If φ satisfies condition (4.3), then the above kernel is equal to (in obvious notations)

$$\begin{aligned}\varphi\left(b_{s_1} \cdots b_{s_{m-1}} [b_{s_m}, b^\dagger_{t_n} \cdots b^\dagger_{t_2} b^\dagger_{t_1}]\right) &= \sum_{h=1}^{n-1} \varphi\left(b_{s_1} \cdots b_{s_{m-1}} b^\dagger_{t_n} \cdots b^\dagger_{t_h+1} [b_{s_m}, b^\dagger_{t_h}] b^\dagger_{t_{h-1}} \cdots b^\dagger_{t_1}\right)\\ &= \sum_{h=1}^{n-1} \delta(s_m - t_h) \varphi\Big(b_{s_1} \cdots b_{s_{m-1}} b^\dagger_{t_n} \cdots b^\dagger_{t_h+1} b^\dagger_{t_{h-1}} \cdots b^\dagger_{t_1}\Big).\end{aligned}$$

From this, one deduces that the kernel $K\left(s_1, s_2, \ldots, s_m; t_1, t_2, \ldots, t_2, t_n\right)$ can be non zero if and only if $m = n$. Finally, since $\delta(s - t)$ is a positive definite distribution kernel, the positivity of φ follows, by induction, from Schur's lemma. ■

Definition 4.1. The unique state φ on $\mathcal{A}$, defined by Theorem 4.1 above, is called the Fock (or lowest weight) state.

The GNS representation $\{\mathcal{H}, \Phi, \pi\}$ of the pair $\{\mathcal{A}, \varphi\}$ is characterized by:

$$\pi(b_t)\Phi = 0 \tag{4.5}$$

in the operator valued distribution sense.

Definition 4.2. In the notations of Definition 4.1 the operator (more precisely, the operator valued distribution) process in $\{\mathcal{A}, \varphi\}$, defined by

$$\{\pi(b^\dagger_t), \pi(b_t)\}$$

is called the *Boson Fock (or standard quantum) white noise* on $\mathbb{R}$.

Definition 4.2 is motivated by the following theorem.

Theorem 4.2. *In the notation* (4.1)*, the two operator subprocesses in* $\{\mathcal{A}, \varphi\}$*:*

$$\{\pi(q_t)\} \qquad \textit{and} \qquad \{\pi(p_t)\} \tag{4.6}$$

are classical processes stochastically isomorphic to the standard classical white noise on $\mathbb{R}$.

Proof. The idea of the proof is that the Fock state has clearly mean zero. Using a modification of the argument used in the proof of Theorem 4.1 one shows that it is Gaussian and delta-correlated, i.e. it is by definition a standard classical white noise. ■

Remark 4.1. Theorem 4.2 and the relations (4.1) show that the Boson Fock white noise is equivalent to the pair (4.6), of standard classical white noises on $\mathbb{R}$. However the commutation relations (2.10) show that these two classical white noises do not commute so that classical probability does not determine their mixed moments: this additional information is provided by quantum probability.

In the following, when no confusion can arise, we omit from the notations the symbol π of the representation.

5 Current algebras over $\mathbb{R}^d$

Current algebras are associated to pairs: ($*$-Lie algebra, set of generators) as follows. Let $\mathcal{G}$ be a $*$-Lie algebra with a set of generators

$$\{l_\alpha^+, l_\alpha^-, l_\beta^0 :\ \alpha \in I,\ \beta \in I_0\},$$

where I, I_0 are sets satisfying

$$I \cap I_0 = \varnothing$$

and $(c_{\alpha\beta}^\gamma(\varepsilon, \varepsilon', \varepsilon''))$ are the structure constants corresponding to the given set of generators, so that:

$$[l_\alpha^\varepsilon, l_\beta^{\varepsilon'}] = c_{\alpha\beta}^\gamma(\varepsilon, \varepsilon', \varepsilon'') l_\gamma^{\varepsilon''}. \tag{5.1}$$

Here and in the following, summation over repeated indices is understood. The sets I, I_0 can, and in the examples below will, be infinite. However, here and in the following, we require that, in the summation on the right hand side of (5.1), only a finite number of terms are non-zero or equivalently that the structure constants are almost all zero (also this condition is automatically satisfied in the examples below). We assume that

$$(l_\alpha^+)^* = l_\alpha^-, \qquad \forall\, \alpha \in I, \qquad (l_\beta^0)^* = l_\beta^0, \qquad \forall\, \beta \in I_0.$$

The transition to the current algebra of $\mathcal{G}$ over $\mathbb{R}^d$ is obtained by replacing the generators by $\mathcal{G}$-valued distributions on $\mathbb{R}^d$

$$l_\alpha^\varepsilon \to l_\alpha^\varepsilon(x), \qquad x \in \mathbb{R}^d$$

and the corresponding relations by

$$\begin{aligned}
&l_\alpha^+(x)^* = l_\alpha^-(x), \qquad l_\beta^0(x)^* = l_\beta^0(x), \qquad \forall\, \alpha \in I, \qquad \forall\, \beta \in I_0,\\
&[l_\alpha^\varepsilon(x), l_\beta^{\varepsilon'}(y)] = c_{\alpha\beta}^\gamma(\varepsilon, \varepsilon', \varepsilon'') l_\gamma^{\varepsilon''}(x)\delta(x-y).
\end{aligned}$$

This means that the structure constants are replaced by

$$c_{\alpha\beta}^\gamma(\varepsilon, \varepsilon', \varepsilon'') \to c_{\alpha\beta}^\gamma(\varepsilon, \varepsilon', \varepsilon'')\delta(x-y).$$

In terms of test functions this can be equivalently formulated as follows.

Definition 5.1. Let $\mathcal{G}$ be a $*$-Lie algebra with generators

$$\{l_\alpha^+, l_\alpha^-, l_\beta^0 :\ \alpha \in I,\ \beta \in I_0\}$$

and let $\mathcal{C}$ be a vector space of functions from $\mathbb{R}^d$ to $\mathbb{C}$ called the test function space. A current algebra of $\mathcal{G}$ over $\mathbb{R}^d$ with test function space $\mathcal{C}$ is a $*$-Lie algebra with generators

$$\{l_\alpha^+(f), l_\alpha^-(g), l_\beta^0(h) : \ \alpha \in I, \ \beta \in I_0, \ f, g, h \in \mathcal{C}\}$$

such that the maps

$$f \in \mathcal{G} \mapsto l_\alpha^+(f), \ l_\beta^0(f)$$

are complex linear, the involution satisfies

$$(l_\alpha^+(f))^* = l_\alpha^-(f), \qquad (l_\beta^0(f))^* = l_\beta^0(\bar{f}), \qquad \forall\, \alpha \in I, \qquad \forall\, \beta \in I_0, \qquad \forall\, f \in \mathcal{C}$$

and the commutation relations are given by:

$$[l_\alpha^\varepsilon(f), l_\beta^{\varepsilon'}(g)] := c_{\alpha\beta}^\gamma(\varepsilon, \varepsilon', \varepsilon'') l_\gamma^{\varepsilon''}(f^\varepsilon g^{\varepsilon'})$$

with the convention:

$$f^\varepsilon = \begin{cases} \overline{f}, & \text{if} \quad \varepsilon = -, \\ f, & \text{if} \quad \varepsilon \in \{0, +\}. \end{cases}$$

Remark 5.1. By restriction of the test function space $\mathcal{C}$ to real valued functions, or more generally by restricting one's attention to real Lie algebras, one could avoid the introduction of generators and use an intrinsic definition. We have chosen the non-intrinsic formulation because we want to emphasize the intuitive analogy between the generators $l_\alpha^\pm$ and the powers of the creation/annihilation operators and between the generators l_α^0 and the powers of the number operator. Thus, for example, in the RHPWN $*$-Lie algebra with generators B_k^n, the indices $\alpha \in I$ are the pairs (n, k) with $n > k$ and the indices $\beta \in I_0$ are the diagonal pairs (n, n).

6 Connection with independent increment processes

Let $\mathcal{G}(\mathcal{C})$ be Lie algebra whose elements depend on test functions belonging to a certain space $\mathcal{C}$ of functions $f : \mathbb{R} \to \mathbb{C}$. Then $\mathcal{G}$ has a natural localization, obtained by fixing a family $\mathcal{F}$ of subsets of $\mathbb{R}$, e.g. intervals, and defining the subalgebra

$$\mathcal{G}_I := \{L(f) \in \mathcal{G} : \ \mathrm{supp}(f) \subseteq I\}.$$

Suppose that $\mathcal{G}$ enjoys the following property: $\forall\, I, J \subseteq \mathbb{R}$

$$I \cap J = \varnothing \ \Rightarrow \ [\mathcal{G}_I, \mathcal{G}_J] = 0$$

(notice that, if $\mathcal{G}(\mathcal{C})$ is a current algebra over $\mathbb{R}$ of a Lie algebra $\mathcal{G}$, then this property is automatically satisfied). If this is the case, denoting $\mathcal{A}_I$ the algebraic linear span of (the image of) $\mathcal{G}_I$ in any representation, the (associative) $*$-algebra generated by $\mathcal{A}_I$ and $\mathcal{A}_J$ is the linear span of the products of the form $a_I a_J$ where a_I (resp. a_J) is in $\mathcal{A}_I$ (resp. $\mathcal{A}_J$).

A similar conclusion holds if, instead of two disjoint sets, one considers an arbitrary finite number of disjoint sets. Denote $\mathcal{A}$ the algebraic linear span of $\mathcal{G}$ in a representation with cyclic vector Φ and φ, the restriction of the state $\langle \Phi, (\,\cdot\,)\Phi \rangle$ to $\mathcal{A}$. The given cyclic representation and the state φ are called *factorizable* if for any finite family $I_1, \ldots, I_n$, of mutually disjoint intervals of $\mathbb{R}$ one has

$$\varphi(a_{I_1} \ldots a_{I_n}) = \prod_{j=1}^n \varphi(a_{I_j}), \qquad a_{I_j} \in \mathcal{A}_{I_j}, \qquad j \in \{1, \ldots, n\}.$$

The Fock representation, and all its generalizations we have considered so far, have this property.

By restriction to Abelian subalgebras, factorizable representations give rise to classical (polynomially) independent increment processes. The above definition of factorizability applies to general linear functionals (i.e. not necessarily positive or normalized). In the following we will make use of this remark.

Given a cyclic representation $\{\mathcal{H}, \pi, \Phi\}$ of $\mathcal{G}$ (we omit π from notations), for any interval I one defines the subspace $\mathcal{H}_I$ of $\mathcal{H}$ as the closed subspace containing Φ and invariant under $\mathcal{G}_I$.

7 Quadratic powers: brief historical survey

The commutation relations imply that

$$[b_s^2, b_t^{+2}] = 4\delta(t-s)b_s^+ b_t + 2\delta(t-s)^2$$

and the appearance of the term $\delta(t-s)^2$ shows that b_t^{+2} and b_t^2 are not well defined even as operator valued distributions. The following formula, due to Ivanov, for the square of the delta function (cf. [46] for a discussion of its precise meaning)

$$\delta^2(t) = c\,\delta(t), \qquad c \text{ is arbitrary constant,} \tag{7.1}$$

was used by Accardi, Lu and Volovich to realize the program discussed in Section 8 for the second powers of WN.

Using this we find the renormalized commutation relation:

$$[b_s^2, b_t^{+2}] = 4\delta(t-s)b_s^+ b_t + 2c\delta(t-s). \tag{7.2}$$

Moreover (without any renormalization!)

$$[b_s^2, b_t^+ b_t] = 2\delta(t-s)b_t^2. \tag{7.3}$$

Introducing a test function space (e.g. the complex valued step functions on $\mathbb{R}$ with finitely many values), one verifies that the smeared operators (see the comments at the beginning of Section 8 about their meaning)

$$b_\varphi^+ = \int dt \varphi(t) b_t^2, \qquad b_\varphi = (b_\varphi^+)^+, \qquad n_\varphi^* = n_\varphi = \int dt \varphi(t) b_t^+ b_t \tag{7.4}$$

satisfy the commutation relations

$$[b_\varphi, b_\psi^+] = c\langle\varphi,\psi\rangle + n_{\overline{\varphi}\psi}, \qquad [n_\varphi, b_\psi] = -2b_{\overline{\varphi}\psi}, \qquad [n_\varphi, b_\psi^+] = 2b_{\varphi\psi}^+,$$
$$(b_\varphi^+)^+ = b_\varphi, \qquad n_\varphi^+ = n_{\overline{\varphi}}.$$

The relations (7.2), (7.3), or their equivalent formulation in terms of the generators (7.4), are then taken as the definition of the renormalized square of white noise (RSWN) $*$-Lie algebra. Recalling that $sl(2,\mathbb{R})$ is the $*$-Lie algebra with 3 generators B^-, B^+, M and relations

$$[B^-, B^+] = M, \qquad [M, B^\pm] = \pm 2B^\pm, \qquad (B^-)^* = B^+, \qquad M^* = M$$

one concludes that the RSWN $*$-Lie algebra is isomorphic to a current algebra, over $\mathbb{R}$, of a central extension of $sl(2,\mathbb{R})$. Notice that this central extension is trivial (like all those of $sl(2,\mathbb{R})$), but its role is essential because without it, i.e. putting $c = 0$ in the commutation relations (7.2), the Fock representation discussed below reduces to the zero representation.

Keeping in mind the intuitive expressions (7.4) of the generators, a natural analogue of the characterizing property (4.3), of the Fock state for this algebra, would be

$$\varphi(b_t^{\dagger 2} x) = \varphi(x b_t^2) = \varphi(b_t^\dagger b_t x) = \varphi(x b_t^\dagger b_t), \qquad \forall\, x \in U(sl(2,\mathbb{R})) \tag{7.5}$$

(let us emphasize that (7.5) is only in an informal sense a particular case of (4.3) where diagonal terms were not included) or, using test functions and the equivalent characterization (4.5) of the first order Fock state:

$$b_\varphi \Phi = n_\varphi \Phi = 0.$$

Using this property as the definition of the *quadratic Fock state*, Accardi, Lu and Volovich proved in [25] the existence of the quadratic Fock representation $\{\mathcal{H}, \Phi, \pi\}$ and formulated the programme to achieve a similar result for higher powers, using a natural generalization of the renormalization used for the square (see Section 8.1).

The paper [25] opened a research programme leading to several investigations in different directions. Among them we mention below only those directly related to the representation theory of Lie algebras and we refer, for more analytical and probabilistic directions, to [7, 10, 13, 15, 16, 19, 21, 27]. The latter paper also includes a discussion of previous attempts to give a meaning to the squares of free fields.

(*i*) Accardi and Skeide introduced in [28] the quadratic exponential (coherent) vectors for the RSWN and noticed that the kernel defined by the scalar product of two such vectors coincided with the kernel used by Boukas and Feinsilver in [37, 39] and [40] to construct unitary representations of the so-called Finite Difference Lie Algebra. Moreover, they proved that the Fock representation of the RSWN $*$-Lie algebra, constructed in [25], gave rise to a type-I product system of Hilbert spaces in the sense of Arveson (cf. [30]).

(*ii*) Accardi, Franz and Skeide realized in [20] that the RSWN $*$-Lie algebra is a current algebra of $sl(2,\mathbb{R})$ over $\mathbb{R}$ and that the factorization property mentioned in item (*i*) above naturally suggested a connection with the theory of infinitely divisible stochastic processes along the lines described in the monographs [44] and [55]. In particular they were able to identify the infinitely divisible classical stochastic processes, arising as vacuum distributions of the generalized field operators of the RSWN, with the three non-standard classes of Meixner laws:

- Gamma,
- Negative binomial (or Pascal),
- Meixner.

Since it was well known that the remaining two classes of Meixner laws, i.e. the Gaussian and Poisson classes, arise as vacuum distributions of the generalized field operators of the usual first order white noise (free boson field), this result showed that the quantum probabilistic approach provided a nice unified view to the 5 Meixner classes which were discovered in 1934 (cf. [53]) in connection with a completely different problem (a survey of this development is contained in [17]).

For a concrete example on how some Meixner laws can appear as vacuum distributions of quantum observables, see Section 13.1 below.

(*iii*) P. Sniady constructed in [58] the free analogue the Fock representation of the RSWN $*$-Lie algebra obtained in [25] and proved the first no-go theorem concerning the impossibility of combining together in a nontrivial way the Fock representations of the first and second order white noise $*$-Lie algebras. This opened the way to a series of no-go theorems which paralleled, in a quite different context and using different techniques, a series of such theorems obtained in the physical literature.

A stronger form of Sniady's result, still dealing with the first and second order case, was later obtained in [20]; in [18] this result was extended to the higher powers, defined with the renormalization used in [25], and further extended to the higher powers of the q-deformed white noise [6].

(*iv*) The attempt to go beyond the Fock representation by constructing more general representations, such as the finite temperature one, related to KMS states, was initiated in [1] where the analogue of the Bogolyubov transformations for the RSWN was introduced (i.e. those transformations on the test function space which induce endomorphisms of the quadratic $*$-Lie algebra) and a (very particular) class of KMS states on the RSWN algebra was constructed.

The problem of constructing the most general KMS states (for the free quadratic evolution) on the RSWN algebra was attacked with algebraic techniques in the paper [26] but its solution was obtained later, with a purely analytical approach by Prohorenko [57].

(*v*) The quadratic Fermi case was investigated by Accardi, Arefeva and Volovich in [2] and led to the rather surprising conclusion that, while the quadratic Bose case leads to the representation theory of the compact form of the real Lie algebra $SL(2,\mathbb{R})$, the corresponding Fermi case leads to the non compact form of the same real Lie algebra.

8 Higher powers of white noise

In order to realize, for the higher powers of white noise, what has been achieved for the square, we define the smeared operators (we will often use this terminology which can be justified only a posteriori by the realization of these objects as linear operators on Hilbert spaces):

$$B_k^n(f) := \int_{\mathbb{R}} f(t) {b_t^\dagger}^n b_t^k dt. \tag{8.1}$$

Notice that the above integral is normally ordered in $b_t^\dagger$, b_t therefore it always has a meaning as a sesquilinear form on the (first order) exponential vectors with test function in $L^1 \cap L^\infty(\mathbb{R})$ independently of any renormalization rule. This allows to consider the symbols $B_k^n(f)$ as generators of a complex vector space (in fact a $*$-vector space) and also to introduce a topology.

It is only when we want to introduce an additional Lie algebra structure, which *keeps some track* of the discrete version of the symbolic commutation relations written below (see formula (8.2)), that a renormalization rule is needed.

In terms of creation and annihilation operators, the commutation relations (2.14) take the form (see [18] for a proof):

$$\begin{aligned} &[{b_t^\dagger}^n b_t^k, {b_s^\dagger}^N b_s^K] \\ &\quad = \sum_{L \geq 1} \binom{k}{L} N^{(L)} \left\{ \epsilon_{k,0} \epsilon_{N,0} {b_t^\dagger}^n {b_s^\dagger}^{N-L} b_t^{k-L} b_s^K - [(k,n) \leftrightarrow (K,N)] \right\} \delta^L(t-s), \end{aligned} \tag{8.2}$$

where $n, k, N, K \in \mathbb{N}$, $x^{(y)}$ is the Pochammer symbol defined in formula (2.3), $[(k,n) \leftrightarrow (K,N)]$ denotes the result obtained by exchanging the roles of (k,n) and (K,N) in the expression $\epsilon_{k,0}\, \epsilon_{N,0} {b_t^\dagger}^n \, {b_s^\dagger}^{N-L} \, b_t^{k-L} \, b_s^K$ and, by definition:

$$\epsilon_{n,k} := 1 - \delta_{n,k} \qquad \text{(Kroneker's delta)}.$$

In order to give a meaning to the powers $\delta^L(t-s)$, of Dirac's delta, for $L \geq 2$, we fix a renormalization rule and a space of test functions $\mathcal{T}$. After that, from the distribution-form of the commutation relations (8.2), one deduces the corresponding commutation rules and involution for these operators.

Notice that it is not *a priori* obvious that, after the modifications introduced by the renormalization rule, the resulting brackets still define a $*$-Lie algebra. This fact has to be checked case by case. The following section gives a first illustration of the procedure described above.

8.1 Renormalization based on Ivanov's formula

The obvious generalization of Ivanov's formula (7.1) to powers strictly higher than two

$$\delta^l(t) = c^{l-1}\delta(t), \qquad l = 2, 3, \ldots, \qquad c > 0$$

leads to the following commutation relations and involution:

$$[b_t^{\dagger^n} b_t^k, b_s^{\dagger^N} b_s^K] \\ = \sum_{L\geq 1} \binom{k}{L} N^{(L)} \left\{\epsilon_{k,0}\,\epsilon_{N,0} b_t^{\dagger^n}\, b_s^{\dagger^{N-L}}\, b_t^{k-L}\, b_s^K - \ [(k,n) \leftrightarrow (K,N)]\right\} c^{L-1}\,\delta(t-s) \tag{8.3}$$

or equivalently in terms of test functions

$$[B_K^N(g), B_k^n(f)] = \sum_{L=1}^{(K\wedge n)\vee(k\wedge N)} \theta_L(N, K; n, k) c^{L-1} B_{K+k-L}^{N+n-L}(gf), \tag{8.4}$$

$$B_K^N(\bar{g})^* = B_N^K(\bar{g}), \tag{8.5}$$

where by definition:

$$\theta_L(N, K; n, k) := H(L-1)\left(\epsilon_{K,0}\,\epsilon_{n,0}\binom{K, n}{L} - \epsilon_{k,0}\epsilon_{N,0}\binom{k, N}{L}\right),$$

$$\binom{y, z}{x} := \binom{y}{x} z^{(x)}, \qquad H(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

One can prove that the brackets and involution (8.4) and (8.5) effectively define a $*$-Lie algebra: the renormalized higher powers of white noise (RHPWN) $*$-Lie algebra. This can be proved directly, but the simplest proof is based on the remark that an inspection of formula (8.3) shows that it coincides with the prescription to construct a current algebra over $\mathbb{R}$ of the universal enveloping algebra of the 1-dimensional Heisenberg algebra discussed in Section 2.1 and denoted $\mathcal{A}(\mathbb{R})$ (cf. [8]).

In the following section we discuss the notion of current algebra in some generality because, in the probabilistic interpretation of $*$-representations of Lie algebras, the transition from a Lie algebra to an associated current algebras corresponds to the transition from a random variable to a stochastic process (or random field).

8.2 Fock representation for RHPWN defined using Ivanov's renormalization and corresponding no-go theorems

In this section the test function space will be the space of complex valued step functions on $\mathbb{R}$ with finitely many values. Using the linear (or anti-linear) dependence of the generators on the test functions, one can restrict to characteristic functions χ_I of intervals $I \subseteq \mathbb{R}$ (i.e. $\chi_I(x) = 1$ if $x \in I$, $\chi_I(x) = 0$ if $x \notin I$) and often we simply write

$$B_k^n := B_k^n(\chi_I). \tag{8.6}$$

Given the expression (8.1) of the generators $B_k^n(f)$ a natural way to extend to them the notion of Fock representation is the following:

Definition 8.1. A cyclic representation $\{\mathcal{H}, \pi, \Phi\}$ of the RHPWN $*$-Lie algebra is called Fock if (omitting as usual π from the notations)

$$B_k^n(f)\Phi = 0, \qquad \forall\, f, \qquad \forall\, k \geq n. \tag{8.7}$$

One can prove that condition (8.7) effectively defines a linear functional on the universal enveloping algebra of the RHPWN $*$-Lie algebra and that this functional is factorizable. This allows to restrict the proof of positivity to the $*$-algebra generated by a single operator of the form (8.6).

The obstructions to the positivity requirements are illustrated by the following no-go theorem.

Theorem 8.1 (no-go theorem [18]). *Let $\mathcal{L}$ be a Lie $*$-subalgebra of RHPWN. Suppose that*

(i) for some $n \geq 1$, $\mathcal{L}$ contains B_0^n (n-th creator power) and B_0^{2n},

(ii) the test function space includes functions whose support has Lebesgue measure smaller than $1/c$ (the inverse of the renormalization constant).

Then $\mathcal{L}$ does not admit a Fock representation.

Proof. The idea of the proof is the following. Assuming that a Fock representation exists and using the above assumptions, one constructs negative norm vectors (ghosts) by considering linear combinations of $B_0^n\Phi$ and $B_0^{2n}\Phi$. ■

Corollary 8.1. *In the notations of Theorem 8.1, if $\mathcal{L}$ contains B_0^3 then it does not admit a Fock representation.*

Proof. The idea of the proof is that, if $\mathcal{L}$ contains B_0^3, then its cyclic space must contain also $B_0^6\Phi$ and then applies Theorem 8.1. ■

Corollary 8.2. *The current algebra over $\mathbb{R}$ of the Schrödinger algebra, i.e. the $*$-Lie algebra with generators*

$$\{a^+, a, a^{+2}, a^2, a^+a, 1\}$$

does not admit a Fock representation if the test function space includes functions whose support has Lebesgue measure smaller than the inverse of the renormalization constant.

Proof. The idea of the proof is that the Schrödinger algebra contains a^+ and a^{+2}. Therefore the associated current algebra over $\mathbb{R}$ contains $B_0^1(\chi_I)$ and $B_0^2(\chi_I)$ for arbitrary small intervals $I \subseteq \mathbb{R}$. The thesis then follows from Theorem 8.1. ■

9 A new renormalization

The no-go theorems, mentioned in Section 8.2 above, emphasize the necessity to investigate other renormalization procedures in order to go beyond the square and construct explicitly the (or better a) $*$-Lie algebra canonically associated with the renormalized higher powers of white noise.

In the attempt to overcome the no-go theorems, Accardi and Boukas introduced another, convolution type, renormalization of $\delta^l(t)$:

$$\delta^l(t-s) = \delta(s)\,\delta(t-s), \qquad l = 2, 3, \dots. \tag{9.1}$$

We refer to [14] for the motivations which led to this special choice. The involution is the same as in (8.5) while the commutation relations resulting from the renormalization prescription (9.1) are:

$$\begin{aligned}&[b_t^{\dagger^n} b_t^k, b_s^{\dagger^N} b_s^K]\\ &\quad = \epsilon_{k,0}\epsilon_{N,0}(kN b_t^{\dagger^n} b_s^{\dagger^{N-1}} b_t^{k-1} b_s^k \delta(t-s) + \sum_{L\geq 2}\binom{k}{L} N^{(L)} b_t^{\dagger^n} b_s^{\dagger^{N-L}} b_t^{k-L} b_s^K \delta(s)\delta(t-s))\end{aligned}$$

$$- \epsilon_{K,0}\epsilon_{n,0}(Knb_s^{\dagger N} b_t^{\dagger n-1} b_s^{K-1} b_t^k \delta(t-s) + \sum_{L\geq 2} \binom{K}{L} n^{(L)} b_s^{\dagger N} b_t^{\dagger n-L} b_s^{K-L} b_t^k \delta(s)\delta(t-s)),$$

which, after multiplying both sides by $f(t)g(s)$ and integrating the resulting identity, yield the commutation relations

$$[B_k^n(g), B_K^N(f)] = (\epsilon_{k,0}\epsilon_{N,0}kN - \epsilon_{K,0}\epsilon_{n,0}Kn)\, B_{K+k-1}^{N+n-1}(gf) + \sum_{L=2}^{(K\wedge n)\vee(k\wedge N)} \theta_L(n,k;N,K)\bar{g}(0)f(0) b_0^{\dagger N+n-l} b_0^{K+k-l},$$

where $\theta_L(n,k;N,K)$ is as in (8.4). By restricting the test function space to functions f, g that satisfy the boundary condition

$$f(0) = g(0) = 0 \tag{9.2}$$

we eliminate the singular terms $b_0^{\dagger N+n-l} b_0^{K+k-l}$. The resulting commutation relations are given in the following definition.

Definition 9.1. The RHPWN commutation relations are:

$$[B_k^n(g), B_K^N(f)]_{\mathrm{RHPWN}} := (kN - Kn)\, B_{k+K-1}^{n+N-1}(gf). \tag{9.3}$$

These commutation relations exhibit a striking similarity to those of the w_∞ Lie algebra with generators $\hat{B}_k^n$, arising in conformal field theory (cf. [32, 50, 56] and [33]):

$$[\hat{B}_k^n, \hat{B}_K^N]_{w_\infty} = (k(N-1) - K\,(n-1))\, \hat{B}_{k+K}^{n+N-2}. \tag{9.4}$$

However, at odds with what happens in the RHPWN algebra, here $n, k \in \mathbb{Z}$, with $n \geq 2$, and the involution is given by

$$\left(\hat{B}_k^n\right)^* = \hat{B}_{-k}^n.$$

In particular, for $n = 2$, one finds the centerless Virasoro (or Witt) Lie algebra commutation relations

$$[\hat{B}_k^2, \hat{B}_K^2]_{Vir} := (k - K)\hat{B}_{k+K}^2.$$

Even with these differences, the similarity of the commutation relations (9.3) and (9.4), was too strong to be a chance. This motivated several papers attempting to prove the identity of the two algebras (cf. [11]). The basic idea to identify the two algebras arose from an analysis of their classical realizations in terms of Poisson brackets which is outlined in the following section.

10 Classical representations of the RHPWN and w_∞ Lie algebras

We say that a Lie algebra $\mathcal{G}$ with generators $(l_\alpha)_{\alpha\in F}$ (F a set) and structure constants $(c_{\alpha,\beta}^\gamma)$ (with the properties specified in Section 5) admits a classical representation if there exists a space of functions $\hat{\mathcal{G}}$ from some $\mathbb{R}^d$ (with even d) and with values in $\mathbb{C}$ such that

(*i*) $\hat{\mathcal{G}}$, as a linear space, has a set of algebraic generators $(\hat{l}_\alpha)_{\alpha\in F}$ (i.e. any element of $\hat{\mathcal{G}}$ is a linear combination of a finite subset of the $(\hat{l}_\alpha)_{\alpha\in F}$), (*ii*) $\hat{\mathcal{G}}$ is a Lie algebra with brackets given by the Poisson-brackets:

$$[f,g]_{\hat{\mathcal{G}}} := \frac{\hbar}{i}\{f,g\} = \frac{\hbar}{i}\left(\frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial x}\right),$$

(*iii*) the structure constants of $\hat{\mathcal{G}}$ in the basis $(\hat{l}_\alpha)_{\alpha\in F}$ are the $(c^{\gamma}_{\alpha,\beta})$, i.e.

$$[\hat{l}_\alpha, \hat{l}_\beta]_{\hat{\mathcal{G}}} = c^{\gamma}_{\alpha,\beta}\hat{l}_\gamma,$$

equivalently: the map $l_\alpha \mapsto \hat{l}_\alpha$ extends to a Lie algebra isomorphism.

The following classical representation of the w_∞ Lie algebra was known in the literature (cf. [38])

$$\hat{w}_\infty := \text{linear span of } \left\{ f_{n,k}(x,y) := e^{ikx}y^{n-1} \,:\, n,k\in\mathbb{Z},\ n\geq 2,\ x,y\in\mathbb{R}\right\}. \tag{10.1}$$

In fact one verifies that:

$$\{f_{n,k}(x,y), f_{N,K}(x,y)\} = i\left(k(N-1) - K(n-1)\right) f_{n+N-2,k+K}(x,y).$$

Notice that, for $n = N = 2$ one recovers a classical representation, in the sense defined at the beginning of the present section, of the Witt–Virasoro algebra, in which the space $\hat{\mathcal{G}}$ is a space of trigonometric polynomials in two real variables. The usual realization of the Witt–Virasoro algebra is in terms of vector fields on the unit circle.

The analogue classical representation of the RHPWN Lie algebra was introduced in the papers [12] and [14]

$$\widehat{\text{RHPWN}} := \text{linear span of } \left\{ g_{n,k} := \left(\frac{x+iy}{\sqrt{2}}\right)^n \left(\frac{x-iy}{\sqrt{2}}\right)^k \,:\, n,k\in\mathbb{N},\ x,y\in\mathbb{R}\right\}. \tag{10.2}$$

In fact one verifies that:

$$\{g_{n,k}(x,y), g_{N,K}(x,y)\} = i\left(kN - nK\right) g_{n+N-1,k+K-1}(x,y).$$

11 White noise form of the w_∞ generators

Comparing (10.1) and (10.2) one realizes that, although the two algebras are different (because the coefficients of one are generated by monomials and those of the other by trigonometric polynomials), their closures in many natural topologies are the same. Therefore it is natural to conjecture that a similar relationship holds also in the quantum case.

After some guessing and corresponding trial and error attempts the following (quantum and continuum) generalization of (10.1) was established in [11]

$$\hat{B}^n_k(f) := \int_{\mathbb{R}} f(t) e^{\frac{k}{2}(b_t - b_t^\dagger)} \left(\frac{b_t + b_t^\dagger}{2}\right)^{n-1} e^{\frac{k}{2}(b_t - b_t^\dagger)} dt. \tag{11.1}$$

For $n = 2$ one obtains the Boson representation of the centerless Virasoro (or Witt) Lie algebra generators

$$\hat{B}^2_k(f) := \int_{\mathbb{R}} f(t) e^{\frac{k}{2}(b_t - b_t^\dagger)} \left(\frac{b_t + b_t^\dagger}{2}\right) e^{\frac{k}{2}(b_t - b_t^\dagger)} dt.$$

Expressions like (11.1) are symbolic expressions which involve ill defined quantities such as exponentials and products of operator valued distribution. In order to give them a precise meaning one adopts the usual strategy in the theory of distributions: the symbols are manipulated by formally applying to them the first order commutation relations, then applying the renormalization prescription (9.1) and finally integrating over test functions which satisfy the boundary condition (9.2).

After having played this game with (11.1), one arrives to the identity

$$\hat{B}_k^n(f) = \frac{1}{2^{n-1}} \sum_{m=0}^{n-1} \binom{n-1}{m} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^p \frac{k^{p+q}}{p!q!} B_{n-1-m+q}^{m+p}(f). \tag{11.2}$$

The series on the right hand side (11.2) is convergent in the natural topology mentioned at the end of Section 8 (for example its matrix element, in an arbitrary pair of first order number vectors, reduces to a finite sum, see [11] for more details). This gives a natural meaning to its analytic continuation in a neighborhood of $k = 0$, which is used in the following inversion formula:

$$B_k^n(f) = \sum_{\rho=0}^{k} \sum_{\sigma=0}^{n} \binom{k}{\rho} \binom{n}{\sigma} \frac{(-1)^\rho}{2^{\rho+\sigma}} \frac{\partial^{\rho+\sigma}}{\partial z^{\rho+\sigma}}\bigg|_{z=0} \hat{B}_z^{k+n+1-(\rho+\sigma)}(f).$$

The conclusion is that, as suggested by the analogy with the classical case, even though the two $*$-Lie algebras w_∞ and RHPWN are different from a purely algebraic point of view, their closure in a natural topology coincide. Moreover the explicit representations given above provide a concrete realization of the RHPWN as sesquilinear forms on the space of the first order white noise.

The problem of realizing them as bona fide closable operators on some Hilbert space is largely open. For example in the case of the w_∞-operators $\hat{B}_k^n$, only for $n = 2$ the Witt–Virasoro algebra, such a representation is available (see the comment at the end of Section 13).

12 Fighting the no-go theorems

A possible way out from the no-go theorems is to look for a modification of the notion of Fock representation that keeps its main property (algebra implies statistics) but avoids ghosts. There is no standard rule for producing such modifications: the best one can do is to conjecture a possible candidate through heuristic manipulations and then try to prove that it has the desired properties.

In the following we illustrate this procedure by a two step modification of the notion of Fock representation: the first step is not sufficient to avoid ghosts (this section) while the second one (following section), which improves the first one by adding a diagonal prescription, leads to *bona fide* Hilbert space representations.

We feel that the comparison between the positive and negative result may help the reader get an intuition of "what makes things work".

One can show that the symbolic expressions

$$B_k^n(f) = \int_{\mathbb{R}} f(t)(b_t^\dagger)^{n-k}(b_t b_t^\dagger)^k dt \tag{12.1}$$

for natural integers $n \geq k \geq 0$ and functions f in the test function space defined by (9.2), provide, together with their adjoints which by definition are denoted $B_n^k(f)$ (recall that $n \geq k \geq 0$), a set of generators of the RHPWN algebra.

This means that, applying to these symbolic expressions the known formulae on the combinatorics of the (Boson) creation/annihilation operators combined with the new renormalization prescription (9.1), one obtains the RHPWN $*$-Lie algebra, that was obtained by applying the same procedure to the symbolic expressions (8.1).

Now, we want to keep track of the heuristic interpretation of $b_t b_t^\dagger$ as a white noise operator. Therefore its action on the modified Fock vacuum Φ should be compatible with the usual Fock prescription $b_t\Phi = 0$, i.e. it should be of the form

$$b_t b_t^\dagger \Phi = c_t \Phi,$$

where c_t is a renormalization constant. Proposition 2 of [5] shows that c_t must be independent of t so that, up to a multiplicative constant, the action of $B_k^n(f)$ on Φ should be a multiple of $B_0^{n-k}(f)\Phi$ for $n \geq k \geq 0$ and zero for $0 \leq n < k$. This suggests the following definition.

Definition 12.1. A generalized Fock representation of the RHPWN $*$-Lie algebra with generators (12.1) is a triple $\{\mathcal{H}, \Phi, \pi\}$ such that $\mathcal{H}$ is a Hilbert space, Φ a unit vector cyclic for the generators (12.1), and π is a $*$-representation of RHPWN (from now on omitted from notations for simplicity) with the following properties:

$$B_k^n(f)\Phi := \begin{cases} 0 & \text{if } n < k \text{ or } n \cdot k < 0, \\ \sigma_1^k B_0^{n-k}(f)\Phi & \text{if } n > k \geq 0, \\ \dfrac{\sigma_1^k}{k+1} \displaystyle\int_{\mathbb{R}} f(t)dt\Phi & \text{if } n = k, \end{cases} \tag{12.2}$$

where σ_1 is a real number depending on the renormalization.

Remark 12.1. Up to a rescaling we can always assume that in (12.2) one has $\sigma_1 = 1$ so that, for $n, k \in \mathbb{N}$ and test functions f:

$$B_k^n(f)\Phi := 0 \qquad \text{if} \quad n < k, \tag{12.3}$$

$$B_k^n(f)\Phi := \frac{1}{n+1}\int_{\mathbb{R}} f(t)dt\Phi \qquad \text{if} \quad n = k, \tag{12.4}$$

$$B_k^n(f)\Phi := B_0^{n-k}(f)\Phi \qquad \text{if} \quad n > k \geq 0. \tag{12.5}$$

In the following we will assume that the test function χ_I, in (8.6), is such that $I \subseteq \mathbb{R}\setminus\{0\}$ is an interval and $\chi_I(x) = 1$ if $x \in I$, $\chi_I(x) = 0$ if $x \notin I$. We also suppose that the Lebesgue measure of I is sufficiently large so that the no-go theorems do not apply. Under these assumptions the $*$-Lie algebra generated by the B_k^n and B_n^k can be considered as a "one-mode" realization of the RHPWN $*$-Lie algebra.

Definition 12.2.

(*i*) $\mathcal{L}_1$ is the $*$-Lie algebra generated by B_0^1 and B_1^0, i.e., $\mathcal{L}_1$ is the linear span of $\{B_0^1, B_1^0, B_0^0\}$ (the usual oscillator algebra).

(*ii*) $\mathcal{L}_2$ is the $*$-Lie algebra generated by B_0^2 and B_2^0, i.e., $\mathcal{L}_2$ is the linear span of $\{B_0^2, B_2^0, B_1^1\}$ (the usual quadratic algebra, isomorphic to $sl(2,\mathbb{R})$).

(*iii*) For $n \in \{3, 4, \dots\}$, $\mathcal{L}_n$ is the $*$-Lie subalgebra of RHPWN generated by B_0^n and B_n^0. It is the linear span of the operators of the form B_y^x where $x - y = kn$, $k \in \mathbb{Z}\setminus\{0\}$, and of the number operators B_x^x with $x \geq n-1$.

The proof of the following theorem can be found in [14].

Theorem 12.1. *Let $n \geq 3$ and suppose that a generalized Fock representation $\{\mathcal{F}_n, \Phi\}$, of $\mathcal{L}_n$, in the sense of Definition* 12.1, *exists. Then it contains both $B_0^n\Phi$ and $B_0^{2n}\Phi$. In particular, if the test function space includes functions whose support has arbitrarily small Lebesgue measure, then $\mathcal{L}_n$ does not admit a generalized Fock representation in the sense of Definition* 12.1.

13 Further generalizations of the Fock representation for the RHPWN algebra

In this section we prove that a further strengthening of the notion of Fock representation for the RHPWN algebra leads to well defined unitary representations. The idea of the construction is the following.

Definition 13.1. A generalized Fock representation of the RHPWN $*$-Lie algebra with generators (12.1) is defined, as in Definition 12.1, to be a triple $\{\mathcal{H}, \Phi, \pi\}$ satisfying the two conditions (12.3) and (12.4) and replacing (12.5) by (in the notation (8.6)):

$$B_x^{n+x}(B_0^n)^N \Phi = B_0^n (B_0^n)^N \Phi, \qquad \forall\, n, x, N \in \mathbb{N}. \tag{13.1}$$

One easily verifies that condition (13.1) implies that:

$$B_{n-1}^{n-1}(B_0^n)^k \Phi = \left(\frac{\mu(I)}{n} + kn(n-1)\right)(B_0^n)^k \Phi$$

or, writing explicitly the test function, $\forall\, k, n \in \mathbb{N}$:

$$B_{n-1}^{n-1}(\chi_I)(B_0^n(\chi_I))^k \Phi := \left(\frac{\mu(I)}{n} + kn(n-1)\right)(B_0^n(\chi_I))^k \Phi.$$

The prescription that the vectors $B_0^n(\chi_I)^k \Phi$ are total in $\mathcal{H}$ uniquely determines this representation up to isomorphism. In fact these prescriptions uniquely fix the inner product among the higher order particle vectors to be given by:

$$\langle (B_0^n(\chi_I))^k \Phi, (B_0^n(\chi_I))^m \Phi\rangle = \delta_{m,k} k! n^k \prod_{i=0}^{k-1}\left(\mu(I) + \frac{n^2(n-1)}{2} i\right)$$

with

$$\langle (B_0^n(\chi_I))^k \Phi, (B_0^n(\chi_J))^m \Phi\rangle = 0$$

if I and J are disjoint.

Defining the n-th order exponential vectors by analogy with the first and the second order case, i.e.

$$\psi_n(\phi) := \prod_{i=1}^m e^{b_i B_0^n(\chi_{I_i})}\Phi,$$

where ϕ is the compact support step function

$$\phi := \sum_{i=1}^m b_i \chi_{I_i}$$

for $n = 1$ one finds

$$\langle \psi_1(f), \psi_1(g)\rangle_1 := e^{\int_{\mathbb{R}^d} \bar{f}(t) g(t) dt} = e^{\langle f, g\rangle_{L^2(\mathbb{R}^d)}} \tag{13.2}$$

and for $n \geq 2$

$$\langle \psi_n(f), \psi_n(g)\rangle := e^{-\frac{2}{n^2(n-1)}\int_{\mathbb{R}} \ln\left(1 - \frac{n^3(n-1)}{2}\bar{f}(t)g(t)\right)dt}, \tag{13.3}$$

where the integral in (13.3) exists under the condition

$$\sup_{t\in\mathbb{R}^d}|f(t)| < \frac{1}{n}\sqrt{\frac{2}{n(n-1)}}, \qquad \sup_{t\in\mathbb{R}^d}|g(t)| < \frac{1}{n}\sqrt{\frac{2}{n(n-1)}}.$$

The positivity of the scalar product is now clear because (13.2) is the formula for the inner product of two exponential vectors in the usual (first order) Fock space while (13.3) coincides (up to rescalings) with the formula, found in [28], for the inner product of two exponential vectors in the quadratic case.

The action of the RHPWN operators on the exponential vectors is given by:

$$\begin{aligned}
B_0^n(f)\psi_n(g) &= \frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}\psi_n(g+\epsilon f),\\
B_n^0(f)\psi_n(g) &= n\int_{\mathbb{R}} f(t)g(t)dt\,\psi_n(g) + \frac{n^3(n-1)}{2}\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}\psi_n(g+\epsilon fg^2)
\end{aligned}$$

for all test functions $f := \sum_i a_i\chi_{I_i}$ and $g := \sum_i b_i\chi_{I_i}$ with $I_i\cap I_j = \oslash$ for $i\neq j$, and for all $n\geq 1$.

The emergence of the quadratic Fock space in this context can be explained, a posteriori, as follows. Denote

$$c := \frac{\mu(I)}{n}, \qquad q_k := kn(n-1), \qquad (B_0^n)^k\Phi =: |k\rangle,$$

so that

$$\begin{aligned}
B_0^n|k\rangle &= |k+1\rangle,\\
B_{n-1}^{n-1}|k\rangle &= B_{n-1}^{n-1}(B_0^n)^k\Phi = (c+q_k)(B_0^n)^k\Phi = (c+q_k)|k\rangle,
\end{aligned}$$

i.e. the restriction of B_{n-1}^{n-1} on the algebraic linear span $\mathcal{F}_n$ of the vectors $(|k\rangle)$, which are clearly mutually orthogonal, takes the form

$$B_{n-1}^{n-1} = \sum(c+q_k)|k\rangle\langle k| = c\cdot 1 + q_N,$$

where

$$q_N|k\rangle := q_k|k\rangle$$

Consequently

$$\begin{aligned}
[B_{n-1}^{n-1}, B_0^n]|k\rangle &= B_{n-1}^{n-1}B_0^n|k\rangle - B_0^nB_{n-1}^{n-1}|k\rangle = B_{n-1}^{n-1}|k+1\rangle - q_kB_0^n|k\rangle\\
&= (q_{k+1}-q_k)|k+1\rangle = n(n-1)B_0^n|k\rangle.
\end{aligned}$$

In conclusion, on $\mathcal{F}_n$ the following commutation relations hold:

$$\begin{aligned}
&[B_n^0, B_0^n] = n^2B_{n-1}^{n-1} = n^2c1 + n^2q_N,\\
&[B_{n-1}^{n-1}, B_0^n] = n(n-1)B_0^n
\end{aligned}$$

and, up to rescalings, these are the relations defining a central extension of $sl(2,\mathbb{R})$, which is precisely the one-particle algebra of the quadratic white noise.

The construction given in [5] of the generalized Fock representation (in the sense specified above) and the determination of the corresponding statistics was based on algebraic techniques.

An analytical construction of the corresponding generalized Fock space, within an extension of Hida theory of white noise calculus to the negative binomial process, has recently been obtained by Barhoumi, Ouerdiane and Rihai [34]. This analytical construction is particularly interesting because it shows that the reduction with the quadratic case does not destroy the connection with the Virasoro algebra. In fact these authors prove that, for $n = 2$, the symbols of the operators $\hat{B}^n_k(f)$, defined by (11.2) (i.e. their matrix elements in the higher order exponential vectors) effectively define closable operators in the quadratic Fock space while this is not true for $n > 2$.

13.1 Classical stochastic processes associated with the representation

In this section we work in dimension 1 (i.e. $d = 1$). For $n \geq 1$ consider the operator process:

$$\{B^n_0(\chi_{[0,t]}) + B^0_n(\chi_{[0,t]}) :\ t \in \mathbb{R}_+\}. \tag{13.4}$$

It is not difficult to verify that the family (13.4) is commutative. Therefore its vacuum distributions define, as described in Section 6, a classical stochastic process. The following theorem identifies these processes as continuous binomial (or Beta) processes. This is a subclass of the Meixner processes which, in their turn, are a special family of stationary independent increment (or Lévy) processes.

Theorem 13.1. *The vacuum moment generating functions of the operator process* (13.4) *is given, for* $n = 1$*, by:*

$$\langle \Phi, e^{s\,(B^1_0(t)+B^0_1(t))}\Phi\rangle_1 = e^{\frac{s^2}{2}t}, \qquad s \in [0, \infty),$$

i.e. the process $\{B^1_0(t) + B^0_1(t) :\ t \geq 0\}$ *is the standard classical Brownian motion. While for* $n \geq 2$ *one finds:*

$$\langle \Phi, e^{s(B^n_0(t)+B^0_n(t))}\Phi\rangle_n = \left(\sec\left(\sqrt{\frac{n^3(n-1)}{2}}\,s\right)\right)^{\frac{2nt}{n^3(n-1)}}, \qquad s \in [0, \infty),$$

i.e. $\{B^n_0(t) + B^0_n(t) :\ t \geq 0\}$ *is for each* n *a continuous binomial (or Beta) process with density*

$$\mu_{t,n}(x) = p_{\frac{2nt}{n^3(n-1)}}(x) = \frac{2^{\frac{2nt}{n^3(n-1)}-1}}{2\pi} B\left(\frac{\frac{2nt}{n^3(n-1)} + ix}{2}, \frac{\frac{2nt}{n^3(n-1)} - ix}{2}\right),$$

where $B(a, c)$ *is the Beta function with parameters* a*,* c*:*

$$B(a, c) = \frac{\Gamma(a)\Gamma(c)}{\Gamma(a + c)} = \int_0^1 x^{a-1}(1 - x)^{c-1}dx, \qquad \Re a > 0, \qquad \Re c > 0.$$

Notice incidentally that the formula postulated by Veneziano for the scattering amplitude of some strong interactions was also defined in terms of the Euler beta function (see [43, p. 373]).

14 Central extensions and renormalization

The analogy with the Virasoro algebra naturally suggests the investigation of the existence of Hilbert space representations not of the RHPWN $*$-Lie algebra itself but of some of its central extensions.

But the RHPWN $*$-Lie algebra is a second quantized and renormalized version of the full oscillator algebra (FOA). Thus a natural preliminary problem is to construct central extensions of the FOA.

Since, in its turn, the FOA is the universal enveloping algebra of the Heisenberg algebra and since a central extension of an algebra automatically provides an extension (even if usually not central) of its universal enveloping algebra, this leads to the further preliminary problem to construct central extensions of the Heisenberg algebra.

In the paper [3] the following results were established:

(i) There exists a one (complex) parameter family of nontrivial central extensions of the Heisenberg $*$-Lie algebra.

(ii) These nontrivial central extensions of the Heisenberg $*$-Lie algebra have a boson realization within the Schrödinger algebra.

(iii) This boson realization can be used to compute the vacuum characteristic function of the field operators in the Fock representation of the Schrödinger algebra.

(iv) The above mentioned boson realization can also be used to obtain a second quantized version of a quadratic boson algebra which cannot be deduced from the constructions in [25] and [20]. This consequence is quite nontrivial due to the no-go theorems: it confirms the strict connection between central extensions and renormalization even if the deep roots of this connection have yet to be clarified. In the following we briefly recall the central extensions mentioned in item (i) above.

Recall (cf. [42, 59]) that, if L and $\widetilde{L}$ are two complex Lie algebras, $\widetilde{L}$ is called a one-dimensional *central extension* of L with *central element* E if

$$[l_1, l_2]_{\widetilde{L}} = [l_1, l_2]_L + \phi(l_1, l_2)E, \qquad [l_1, E]_{\widetilde{L}} = 0, \qquad \forall\, l_1, l_2 \in L,$$

where $[\cdot,\cdot]_{\widetilde{L}}$ and $[\cdot,\cdot]_L$ are the Lie brackets in $\widetilde{L}$ and L respectively and $\phi : L \times L \mapsto \mathbb{C}$ is a 2-cocycle on L, i.e. a bilinear form satisfying the skew-symmetry condition

$$\phi(l_1, l_2) = -\phi(l_2, l_1)$$

and the Jacobi identity

$$\phi([l_1, l_2]_L, l_3) + \phi([l_2, l_3]_L, l_1) + \phi([l_3, l_1]_L, l_2) = 0.$$

A central extension is *trivial* if there exists a linear function $f : L \mapsto \mathbb{C}$ satisfying for all $l_1, l_2 \in L$

$$\phi(l_1, l_2) = f([l_1, l_2]_L).$$

14.1 Central extensions of the Heisenberg algebra

While all central extensions of the Oscillator algebra (i.e. the $*$-Lie subalgebra of the full oscillator algebra generated by B_0^1, B_1^0, B_0^0, and B_1^1) as well as those of the Square of White Noise algebra (generated by B_0^2, B_2^0, and B_1^1 and isomorphic to $sl(2,\mathbb{R})$) are trivial, this is not true for the Heisenberg algebra, i.e. the $*$-Lie subalgebra of the full oscillator algebra generated by B_0^1, B_1^0 and B_0^0. In fact all central extensions of the Heisenberg algebra are described as follows:

$$\left(B_0^1\right)^* = B_1^0, \qquad \left(B_1^0\right)^* = B_0^1, \qquad \left(B_0^0\right)^* = B_0^0,$$
$$[B_1^0, B_0^1] = B_0^0 + \lambda E, \qquad [B_0^0, B_0^1] = zE, \qquad [B_1^0, B_0^0] = \bar{z}E,$$

where $z \in \mathbb{C}$ and $\lambda \in \mathbb{R}$ are arbitrary constants. A central extension of the Heisenberg $*$-Lie algebra is trivial if and only if $z = 0$.

Up to algebraic (but not stochastic) isomorphism there is one nontrivial central extension which belongs to the list of 15 real 4-dimensional Lie algebras in the classification due to Kruchkovich [51, 54]. One boson realization of this algebra (see [41]) is generated by $\{q^2, q, p, 1\}$. This shows that it can be identified to a subalgebra of the Schrödinger algebra (for whose current algebra over $\mathbb{R}$ we know that a no-go theorem holds). The presence of q^2 shows that a renormalization is required to give a meaning to the associated current algebra. However the simultaneous presence of q and p shows that, if a Fock representation of this algebra exists, then it cannot be realized in the RSWN $*$-Lie algebra discussed in Section 7. However from the end of Section 2.3 we know that the associated $*$-Lie algebra is well defined and one can show that the general construction of [44] and [55] can be applied, hence $*$-representations can be constructed. It is however not known if any of these representations can be identified in a natural way to the Fock representation of this algebra (which, if existing as a $*$-representation, is uniquely determined up to unitary isomorphism).

14.2 Central extensions of the RHPWN and of the w_∞ algebra

The $*$-algebras RHPWN and w_∞ are too large to admit nontrivial central extensions. In [4] the following results were proved:

(*i*) All central extensions of the RHPWN $*$-Lie algebra are trivial.

(*ii*) All central extensions of the higher order w_∞ $*$-Lie algebra (i.e. with the Witt–Virasoro sector removed) are trivial.

The statement (*i*) is new, the statement (*ii*) provides a new proof of a result due to Bakas [32] who proved that the coefficients of the central terms of a suitable contraction of the Zamolodchikov Lie algebra W_N go to zero (cf. [56, 60]) as $N \to \infty$. From this he could conclude that all cocycles, which arise from that finite dimensional approximation, are trivial. Our proof, being based on a purely algebraic analysis of the 2-cocycles of w_∞, is more general and direct.

One might hope that the $*$-Lie algebras $\mathcal{L}_n$, of the RHPWN $*$-Lie algebra, introduced in Section 12 above, admit nontrivial central extensions. This seems to be unlikely due to the recent discovery (see [4]) that, in the family of natural subalgebras of w_∞, only the Virasoro algebra admits nontrivial central extensions. A similar result for subalgebras of RHPWN is not available at the moment, but we hope to come back to this point soon.

15 Conclusions

The identification of the (closures of) the RHPWN and the w_∞-$*$-Lie algebras suggests that these algebras have a canonical mathematical meaning. However our program of identifying the elements of these algebras to renormalized powers of white noise can be considered realized only in the quadratic case. An important guiding principle that we have learned from this case is that, if this program can be realized, then in the representation space there should be a unit vector Φ such that the Φ-moments of the classical process given by the renormalized n-th power of $(b_t^+ + b_t)$ should be an independent increment process whose distribution is the n-th power of the standard Gaussian. This brings a connection with an old open problem of classical probability. In fact, while it is known that even powers of the standard Gaussian are infinitely divisible, the same statement for odd powers (≥ 3) is not known and experts conjecture that the answer is negative. This suggests that one could use the no-go theorems to deduce a negative answer to this classical conjecture based on quantum probabilistic techniques. Moreover fortunately, both for RHPWN and for w_∞, the even powers form a $*$-Lie subalgebra, so that one can restrict one's attention to even powers. This gives the advantage that one knows a priori a natural candidate for the representation space: i.e. for each $n \in \mathbb{N}$, the L^2-space of the independent increment stationary process corresponding to the $2n$-th power of the standard Gaussian.

Our strategy to take advantage of this information consists in the complete inversion of the strategy pursued from 1999 up to now, namely: up to now we have pursued one of the basic tenets of quantum probability, *algebra implies statistics*, in the sense that we have tried to guess a reasonable definition of *Fock* (lowest weight) representation and to deduce the statistics from it on the lines outlined in Section 4 and after equation (7.5).

From now on we will pursue the other basic tenet of quantum probability, *statistics implies algebra*, and starting from the L^2-space of the independent increment stationary process corresponding to the $2n$-th power of the standard Gaussian we will apply the theory of interacting Fock spaces to deduce the quantum decomposition of this classical process and the commutation relations canonically associated to the principal Jacobi sequence of this distribution (which is symmetric so that the secondary Jacobi sequence vanishes identically).

In this direction we will surely benefit from the results developed by Y. Berezansky and his school on the extension of classical white noise theory to a general class of Lévy processes (Jacobi fields, see [35, 52]). The L^2-spaces of these processes are naturally isomorphic to a class of 1-mode type interacting Fock spaces which includes the even powers of the Gaussian. Although naturally isomorphic the two realizations are different and the interacting Fock space one is simpler to handle from the algebraic point of view.

Another direction might be to look for representations different from the Fock one (cf. [47, 48, 49]). This is surely a direction worth investigating. However at the moment our knowledge of such representations is rather limited even in the quadratic case. In fact, as already mentioned in Section 4, even in the first order case our explicit control of these representations is restricted to the Gaussian (or quasi-free) ones.

As often in mathematics what has been understood is a tiny fraction of what one would like to understand. However the landscape that has emerged along this path is so intriguing and promising that it constitutes a stimulus to meet this challenge.

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