Algebraic Topology Foundations of Supersymmetry and Symmetry Breaking in Quantum Field Theory and Quantum Gravity: A Review

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Abstract. A novel algebraic topology approach to supersymmetry (SUSY) and symmetry breaking in quantum field and quantum gravity theories is presented with a view to developing a wide range of physical applications. These include: controlled nuclear fusion and other nuclear reaction studies in quantum chromodynamics, nonlinear physics at high energy densities, dynamic Jahn–Teller effects, superfluidity, high temperature superconductors, multiple scattering by molecular systems, molecular or atomic paracrystal structures, nanomaterials, ferromagnetism in glassy materials, spin glasses, quantum phase transitions and supergravity. This approach requires a unified conceptual framework that utilizes extended symmetries and quantum groupoid, algebroid and functorial representations of non-Abelian higher dimensional structures pertinent to quantized spacetime topology and state space geometry of quantum operator algebras. Fourier transforms, generalized Fourier–Stieltjes transforms, and duality relations link, respectively, the quantum groups and quantum groupoids with their dual algebraic structures; quantum double constructions are also discussed in this context in relation to quasi-triangular, quasi-Hopf algebras, bialgebroids, Grassmann–Hopf algebras and higher dimensional algebra. On the one hand, this quantum algebraic approach is known to provide solutions to the quantum Yang–Baxter equation. On the other hand, our novel approach to extended quantum symmetries and their associated representations is shown to be relevant to locally covariant general relativity theories that are consistent with either nonlocal quantum field theories or local bosonic (spin) models with the extended quantum symmetry of entangled, 'string-net condensed' (ground) states.

Key words: extended quantum symmetries; groupoids and algebroids; quantum algebraic topology (QAT); algebraic topology of quantum systems; symmetry breaking, paracrystals, superfluids, spin networks and spin glasses; convolution algebras and quantum algebroids; nuclear Fréchet spaces and GNS representations of quantum state spaces (QSS); groupoid and functor representations in relation to extended quantum symmetries in QAT; quantization procedures; quantum algebras: von Neumann algebra factors; paragroups and Kac algebras; quantum groups and ring structures; Lie algebras; Lie algebroids; Grassmann–Hopf, weak C^*-Hopf and graded Lie algebras; weak C^*-Hopf algebroids; compact quantum groupoids; quantum groupoid C^*-algebras; relativistic quantum gravity (RQG); supergravity and supersymmetry theories; fluctuating quantum spacetimes; intense gravitational fields; Hamiltonian algebroids in quantum gravity; Poisson–Lie manifolds and quantum gravity theories; quantum fundamental groupoids; tensor products of algebroids and categories; quantum double groupoids and algebroids; higher dimensional quantum symmetries; applications of generalized van Kampen theorem (GvKT) to quantum spacetime invariants

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Contents

1 Introduction

The theory of scattering by partially ordered, atomic or molecular, structures in terms of *paracrystals* and *lattice convolutions* was formulated by Hosemann and Bagchi in [121] using basic techniques of Fourier analysis and convolution products. A natural generalization of such molecular, partial symmetries and their corresponding analytical versions involves convolution algebras – a functional/distribution [197, 198] based theory that we will discuss in the context of a more general and original concept of a *convolution-algebroid of an extended symmetry groupoid of a paracrystal*, of any molecular or nuclear system, or indeed, any quantum system in general, including quantum fields and local quantum net configurations that are endowed with either partially disordered or 'completely' ordered structures. Further specific applications of the paracrystal theory to X-ray scattering, based on computer algorithms, programs and explicit numerical computations, were subsequently developed by the first author [11] for one-dimensional paracrystals, partially ordered membrane lattices [12] and other biological structures with partial structural disorder [18]. Such biological structures, 'quasi-crystals', and the paracrystals, in general, provide rather interesting physical examples of such extended symmetries (cf. [120]).

Further statistical analysis linked to structural symmetry and scattering theory considerations shows that a real paracrystal can be defined by a three dimensional convolution polynomial with a semi-empirically derived composition law, $*$, [122]. As was shown in [11, 12] – supported with computed specific examples – several systems of convolution can be expressed analytically, thus allowing the numerical computation of X-ray, or neutron, scattering by partially disordered layer lattices via complex Fourier transforms of one-dimensional structural models using fast digital computers. The range of paracrystal theory applications is however much wider than the one-dimensional lattices with disorder, thus spanning very diverse non-crystalline systems, from metallic glasses and spin glasses to superfluids, high-temperature superconductors, and extremely hot anisotropic plasmas such as those encountered in controlled nuclear fusion (for example, JET) experiments. Other applications – as previously suggested in [10] – may also include novel designs of 'fuzzy' quantum machines and quantum computers with extended symmetries of quantum state spaces.

1.1 Convolution product of groupoids and the convolution algebra of functions

A salient, and well-fathomed concept from the mathematical perspective concerns that of a C^*-algebra of a (discrete) group (see, e.g., [73]). The underlying vector space is that of complex valued functions with finite support, and the multiplication of the algebra is the fundamental *convolution product* which it is convenient for our purposes to write slightly differently from the common formula as

$$(f * g)(z) = \sum_{xy=z} f(x)g(y),$$

and $*$-operation

$$f^*(x) = \overline{f(x^{-1})}.$$

The more usual expression of these formulas has a sum over the elements of the group. For topological groups, where the underlying vector space consists of continuous complex valued functions, this product requires the availability of some structure of measure and of measurable functions, with the sum replaced by an integral. Notice also that this algebra has an identity, the distribution function δ_1, which has value 1 on the identity 1 of the group, and has zero value elsewhere.

Given this convolution/distribution representation that combines crystalline ('perfect' or global-group, and/or group-like symmetries) with partial symmetries of paracrystals and glassy solids on the one hand, and also with non-commutative harmonic analysis [151] on the other hand, we propose that several extended quantum symmetries can be represented algebraically in terms of certain structured *groupoids*, their *C^*-convolution quantum algebroids*, paragroup/*quantized groups* and/or other more general mathematical structures that will be introduced in this report. It is already known that such extensions to groupoid and algebroid/coalgebroid symmetries require also a generalization of non-commutative harmonic analysis which involves certain Haar measures, generalized Fourier–Stieltjes transforms and certain categorical duality relationships representing very general mathematical symmetries as well. Proceeding from the abstract structures endowed with extended symmetries to numerical applications in quantum physics always involves representations through specification of concrete elements, objects and transformations. Thus, groupoid and functorial representations that generalize group representations in several, meaningful ways are key to linking abstract, quantum operator algebras and symmetry properties with actual numerical computations of quantum eigenvalues and their eigenstates, as well as a wide variety of numerical factors involved in computing quantum dynamics. The well-known connection between groupoid convolution representations and matrices [212] is only one of the several numerical computations made possible via groupoid representations. A very promising approach to nonlinear (anharmonic) analysis of aperiodic quantum systems represented by rigged Hilbert space bundles may involve the computation of representation coefficients of Fourier–Stieltjes groupoid transforms that we will also discuss briefly in Section 7.

Currently, however, there are important aspects of quantum dynamics left out of the invariant, simplified picture provided by group symmetries and their corresponding representations of quantum operator algebras [102]. An alternative approach proposed in [114] employs differential forms to find symmetries.

Often physicists deal with such problems in terms of either spontaneous symmetry breaking or approximate symmetries that require underlying explanations or ad-hoc dynamic restrictions that are semi-empirical. A well-studied example of this kind is that of the dynamic Jahn–Teller effect and the corresponding 'theorem' (Chapter 21 on pp. 807–831, as well as p. 735 of [1]) which in its simplest form stipulates that a quantum state with electronic non-Kramers degeneracy may be unstable against small distortions of the surroundings, that would lower the symmetry of the crystal field and thus lift the degeneracy (i.e., cause observable splitting of the corresponding energy levels); this effect occurs in certain paramagnetic ion systems *via* dynamic distortions of the crystal field symmetries around paramagnetic or high-spin centers by moving ligands that are diamagnetic. The established physical explanation is that the Jahn–Teller coupling replaces a purely electronic degeneracy by a vibronic degeneracy (of *exactly the same* symmetry!). The dynamic, or spontaneous breaking of crystal field symmetry (for example, distortions of the octahedral or cubic symmetry) results in certain systems in the appearance of doublets of symmetry γ_3 or singlets of symmetry γ_1 or γ_2. Such dynamic systems could be locally expressed in terms of symmetry representations of a Lie algebroid, or globally in terms of a special Lie (or Lie–Weinstein) symmetry groupoid representations that can also take into account the spin exchange interactions between the Jahn–Teller centers exhibiting such quantum dynamic effects. Unlike the simple symmetries expressed by group representations, the latter can accommodate a much wider range of possible or approximate symmetries that are indeed characteristic of real, molecular systems with varying crystal field symmetry, as for example around certain transition ions dynamically bound to ligands in liquids where motional narrowing becomes very important. This well known example illustrates the importance of the interplay between symmetry and dynamics in quantum processes which is undoubtedly involved in many other instances including: *quantum chromodynamics, superfluidity, spontaneous symmetry breaking, quantum gravity and Universe dynamics* (i.e., the inflationary Universe).

Therefore, the various interactions and interplay between the symmetries of quantum operator state space geometry and quantum dynamics at various levels leads to both algebraic and topological structures that are variable and complex, well beyond symmetry groups and well-studied group algebras (such as Lie algebras, see for example [102]). A unified treatment of quantum phenomena/dynamics and structures may thus become possible with the help of algebraic topology, non-Abelian treatments; such powerful mathematical tools are capable of revealing novel, fundamental aspects related to extended symmetries and quantum dynamics through a detailed analysis of the variable geometry of (quantum) operator algebra state spaces. At the center stage of non-Abelian algebraic topology are groupoid and algebroid structures with their internal and external symmetries [212] that allow one to treat physical spacetime structures and dynamics within an unified categorical, higher dimensional algebra framework [44]. As already suggested in our previous report, the interplay between extended symmetries and dynamics generates higher dimensional structures of quantized spacetimes that exhibit novel properties not found in lower dimensional representations of groups, group algebras or Abelian groupoids.

It is also our intention here to explore, uncover, and then develop, new links between several important but seemingly distinct mathematical approaches to extended quantum symmetries that were not considered in previous reports.

2 Quantum groups, quantum operator algebras, Ocneanu paragroups, quantum groupoids and related symmetries

Quantum theories adopted a new lease of life post 1955 when von Neumann beautifully reformulated quantum mechanics (QM) in the mathematically rigorous context of Hilbert spaces and operator algebras. From a current physics perspective, von Neumann's approach to quantum mechanics has done however much more: it has not only paved the way to expanding the role of symmetry in physics, as for example with the Wigner–Eckhart theorem and its applications, but also revealed the fundamental importance in quantum physics of the state space geometry of (quantum) operator algebras.

The basic definition of von Neumann and Hopf algebras (see for example [154]), as well as the topological groupoid definition, are recalled in the Appendix to maintain a self-contained presentation. Subsequent developments of the quantum operator algebra were aimed at identifying more general quantum symmetries than those defined for example by symmetry groups, groups of unitary operators and Lie groups, thus leading to the development of theories based on various quantum groups [81]. Several, related quantum algebraic concepts were also fruitfully developed, such as: the Ocneanu *paragroups*-later found to be represented by Kac–Moody algebras, quantum groups represented either as Hopf algebras or locally compact groups with Haar measure, 'quantum' groupoids represented as weak Hopf algebras, and so on. The Ocneanu paragroups case is particularly interesting as it can be considered as an extension through quantization of certain finite group symmetries to infinitely-dimensional von Neumann type II_1 algebras, and are, in effect, *quantized groups* that can be nicely constructed as Kac algebras; in fact, it was recently shown that a paragroup can be constructed from a crossed product by an outer action of a Kac(–Moody) algebra. This suggests a relation to categorical aspects of paragroups (rigid monoidal tensor categories [208, 227]). The strict symmetry of the group of (quantum) unitary operators is thus naturally extended through paragroups to the symmetry of the latter structure's unitary representations; furthermore, if a subfactor of the von Neumann algebra arises as a crossed product by a finite group action, the paragroup for this subfactor contains a very similar group structure to that of the original finite group, and also has a unitary representation theory similar to that of the original finite group. Last-but-not least, a paragroup yields a *com-*

plete invariant for irreducible inclusions of AFD von Neumannn II_1 factors with finite index and finite depth (Theorem 2.6 of [196]). This can be considered as a kind of internal, 'hidden' quantum symmetry of von Neumann algebras.

On the other hand, unlike paragroups, quantum locally compact groups are not readily constructed as either Kac or Hopf C^*-algebras. In recent years the techniques of Hopf symmetry and those of weak Hopf C^*-algebras, sometimes called *quantum groupoids* (cf. Böhm et al. [34]), provide important tools – in addition to the paragroups – for studying the broader relationships of the Wigner fusion rules algebra, $6j$-symmetry [188], as well as the study of the noncommutative symmetries of subfactors within the Jones tower constructed from finite index depth 2 inclusion of factors, also recently considered from the viewpoint of related Galois correspondences [172].

We shall proceed at first by pursuing the relationships between these mainly algebraic concepts and their extended quantum symmetries, also including relevant computation examples; then we shall consider several further extensions of symmetry and algebraic topology in the context of local quantum physics/algebraic quantum field theory, symmetry breaking, quantum chromodynamics and the development of novel supersymmetry theories of quantum gravity. In this respect one can also take spacetime 'inhomogeneity' as a criterion for the comparisons between physical, partial or local, symmetries: on the one hand, the example of paracrystals reveals thermodynamic disorder (entropy) within its own spacetime framework, whereas in spacetime itself, whatever the selected model, the inhomogeneity arises through (super) gravitational effects. More specifically, in the former case one has the technique of the generalized Fourier–Stieltjes transform (along with convolution and Haar measure), and in view of the latter, we may compare the resulting 'broken'/paracrystal-type symmetry with that of the supersymmetry predictions for weak gravitational fields (e.g., 'ghost' particles) along with the broken supersymmetry in the presence of intense gravitational fields. Another significant extension of quantum symmetries may result from the superoperator algebra/algebroids of Prigogine's quantum *superoperators* which are defined only for irreversible, infinite-dimensional systems [183].

2.1 Solving quantum problems by algebraic methods: applications to molecular structure, quantum chemistry and quantum theories

As already discussed in the Introduction, one often deals with continuity and continuous transformations in natural systems, be they physical, chemical or self-organizing. Such continuous 'symmetries' often have a special type of underlying continuous group, called a *Lie group*. Briefly, a *Lie group* G is generally considered having a (smooth) C^∞ manifold structure, and acts upon itself smoothly. Such a *globally smooth* structure is surprisingly simple in two ways: it always admits an *Abelian fundamental group*, and seemingly also related to this global property, it admits an associated, *unique* – as well as *finite* – *Lie algebra* that completely specifies *locally* the properties of the Lie group everywhere.

2.1.1 The finite Lie algebra of quantum commutators and their unique (continuous) Lie groups

Lie algebras can greatly simplify quantum computations and the initial problem of defining the form and symmetry of the quantum Hamiltonian subject to boundary and initial conditions in the quantum system under consideration. However, unlike most regular abstract algebras, a Lie algebra is *not associative*, and it is in fact a *vector space* [116]. It is also perhaps this feature that makes the Lie algebras somewhat compatible, or 'consistent', with quantum logics that are also thought to have *non-associative, non-distributive and non-commutative lattice* structures. Nevertheless, the need for 'quantizing' Lie algebras in the sense of a certain *non-commutative* 'deformation' apparently remains for a quantum system, especially if one starts with a 'classical'

Poisson algebra [140]. This requirement remains apparently even for the generalized version of a Lie algebra, called a *Lie algebroid* (see its definition and related remarks in Sections 4 and 5).

The presence of Lie groups in many classical physics problems, in view of its essential *continuity* property and its *Abelian* fundamental group, is not surprising. However, what is surprising in the beginning, is the appearance of *Lie groups and Lie algebras* in the context of commutators of observable operators even in quantum systems *with no classical analogue observables such as the spin*, as – for example – the SU(2) and its corresponding, unique $\mathfrak{su}(2)$ algebra.

As a result of quantization, one would expect to deal with an algebra such as the Hopf (quantum group) which is *associative*. On the other hand, the application of the correspondence principle to the simple, classical harmonic oscillator system leads to a quantized harmonic oscillator and remarkably simple *commutator* algebraic expressions, which correspond precisely to the definition of a Lie algebra. Furthermore, this (*Lie*) algebraic procedure of assembling the quantum Hamiltonian from simple observable operator commutators is readily extended to *coupled, quantum harmonic oscillators*, as shown in great detail by Fernandez and Castro in [94].

2.2 Some basic examples

Example 2.1 (The Lie algebra of a quantum harmonic oscillator). Here one aims to solve the time-independent Schrödinger equations of motion in order to determine the stationary states of the quantum harmonic oscillator which has a quantum Hamiltonian of the form:

$$\mathbf{H} = \left(\frac{1}{2m}\right) \cdot P^2 + \frac{k}{2} \cdot X^2,$$

where X and P denote, respectively, the coordinate and conjugate momentum operators. The terms X and P satisfy the Heisenberg commutation/uncertainty relations $[X, P] = \iota\hbar I$, where the identity operator I is employed to simplify notation. A simpler, equivalent form of the above Hamiltonian is obtained by defining physically dimensionless coordinate and momentum:

$$\mathbf{x} = \left(\frac{X}{\alpha}\right), \qquad \mathbf{p} = \left(\frac{\alpha P}{\hbar}\right) \qquad \text{and} \qquad \alpha = \sqrt{\frac{\hbar}{mk}}.$$

With these new dimensionless operators, $\mathbf{x}$ and $\mathbf{p}$, the quantum Hamiltonian takes the form:

$$\mathbf{H} = \left(\frac{\hbar\omega}{2}\right) \cdot (\mathbf{p}^2 + \mathbf{x}^2),$$

which in units of $\hbar \cdot \omega$ is simply:

$$\mathbf{H}' = \frac{1}{2}(\mathbf{p}^2 + \mathbf{x}^2).$$

The commutator of $\mathbf{x}$ with its conjugate operator $\mathbf{p}$ is simply $[\mathbf{x}, \mathbf{p}] = \iota$.

Next one defines the superoperators $S_{Hx} = [H, x] = -\iota \cdot p$, and $S_{Hp} = [H, p] = \iota \cdot \mathbf{x}$ that will lead to new operators that act as generators of a Lie algebra for this quantum harmonic oscillator. The eigenvectors Z of these superoperators are obtained by solving the equation $S_H \cdot Z = \zeta Z$, where ζ are the eigenvalues, and Z can be written as $(c_1 \cdot x + c_2 \cdot p)$. The solutions are:

$$\zeta = \pm 1, \qquad \text{and} \qquad c_2 = \mp\iota \cdot c_1.$$

Therefore, the two eigenvectors of S_H can be written as:

$$a^\dagger = c_1 * (x - \iota p) \qquad \text{and} \qquad a = c_1(x + \iota p),$$

respectively for $\zeta = \pm 1$. For $c_1 = \sqrt{2}$ one obtains normalized operators H, a and $a^\dagger$ that generate a 4-dimensional Lie algebra with commutators:

$$[H, a] = -a, \qquad [H, a^\dagger] = a^\dagger \qquad \text{and} \qquad [a, a^\dagger] = I.$$

The term a is called the *annihilation* operator and the term $a^\dagger$ is called the *creation* operator. This Lie algebra is solvable and generates after repeated application of $a^\dagger$ all of the eigenvectors of the quantum harmonic oscillator:

$$\Phi_n = \frac{(a^\dagger)^n}{\sqrt{(n!)}} \Phi_0.$$

The corresponding, possible eigenvalues for the energy, derived then as solutions of the Schrödinger equations for the quantum harmonic oscillator are:

$$E_n = \hbar\omega \left(n + \tfrac{1}{2}\right), \qquad \text{where} \qquad n = 0, 1, \ldots, N.$$

The position and momentum eigenvector coordinates can be then also computed by iteration from (*finite*) matrix representations of the (*finite*) Lie algebra, using perhaps a simple computer programme to calculate linear expressions of the annihilation and creation operators. For example, one can show analytically that:

$$[a, x^k] = \frac{k}{\sqrt{2}} x_{k-1}.$$

One can also show by introducing a *coordinate* representation that the eigenvectors of the harmonic oscillator can be expressed as *Hermite polynomials* in terms of the coordinates. In the coordinate representation the quantum *Hamiltonian* and *bosonic* operators have, respectively, the simple expressions:

$$H = \frac{1}{2}\left[-\frac{d^2}{dx^2} + x^2\right], \qquad a = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right), \qquad a^\dagger = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right).$$

The ground state eigenfunction normalized to unity is obtained from solving the simple first-order differential equation $a\Phi_0(x) = 0$, thus leading to the expression:

$$\Phi_0(x) = \pi^{-\frac{1}{4}} \exp\left(-\frac{x^2}{2}\right).$$

By repeated application of the creation operator written as

$$a\dagger = -\frac{1}{\sqrt{2}} \exp\left(\frac{x^2}{2}\right) \frac{d}{dx^2} \exp\left(-\frac{x^2}{2}\right),$$

one obtains the n-th level eigenfunction:

$$\Phi_n(x) = \frac{1}{\sqrt{\pi} 2^n n!} \mathbf{He}_n(x),$$

where $\mathbf{He}_n(x)$ is *the Hermite polynomial* of order n. With the special generating function of the Hermite polynomials

$$F(t, x) = \pi^{-\frac{1}{4}} \left(\exp\left(-\frac{x^2}{2}\right) + tx - \frac{t^2}{4}\right),$$

one obtains explicit analytical relations between the eigenfunctions of the quantum harmonic oscillator and the above special generating function:

$$F(t,x) = \sum_{n=0} \frac{t^n}{\sqrt{2^n n!}} \Phi_n(x).$$

Such applications of the Lie algebra, and the related algebra of the *bosonic* operators as defined above are quite numerous in theoretical physics, and especially for various quantum field carriers in QFT that are all *bosons* (note also additional examples of special Lie superalgebras for gravitational and other fields in Section 6, such as gravitons and Goldstone quanta that are all *bosons* of different spin values and 'Penrose homogeneity').

In the interesting case of a *two-mode* bosonic quantum system formed by the tensor (direct) product of *one-mode* bosonic states:

$$\mid m, n\rangle :=\mid m\rangle \otimes \mid n\rangle,$$

one can generate a 3-dimensional Lie algebra in terms of *Casimir* operators. *Finite*-dimensional Lie algebras are far more tractable and easier to compute than those with an infinite basis set. For example, such a Lie algebra as the 3-dimensional one considered above for the two-mode, bosonic states is quite useful for numerical computations of vibrational (IR, Raman, etc.) spectra of two-mode, *diatomic* molecules, as well as the computation of scattering states. Other perturbative calculations for more complex quantum systems, as well as calculations of exact solutions by means of Lie algebras have also been developed (see e.g. [94]).

Example 2.2 (The SU(2) quantum group). Let us consider the structure of the ubiquitous quantum group SU(2) [224, 68]. Here A is taken to be a C^*-algebra generated by elements α and β subject to the relations:

$$\alpha\alpha^* + \mu^2\beta\beta^* = 1, \qquad \alpha^*\alpha + \beta^*\beta = 1,$$
$$\beta\beta^* = \beta^*\beta, \qquad \alpha\beta = \mu\beta\alpha, \qquad \alpha\beta^* = \mu\beta^*\alpha,$$
$$\alpha^*\beta = \mu^{-1}\beta\alpha^*, \qquad \alpha^*\beta^* = \mu^{-1}\beta^*\alpha^*,$$

where $\mu \in [-1,1]\backslash\{0\}$. In terms of the matrix

$$u = \begin{bmatrix} \alpha & -\mu\beta^* \\ \beta & \alpha^* \end{bmatrix}$$

the coproduct Δ is then given via

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}.$$

As will be shown in our later sections, such quantum groups and their associated algebras need be extended to more general structures that involve *supersymmetry*, as for example in the case of quantum gravity or *supergravity* and superfield theories. Another important example of such *quantum supergroups* involves Drinfel'd 's quantum double construction and the R-matrix (e.g., as developed in [133] and subsequent reports related to quantum quasi-algebras [5, 228, 229]).

Numerous quantum supergroup examples also emerge in the cases presented in the next subsection of either molecular groups of spins or nuclear quasi-particles coupled, respectively, by either dipolar (magnetic) or colour-charge and dipolar interactions. In such cases, the simple Lie algebras considered above in the first example need to be extended to Lie *superalgebras* exhibiting supersymmetry that includes both fermionic and bosonic symmetries (as explained in Section 6.2). Furthermore – as discussed next – local bosonic models (or *spin models*) were reported to lead to quantum gravity as well as the emergence of certain near massless fermions such as the electron.

Example 2.3 (Quantum supergroups of dipolar-coupled spins). An important example for either nuclear magnetic or electron spin resonances in solids is that of (magnetic) *dipolar-coupled (molecular) groups of spins*. Among such systems in which an understanding of the dipole-dipole interactions is essential are molecular groups of dipolar-coupled spin-1/2 particles (fermions) with local symmetry, or symmetries, such as groups of dipolar-coupled protons or magnetically-coupled microdomains of unpaired electrons in solids [13, 14, 15, 16, 17, 19]. Although one might expect such systems of fermions to follow the Fermi statistics, in fact, the dipolar-coupled groups of spin-1/2 particles behave much more like quasi-quadrupolar (quasi) particles of spin-1 for proton pairs (as for example in ice or dihydrate gypsum crystals), or as spin-3/2 quasi-particles in the case of hydrogen nuclei in methyl groups and hydronium ions in solids, or coupled ^{19}F (spin-1/2) nuclei in $-CF_3$ molecular groups in polycrystalline solids [19]. (Interestingly, quantum theories of fermions were also recently proposed that do not require the presence of fermion fields [27].) A partially symmetric local structure was reported from such ^1H NMR studies which involved the determination of both inter- and intra-molecular van Vleck second moments of proton dipolar interactions of water in strongly ionic $\mathrm{LiCl} \times n\,\mathrm{H_2O}$ and $\times n\,\mathrm{D_2O}$ electrolyte glasses (with $2.6 < n \leq 12$) at low temperature. Thus, the local symmetry of the hydration sphere for Li^+ cations in such glasses at low temperatures was reported to be *quasi-tetrahedral* [15], and this was subsequently confirmed by independent neutron scattering and electron tunneling/spectroscopic studies of the same systems at low temperatures, down to 4K; on the other hand, for $n \geq 4$ water molecules bridged between hydrated Li^+ clusters and a Cl^- anion, the local symmetry approached quasi-octahedral around the anion. Similar studies were carried out for $\mathrm{Ca(NO_3)_2} \times n\,\mathrm{H_2O}$, $\mathrm{Zn(NO_3)_2} \times n\,\mathrm{H_2O}$, $\mathrm{Cd(NO_3)_2} \times m\,\mathrm{H_2O}$ and $\mathrm{La(NO_3)_3} \times k\,\mathrm{H_2O}$ electrolyte glasses (with $3 < m \leq 20$ and, respectively, $3 < k \leq 30$), and the hydration local symmetries were found, respectively, to be: quasi-octahedral for both Ca^{+2} and Zn^{+2} (divalent) hydrated cations, *quasi-icosahedral* for the (divalent) Cd^{+2} hydrated cation, and quasi-dodecahedral for the (trivalent) La^{+3} hydrated cation. Interstitial water molecules between hydrated cation clusters exhibited however much lower local symmetry, and it was reported to be very close to that of water monomers and dimers in the vapor phase [15].

The NMR behaviour of such proton and ^{19}F *quasi-particles* in solids [19] suggests therefore the use of an unified supersymmetry approach using Lie superalgebras [211] and *quantum supergroups*.

Quasi-particles were also recently reported for Anderson-localized electrons in solids with partial disorder, and in the case of sufficiently strong disorder, "the Mott–Anderson transition was characterized by a precisely defined *two-fluid* behaviour, in which only a fraction of the electrons undergo a 'site selective' Mott localization" [3] (see also related previous articles in [166, 163, 164, 165, 203, 207, 6, 66]). Thus, in any non-crystalline system – such as a glass – the lowest states in the conduction band are "localized", or they act as traps, and "on the energy scale there is a continuous range of such localized states leading from the bottom of the band up to a critical energy E_c, called the mobility edge, where states become non-localized or extended" [163]. Recently, a concept of "quantum glassiness" was also introduced [70].

Similarly to spin-1/2 dipolar-coupled pairs, dipolar-coupled linear chains of either spin-1 or spin-0 bosons exhibit most remarkable properties that also depend on the strength of *dipole-dipole* interactions among the neighbour bosons in the chain, as well as the overall, *extended quantum symmetry* (EQS) of the chain. On the other hand, *local bosonic models, or spin models,* may also provide a unified origin for identical particles, gauge interactions, Fermi statistics and near masslessness of certain fermions [144]. Gauge interactions and Fermi statistics were also suggested to be unified under the point of view of emergence of identical particles; furthermore, a local bosonic model was constructed from which gravitons also emerge [144], thus leading to *quantum gravity*. Spin-2 boson models on a lattice are therefore being studied in such theories of quantum gravity [111].

Examples of dipolar-coupled and colour charge-coupled spin-0 bosons may be very abundant in nuclear physics where quark pairs provide a better model than the often used, 'quark bag' model. Such spin-0 boson models of coupled quark pairs may also provide new insights into how to achieve *controlled thermonuclear fusion* [129]. An example of a system of dipolar-coupled spin-1 bosons is that of an array of deuterons (^{2}H) in *deuteriated* long chain molecules such as phospholipids or fluorinated aliphatic chains in liquid crystals (e.g., perfluorooctanoate). For such systems it is possible to set up an explicit form of the Hamiltonian and to digitally compute all the spin energy levels and the nuclear magnetic resonance properties (including the phase coherence and spin correlations) for the entire chain of *dipolar-coupled spin-*1 *bosons* (see for example the simple "bosonization" computations in [157]).

The case of dipolar-coupled and also ion-coupled (or phonon-coupled), spin-0 bosons is also most remarkable in its *long-range* correlations/coherence properties, as well as the temperature dependent symmetry breaking behaviour which is well-established; consider, for example, the Cooper (electron) pairs in superconductors [141, 192] that share this behaviour with other superfluids (e.g., liquid ^{3}He). Somewhat surprisingly, *long-range* magnetic correlations involving nonlinear magnon dispersion also occur in ferromagnetic metallic glasses that have only short-range (or local) atomic structures of very low, or *broken* symmetry, but exhibit microwave resonance absorption spectra caused by (long-range, coupled electron) *spin wave excitations* as reported in [17]. The corresponding, explicit form of the Hamiltonian for the latter systems – including magnetic dipolar coupling, exchange and magnon interactions – has also been specified in [17], and the short-range local structure present in such metallic glasses – noncrystalline systems with broken, local symmetry – was reported from X-ray scattering and ferromagnetic resonance studies [16]. Such noncrystalline systems with long-range coupling may be therefore more amenable to descriptions in terms of topological order theories as pointed out in [7, 163, 164] rather than Landau symmetry-breaking models. Topological order theories and topological quantum computation were also recently reported to be of interest for the design of quantum computers [171, 202, 96, 134, 79], and thus such fundamental topological order theories might, conceivably, also lead to practical applications in developing ultra-fast quantum supercomputers.

2.3 Hopf algebras

Firstly, a unital associative algebra consists of a linear space A together with two linear maps

$$m:\ A\otimes A\longrightarrow A \quad \text{(multiplication)}, \qquad \eta:\ \mathbb{C}\longrightarrow A \quad \text{(unity)}$$

satisfying the conditions

$$m(m\otimes \mathbf{1}) = m(\mathbf{1}\otimes m), \qquad m(\mathbf{1}\otimes\eta) = m(\eta\otimes\mathbf{1}) = \mathrm{id}.$$

This first condition can be seen in terms of a commuting diagram:

$$\begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{m\otimes \mathrm{id}} & A\otimes A \\ {\scriptstyle \mathrm{id}\otimes m}\big\downarrow & & \big\downarrow{\scriptstyle m} \\ A\otimes A & \xrightarrow{m} & A \end{array}$$

Next let us consider 'reversing the arrows', and take an algebra A equipped with a linear homorphisms $\Delta : A\longrightarrow A\otimes A$, satisfying, for $a, b\in A$:

$$\Delta(ab) = \Delta(a)\Delta(b), \qquad (\Delta\otimes \mathrm{id})\Delta = (\mathrm{id}\otimes\Delta)\Delta. \tag{2.1}$$

We call Δ a *comultiplication*, which is said to be *coassociative* in so far that the following diagram commutes

$$\begin{array}{ccc} A\otimes A\otimes A & \xleftarrow{\Delta\otimes\mathrm{id}} & A\otimes A \\ {\scriptstyle \mathrm{id}\otimes\Delta}\uparrow & & \uparrow{\scriptstyle \Delta} \\ A\otimes A & \xleftarrow{\Delta} & A \end{array}$$

There is also a counterpart to η, the *counity* map $\varepsilon : A \longrightarrow \mathbb{C}$ satisfying

$$(\mathrm{id}\otimes\varepsilon)\circ\Delta = (\varepsilon\otimes\mathrm{id})\circ\Delta = \mathrm{id}.$$

A *bialgebra* $(A, m, \Delta, \eta, \varepsilon)$ is a linear space A with maps m, Δ, η, ε satisfying the above properties.

Now to recover anything resembling a group structure, we must append such a bialgebra with an antihomomorphism $S : A\longrightarrow A$, satisfying $S(ab) = S(b)S(a)$, for $a, b \in A$. This map is defined implicitly via the property:

$$m(S\otimes\mathrm{id})\circ\Delta = m(\mathrm{id}\otimes S)\circ\Delta = \eta\circ\varepsilon.$$

We call S the *antipode map*. A *Hopf algebra* is then a bialgebra $(A, m, \eta, \Delta, \varepsilon)$ equipped with an antipode map S.

Commutative and non-commutative Hopf algebras form the backbone of quantum groups [69] and are essential to the generalizations of symmetry. Indeed, in most respects a quantum group is identifiable with a Hopf algebra. When such algebras are actually associated with proper groups of matrices there is considerable scope for their representations on both finite and infinite dimensional Hilbert spaces.

Example 2.4 (The $\mathbf{SL}_q(2)$ Hopf algebra). This algebra is defined by the generators a, b, c, d and the following relations:

$$ba = qab, \qquad db = qbd, \qquad ca = qac, \qquad dc = qcd, \qquad bc = cb,$$

together with

$$adda = \left(q^{-1} - q\right) bc, \qquad adq^{-1}bc = 1,$$

and

$$\Delta \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \epsilon \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad S \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -qb \\ -q^{-1}c & a \end{bmatrix}.$$

2.4 Quasi-Hopf algebra

A quasi-Hopf algebra is an extension of a Hopf algebra. Thus, a quasi-Hopf algebra is a *quasi-bialgebra* $\mathcal{B}_{\mathcal{H}} = (\mathcal{H}, \Delta, \varepsilon, \Phi)$ for which there exist $\alpha, \beta \in \mathcal{H}$ and a bijective antihomomorphism S (the 'antipode') of $\mathcal{H}$ such that $\sum_i S(b_i)\alpha c_i = \varepsilon(a)\alpha$, $\sum_i b_i\beta S(c_i) = \varepsilon(a)\beta$ for all $a \in \mathcal{H}$, with $\Delta(a) = \sum_i b_i \otimes c_i$, and the relationships

$$\sum_i X_i\beta S(Y_i)\alpha Z_i = \mathbf{I}, \qquad \sum_j S(P_j)\alpha Q_j\beta S(R_j) = \mathbf{I},$$

where the expansions for the quantities Φ and Φ^{-1} are given by

$$\Phi = \sum_i X_i \otimes Y_i \otimes Z_i, \qquad \Phi^{-1} = \sum_j P_j \otimes Q_j \otimes R_j.$$

As in the general case of a quasi-bialgebra, the property of being quasi-Hopf is unchanged by "twisting". Thus, twisting the comultiplication of a coalgebra

$$\mathcal{C} = (C, \Delta, \epsilon)$$

over a field k produces another coalgebra $\mathcal{C}^{\mathrm{cop}}$; because the latter is considered as a vector space over the field k, the new comultiplication of $\mathcal{C}^{\mathrm{cop}}$ (obtained by "twisting") is defined by

$$\Delta^{\mathrm{cop}}(c) = \sum c_{(2)} \otimes c_{(1)},$$

with $c \in \mathcal{C}$ and

$$\Delta(c) = \sum c_{(1)} \otimes c_{(2)}.$$

Note also that the linear dual $\mathcal{C}^*$ of $\mathcal{C}$ is an algebra with unit ϵ and the multiplication being defined by

$$\langle c^* {*} d^*, c\rangle = \sum \langle c^*, c_{(1)}\rangle \langle d^*, c_{(2)}\rangle,$$

for $c^*, d^* \in \mathcal{C}^*$ and $c \in \mathcal{C}$ (see [136]).

Quasi-Hopf algebras emerged from studies of Drinfel'd twists and also from F-matrices associated with finite-dimensional irreducible representations of a quantum affine algebra. Thus, F-matrices were employed to factorize the corresponding R-matrix. In turn, this leads to several important applications in statistical quantum mechanics, in the form of quantum *affine* algebras; their representations give rise to solutions of the quantum Yang–Baxter equation. This provides solvability conditions for various quantum statistics models, allowing characteristics of such models to be derived from their corresponding quantum affine algebras. The study of F-matrices has been applied to models such as the so-called Heisenberg 'XXZ model', in the framework of the algebraic *Bethe ansatz*. Thus F-matrices and quantum groups together with quantum affine algebras provide an effective framework for solving two-dimensional integrable models by using the quantum inverse scattering method as suggested by Drinfel'd and other authors.

2.5 Quasi-triangular Hopf algebra

We begin by defining the quasi-triangular Hopf algebra, and then discuss its usefulness for computing the R-matrix of a quantum system.

Definition 2.1. A Hopf algebra, H, is called *quasi-triangular* if there is an invertible element R, of $H \otimes H$ such that:

(1) $R\,\Delta(x) = (T \circ \Delta)(x)\,R$ for all $x \in H$, where Δ is the coproduct on H, and the linear map $T : H \otimes H \to H \otimes H$ is given by

$$T(x \otimes y) = y \otimes x,$$

(2) $(\Delta \otimes 1)(R) = R_{13}R_{23}$,

(3) $(\mathbf{1} \otimes \Delta)(R) = R_{13}R_{12}$, where $R_{12} = \phi_{12}(R)$,

(4) $R_{13} = \phi_{13}(R)$, and $R_{23} = \phi_{23}(R)$, where $\phi_{12} : H \otimes H \to H \otimes H \otimes H$,

(5) $\phi_{13} : H \otimes H \to H \otimes H \otimes H$, and $\phi_{23} : H \otimes H \to H \otimes H \otimes H$, are algebra morphisms determined by

$$\phi_{12}(a \otimes b) = a \otimes b \otimes 1, \qquad \phi_{13}(a \otimes b) = a \otimes 1 \otimes b, \qquad \phi_{23}(a \otimes b) = 1 \otimes a \otimes b.$$

R is called the *R-matrix*.

An important part of the above algebra can be summarized in the following commutative diagrams involving the algebra morphisms, the coproduct on H and the identity map id:

$$\begin{tikzcd}
H \otimes H \otimes H & H \otimes H \arrow[l, "\phi_{12},\ \phi_{13}"'] \\
H \otimes H \otimes H \arrow[u, "\mathrm{id}\otimes\mathrm{id}\otimes\Delta"] & H \otimes H \arrow[l, "\phi_{23},\ \mathrm{id}\otimes\Delta"'] \arrow[u, "\Delta"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
H \otimes H \otimes H & H \otimes H \arrow[l, "\Delta\otimes\mathrm{id}"'] \\
H \otimes H \arrow[u, "\mathrm{id}\otimes\Delta"] & H \arrow[l, "\Delta"'] \arrow[u, "\Delta"']
\end{tikzcd}$$

Because of this property of quasi-triangularity, the R-matrix, R, becomes a solution of *the Yang–Baxter equation*. Thus, a module M of H can be used to determine quasi-invariants of links, braids, knots and higher dimensional structures with similar quantum symmetries. Furthermore, as a consequence of the property of quasi-triangularity, one obtains:

$$(\epsilon \otimes 1)R = (1 \otimes \epsilon)R = 1 \in H.$$

Finally, one also has:

$$R^{-1} = (S \otimes 1)(R), \qquad R = (1 \otimes S)\big(R^{-1}\big) \qquad \text{and} \qquad (S \otimes S)(R) = R.$$

One can also prove that the antipode S is a linear isomorphism, and therefore S^2 is an automorphism: S^2 is obtained by conjugating by an invertible element, $S(x) = uxu^{-1}$, with

$$u = m(S \otimes 1)R^{21}.$$

By employing Drinfel'd's quantum double construction one can assemble a quasi-triangular Hopf algebra from a Hopf algebra and its dual.

2.5.1 Twisting a quasi-triangular Hopf algebra

The property of being a quasi-triangular Hopf algebra is invariant under twisting *via* an invertible element $F = \sum_i f^i \otimes f_i \in \mathcal{A} \otimes \mathcal{A}$ such that $(\varepsilon \otimes \mathrm{id})F = (\mathrm{id} \otimes \varepsilon)F = 1$, and also such that the following cocycle condition is satisfied:

$$(F \otimes 1) \circ (\Delta \otimes \mathrm{id})F = (1 \otimes F) \circ (\mathrm{id} \otimes \Delta)F.$$

Moreover, $u = \sum_i f^i S(f_i)$ is invertible and the twisted antipode is given by $S'(a) = uS(a)u^{-1}$, with the twisted comultiplication, R-matrix and co-unit change according to those defined for the quasi-triangular quasi-Hopf algebra. Such a twist is known as *an admissible, or Drinfel'd, twist*.

2.6 Quasi-triangular quasi-Hopf algebra (QTQH)

A quasi-triangular quasi-Hopf algebra as defined by Drinfel'd in [84] is an extended form of a quasi-Hopf algebra, and also of a quasi-triangular Hopf algebra. Thus, a quasi-triangular quasi-Hopf algebra is defined as a *quintuple* $\mathcal{B}_{\mathcal{H}} = (\mathcal{H}, R, \Delta, \varepsilon, \Phi)$ where the latter is a quasi-Hopf algebra, and $R \in \mathcal{H} \otimes \mathcal{H}$ referred to as the R-matrix (as defined above), which is an invertible element such that:

$$R\Delta(a) = \sigma \circ \Delta(a)R, \qquad a \in \mathcal{H}, \qquad \sigma : \ \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}, \qquad x \otimes y \to y \otimes x,$$

so that σ is the switch map and

$$(\Delta \otimes \mathrm{id})R = \Phi_{321} R_{13} \Phi_{132}^{-1} R_{23} \Phi_{123}, \qquad (\mathrm{id} \otimes \Delta)R = \Phi_{231}^{-1} R_{13} \Phi_{213} R_{12} \Phi_{123}^{-1},$$

where $\Phi_{abc} = x_a \otimes x_b \otimes x_c$, and $\Phi_{123} = \Phi = x_1 \otimes x_2 \otimes x_3 \in \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$. The quasi-Hopf algebra becomes triangular if in addition one has $R_{21} R_{12} = 1$.

The twisting of $\mathcal{B}_{\mathcal{H}}$ by $F \in \mathcal{H} \otimes \mathcal{H}$ is the same as for a quasi-Hopf algebra, with the additional definition of the twisted R-matrix. A quasi-triangular, quasi-Hopf algebra with $\Phi = 1$ is a quasi-triangular Hopf algebra because the last two conditions in the definition above reduce to the quasi-triangularity condition for a Hopf algebra. Therefore, just as in the case of the twisting of a quasi-Hopf algebra, the property of being quasi-triangular of a quasi-Hopf algebra is preserved by twisting.

2.7 Yang–Baxter equations

2.7.1 Parameter-dependent Yang–Baxter equation

Consider A to be an unital associative algebra. Then, *the parameter-dependent Yang–Baxter equation* is an equation for $R(u)$, the parameter-dependent invertible element of the tensor product $A \otimes A$ (here, u is the parameter, which usually ranges over all real numbers in the case of an additive parameter, or over all positive real numbers in the case of a multiplicative parameter; for the dynamic Yang–Baxter equation see also [89]). The Yang–Baxter equation is usually stated as:

$$R_{12}(u) R_{13}(u+v) R_{23}(v) = R_{23}(v) R_{13}(u+v) R_{12}(u),$$

for all values of u and v, in the case of an additive parameter, and

$$R_{12}(u) R_{13}(uv) R_{23}(v) = R_{23}(v) R_{13}(uv) R_{12}(u),$$

for all values of u and v, in the case of a multiplicative parameter, where

$$R_{12}(w) = \phi_{12}(R(w)), \qquad R_{13}(w) = \phi_{13}(R(w)), \qquad R_{23}(w) = \phi_{23}(R(w))$$

for all values of the parameter w, and

$$\begin{aligned} &\phi_{12} : \ H \otimes H \to H \otimes H \otimes H, \qquad \phi_{13} : \ H \otimes H \to H \otimes H \otimes H, \\ &\phi_{23} : \ H \otimes H \to H \otimes H \otimes H \end{aligned}$$

are algebra morphisms determined by the following (strict) conditions:

$$\phi_{12}(a \otimes b) = a \otimes b \otimes 1, \qquad \phi_{13}(a \otimes b) = a \otimes 1 \otimes b, \qquad \phi_{23}(a \otimes b) = 1 \otimes a \otimes b.$$

2.7.2 The parameter-independent Yang–Baxter equation

Let A be a unital associative algebra. The parameter-independent Yang–Baxter equation is an equation for R, an invertible element of the tensor product $A \otimes A$. The Yang–Baxter equation is:

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12},$$

where $R_{12} = \phi_{12}(R)$, $R_{13} = \phi_{13}(R)$, and $R_{23} = \phi_{23}(R)$.

Let V be a module over A. Let $T : V \otimes V \to V \otimes V$ be the linear map satisfying $T(x \otimes y) = y \otimes x$ for all $x, y \in V$. Then a representation of the braid group B_n, can be constructed on $V^{\otimes n}$ by $\sigma_i = 1^{\otimes i-1} \otimes \check{R} \otimes 1^{\otimes n-i-1}$ for $i = 1, \dots, n-1$, where $\check{R} = T \circ R$ on $V \otimes V$. This representation may thus be used to determine quasi-invariants of braids, knots and links.

2.7.3 Generalization of the quantum Yang–Baxter equation

The quantum Yang–Baxter equation was generalized in [136] to:

$$R = qb\sum_{i=1}^{n} e_{ii}\otimes e_{ii} + b\sum_{i>j} e_{ii}\otimes e_{jj} + c\sum_{i<j} e_{ii}\otimes e_{jj} + \left(qb - q^{-1}c\right)\sum_{i>j} e_{ij}\otimes e_{ji},$$

for $b, c \neq 0$. A solution of the quantum Yang–Baxter equation has the form $R : M\otimes M \to M\otimes M$, with M being a finite dimensional vector space over a field k. Most of the solutions are stated for a given ground field but in many cases a commutative ring with unity may instead be sufficient. (See also the classic paper by Yang and Mills [226].)

2.8 SU(3), SU(5), SU(10) and E_6 representations in quantum chromodynamics and unified theories involving spontaneous symmetry breaking

There have been several attempts to take into consideration extended quantum symmetries that would include, or embed, the SU(2) and SU(3) symmetries in larger symmetry groups such as SU(5), SU(10) and the exceptional Lie group E_6, but so far with only limited success as their representations make several predictions that are so far unsupported by high energy physics experiments [102]. To remove unobserved particles from such predictions, one has invariably to resort to ad-hoc spontaneous symmetry breaking assumptions that would require still further explanations, and so on. So far the only thing that is certain is the fact that the $\mathrm{U}(1)\times \mathrm{SU}(2)\times \mathrm{SU}(3)$ symmetry is broken in nature, presumably in a 'spontaneous' manner. Due to the nonlocal character of quantum theories combined with the restrictions imposed by relativity on the 'simultaneity' of events in different reference systems, a global or universal, spontaneous symmetry breaking mechanism appears contrived, with the remaining possibility that it does however occur locally, thus resulting in quantum theories that use local approximations for broken symmetries, and thus they are not unified, as it was intended. Early approaches to space-time were made in non-relativistic quantum mechanics [95], and were subsequently followed by relativistic and axiomatic approaches to quantum field theory [219, 221, 220].

On the one hand, in GR all interactions are local, and therefore spontaneous, local symmetry breaking may appear not to be a problem for GR, except for the major obstacle that it does severely limit the usefulness of the Lorentz group of transformations which would have to be modified accordingly to take into account the *local* $\mathrm{SU}(2)\times\mathrm{SU}(3)$ spontaneous symmetry breaking. This seems to cause problems with the GR's equivalence principle for all reference systems; the latter would give rise to an equivalence class, or possibly a set, of reference systems. On the other hand, local, spontaneous symmetry breaking generates a *groupoid of equivalence classes of reference systems*, and further, through quantization, to a category of groupoids of such reference systems, $\mathbf{Grpd}_{\mathfrak{R}}$, and their transformations defined as groupoid homomorphisms. Functor representations of $\mathbf{Grpd}_{\mathfrak{R}}$ into the category **BHilb** of rigged Hilbert spaces $\mathcal{H}_r$ would then allow the computation of *local* quantum operator eigenvalues and their eigenstates, in a manner invariant to the local, broken symmetry transformations. One might call such a theory, a locally covariant – quantized GR (lcq-GR), as it would be locally, but not necessarily, globally quantized. Obviously, such a locally covariant GR theory is consistent with AQFT and its operator nets of local quantum observables. Such an extension of the GR theory to a locally covariant GR in a quantized form may not require the 'universal' or global existence of Higgs bosons as a compelling property of the expanding Universe; thus, any lcq-GR theory can allow for the existence of inhomogeneities in spacetime caused by distinct local symmetries in the presence of very intense gravitational fields, dark matter, or other condensed quantum systems such as neutron stars and black holes (with or without 'hair' – cf. J. Wheeler). The GR principle of equivalence is then replaced in lcq-GR by the representations of the quantum fundamental groupoid functor that will be introduced in Section 9.

In view of the existing problems and limitations encountered with group quantum symmetries and their group (or group algebra) representations, current research into the geometry of state spaces of quantum operator algebras leads to extended symmetries expressed as topological groupoid representations that were shown to link back to certain C^*-algebra representations [80] and the dual spaces of C^*-algebras [93]. Such extended symmetries will be discussed in the next sections in terms of quantum groupoid representations involving the notion of measure Haar systems associated with locally compact quantum groupoids.

3 Quantum groupoids and the groupoid C^*-algebra

Quantum groupoid (e.g., weak Hopf algebras) and algebroid symmetries figure prominently both in the theory of dynamical deformations of quantum groups [83] (e.g., Hopf algebras) and the quantum Yang–Baxter equations [87, 88]. On the other hand, one can also consider the natural extension of locally compact (quantum) groups to locally compact (proper) *groupoids* equipped with a Haar measure and a corresponding groupoid representation theory [54] as a major, potentially interesting source for locally compact (but generally *non-Abelian*) *quantum groupoids.* The corresponding quantum groupoid representations on bundles of Hilbert spaces extend quantum symmetries well beyond those of quantum groups/Hopf algebras and simpler operator algebra representations, and are also consistent with the locally compact quantum group representations that were recently studied in some detail by Kustermans and Vaes (see [135] and references cited therein). The latter quantum groups are neither Hopf algebras, nor are they equivalent to Hopf algebras or their dual coalgebras. As pointed out in the previous section, quantum groupoid representations are, however, the next important step towards unifying quantum field theories with general relativity in a locally covariant and quantized form. Such representations need not however be restricted to weak Hopf algebra representations, as the latter have no known connection to any type of GR theory and also appear to be inconsistent with GR.

We are also motivated here by the quantum physics examples mentioned in the previous sections to introduce through several steps of generality, a framework for quantum symmetry breaking in terms of either locally compact quantum groupoid or related algebroid representations, such as those of *weak Hopf C^*-algebroids with convolution* that are realized in the context of *rigged Hilbert spaces* [35]. A novel extension of the latter approach is also now possible *via* generalizations of Grassman–Hopf algebras ($\mathcal{G}_H$), gebras [205, 92] and co-algebra representations to those of graded Grassman–Hopf algebroids. Grassman–Hopf algebras and gebras not only are bi-connected in a manner somewhat similar to Feynman diagrams but also possess a unique left/right integral μ [92, p. 288], whereas such integrals in general do not exist in Clifford–Hopf algebras [91]. This unique integral property of Grassman–Hopf algebras makes them very interesting candidates, for example, in physical applications that require either *generalized convolution* and measure concepts, or generalizations of quantum groups/algebras to structures that are more amenable than weak Hopf C^*-algebras. Another important point made by Fauser [92] is that – unlike Hopf and weak Hopf algebras that have no direct physical visualization either in quantum dynamics or in the Feynman interaction representation of Quantum Electrodynamics – the duals, or tangles, of Grassman–Hopf algebras, such as respectively G–H co-algebras and Grassman–Hopf 'algebras' [92] provide direct visual representations of physical interactions and quantum dynamics in Feynman-like diagrams that utilize directly the dual/tangled, or 'co-algebraic', structure elements. Such visual representations can greatly facilitate exact computations in quantum chromodynamics for the difficult case of strong, nuclear interactions where approximate perturbation methods usually fail. The mathematical definitions and grading of Grassman–Hopf algebroids, (tangled/mirror) algebroids and co-algebroids then follow naturally for supersymmetry, symmetry breaking, and other physical theories. Furthermore, with regard to a unified and global framework for symmetry breaking, as well as higher order quantum symmetries, we look towards the *double groupoid* structures of Brown and Spencer [52], and introduce the concepts of *quantum and graded Lie bi-algebroids* which are expected to carry a distinctive C^*-algebroid convolution structure. The extension to *supersymmetry* leads then naturally to superalgebra, superfield symmetries and their involvement in supergravity or quantum gravity (QG) theories for intense gravitational fields in fluctuating, quantized spacetimes. Our self-contained approach, leads to several novel concepts which exemplify a certain *non-reductionist* viewpoint and theories of the nature of physical spacetime structure [44, 23].

A natural extension in higher dimensional algebra (HDA) of quantum symmetries may involve both *quantum double groupoids* defined as locally compact double groupoids equipped with Haar measures via convolution, and an extension to double algebroids, (that are naturally more general than the Lie double algebroids defined in [150]).

We shall now proceed to formally define several quantum algebraic topology concepts that are needed to express the extended quantum symmetries in terms of proper quantum groupoid and quantum algebroid representations. Hidden, higher dimensional quantum symmetries will then also emerge either *via* generalized quantization procedures from higher dimensional algebra representations or will be determined as global or local invariants obtainable – at least in principle – through non-Abelian algebraic topology (NAAT) methods [46] (see also the earlier classic paper by Fröhlich [97]).

3.1 The weak Hopf algebra

In this, and the following subsections, we proceed through several stages of generality by relaxing the axioms for a Hopf algebra as defined above. The motivation begins with the more restrictive notion of a quantum group in relation to a Hopf algebra where the former is often realized as an automorphism group for a quantum space, that is, an object in a suitable category of generally noncommutative algebras. One of the most common guises of a quantum 'group' is as the dual of a noncommutative, nonassociative Hopf algebra. The Hopf algebras (cf. [68, 154]), and their generalizations [130], are some of the fundamental building blocks of quantum operator algebra, even though they cannot be generally 'integrated' to groups like the 'integration' of Lie algebras to Lie groups, or the Fourier transformation of certain commutative Hopf algebras to their dual, finite commutative groups. However, Hopf algebras are linked and limited only to certain quantum symmetries that are represented by finite compact quantum groups (CQGs).

In order to define a *weak Hopf algebra*, one can relax certain axioms of a Hopf algebra as follows:

(1) The comultiplication is not necessarily unit-preserving.

(2) The counit ε is not necessarily a homomorphism of algebras.

(3) The axioms for the antipode map $S : A \longrightarrow A$ with respect to the counit are as follows. For all $h \in H$,

$$\begin{aligned} &m(\mathrm{id} \otimes S)\Delta(h) = (\varepsilon \otimes \mathrm{id})(\Delta(1)(h \otimes 1)), \\ &m(S \otimes \mathrm{id})\Delta(h) = (\mathrm{id} \otimes \varepsilon)((1 \otimes h)\Delta(1)), \qquad S(h) = S(h_{(1)})h_{(2)}S(h_{(3)}). \end{aligned}$$

These axioms may be appended by the following commutative diagrams

$$\begin{array}{ccc} A \otimes A & \xrightarrow{S \otimes \mathrm{id}} & A \otimes A \\ {\scriptstyle\Delta}\uparrow & & \downarrow{\scriptstyle m} \\ A & \xrightarrow{u \circ \varepsilon} & A \end{array} \qquad \begin{array}{ccc} A \otimes A & \xrightarrow{\mathrm{id} \otimes S} & A \otimes A \\ {\scriptstyle\Delta}\uparrow & & \downarrow{\scriptstyle m} \\ A & \xrightarrow{u \circ \varepsilon} & A \end{array}$$

along with the counit axiom:

$$\begin{tikzcd}
A \otimes A \arrow[d, "\varepsilon\otimes 1"'] & A \arrow[l, "\Delta"'] \arrow[d, "\Delta"] \arrow[dl, "\mathrm{id}_A"'] \\
A & A \otimes A \arrow[l, "1\otimes\varepsilon"]
\end{tikzcd}$$

Several mathematicians substitute the term *quantum groupoid* for a weak Hopf algebra, although this algebra in itself is not a proper groupoid, but it may have a component *group* algebra as in the example of the quantum double discussed next; nevertheless, weak Hopf algebras generalize Hopf algebras – that with additional properties – were previously introduced as' quantum group' by mathematical physicists. (The latter are defined in the Appendix A and, as already discussed, are not mathematical groups but algebras). As it will be shown in the next subsection, quasi-triangular quasi-Hopf algebras are directly related to quantum symmetries in conformal (quantum) field theories. Furthermore, weak C^*-Hopf quantum algebras lead to weak C^*-Hopf algebroids that are linked to quasi-group quantum symmetries, and also to certain Lie algebroids (and their associated Lie–Weinstein groupoids) used to define Hamiltonian (quantum) algebroids over the phase space of (quantum) W_N-gravity.

3.1.1 Examples of weak Hopf algebras

(1) We refer here to [26]. Let G be a non-Abelian group and $H \subset G$ a discrete subgroup. Let $F(H)$ denote the space of functions on H and $\mathbb{C}H$ the group algebra (which consists of the linear span of group elements with the group structure). *The quantum double* $D(H)$ [83] is defined by

$$D(H) = F(H)\,\widetilde{\otimes}\,\mathbb{C}H,$$

where, for $x \in H$, the 'twisted tensor product' is specified by

$$\widetilde{\otimes} \mapsto (f_1 \otimes h_1)(f_2 \otimes h_2)(x) = f_1(x) f_2\left(h_1 x h_1^{-1}\right) \otimes h_1 h_2.$$

The physical interpretation is often to take H as the 'electric gauge group' and $F(H)$ as the 'magnetic symmetry' generated by $\{f \otimes e\}$. In terms of the counit ε, the double $D(H)$ has a trivial representation given by $\varepsilon(f \otimes h) = f(e)$. We next look at certain features of this construction.

For the purpose of braiding relations there is an R matrix, $R \in D(H) \otimes D(H)$, leading to the operator

$$\mathcal{R} \equiv \sigma \cdot \left(\Pi^A_\alpha \otimes \Pi^B_\beta\right)(R),$$

in terms of the Clebsch–Gordan series $\Pi^A_\alpha \otimes \Pi^B_\beta \cong N^{AB\gamma}_{\alpha\beta C} \Pi^C_\gamma$, and where σ denotes a flip operator. The operator $\mathcal{R}^2$ is sometimes called the *monodromy* or *Aharanov–Bohm phase factor*. In the case of a condensate in a state $|v\rangle$ in the carrier space of some representation Π^A_α one considers the maximal Hopf subalgebra T of a Hopf algebra A for which $|v\rangle$ is T-invariant; specifically:

$$\Pi^A_\alpha(P)|v\rangle = \varepsilon(P)|v\rangle, \qquad \forall\, P \in T.$$

(2) For the second example, consider $A = F(H)$. The algebra of functions on H can be broken to the algebra of functions on H/K, that is, to $F(H/K)$, where K is normal in H, that is, $HKH^{-1} = K$. Next, consider $A = D(H)$. On breaking a purely electric condensate $|v\rangle$, the

magnetic symmetry remains unbroken, but the electric symmetry $\mathbb{C}H$ is broken to $\mathbb{C}N_v$, with $N_v \subset H$, the stabilizer of $|v\rangle$. From this we obtain $T = F(H)\widetilde{\otimes}\mathbb{C}N_v$.

(3) In [172] quantum groupoids (considered as weak C^*-Hopf algebras, see below) were studied in relationship to the noncommutative symmetries of depth 2 von Neumann subfactors. If

$$A \subset B \subset B_1 \subset B_2 \subset \cdots$$

is the Jones extension induced by a finite index depth 2 inclusion $A \subset B$ of II_1 factors, then $Q = A' \cap B_2$ admits a quantum groupoid structure and acts on B_1, so that $B = B_1^Q$ and $B_2 = B_1 \rtimes Q$. Similarly, in [188] 'paragroups' (derived from weak C^*-Hopf algebras) comprise (quantum) groupoids of equivalence classes such as those associated with $6j$-symmetry groups (relative to a fusion rules algebra). They correspond to type II von Neumann algebras in quantum mechanics, and arise as symmetries where the local subfactors (in the sense of containment of observables within fields) have depth 2 in the Jones extension. A related question is how a von Neumann algebra N, such as of finite index depth 2, sits inside a weak Hopf algebra formed as the crossed product $N \rtimes A$ [34].

(4) Using a more general notion of the Drinfel'd construction, Mack and Schomerus developed in [148] the notion of a *quasi-triangular quasi-Hopf algebra* (QTQHA) with the aim of studying a range of essential symmetries with special properties, such as the quantum group algebra $\mathrm{U}_q(\mathrm{sl}_2)$ with $|q| = 1$. If $q^p = 1$, then it is shown that a QTQHA is canonically associated with $\mathrm{U}_q(\mathrm{sl}_2)$. Such QTQHAs are claimed as the true symmetries of minimal conformal field theories.

3.1.2 The weak Hopf C^*-algebra in relation to quantum symmetry breaking

In our setting, a *weak C^*-Hopf algebra* is a weak $*$-Hopf algebra which admits a faithful $*$-representation on a Hilbert space. The weak C^*-Hopf algebra is therefore much more likely to be closely related to a quantum groupoid representation than any weak Hopf algebra. However, one can argue that locally compact groupoids equipped with a Haar measure (after quantization) come even closer to defining quantum groupoids. There are already several, significant examples that motivate the consideration of weak C^*-Hopf algebras which also deserve mentioning in the context of 'standard' quantum theories. Furthermore, notions such as (proper) *weak C^*-algebroids* can provide the main framework for symmetry breaking and quantum gravity that we are considering here. Thus, one may consider the quasi-group symmetries constructed by means of special transformations of the coordinate space M. These transformations along with the coordinate space M define certain Lie groupoids, and also their infinitesimal version – the Lie algebroids $\mathbf{A}$, when the former are Weinstein groupoids. If one then lifts the algebroid action from M to the principal homogeneous space $\mathcal{R}$ over the cotangent bundle $T^*M \longrightarrow M$, one obtains a physically significant algebroid structure. The latter was called the Hamiltonian algebroid, $\mathcal{A}^H$, related to the Lie algebroid, $\mathbf{A}$. The Hamiltonian algebroid is an analog of the Lie algebra of symplectic vector fields with respect to the canonical symplectic structure on $\mathcal{R}$ or T^*M. In this recent example, the Hamiltonian algebroid, $\mathcal{A}^H$ over $\mathcal{R}$, was defined over the phase space of W_N-gravity, with the anchor map to Hamiltonians of canonical transformations [142]. Hamiltonian algebroids thus generalize Lie algebras of canonical transformations; canonical transformations of the Poisson sigma model phase space define a *Hamiltonian algebroid* with the Lie brackets related to such a Poisson structure on the target space. The Hamiltonian algebroid approach was utilized to analyze the symmetries of generalized deformations of complex structures on Riemann surfaces $\sum_{g,n}$ of genus g with n marked points. However, its implicit algebraic connections to von Neumann $*$-algebras and/or *weak C^*-algebroid representations* have not yet been investigated. This example suggests that algebroid (quantum) symmetries are implicated in the foundation of relativistic quantum gravity theories and supergravity that we shall consider in further detail in Sections 6–9.

3.2 Compact quantum groupoids

Compact quantum groupoids were introduced in [138] as a simultaneous generalization of a compact groupoid and a quantum group. Since this construction is relevant to the definition of locally compact quantum groupoids and their representations investigated here, its exposition is required before we can step up to the next level of generality. Firstly, let $\mathfrak{A}$ and $\mathfrak{B}$ denote C^*-algebras equipped with a $*$-homomorphism $\eta_s : \mathfrak{B} \longrightarrow \mathfrak{A}$, and a $*$-antihomomorphism $\eta_t : \mathfrak{B} \longrightarrow \mathfrak{A}$ whose images in $\mathfrak{A}$ commute. A non-commutative Haar measure is defined as a completely positive map $P : \mathfrak{A} \longrightarrow \mathfrak{B}$ which satisfies $P(A\eta_s(B)) = P(A)B$. Alternatively, the composition $\mathcal{E} = \eta_s \circ P : \mathfrak{A} \longrightarrow \eta_s(B) \subset \mathfrak{A}$ is a faithful conditional expectation.

Next consider G to be a (topological) groupoid as defined in the Appendix A. We denote by $C_c(\mathsf{G})$ the space of smooth complex-valued functions with compact support on G. In particular, for all $f, g \in C_c(\mathsf{G})$, the function defined via convolution

$$(f * g)(\gamma) = \int_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1) g(\gamma_2),$$

is again an element of $C_c(\mathsf{G})$, where the convolution product defines the composition law on $C_c(\mathsf{G})$. We can turn $C_c(\mathsf{G})$ into a $*$-algebra once we have defined the involution $*$, and this is done by specifying $f^*(\gamma) = \overline{f(\gamma^{-1})}$. This $*$-algebra whose multiplication is the convolution becomes a groupoid C^*-convolution algebra, or groupoid C^*-algebra, G_{CA}, when G is a measured groupoid and the C^*-algebra has a smallest C^*-norm which makes its representations continuous [189, 170].

We recall that following [139] a *representation* of a groupoid $\mathcal{G}$, consists of a family (or field) of Hilbert spaces $\{\mathcal{H}_x\}_{x \in X}$ indexed by $X = \operatorname{Ob} \mathcal{G}$, along with a collection of maps $\{U(\gamma)\}_{\gamma \in \mathcal{G}}$, satisfying:

1) $U(\gamma) : \mathcal{H}_{s(\gamma)} \longrightarrow \mathcal{H}_{r(\gamma)}$, is unitary;
2) $U(\gamma_1 \gamma_2) = U(\gamma_1) U(\gamma_2)$, whenever $(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)}$ (the set of arrows);
3) $U(\gamma^{-1}) = U(\gamma)^*$, for all $\gamma \in \mathcal{G}$.

Suppose now G_{lc} is a Lie groupoid. Then the isotropy group G_x is a Lie group, and for a (left or right) Haar measure μ_x on G_x, we can consider the Hilbert spaces $\mathcal{H}_x = L^2(\mathsf{G}_x, \mu_x)$ as exemplifying the above sense of a representation. Putting aside some technical details which can be found in [73, 139], the overall idea is to define an operator of Hilbert spaces

$$\pi_x(f) : \ L^2(\mathsf{G}_x, \mu_x) \longrightarrow L^2(\mathsf{G}_x, \mu_x),$$

given by

$$(\pi_x(f)\xi)(\gamma) = \int f(\gamma_1) \xi(\gamma_1^{-1}\gamma) d\mu_x,$$

for all $\gamma \in \mathsf{G}_x$, and $\xi \in \mathcal{H}_x$. For each $x \in X = \operatorname{Ob} \mathsf{G}$, π_x defines an involutive representation $\pi_x : C_c(\mathsf{G}) \longrightarrow \mathcal{H}_x$. We can define a norm on $C_c(\mathsf{G})$ given by

$$\|f\| = \sup_{x \in X} \|\pi_x(f)\|,$$

whereby the completion of $C_c(\mathsf{G})$ in this norm, defines *the reduced C^*-algebra $C_r^*(\mathsf{G})$ of G_{lc}*. It is perhaps the most commonly used C^*-algebra for Lie groupoids (groups) in noncommutative geometry [210, 72, 73].

The next step requires a little familiarity with the theory of Hilbert modules (see e.g. [137]). We define a left $\mathfrak{B}$-action λ and a right $\mathfrak{B}$-action ρ on $\mathfrak{A}$ by $\lambda(B)A = A\eta_t(B)$ and $\rho(B)A =$

$A\eta_s(B)$. For the sake of localization of the intended Hilbert module, we implant a $\mathfrak{B}$-valued inner product on $\mathfrak{A}$ given by $\langle A, C\rangle_{\mathfrak{B}} = P(A^*C)$. Let us recall that P is defined as a *completely positive map*. Since P is faithful, we fit a new norm on $\mathfrak{A}$ given by $\|A\|^2 = \|P(A^*A)\|_{\mathfrak{B}}$. The completion of $\mathfrak{A}$ in this new norm is denoted by $\mathfrak{A}^-$ leading then to a Hilbert module over $\mathfrak{B}$.

The tensor product $\mathfrak{A}^- \otimes_{\mathfrak{B}} \mathfrak{A}^-$ can be shown to be a Hilbert bimodule over $\mathfrak{B}$, which for $i = 1, 2$, leads to $*$-homorphisms $\varphi^i : \mathfrak{A} \longrightarrow \mathcal{L}_{\mathfrak{B}}(\mathfrak{A}^- \otimes \mathfrak{A}^-)$. Next is to define the (unital) C^*-algebra $\mathfrak{A} \otimes_{\mathfrak{B}} \mathfrak{A}$ as the C^*-algebra contained in $\mathcal{L}_{\mathfrak{B}}(\mathfrak{A}^- \otimes \mathfrak{A}^-)$ that is generated by $\varphi^1(\mathfrak{A})$ and $\varphi^2(\mathfrak{A})$. The last stage of the recipe for defining a compact quantum groupoid entails considering a certain coproduct operation $\Delta : \mathfrak{A} \longrightarrow \mathfrak{A} \otimes_{\mathfrak{B}} \mathfrak{A}$, together with a coinverse $Q : \mathfrak{A} \longrightarrow \mathfrak{A}$ that it is both an algebra and bimodule antihomomorphism. Finally, the following axiomatic relationships are observed:

$$(\mathrm{id} \otimes_{\mathfrak{B}} \Delta) \circ \Delta = (\Delta \otimes_{\mathfrak{B}} \mathrm{id}) \circ \Delta, \qquad (\mathrm{id} \otimes_{\mathfrak{B}} P) \circ \Delta = P, \qquad \tau \circ (\Delta \otimes_{\mathfrak{B}} Q) \circ \Delta = \Delta \circ Q,$$

where τ is a flip map: $\tau(a \otimes b) = (b \otimes a)$.

There is a natural extension of the above definition of a quantum compact groupoid to a *locally compact* quantum groupoid by taking G_{lc} to be a locally compact groupoid (instead of a compact groupoid), and then following the steps in the above construction with the topological groupoid G being replaced by G_{lc}. Additional integrability and Haar measure system conditions need however be also satisfied as in the general case of locally compact groupoid *representations* (for further details, see for example the monograph [54], the Appendix, and also our subsequent sections on groupoid and categorical/functor representations). In the last three sections we shall tacitly consider quantum groupoids to be, in general, *locally compact* quantum groupoids that are endowed with a Haar measure system (as described in [54] and references cited therein), and also generated through the construction method recalled in this subsection following Landsman [138].

4 Algebroids and their symmetries

By an *algebroid structure* A we shall specifically mean also a ring, or more generally an algebra, but *with several objects* (instead of a single object), in the sense of Mitchell [158]. Thus, an algebroid has been defined by Mosa in [162] and by Brown and Mosa [51] as follows.

An *R-algebroid* A on a set of 'objects' A_0 is a directed graph over A_0 such that for each $x, y \in A_0$, $A(x, y)$ has an R-module structure and there is an R-bilinear function

$$\circ : \ A(x, y) \times A(y, z) \longrightarrow A(x, z),$$

where $(a, b) \mapsto a \circ b$ is the composition, that satisfies the associativity condition, and the existence of identities. A *pre-algebroid* has the same structure as an algebroid and the same axioms except for the fact that the existence of identities $1_x \in A(x, x)$ is not assumed. For example, if A_0 has exactly one object, then an R-algebroid A over A_0 is just an R-algebra. An ideal in A is then an example of a pre-algebroid. Let now R be a commutative ring.

An *R-category* $\mathcal{A}$ is a category equipped with an R-module structure on each Hom set such that the composition is R-bilinear. More precisely, let us assume for instance that we are given a commutative ring R with identity. Then a small R-category – or equivalently an *R-algebroid* – will be defined as a category enriched in the monoidal category of R-modules, with respect to the monoidal structure of tensor product. This means simply that for all objects b, c of $\mathcal{A}$, the set $\mathcal{A}(b, c)$ is given the structure of an R-module, and composition $\mathcal{A}(b, c) \times \mathcal{A}(c, d) \longrightarrow \mathcal{A}(b, d)$ is R-bilinear, or is a morphism of R-modules $\mathcal{A}(b, c) \otimes_R \mathcal{A}(c, d) \longrightarrow \mathcal{A}(b, d)$.

If $\mathcal{G}$ is a groupoid (or, more generally, a category) then we can construct an *R-algebroid* $R\mathcal{G}$ as follows. The object set of $R\mathcal{G}$ is the same as that of $\mathcal{G}$ and $R\mathcal{G}(b, c)$ is the free R-module on the set $\mathcal{G}(b,c)$, with composition given by the usual bilinear rule, extending the composition of $\mathcal{G}$.

Alternatively, we can define $\bar{R}\mathcal{G}(b,c)$ to be the set of functions $\mathcal{G}(b,c) \longrightarrow R$ with finite support, and then we define the *convolution product* as follows:

$$(f * g)(z) = \sum \{(fx)(gy) \mid z = x \circ y\}. \tag{4.1}$$

As is well known, it is the second construction which is natural for the topological case, when we need to replace 'function' by 'continuous function with compact support' (or *locally compact support* for the QFT extended symmetry sectors), and in this case $R \cong \mathbb{C}$. The point we are making here is that to make the usual construction and end up with an algebra rather than an algebroid, is a procedure analogous to replacing a groupoid $\mathcal{G}$ by a semigroup $G' = \mathcal{G} \cup \{0\}$ in which the compositions not defined in $\mathcal{G}$ are defined to be 0 in G'. We argue that this construction removes the main advantage of groupoids, namely the spatial component given by the set of objects.

At present, however, the question of how one can use categorical duality in order to find the analogue of the diagonal of a Hopf algebra remains open. Such questions require further work and also future development of the theoretical framework proposed here for extended symmetries and the related fundamental aspects of quantum field theories. Nevertheless, for Fourier–Stieltjes groupoid representations, there has already been substantial progress made [180] with the specification of their dual Banach algebras (but not algebroids), in a manner similar to the case of locally compact groups and their associated Fourier algebras. Such progress will be further discussed in Section 7.

A related problem that we are addressing next is how the much studied theory of C^*-algebras and their representations would be naturally extended to carefully selected C^*-algebroids so that novel applications in quantum physics become possible. This is indeed a moot point because the classification problem for C^*-algebra representations is more complex and appears much more difficult to solve in the general case than it is in the case of von Neumann algebra representations. On the other hand, the extended symmetry links that we shall also discuss next, between locally compact groupoid *unitary* representations and their induced C^*-algebra representations, also warrant further careful consideration.

4.1 The weak C^*-Hopf algebroid and its symmetries

Progressing to the next level of generality, let A denote an algebra with local identities in a commutative subalgebra $R \subset A$. We adopt the definition of a *Hopf algebroid structure on A over R* following [168]. Relative to a ground field $\mathbb{F}$ (typically $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$), the definition commences by taking three $\mathbb{F}$-linear maps, the *comultiplication* $\Delta : A \longrightarrow A \otimes_R A$, the *counit* $\varepsilon : A \longrightarrow R$, and the *antipode* $S : A \longrightarrow A$, such that:

(*i*) Δ and ε are homomorphisms of left R-modules satisfying $(\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}$ and $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}$.

(*ii*) $\varepsilon|_R = \mathrm{id}$, $\Delta|_R$ is the canonical embedding $R \cong R \otimes_R R \subset A \otimes_R A$, and the two right R-actions on $A \otimes_R A$ coincide on ΔA.

(*iii*) $\Delta(ab) = \Delta(a)\Delta(b)$ for any $a, b \in A$.

(*iv*) $S|_R = \mathrm{id}$ and $S \circ S = \mathrm{id}$.

(*v*) $S(ab) = S(a)S(b)$ for any $a, b \in A$.

(*vi*) $\mu \circ (S \otimes \mathrm{id}) \circ \Delta = \varepsilon \circ S$, where $\mu : A \otimes_R A \longrightarrow A$ denotes the multiplication.

If R is a commutative subalgebra with local identities, then a *Hopf algebroid over* R is a *quadruple* $(A, \Delta, \varepsilon, S)$ where A is an algebra which has R for a subalgebra and has local identities in R, and where (Δ, ε, S) is a Hopf algebroid structure on A over R. Our interest lies in the fact that a Hopf-algebroid comprises a (universal) enveloping algebra for a quantum groupoid, thus hinting either at an adjointness situation or duality between the Hopf-algebroid and such a quantum groupoid.

Definition 4.1. Let $(A, \Delta, \varepsilon, S)$ be a Hopf algebroid as above. We say that $(A, \Delta, \varepsilon, S)$ is a *weak C^*-Hopf algebroid* when the following axioms are satisfied:

$(w1)$ A is a unital C^*-algebra. We set $\mathbb{F} = \mathbb{C}$.

$(w2)$ The comultiplication $\Delta : A \longrightarrow A \otimes A$ is a coassociative $*$-homomorphism. The counit is a positive linear map $\varepsilon : A \longrightarrow R$ satisfying the above compatibility condition. The antipode S is a complex-linear anti-homomorphism and anti-cohomorphism $S : A \longrightarrow A$ (that is, it reverses the order of the multiplication and comultiplication), and is inverted under the $*$-structure: $S^{-1}(a) = S(a^*)^*$.

$(w3)$

$$\Delta(\mathbf{1}) \equiv \mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)} = \text{projection}, \qquad \varepsilon(ap) = \varepsilon(a\mathbf{1}_{(1)}) \cdot \varepsilon(\mathbf{1}_{(2)}p),$$
$$S(a_{(1)})a_{(2)} \otimes a_{(3)} = (\mathbf{1} \otimes a) \cdot \Delta(\mathbf{1}).$$

Here $a_{(1)} \otimes a_{(2)}$ is shorthand notation for the expansion of $\Delta(a)$.

$(w4)$ The dual $\widehat{A}$ is defined by the linear maps $\hat{a} : A \longrightarrow \mathbb{C}$. The structure of $\widehat{A}$ is canonically dualized via the pairing and $\widehat{A}$ is endowed with a dual $*$-structure via $\langle \hat{a}^*, a\rangle_A = \overline{\langle \hat{a}, S(a)^*\rangle_A}$. Further, $(\widehat{A}, \widehat{\Delta}, \hat{\varepsilon}, \widehat{S})$ with $*$ and $\varepsilon = \widehat{\mathbf{1}}$, is a weak C^*-Hopf algebroid.

5 Comparing groupoid and algebroid quantum symmetries: weak Hopf C^*-algebroid vs. locally compact quantum groupoid symmetry

At this stage, we make a comparison between the Lie group 'classical' symmetries discussed in Section 2 and a schematic representation for the extended groupoid and algebroid symmetries considered in Sections 3 and 4, as follows:

Standard classical and quantum group/algebra symmetries:

Lie groups $\Longrightarrow$ Lie algebras $\Longrightarrow$ universal enveloping algebra $\Longrightarrow$ quantization $\rightarrow$ quantum group symmetry (or noncommutative (quantum) geometry).

Extended quantum, groupoid and algebroid, symmetries:

Quantum groupoid/algebroid $\leftarrow$ weak Hopf algebras $\longleftarrow$ representations $\leftarrow$ quantum groups.

Our intention here is to view the latter scheme in terms of *weak Hopf C^*-algebroid* – and/or other – extended symmetries, which we propose to do, for example, by incorporating the concepts of *rigged Hilbert spaces* and *sectional functions for a small category*. We note, however, that an alternative approach to quantum groupoids has already been reported in [155] (perhaps also related to non-commutative geometry); this was later expressed in terms of deformation-quantization: the Hopf algebroid deformation of the universal enveloping algebras of Lie algebroids [225] as the classical limit of a quantum 'groupoid'; this also parallels the introduction of quantum 'groups' as the deformation-quantization of Lie bialgebras. Furthermore, such a Hopf algebroid approach [147] leads to categories of Hopf algebroid modules [225] which are monoidal, whereas the links between Hopf algebroids and monoidal bicategories were investigated in [76].

As defined in Section 7 and the Appendix A, let (G_{lc}, τ) be a locally compact groupoid endowed with a (left) Haar system, and let $A = C^*(\mathsf{G}_{lc}, \tau)$ be the convolution C^*-algebra (we append A with $\mathbf{1}$ if necessary, so that A is unital). Then consider such a *groupoid representation*

$$\Lambda : \ (\mathsf{G}_{lc}, \tau) \longrightarrow \{\mathcal{H}_x, \sigma_x\}_{x \in X}, \tag{5.1}$$

that respects a compatible measure σ_x on $\mathcal{H}_x$ [54]. On taking a state ρ on A, we assume a parametrization

$$(\mathcal{H}_x, \sigma_x) := (\mathcal{H}_\rho, \sigma)_{x \in X}. \tag{5.2}$$

Furthermore, each $\mathcal{H}_x$ is considered as a *rigged Hilbert space* [35], that is, one also has the following nested inclusions:

$$\Phi_x \subset (\mathcal{H}_x, \sigma_x) \subset \Phi_x^{\times},$$

in the usual manner, where Φ_x is a dense subspace of $\mathcal{H}_x$ with the appropriate locally convex topology, and $\Phi_x^{\times}$ is the space of continuous antilinear functionals of Φ. For each $x \in X$, we require Φ_x to be invariant under Λ and $\operatorname{Im} \Lambda | \Phi_x$ is a continuous representation of G_{lc} on Φ_x. With these conditions, representations of (proper) quantum groupoids that are derived for weak C^*-Hopf algebras (or algebroids) modeled on rigged Hilbert spaces could be suitable generalizations in the framework of a Hamiltonian generated semigroup of time evolution of a quantum system via integration of Schrödinger's equation $\imath\hbar\frac{\partial\psi}{\partial t} = H\psi$ as studied in the case of Lie groups [218]. The adoption of the rigged Hilbert spaces is also based on how the latter are recognized as reconciling the Dirac and von Neumann approaches to quantum theories [35].

Next let G_{lc} be a locally compact Hausdorff groupoid and X a locally compact Hausdorff space. In order to achieve a small C^*-category we follow a suggestion of A. Seda (private communication) by using a general principle in the context of Banach bundles [199, 200]. Let

$$q = (q_1, q_2) : \ \mathsf{G}_{lc} \longrightarrow X \times X,$$

be a continuous, open and surjective map. For each $z = (x, y) \in X \times X$, consider the fibre $\mathsf{G}_{\mathsf{z}} = \mathsf{G}_{lc}(x, y) = q^{-1}(z)$, and set $\mathcal{A}_z = C_0(\mathsf{G}_{\mathsf{z}}) = C_0(\mathsf{G}_{lc})$ equipped with a uniform norm $\| \ \|_z$. Then we set $\mathcal{A} = \bigcup_z \mathcal{A}_z$. We form a Banach bundle $p : \mathcal{A} \longrightarrow X \times X$ as follows. Firstly, the projection is defined via the typical fibre $p^{-1}(z) = \mathcal{A}_z = \mathcal{A}_{(x,y)}$. Let $C_c(\mathsf{G}_{lc})$ denote the continuous complex valued functions on G_{lc} with compact support. We obtain a *sectional function* $\widetilde{\psi} : X \times X \longrightarrow \mathcal{A}$ defined via restriction as $\widetilde{\psi}(z) = \psi | \mathsf{G}_{\mathsf{z}} = \psi | \mathsf{G}_{lc}$. Commencing from the vector space $\gamma = \{\widetilde{\psi} : \psi \in C_c(\mathsf{G}_{lc})\}$, the set $\{\widetilde{\psi}(z) : \widetilde{\psi} \in \gamma\}$ is dense in $\mathcal{A}_z$. For each $\widetilde{\psi} \in \gamma$, the function $\|\widetilde{\psi}(z)\|_z$ is continuous on X, and each $\widetilde{\psi}$ is a continuous section of $p : \mathcal{A} \longrightarrow X \times X$. These facts follow from [200, Theorem 1]. Furthermore, under the convolution product $f * g$, *the space* $C_c(\mathsf{G}_{lc})$ *forms an associative algebra over* $\mathbb{C}$ (cf. [200, Theorem 3]).

Definition 5.1. The data proposed for a *weak C^*-Hopf symmetry* consists of:

(1) A weak C^*-Hopf algebroid $(A, \Delta, \varepsilon, S)$, where as above, $A = C^*(\mathcal{G}, \tau)$ is constructed via sectional functions over a small category.

(2) A family of GNS representations

$$(\pi_\rho)_x : \ A \longrightarrow (\mathcal{H}_\rho)_x := \mathcal{H}_x,$$

where for each, $x \in X$, $\mathcal{H}_x$ is a rigged Hilbert space.

5.1 Grassmann–Hopf algebra and the Grassmann–Hopf algebroid

Let V be a (complex) vector space ($\dim_{\mathbb{C}} V = n$) and let $\{e_0, e_1, \dots,\}$ with identity $e_0 \equiv 1$, be the generators of a Grassmann (exterior) algebra

$$\Lambda^* V = \Lambda^0 V \oplus \Lambda^1 V \oplus \Lambda^2 V \oplus \cdots$$

subject to the relation $e_i e_j + e_j e_i = 0$. Following [218, 92] we append this algebra with a Hopf structure to obtain a 'co-gebra' based on the interchange (or *tangled duality*):

$$(\textit{objects/points},\ \textit{morphisms}) \mapsto (\textit{morphisms},\ \textit{objects/points}).$$

This leads to a *tangled duality* between

(i) the binary product $A \otimes A \xrightarrow{m} A$, and

(ii) the coproduct $C \xrightarrow{\Delta} C \otimes C$.

where the Sweedler notation [205], with respect to an arbitrary basis is adopted:

$$\Delta(x) = \sum_r a_r \otimes b_r = \sum_{(x)} x_{(1)} \otimes x_{(2)} = x_{(1)} \otimes x_{(2)},$$
$$\Delta(x^i) = \sum_i \Delta_i^{jk} = \sum_{(r)} a^j_{(r)} \otimes b^k_{(r)} = x_{(1)} \otimes x_{(2)}.$$

Here the Δ_i^{jk} are called 'section coefficients'. We have then a generalization of associativity to coassociativity

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C \otimes C \arrow[d, "\mathrm{id}\otimes\Delta"] \\ C \otimes C \arrow[r, "\Delta\otimes\mathrm{id}"] & C \otimes C \otimes C \end{tikzcd}$$

inducing a tangled duality between an associative (unital algebra $\mathcal{A} = (A, m)$, and an associative (unital) 'co-gebra' $\mathcal{C} = (C, \Delta)$. The idea is to take this structure and combine the Grassmann algebra $(\Lambda^* V, \wedge)$ with the 'co-gebra' $(\Lambda^* V, \Delta_\wedge)$ (the 'tangled dual') along with the Hopf algebra compatibility rules: 1) the product and the unit are 'co-gebra' morphisms, and 2) the coproduct and counit are algebra morphisms.

Next we consider the following ingredients:

(1) the graded switch $\hat{\tau}(A \otimes B) = (-1)^{\partial A \partial B} B \otimes A$;

(2) the counit ε (an algebra morphism) satisfying $(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon)\Delta$;

(3) the antipode S.

The *Grassmann–Hopf algebra* $\widehat{H}$ thus consists of the *septet* $\widehat{H} = (\Lambda^* V, \wedge, \mathrm{id}, \varepsilon, \hat{\tau}, S)$.

Its generalization to a *Grassmann–Hopf algebroid* $H^\wedge$ is straightforward by defining the $H^\wedge$-*algebroid* over $\widehat{H}$ as a *quadruple* $(\widehat{H}, \Delta, \varepsilon, S)$, with $H^\wedge$ subject to the Hopf algebroid defining axioms, and with $\widehat{H} = (\Lambda^* V, \wedge, \mathrm{id}, \varepsilon, \hat{\tau}, S)$ subject to the standard Grassmann–Hopf algebra axioms stated above. We may also define $(\widehat{H}_w, \Delta, \varepsilon, S)$ as a *weak C^*-Grassmann–Hopf algebroid*, $H^\wedge_w$, when $\widehat{H}_w$ is selected as a unital C^*-algebra, and axioms ($w2$)–($w4$) of the weak C^*-Hopf algebroid are also satisfied by $H^\wedge_w$. We thus set $\mathbb{F} = \mathbb{C}$. Note however that the tangled-duals of Grassman–Hopf algebroids retain the intuitive interactions/dynamic diagram advantages of their physical, extended symmetry representations exhibited by the Grassman–Hopf algebras, gebras and co-gebras over those of either weak C^*-Hopf algebroids or weak Hopf C^*-algebras.

Alternatively, if $\mathcal{G}$ is a groupoid (or, more generally, a category) then we can construct a *Grassmann–Hopf algebroid* $H^\wedge$ as a special case of an *R-algebroid* $H^\wedge\mathcal{G}$. The object set of $H^\wedge\mathcal{G}$ is the same as that of $\mathcal{G}$ and $H^\wedge\mathcal{G}(b,c)$ is the free $H^\wedge$-module on the set $\mathcal{G}(b,c)$, with composition given by the usual bilinear rule, extending the composition of $\mathcal{G}$. Furthermore, can define also define as above $\bar{H}^\wedge\mathcal{G}(b,c)$ to be the set of functions $\mathcal{G}(b,c)\longrightarrow H^\wedge$ with finite support, and then we define the *convolution product* as in equation (4.1)

$$(f * g)(z) = \sum\{(fx)(gy) \mid z = x \circ y\}.$$

As already pointed out, this second, convolution construction is natural for the topological $\mathcal{G}$ case, when we need to replace 'function' by 'continuous function with compact support' – or with *locally compact support* in the case of QFT extended symmetry sectors – and in this case one also has that $H^\wedge \cong \mathbb{C}$, or the definition of a *convolution Grassmann–Hopf algebroid* $H_c^\wedge$. By also making $H_c^\wedge$ subject to axioms $(w1)$–$(w4)$ one obtains a *weak* C^**-convolution Grassmann–Hopf algebroid*. Its duals are the corresponding co-algebroid, $H_c^{\wedge *}$ and also the *tangled weak* C^**-convolution Grassmann–Hopf gebroid*, $\widetilde{H}_c^\wedge$ with distinct mathematical properties and physical significance.

6 Non-Abelian algebroid representations of quantum state space geometry in quantum supergravity fields

Supergravity, in essence, is an extended supersymmetric theory of both matter and gravitation [211]. A first approach to supersymmetry relies on a curved 'superspace' [217], and is analogous to supersymmetric gauge theories (see, for example, Sections 27.1–27.3 of [211]). Unfortunately, a complete non-linear supergravity theory might be forbiddingly complicated and furthermore, the constraints that need be made on the graviton superfield appear somewhat subjective, according to [211]. On the other hand, the second approach to supergravity is much more transparent than the first, albeit theoretically less elegant. The physical components of the gravitational superfield can be identified in this approach based on flat-space superfield methods (Chapters 26 and 27 of [211]). By implementing the weak-field approximation one obtains several of the most important consequences of supergravity theory, including masses for the hypothetical gravitino and gaugino 'particles' whose existence may be expected from supergravity theories. Furthermore, by adding on the higher order terms in the gravitational constant to the supersymmetric transformation, the general coordinate transformations form a closed algebra and the Lagrangian that describes the interactions of the physical fields is invariant under such transformations. Quantization of such a flat-space superfield would obviously involve its 'deformation' as discussed in Section 2 above, and as a result its corresponding *supersymmetry algebra* would become *non-commutative*.

6.1 The metric superfields

Because in supergravity both spinor and tensor fields are being considered, the gravitational fields are represented in terms of *tetrads*, $e^a_\mu(x)$, rather than in terms of the general relativistic metric $g_{\mu\nu}(x)$. The connections between these two distinct representations are as follows:

$$g_{\mu\nu}(x) = \eta_{ab} e^a_\mu(x) e^b_\gamma(x),$$

with the general coordinates being indexed by μ, ν, etc., whereas local coordinates that are being defined in a locally inertial coordinate system are labeled with superscripts a, b, etc.; η_{ab} is the diagonal matrix with elements +1, +1, +1 and −1. The tetrads are invariant to two distinct types of symmetry transformations – the local Lorentz transformations:

$$e^a_\mu(x) \longmapsto \Lambda^a_b(x) e^b_\mu(x),$$

(where Λ^a_b is an arbitrary real matrix), and the general coordinate transformations:

$$x^\mu \longmapsto (x')^\mu(x).$$

In a weak gravitational field the tetrad may be represented as:

$$e^a_\mu(x) = \delta^a_\mu(x) + 2\kappa\Phi^a_\mu(x),$$

where $\Phi^a_\mu(x)$ is small compared with $\delta^a_\mu(x)$ for all x values, and $\kappa = \sqrt{8\pi G}$, where G is Newton's gravitational constant. As it will be discussed next, the supersymmetry algebra (SA) implies that the graviton has a fermionic superpartner, the hypothetical *gravitino*, with helicities $\pm 3/2$. Such a self-charge-conjugate massless particle as the gravitiono with helicities $\pm 3/2$ can only have *low-energy* interactions if it is represented by a Majorana field $\psi_\mu(x)$ which is invariant under the gauge transformations:

$$\psi_\mu(x) \longmapsto \psi_\mu(x) + \delta_\mu\psi(x),$$

with $\psi(x)$ being an arbitrary Majorana field as defined in [105]. The tetrad field $\Phi_{\mu\nu}(x)$ and the graviton field $\psi_\mu(x)$ are then incorporated into a term $H_\mu(x,\theta)$ defined as the *metric superfield.* The relationships between $\Phi_{\mu_\nu}(x)$ and $\psi_\mu(x)$, on the one hand, and the components of the metric superfield $H_\mu(x,\theta)$, on the other hand, can be derived from the transformations of the whole metric superfield:

$$H_\mu(x,\theta) \longmapsto H_\mu(x,\theta) + \Delta_\mu(x,\theta),$$

by making the simplifying – and physically realistic – assumption of a weak gravitational field (further details can be found, for example, in Chapter 31 of Vol. 3 of [211]). The interactions of the entire superfield $H_\mu(x)$ with matter would be then described by considering how a weak gravitational field, $h_{\mu\nu}$ interacts with an energy-momentum tensor $T^{\mu\nu}$ represented as a linear combination of components of a real vector superfield Θ^μ. Such interaction terms would, therefore, have the form:

$$I_{\mathcal{M}} = 2\kappa \int dx^4 [H_\mu \Theta^\mu]_D,$$

($\mathcal{M}$ denotes 'matter') integrated over a four-dimensional (Minkowski) spacetime with the metric defined by the superfield $H_\mu(x,\theta)$. The term Θ^μ, as defined above, is physically a *supercurrent* and satisfies the conservation conditions:

$$\gamma^\mu \mathbf{D}\Theta_\mu = \mathbf{D},$$

where $\mathbf{D}$ is the four-component super-derivative and X denotes a real chiral scalar superfield. This leads immediately to the calculation of the interactions of matter with a weak gravitational field as:

$$I_{\mathcal{M}} = \kappa \int d^4x T^{\mu\nu}(x) h_{\mu\nu}(x),$$

It is interesting to note that the gravitational actions for the superfield that are invariant under the generalized gauge transformations $H_\mu \longmapsto H_\mu + \Delta_\mu$ lead to solutions of the Einstein field equations for a homogeneous, non-zero vacuum energy density ρ_V that correspond to either a de Sitter space for $\rho_V > 0$, or an anti-de Sitter space [223] for $\rho_V < 0$. Such spaces can be represented in terms of the hypersurface equation

$$x_5^2 \pm \eta_{\mu,\nu} x^\mu x^\nu = R^2,$$

in a quasi-Euclidean five-dimensional space with the metric specified as:

$$ds^2 = \eta_{\mu,\nu} x^\mu x^\nu \pm dx_5^2,$$

with "+' for de Sitter space and '−' for anti-de Sitter space, respectively.

The spacetime symmetry groups, or groupoids – as the case may be – are different from the 'classical' Poincaré symmetry group of translations and Lorentz transformations. Such spacetime symmetry groups, in the simplest case, are therefore the $\mathrm{O}(4,1)$ group for the de Sitter space and the $\mathrm{O}(3,2)$ group for the anti-de Sitter space. A detailed calculation indicates that the transition from ordinary flat space to a bubble of anti-de Sitter space is *not* favored energetically and, therefore, the ordinary (de Sitter) flat space is stable (cf. [71]), even though quantum fluctuations might occur to an anti-de Sitter bubble within the limits permitted by the Heisenberg uncertainty principle.

6.2 Supersymmetry algebras and Lie ($\mathbb{Z}_2$-graded) superalgebras

It is well known that *continuous symmetry transformations* can be represented in terms of a *Lie algebra* of linearly independent *symmetry generators* t_j that satisfy the commutation relations:

$$[t_j, t_k] = \iota \Sigma_l C_{jk} t_l,$$

Supersymmetry is similarly expressed in terms of the symmetry generators t_j of a *graded ('Lie') algebra* – which is in fact defined as a *superalgebra* – by satisfying relations of the general form:

$$t_j t_k - (-1)^{\eta_j \eta_k} t_k t_j = \iota \Sigma_l C_{jk}^l t_l.$$

The generators for which $\eta_j = 1$ are fermionic whereas those for which $\eta_j = 0$ are bosonic. The coefficients C_{jk}^l are structure constants satisfying the following conditions:

$$C_{jk}^l = -(-1)^{\eta_j \eta_k} C_{jk}^l.$$

If the generators t_j are quantum Hermitian operators, then the structure constants satisfy the reality conditions $C_{jk}^* = -C_{jk}$. Clearly, such a graded algebraic structure is a superalgebra and not a proper Lie algebra; thus graded Lie algebras are often called *Lie superalgebras* [127].

The standard computational approach in QM utilizes the S-matrix approach, and therefore, one needs to consider the general, *graded* 'Lie algebra' of *supersymmetry generators* that commute with the S-matrix. If one denotes the fermionic generators by Q, then $U^{-1}(\Lambda) Q U(\Lambda)$ will also be of the same type when $U(\Lambda)$ is the quantum operator corresponding to arbitrary, homogeneous Lorentz transformations $\Lambda^{\mu\nu}$. Such a group of generators provide therefore a representation of the homogeneous Lorentz group of transformations $\mathbb{L}$. The irreducible representation of the homogeneous Lorentz group of transformations provides therefore a classification of such individual generators.

6.2.1 Graded 'Lie' algebras and superalgebras

A set of quantum operators Q^{AB}_{jk} form an $\mathbf{A}$, $\mathbf{B}$ representation of the group $\mathbb{L}$ defined above which satisfy the commutation relations:

$$[\mathbf{A}, Q^{AB}_{jk}] = -[\Sigma'_j J^A_{jj'}, Q^{AB}_{j'k}],$$

and

$$[\mathbf{B}, Q^{AB}_{jk}] = -[\Sigma_{j'} J^A_{kk'}, Q^{AB}_{jk'}],$$

with the generators $\mathbf{A}$ and $\mathbf{B}$ defined by $\mathbf{A} \equiv (1/2)(\mathbf{J} \pm i\mathbf{K})$ and $\mathbf{B} \equiv (1/2)(\mathbf{J} - i\mathbf{K})$, with $\mathbf{J}$ and $\mathbf{K}$ being the Hermitian generators of rotations and 'boosts', respectively.

In the case of the two-component Weyl-spinors Q_{jr} the Haag–Lopuszanski–Sohnius (HLS) theorem applies, and thus the fermions form a *supersymmetry algebra* defined by the anti-commutation relations:

$$[Q_{jr}, Q^*_{ks}] = 2\delta_{rs}\sigma^{\mu}_{jk}P_{\mu}, \qquad [Q_{jr}, Q_{ks}] = e_{jk}Z_{rs},$$

where P_μ is the 4-momentum operator, $Z_{rs} = -Z_{sr}$ are the bosonic symmetry generators, and σ_μ and $\mathbf{e}$ are the usual 2×2 Pauli matrices. Furthermore, the fermionic generators commute with both energy and momentum operators:

$$[P_\mu, Q_{jr}] = [P_\mu, Q^*_{jr}] = 0.$$

The bosonic symmetry generators Z_{ks} and Z^*_{ks} represent the set of *central charges* of the supersymmetric algebra:

$$[Z_{rs}, Z^*_{tn}] = [Z^*_{rs}, Q_{jt}] = [Z^*_{rs}, Q^*_{jt}] = [Z^*_{rs}, Z^*_{tn}] = 0.$$

From another direction, the Poincaré symmetry mechanism of special relativity can be extended to new algebraic systems [206]. In [167] in view of such extensions, are considered invariant-free Lagrangians and bosonic multiplets constituting a symmetry that interplays with (Abelian) U(1)-gauge symmetry that may possibly be described in categorical terms, in particular, within the notion of a *cubical site* [104].

We shall proceed to introduce in the next section generalizations of the concepts of Lie algebras and graded Lie algebras to the corresponding Lie *algebroids* that may also be regarded as C^*-convolution representations of *quantum gravity groupoids* and superfield (or supergravity) supersymmetries. This is therefore a novel approach to the proper representation of the *non-commutative geometry of quantum spacetimes* – that are *curved* (or 'deformed') by the presence of *intense* gravitational fields – in the framework of *non-Abelian, graded Lie algebroids*. Their correspondingly *deformed quantum gravity groupoids* (QGG) should, therefore, adequately represent supersymmetries modified by the presence of such intense gravitational fields on the Planck scale. Quantum fluctuations that give rise to quantum 'foams' at the Planck scale may be then represented by *quantum homomorphisms* of such QGGs. If the corresponding graded Lie algebroids are also *integrable*, then one can reasonably expect to recover in the limit of $\hbar \to 0$ the Riemannian geometry of general relativity and the *globally hyperbolic spacetime* of Einstein's classical gravitation theory (GR), as a result of such an integration to the *quantum gravity fundamental groupoid* (QGFG). The following subsection will define the precise mathematical concepts underlying our novel quantum supergravity and extended supersymmetry notions.

6.3 Extending supersymmetry in relativistic quantum supergravity: Lie bialgebroids and a novel graded Lie algebroid concept

Whereas not all Lie algebroids are integrable to Lie groupoids, there is a subclass of the latter called sometimes 'Weinstein groupoids' that are in a one-to-one correspondence with their Lie algebroids.

6.3.1 Lie algebroids and Lie bialgebroids

One can think of a Lie algebroid as generalizing the idea of a tangent bundle where the tangent space at a point is effectively the equivalence class of curves meeting at that point (thus suggesting a groupoid approach), as well as serving as a site on which to study infinitesimal geometry (see, e.g., [150]). Specifically, let M be a manifold and let $\mathfrak{X}(M)$ denote the set of vector fields on M. Recall that a Lie algebroid over M consists of a vector bundle $E \longrightarrow M$, equipped with a Lie bracket $[\ ,\ ]$ on the space of sections $\gamma(E)$, and a bundle map $\Upsilon : E \longrightarrow TM$, usually called the *anchor*. Further, there is an induced map $\Upsilon : \gamma(E) \longrightarrow \mathfrak{X}(M)$, which is required to be a map of Lie algebras, such that given sections $\alpha, \beta \in \gamma(E)$ and a differentiable function f, the following Leibniz rule is satisfied:

$$[\alpha, f\beta] = f[\alpha, \beta] + (\Upsilon(\alpha))\beta.$$

A typical example of a Lie algebroid is when M is a Poisson manifold and $E = T^*M$ (the cotangent bundle of M).

Now suppose we have a Lie groupoid G:

$$r, s : \quad \mathsf{G} \underset{s}{\overset{r}{\rightrightarrows}} \mathsf{G}^{(0)} = M.$$

There is an associated Lie algebroid $\mathcal{A} = \mathcal{A}(\mathsf{G})$, which in the guise of a vector bundle, is in fact the restriction to M of the bundle of tangent vectors along the fibers of s (i.e. the s-vertical vector fields). Also, the space of sections $\gamma(\mathcal{A})$ can be identified with the space of s-vertical, right-invariant vector fields $\mathfrak{X}^s_{\rm inv}(\mathsf{G})$ which can be seen to be closed under $[\ ,\ ]$, and the latter induces a bracket operation on $\gamma(A)$ thus turning $\mathcal{A}$ into a Lie algebroid. Subsequently, a Lie algebroid $\mathcal{A}$ is integrable if there exists a Lie groupoid G inducing $\mathcal{A}$.

6.3.2 Graded Lie bialgebroids and symmetry breaking

A *Lie bialgebroid* is a Lie algebroid E such that $E^* \longrightarrow M$ also has a Lie algebroid structure. Lie bialgebroids are often thought of as the infinitesimal variations of Poisson groupoids. Specifically, with regards to a Poisson structure Λ, if $(G \rightrightarrows M, \Lambda)$ is a Poisson groupoid and if $E\mathsf{G}$ denotes the Lie algebroid of G, then $(E\mathsf{G}, E^*\mathsf{G})$ is a Lie bialgebroid. Conversely, a Lie bialgebroid structure on the Lie algebroid of a Lie groupoid can be integrated to a Poisson groupoid structure. Examples are Lie bialgebras which correspond bijectively with simply connected Poisson Lie groups.

6.4 Graded Lie algebroids and bialgebroids

A grading on a Lie algebroid follows by endowing a graded Jacobi bracket on the smooth functions $C^\infty(M)$ (see [103]). A Graded Jacobi bracket of degree k on a $\mathbb{Z}$-graded associative commutative algebra $\mathcal{A} = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}$ consists of a graded bilinear map

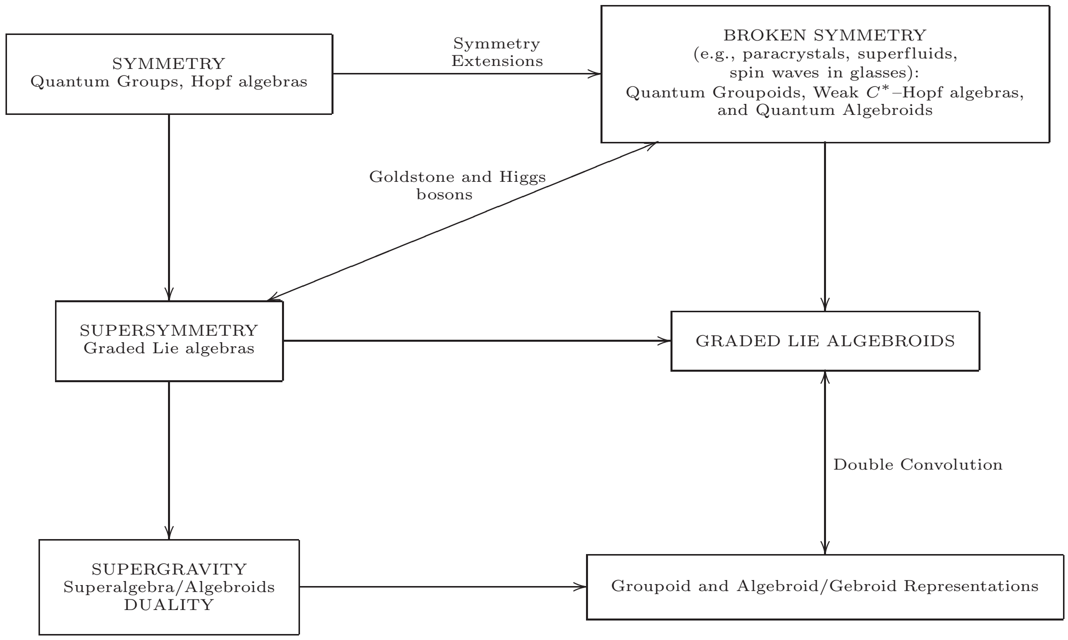
$$\{\cdot, \cdot\} : \ \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A},$$

of degree k (that is, $|\{a, b\}| = |a| + |b| + k$) satisfying:

1) $\{a, b\} = -(-1)^{\langle a+k, b+k \rangle} \{b, a\}$ (graded anticommutativity);
2) $\{a, bc\} = \{a, b\}c + (-1)^{\langle a+k, b \rangle}\ b\{a, c\} - \{a, \mathbf{1}\}bc$ (graded generalized Leibniz rule);
3) $\{\{a, b\}, c\} = \{a, \{b, c\}\} - (-1)^{\langle a+k, b+k \rangle} \{b, \{a, c\}\}$ (graded Jacobi identity),

where $\langle \cdot, \cdot \rangle$ denotes the usual pairing in $\mathbb{Z}^n$. Item 2) says that $\{\ ,\ \}$ corresponds to a first-order bidifferential operator on $\mathcal{A}$, and an odd Jacobi structure corresponds to a generalized *graded Lie bialgebroid*.

Having considered and also introduced several extended quantum symmetries, we are summarizing in the following diagram the key links between such quantum symmetry related concepts; included here also are the groupoid/algebroid representations of quantum symmetry and QG supersymmetry breaking. Such interconnections between quantum symmetries and supersymmetry are depicted in the following diagram in a manner suggestive of novel physical applications that will be reported in further detail in a subsequent paper [25]



The extended quantum symmetries formalized in the next section are defined as representations of the groupoid, algebroid and categorical structures considered in the above sections.

7 Extended quantum symmetries as algebroid and groupoid representations

7.1 Algebroid representations

A definition of a *vector bundle representation* (VBR), (ρ, V), of a Lie algebroid Λ over a manifold M was given in [142] as a vector bundle $V \longrightarrow M$ and a bundle map ρ from Λ to the bundle of order ≤ 1 differential operators $\mathsf{D} : \Gamma(V) \longrightarrow \Gamma(V)$ on sections of V compatible with the anchor map and commutator such that:

(i) for any $\epsilon_1, \epsilon_2 \in \gamma$ the symbol $\mathrm{Symb}(\rho(\epsilon))$ is a scalar equal to the anchor of ϵ:

$$\mathrm{Symb}(\rho(\epsilon)) = \delta_\epsilon \mathrm{Id}_V,$$

(ii) for any $\epsilon_1, \epsilon_2 \in \gamma(\Lambda)$ and $f \in C^\infty(M)$ we have $[\rho(\epsilon_1), \rho(\epsilon_2)] = \rho([\epsilon_1, \epsilon_2])$.

In (ii) $C^\infty(M)$ is the algebra of $\Re$-valued functions on M.

7.2 Hopf and weak Hopf C^*-algebroid representations

We shall begin in this section with a consideration of the Hopf algebra representations that are known to have additional structure to that of a Hopf algebra. If H is a Hopf algebra and A is an algebra with the product operation $\mu : A \otimes A \longrightarrow A$, then a linear map $\rho : H \otimes A \longrightarrow A$ is an *algebra representation* of H if in addition to being a (vector space) representation of H, μ is also an H-intertwiner. If A happens to be unital, it will also be required that there is an H-intertwiner from ϵ_H to A such that the unity of ϵ_H maps to the unit of A.

On the other hand, the Hopf-algebroid H_A over $C_c^\infty(M)$, with M a smooth manifold, is sometimes considered as a quantum groupoid because one can construct its spectral *étale* Lie groupoid $\mathsf{G}_s \to p(H_A)$ representation beginning with the groupoid algebra $C_c(\mathsf{G})$ of smooth functions with compact support on G_{lc}; this is an *étale* Lie groupoid for M's that are not necessarily Hausdorff (cf. [168, 169]). Recently, Konno [132] reported a systematic construction of both finite and infinite-dimensional *dynamical representations of a H-Hopf algebroid*(introduced in [87]), and their parallel structures to the *quantum affine algebra* $U_q(\hat{sl}_2)$. Such generally non-Abelian structures are constructed in terms of the Drinfel'd generators of the quantum affine algebra $U_q(\hat{sl}_2)$ and a Heisenberg algebra. The structure of the tensor product of two evaluation representations was also provided by Konno [132], and an elliptic analogue of the Clebsch–Gordan coefficients was expressed by using certain balanced elliptic hypergeometric series ${}_{12}V_{11}$.

7.3 Groupoid representations

Whereas group representations of quantum unitary operators are extensively employed in standard quantum mechanics, the applications of groupoid representations are still under development. For example, a description of stochastic quantum mechanics in curved spacetime [82] involving a Hilbert bundle is possible in terms of *groupoid representations* which can indeed be defined on such a Hilbert bundle $(X * \mathcal{H}, \pi)$, but cannot be expressed as the simpler group representations on a Hilbert space $\mathcal{H}$. On the other hand, as in the case of group representations, unitary groupoid representations induce associated C^*-algebra representations. In the next subsection we recall some of the basic results concerning groupoid representations and their associated groupoid $*$-algebra representations. For further details and recent results in the mathematical theory of groupoid representations one has also available the succint monograph [54] and references cited therein (`www.utgjiu.ro/math/mbuneci/preprint.html`, [55, 56, 57, 58, 59, 60, 61, 62, 63]).

7.4 Equivalent groupoid and algebroid representations: the correspondence between groupoid unitary representations and the associated C^*-algebra representations

We shall briefly consider here a main result due to Hahn [112] that relates groupoid and associated groupoid algebra representations [113]:

Theorem 7.1 (Theorem 3.4 on p. 50 in [112]). *Any representation of a groupoid G_{lc} with Haar measure (ν, μ) in a separable Hilbert space $\mathcal{H}$ induces a $*$-algebra representation $f \mapsto X_f$ of the associated groupoid algebra $\Pi(\mathsf{G}_{lc}, \nu)$ in $L^2(U_{\mathsf{G}_{lc}}, \mu, \mathcal{H})$ with the following properties:*

(1) *For any $l, m \in \mathcal{H}$, one has that $|\langle X_f(u \mapsto l), (u \mapsto m)\rangle| \le \|f_l\| \, \|l\| \, \|m\|$, and*

(2) *$M_r(\alpha)X_f = X_{f\alpha \circ r}$, where $M_r : L^\infty(U_{\mathcal{G}}, \mu, \mathcal{H}) \longrightarrow \mathcal{L}(L^2(U_{\mathcal{G}}, \mu, \mathcal{H}))$, with $M_r(\alpha)j = \alpha \cdot j$.*

Conversely, any $*$-algebra representation with the above two properties induces a groupoid representation, X, as follows:

$$\langle X_f, j, k\rangle = \int f(x)[X(x)j(d(x)), k(r(x))d\nu(x)].$$

Furthermore, according to Seda [201, p. 116] the continuity of a Haar system is equivalent to the continuity of the convolution product $f * g$ for any pair f, g of continuous functions with compact support. One may thus conjecture that similar results could be obtained for functions with *locally compact* support in dealing with convolution products of either locally compact groupoids or quantum groupoids. Seda' s result also implies that the convolution algebra $C_c(\mathsf{G})$ of a groupoid G is closed with respect to convolution if and only if the fixed Haar system associated with the measured groupoid G is *continuous* [54].

In the case of groupoid algebras of transitive groupoids, [54] and in related [54, 56, 57, 58, 59, 60, 61, 62, 63] showed that representations of a measured groupoid $(\mathsf{G}, [\int \nu^u d\tilde{\lambda}(u)] = [\lambda])$ on a separable Hilbert space $\mathcal{H}$ induce *non-degenerate* $*$-representations $f \mapsto X_f$ of the associated groupoid algebra $\Pi(\mathsf{G}, \nu, \tilde{\lambda})$ with properties formally similar to (1) and (2) above [64]. Moreover, as in the case of groups, *there is a correspondence between the unitary representations of a groupoid and its associated C^*-convolution algebra representations* of [54, p. 182], the latter involving however fiber bundles of Hilbert spaces instead of single Hilbert spaces. Therefore, groupoid representations appear as the natural construct for algebraic quantum field theories in which nets of local observable operators in Hilbert space fiber bundles were introduced by Rovelli in [195].

7.5 Generalized Fourier–Stieltjes transforms of groupoids: Fourier–Stieltjes algebras of locally compact groupoids and quantum groupoids; left regular groupoid representations and the Fourier algebra of a measured groupoid

We shall recall first that the *Fourier–Stieltjes algebra* $\mathbf{B}(G_{lc})$ of a locally compact group G_{lc} is defined by the space of coefficients (ξ, η) of *Hilbert space representations* of G_{lc}. In the special case of left regular representations and a measured groupoid, G, the Fourier–Stieltjes algebra $\mathbf{B}(\mathsf{G}, \nu^u, \mu)$ – defined as an involutive subalgebra of $L^\infty(\mathsf{G})$ – becomes the *Fourier algebra* $\mathbf{A}(\mathsf{G})$ defined by Renault [191]; such algebras are thus defined as a set of *representation coefficients* $(\mu, U_{\mathsf{G}} * H, L)$, which are effectively realized as a function $(\xi, \eta) : \mathsf{G} \longrightarrow \mathbb{C}$, defined by

$$(\xi, \eta)(x) := \Big\langle \xi(r(x)), \hat{L}(x)\eta(d(x)) \Big\rangle ,$$

(see [54, pp. 196–197]).

The Fourier–Stieltjes (FS) and Fourier (FR) algebras, respectively, $\mathbf{B}(G_{lc})$, $\mathbf{A}(G_{lc})$, were first studied by P. Eymard for a general locally compact group G_{lc} in [90], and have since played ever increasing roles in harmonic analysis and in the study of the operator algebras generated by G_{lc}.

Recently, there is also a considerable interest in developing extensions of these two types of algebras for *locally compact groupoids* because, as in the group case, such algebras play a useful role both in the study of the theory of quantum operator algebras and that of groupoid operator algebras. Furthermore, as discussed in the Introduction, there are new links between (physical) scattering theories for paracrystals, or other systems with local/partial ordering such as glasses/ 'non-crystalline' solids, and the generalizations of Fourier transforms that realize the well-established duality between the physical space, S, and the 'diffraction', or *reciprocal*, space, $\mathcal{R} = \tilde{S}$. On the other hand, the duality between the real time of quantum dynamics/resonant processes, T, and the 'spectral space', $\mathcal{F} = \tilde{T}$, of resonance frequencies (and the corresponding quanta of energies, $h\nu$) for electrons, nucleons and other particles in bound configurations is just as well-established by comparison with that occurring between the 'real' and reciprocal spaces in the case of electrons, neutron or emf/X-ray diffraction and scattering by periodic and aperiodic solids. The deep quantum connection between these two fundamental dualities, or symmetries, that seem to be ubiquitous in nature, can possibly lead to an unified quantum theory of *dispersion* in solids, liquids, superfluids and plasmas.

Let X be a locally compact Hausdorff space and $C(X)$ the algebra of bounded, continuous, complex-valued functions on X. Then denote the space of continuous functions in $C(X)$ that vanish at infinity by $C_0(X)$, while $C_c(X)$ is the space of functions in $C(X)$ with compact support. The space of complex, bounded, regular Borel measures on X is then denoted by $M(X)$. The Banach spaces $\mathbf{B}(\mathsf{G}_{lc})$, $\mathbf{A}(\mathsf{G}_{lc})$ (where G_{lc} denotes a locally compact groupoid) as considered here occur naturally in the group case in both non-commutative harmonic analysis and duality theory. Thus, in the case when G is a locally compact group, $\mathbf{B}(G_{lc})$ and $\mathbf{A}(G_{lc})$ are just the well known Fourier–Stieltjes and Fourier algebras discussed above. The need to have available generalizations of these Banach algebras for the case of a *locally compact groupoid* stems from the fact that many of the operator algebras of current interest – as for example in non-commutative geometry and quantum operator algebras – originate from *groupoid*, rather than group, representations, so that one needs to develop the notions of $\mathbf{B}(\mathsf{G}_{lc})$, $\mathbf{A}(\mathsf{G}_{lc})$ in the groupoid case for groupoid operator algebras (or indeed for *algebroids*) that are much more general than $\mathbf{B}(G_{lc})$, $\mathbf{A}(G_{lc})$. One notes also that in the operator space context, $\mathbf{A}(G_{lc})$ is regarded as the *convolution algebra of the dual quantum group* [180].

However, for groupoids and more general structures (e.g., categories and toposes of ŁM–algebras), such an extension of Banach space duality still needs further investigation. Thus, one can also conceive the notion of a measure theory based on Łukasiewicz–Moisil (ŁM) N-valued logic algebras (see [99] and references cited therein), and a corresponding ŁM-topos generalization of harmonic (or anharmonic) analysis by defining extended Haar–LM measures, ŁM-topos representations and $\mathcal{F}_{\text{S–L–M}}$ transforms. This raises the natural question of duality for the catgeory of ŁM-algebras that was introduced by Georgescu and Vraciu [101]. An appropriate framework for such logic ŁM-algebras is provided by algebraic categories [100].

Let us consider first the algebra involved in the simple example of the Fourier transform and then note that its extension to the Fourier–Stieltjes transform involves a convolution, just as it did in the case of the paracrystal scattering theory.

Thus, consider as in [180] the Fourier algebra in the locally compact group case and further assume that G_{lc} is a locally compact Abelian group with character space $\hat{G_{lc}}$; then an element of $\hat{G_{lc}}$ is a continuous homomorphism $t : G_{lc} \to T$, with $\hat{G_{lc}}$ being a locally compact abelian group with pointwise product and the topology of uniform convergence on compacta. Then, the Fourier transform $f \to \hat{f}$ takes $f \in L^1(G_{lc})$ into $C_0(\hat{G_{lc}})$, with $\hat{f}(t) = \int f(x)\overline{t(x)}dx$, where dx is defined as a left Haar measure on G_{lc}. On the other hand, its inverse Fourier transform $\mu \longrightarrow \check{\mu}$ reverses the process by taking $M(\hat{G_{lc}})$ back into $C(G_{lc})$, with $\check{\mu}$ being defined by the (inverse Fourier transform) integral: $\check{\mu}(x) = \int \check{x}(t)d\mu(t)$. For example, when $G_{lc} = \Re$, one also has that $\hat{G_{lc}} = \Re$ so that $t \in \hat{G_{lc}}$ is associated with the character $x \mapsto e^{ixt}$. Therefore, one obtains in this case the usual Fourier transform $\hat{f}(t) = \int f(x)e^{-ixt}dx$ and its inverse (or dual) $\check{\mu}(x) = \int e^{itx}d\mu(t)$. By considering $M(\hat{G_{lc}})$ as a *convolution* Banach algebra (which contains $L^1(\hat{G_{lc}})$ as a closed ideal) one can define the Fourier–Stieltjes algebra $\mathbf{B}(G_{lc})$ by $M(\hat{G_{lc}})\check{}$, whereas the simpler Fourier algebra, $\mathbf{A}(G_{lc})$, is defined as $L^1(\hat{G_{lc}})\check{}$.

Remark 7.1. In the case of a *discrete* Fourier transform, the integral is replaced by summation of the terms of a Fourier series. The discrete Fourier (transform) summation has by far the widest and most numerous applications in digital computations in both science and engineering. Thus, one represents a continuous function by an infinite Fourier series of 'harmonic' components that can be either real or complex, depending on the *symmetry* properties of the represented function; the latter is then approximated to any level of desired precision by truncating the Fourier series to a finite number of terms and then neglecting the remainder. To avoid spurious 'truncation errors' one then applies a 'smoothing' function, such as a negative exponential, that is digitized at closely spaced sample points so that the Nyquist's theorem criterion is met in order to both obtain the highest possible resolution and to drastically reduce the noise in the final, computed fast Fourier transform (FFT). Thus, for example, in the simpler case of a centrosymmetric electron density of a unit cell in a crystalline lattice, the diffracted X-ray, electron or neutron intensity can be shown to be proportional to the modulus squared of the *real* Fourier transform of the (centrosymmetric) electron density of the lattice. In a (digital) FFT computation, the approximate electron density reconstruction of the lattice structure is obtained through truncation to the highest order(s) of diffraction observed, and thus the spatial resolution obtained is limited to a corresponding value in real 3-D space.

Remark 7.2 (Laplace vs 1-D and 2-D Fourier transforms). On the other hand, although Laplace transforms are being used in some engineering applications to calculate transfer functions, they are much less utilized in the experimental sciences than the Fourier transforms even though the former may have advantages over FFT for obtaining both improved resolution and increased signal-to-noise. It seems that the major reason for this strong preference for FFT is the much shorter computation time on digital computers, and perhaps also FFT's relative simplicity when compared with Laplace transforms; the latter may also be one of the main reasons for the presence of very few digital applications in experimental science of the Fourier–Stieltjes transforms which generalize Fourier transforms. Somewhat surprising, however, is the use of FFT also in *algebraic quantum field computations on a lattice* where both FS or Laplace transforms could provide superior results, albeit at the expense of increased digital computation time and substantially more complex programming. On the other hand, one also notes the increasing use of 'two-dimensional' FFT in comparison with one-dimensional FFT in both experimental science and medicine (for example, in 2D-NMR, 2D-chemical (IR/NIR) imaging and MRI cross-section computations, respectively), even though the former require both significantly longer computation times and more complex programming.

7.5.1 Fourier–Stieltjes transforms as generalizations of the classical Fourier transforms in harmonic analysis to extended anharmonic analysis in quantum theories

Not surprisingly, there are several versions of the near-'harmonic' F–S algebras for the locally compact groupoid case that appear at least in three related theories:

1) the *measured groupoid* theory of J. Renault [189, 190, 191],
2) a Borel theory of A. Ramsay and M. Walter [184], and
3) a continuity-based theory of A. Paterson [180, 181].

Ramsay and Walter [184] made a first step towards extending the theory of Fourier–Stieltjes algebras from groups to groupoids, thus paving the way to the extension of F–S applications to generalized anharmonic analysis in Quantum theories *via* quantum algebra and quantum groupoid representations. Thus, if G_{lc} is a locally compact (second countable) groupoid, Ramsay and Walter showed that $\mathbf{B}(\mathsf{G}_{lc})$, which was defined as the linear span of the Borel positive definite functions on G_{lc}, is a *Banach algebra* when represented as an algebra of completely bounded maps on a C^*-algebra associated with the G_{lc} that involves equivalent elements of $\mathbf{B}(\mathsf{G}_{lc})$; positive definite functions will be defined in the next paragraph using the notation of [180]. Corresponding to the universal C^*-algebra, $C^*(G)$, in the group case is the universal $C^*_\mu(\mathsf{G})$ in the measured groupoid G case. The latter is the completion of $C_c(\mathsf{G}_{lc})$ under the largest C^*-norm coming from some measurable G_{lc}-Hilbert bundle $(\mu, \Re, L)$. In the group case, it is known that $\mathbf{B}(G)$ is isometric to the Banach space *dual* of $C^*(G)$. On the other hand, for groupoids, one can consider a representation of $\mathbf{B}(\mathsf{G}_{lc})$ as a Banach space of completely bounded maps from a

C^*-algebra associated with G_{lc} to a C^*-algebra associated with the equivalence relation induced by G_{lc}. Obviously, any Hilbert space $\mathcal{H}$ can also be regarded as an operator space by identifying it with a subspace of $\mathbf{B}(\mathbb{C},\mathcal{H})$: each $\xi \in \mathcal{H}$ is identified with the map $a \longrightarrow a\xi$ for $a \in \mathbb{C}$; thus, $\mathcal{H}^*$ is an operator space as a subspace of $\mathbf{B}(\mathcal{H},\mathbb{C})$. Renault showed for measured groupoids that the operator space $C^*_\mu(\mathsf{G}_{lc})$ is a completely contractive left $L^\infty(\mathsf{G}^0_{lc})$ module. If E is a right, and F is a left, A-operator module, with A being a C^*-algebra, then a Haagerup tensor norm is determined on the algebraic tensor product $E \otimes_A F$ by setting $\| u\| = \sum\limits_{i=1}^{n} \|e_i\| \, \|f_i\|$ over all representations $u = \sum\limits_{i=1}^{n} e_i \otimes_A f_i$.

According to [180], the completion $E \otimes_A F$ of E is called the *module Haagerup tensor product of E and F over A*. With this definition, the module Haagerup tensor product is:

$$X(\mathsf{G}_{lc}) = L^2(\mathsf{G}^0_{lc})^* \otimes C^*_\mu(\mathsf{G}_{lc}) \otimes L^2(\mathsf{G}^0_{lc}),$$

taken over $L^\infty(\mathsf{G}^0_{lc})$. Then, with this tensor product construction, Renault was able to prove that

$$X(\mathsf{G}_{lc})^* = \mathbf{B}_\mu(\mathsf{G}_{lc}).$$

Thus, each $\phi = (\xi, \eta)$ can be expressed by the linear functional $a^* \otimes f \otimes b \longrightarrow \int \overline{a \circ r}(\phi f) b \circ s \, d\nu$ with $f \in C_c(\mathsf{G}_{lc})$.

We shall also briefly discuss here Paterson's generalization to the groupoid case in the form of a *Fourier–Stieltjes algebra of a groupoid*, $\mathbf{B}_\mu(\mathsf{G}_{lc})$, which was defined (e.g., in [180]) as the space of coefficients $\phi = (\xi, \eta)$, where ξ, η are L^∞-sections for some measurable G-Hilbert bundle $(\mu, \Re, L))$. Thus, for $x \in \mathsf{G}_{lc}$,

$$\phi(x) = (L(x)\xi(s(x)), \eta(r(x))).$$

Therefore, ϕ belongs to $L^\infty(\mathsf{G}_{lc}) = L^\infty(\mathsf{G}_{lc}, \nu)$.

Both in the groupoid and group case, the set $P_\mu(\mathsf{G}_{lc})$ of *positive definite functions* in $L^\infty(\mathsf{G}_{lc})$ plays the central role. Thus, a function $\phi \in L^\infty(\mathsf{G}_{lc})$ is called *positive definite* if and only if for all $u \in (\mathsf{G}^0_{lc})$,

$$\iint \phi(y^{-1}x) f(y) \overline{f(x)} \, d\lambda^u(x) d\lambda^u(y) \geq 0.$$

Now, one can define the notion of a *Fourier–Stieltjes transform* as follows:

Definition 7.1 (The Fourier–Stieltjes transform). Given a *positive definite, measurable function* $f(x)$ on the interval $(-\infty, \infty)$ there exists a monotone increasing, real-valued bounded function $\alpha(t)$ such that:

$$f(x) = \int_{-\infty}^{\infty} e^{itx} d\alpha(t),$$

for all x except a small set. When $f(x)$ is defined as above and if $\alpha(t)$ is nondecreasing and bounded then the measurable function defined by the above integral is called *the Fourier–Stieltjes transform of* $\alpha(t)$, and it is *continuous* in addition to being positive definite in the sense defined above.

In [180] is also defined the *continuous* Fourier–Stieltjes algebra $\mathbf{B}(\mathsf{G})$ as follows. Let us consider a *continuous* G-Hilbert bundle $\mathcal{H}_\Re$, and the Banach space Δ_b of continuous, bounded sections of $\mathcal{H}_\Re$. For $\xi, \eta \in \Delta_b$, the coefficient $(\xi, \eta) \in C(\mathsf{G})$ is defined by:

$$(\xi, \eta)(u) = (L_x \xi(s(x)), \eta(r(x))),$$

where $x \longrightarrow L_x$ is the G-action on $\mathcal{H}_{\mathfrak{R}}$. Then, the *continuous Fourier–Stieltjes algebra* $\mathbf{B}(\mathsf{G})$ is defined to be the set of all such coefficients, coming from all possible continuous G-Hilbert bundles. Thus, $\mathbf{B}(\mathsf{G})$ is an algebra over $\mathbb{C}$ and the norm of $\phi \in \mathbf{B}(\mathsf{G})$ is defined to be inf $\|\xi\| \, \|\eta\|$, with the *infimum* inf being taken over all G representations $\phi = (\xi, \eta)$. Then $\mathbf{B}(\mathsf{G}) \subset C(\mathsf{G})$, and $\|\cdot\|_\infty = \|\cdot\|$.

Paterson in [180] showed that $\mathbf{B}(\mathsf{G})$ thus defined – just as in the group case – is a *commutative Banach algebra*. He also defined for a general group G the *left regular representation* π_2 of G on $L^2(G)$ by: $\pi_2(x)f(t) = f(x^{-1}t)$. One also has the *universal representation* $\pi_{2,\mathrm{univ}}$ of G which is defined on a Hilbert space $\mathcal{H}_{\mathrm{univ}}$. Moreover, every *unitary* representation of G determines by integration a *non-degenerate* π_2-representation of $C_c(G)$. The norm closure of $\pi_2(C_c(G))$ then defines the *reduced* C^*-algebra $C^*_{\mathrm{red}}(G)$ of G, whereas the norm closure of $\pi_{2,\mathrm{univ}}(C_c(G))$ was defined as the *universal C^*-algebra* of G (*loc. cit.*). The algebra $C^*_{\mathrm{red}}(G) \subset B(L^2(G))$ generates a von Neumann algebra denoted by $V_N(G)$. Thus, $C^*_{\mathrm{red}}(G)$ representations generate $V_N(G)$ representations that have a much simpler classification through their V_N factors than the representations of general C^*-algebras; consequently, the classification of $C^*_{\mathrm{red}}(G)$ representations is closer linked to that of V_N factors than in the general case of C^*-algebras. One would expect that a similar simplification may not be available when group G symmetries (and, respectively, their associated $C^*_{\mathrm{red}}(G)$ representations) are extended to the more general groupoid symmetries (and their associated groupoid C^*-convolution algebra representations relative to Hilbert bundles).

Recently, however, Bos in [37, 40] reported that one can extend – with appropriate modifications and conditions added – the Schur's lemma and Peter–Weyl theorems from group representations to corresponding theorems for (continuous) internally irreducible representations of *continuous* groupoids in the case of Schur's lemma, and restriction maps in the case of two Peter–Weyl theorems, (one of the latter theorems being applicable only to *compact*, proper groupoids and their isomorphism classes of irreducible unitary (or internally irreducible) representations (IrRep(G) and IrRep^i(G), respectively)). It is well established that using Schur's lemma for groups one can prove that if a matrix commutes with every element in an *irreducible* representation of a group that matrix must be a multiple of the *identity*. A continuous groupoid representation $(\pi, \mathcal{H}, \Delta)$ of a continuous groupoid $\mathsf{G}\rightrightarrows M$ was called *internally irreducible* by Ros if the restriction of π to each of the isotropy groups is an irreducible representation. Thus, in the case of *continuous* groupoids $\mathsf{G}\rightrightarrows M$ (endowed with a Haar system), *irreducible representations are also internally irreducible* but the converse does not hold in general (see also the preprints of R.D. Bos [40]). Bos also introduced in [37, 40] the *universal enveloping C^*-category* of a Banach $*$-category, and then used this to define the C^*-category, $\mathbf{C}^*(\mathsf{G}, \mathsf{G})$, of a groupoid. Then, he found that *there exists a bijection between the continuous representations of* $\mathbf{C}^*(\mathsf{G}, \mathsf{G})$ *and the continuous representations of* $\mathsf{G}\rightrightarrows M$.

7.6 Categorical representations of categorical groups in monoidal bicategories

Barrett pointed out in [28] that monoidal categories play an important role in the construction of *invariants* of three-manifolds (as well as knots, links and graphs in three-manifolds). The approach is based on constructions inspired by strict categorical groups which lead to monoidal categories [29]. A categorical group was thus considered in this recent report as a group-object in the category of groupoids, and it can also be shown that categorical groups are equivalent to crossed modules of groups. (A *crossed module* is a homomorphism of groups $\partial : E \to G$, together with an action $\triangleright$ of G on E by automorphisms, such that $\partial(X \triangleright e) = X(\partial)X^{-1}$, and $\partial(e) \triangleright e' = ee'e^{-1}$, where E denotes the *principal* group and G is the *base* group.)

Specifically, a *categorical group* was defined in [29] as a groupoid $\mathcal{G}$, with a set of objects $G^0 \subset \mathcal{G}$, together with functors which implement the group product, $\circ : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$, and the inverse $\iota^{-1} : \mathcal{G} \to \mathcal{G}$, together with an identity object $\mathbf{1} \in \mathcal{G}^0$; these satisfy the usual group laws:

$$a \circ (b \circ c) = (a \circ b) \circ c, \quad a \circ \mathbf{1} = \mathbf{1} \circ a = a, \quad a \circ a^{-1} = a^{-1} \circ a = \mathbf{1},$$

for all $a, b, c \in \mathcal{G}$. Furthermore, a *functorial homomorphism* between two such categorical groups was defined as a strict monoidal functor.

In particular, $\mathcal{G}$ is a strict monoidal category (or *tensor category*, that is, a category $\mathcal{C}$ equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ which is associative and an object which is both left and right identity for the bifunctor $\otimes$, up to a natural isomorphism; see also Section 8 for further details on tensor products and tensor categories).

One of many physical representations of such monoidal categories is the *topological order* in condensed matter theory, quantum field theory [222] and string models [111, 143, 144, 215, 216]. One of the first theoretical reports of topological order in metallic glasses with ferromagnetic properties was made by P.W. Anderson in 1977 [7]. In the recent quantum theory of condensed matter, *topological order* is a pattern of *long-range* entanglement of quantum states (such as the "string-net condensed" states [145]) defined by a new set of quantum numbers such as quasi-particle fractional statistics, edge states, ground state degeneracy, topological entropy [144], and so on; therefore, topological order also introduces new extended quantum symmetries [185] that are beyond the Landau symmetry-breaking model. Such string-net condensed models can have excitations that behave like gluons, quarks and the other particles already present in the standard model (SUSY).

On the other hand, *braided* monoidal categories are being applied to both quantum field theory and string models. A *braided monoidal category* is defined as a monoidal category equipped with a braiding, that is, a natural isomorphism $\gamma_{A,B} : A \otimes B \longrightarrow B \otimes A$ such that the braiding morphisms $\gamma : A \otimes (B \otimes C) \to B \otimes (C \otimes A)$ commute with the associativity isomorphism in two hexagonal diagrams containing all associative permutations of objects A, B and C. An alternative definition of a braided monoidal category is as a tricategory, or 3-category, with a single *one*-cell and also a single 2-cell [126, 8]; thus, it may be thought of as a 'three-dimensional' categorical structure.

As an example of bicategory associated with categorical group representations, consider the closure of $\mathcal{G}$, denoted by $\overline{\mathcal{G}}$, to be the 2-category with one object denoted by $\bullet$, such that $\overline{\mathcal{G}}(\bullet, \bullet) = \mathcal{G}$. The horizontal composition is then defined by the monoidal structure in $\mathcal{G}$.

In the current literature, the notion of categorical group is used in the sense of a strict monoidal category in which multiplication by an object on either side is an equivalence of categories, and the definition of a categorical group was seemingly first published by Brown and Spencer in 1976 [53]).

From a quantum physics perspective, the monoidal categories determined by quantum groups seen as Hopf algebras, generalize the notion that the representations of a group form a monoidal category. One particular example of a monoidal category was reported to provide a state-sum model for quantum gravity in three-dimensional spacetime [208, 28]. The motivation for such applications was the search for more realistic categorical models of four-dimensional, relativistic spacetimes. Thus, it was proposed to construct the monoidal 2-category of representations for the case of the categorical Lie group determined by the group of Lorentz transformations and its action on the translation group of Minkowski space.

In higher dimensions than three, the complexity of such algebraic representations increases dramatically. In the case of four-manifolds there are several examples of applications of categorical algebra [152] to four-dimensional topology; these include: Hopf categories [74], categorical groups [227] or monoidal 2-categories [67, 9], and the representation theory of quasi-triangular Hopf algebras. Invariants of four-manifolds were derived by Crane and Kauffman, as well as Roberts [193, 194], who give information on the homotopy type of the four-manifold [75, 194, 187, 149]. Recently, Martin and Porter [156] presented results concerning the Yetter invariants [227], and an extension of the Dijkgraaf–Witten invariant to categorical groups. Other types of categorical invariants and extended symmetries are also expected to emerge in higher dimensions as illustrated here in Section 9. Barrett [28] also introduced a definition of categorical representations and the functors between them [29]. Such definitions are analogues of Neuchl's definitions for *Hopf categories* [174]. Consider first the specific example of a categorical group $\mathcal{G}$ and its closure $\overline{\mathcal{G}}$ as defined above. One can check that a representation of $\mathcal{G}$ is precisely a functor $\mathcal{R} : \overline{\mathcal{G}} \to \mathbf{Vect}$, and that an *intertwiner* is precisely a natural transformation between two such functors (or representations). This motivates the categorical representation of categorical groups in the monoidal bicategory **2-Vect** of 2-vector spaces. If $\mathcal{G}$ is an arbitrary categorical group and $\overline{\mathcal{G}}$ its closure, the *categorical representation* of $\mathcal{G}$ is a strictly unitary homomorphism $(R, \tilde{R}) : \overline{\mathcal{G}} \to \mathbf{2\text{-}Vect}$.

The non-negative integer $R(\bullet) \in \mathbf{2\text{-}Vect}_0$ was called the *dimension* of the categorical representation. The categorical group representation can be equivalently described as a homomorphism between the corresponding crossed modules. The possibility of generalizing such categorical representations to monoidal categories other than **2-Vect** was also considered.

A theorem proven by Verdier states that the category of categorical groups and functorial homomorphisms $\mathbf{C}_{\mathcal{G}}$ and the category **CM** of crossed modules of groups and homomorphisms between them are equivalent. Based on this theorem, Barrett in [28] showed that each categorical group determines a monoidal bicategory of representations. Typically, such bicategories were shown to contain representations that are indecomposable but *not irreducible*.

In the following sections we shall consider even wider classes of representations for groupoids, arbitrary categories and functors.

7.7 General definition of extended symmetries as representations

We aim here to define extended quantum symmetries as general representations of mathematical structures that have as many as possible physical realizations, i.e. *via* unified quantum theories. In order to be able to extend this approach to very large ensembles of composite or complex quantum systems one requires general procedures for quantum 'coupling' of component quantum systems; we propose to approach this important 'composition', or scale up/assembly problem in a formal manner as described in the next section.

Because a group G can be viewed as a category with a single object, whose morphisms are just the elements of G, a *general representation* of G in an arbitrary category $\mathbf{C}$ is a functor R_G from G to $\mathbf{C}$ that selects an object X in $\mathbf{C}$ and a group homomorphism from γ to $\mathrm{Aut}(X)$, the automorphism group of X. Let us also define an *adjoint representation* by the functor $R^*_{\mathbf{C}} : \mathbf{C} \longrightarrow \mathbf{G}$. If $\mathbf{C}$ is chosen as the category **Top** of topological spaces and homeomorphisms then representations of G in **Top** are homomorphisms from G to the homeomorphism group of a topological space X. Similarly, a *general representation of a groupoid* $\mathcal{G}$ (considered as a category of invertible morphisms) in an arbitrary category $\mathbf{C}$ is a functor $\mathbf{R}_{\mathcal{G}}$ from $\mathcal{G}$ to $\mathbf{C}$, defined as above simply by substituting $\mathcal{G}$ for G. In the special case of a Hilbert space, this categorical definition is consistent with the representation of the groupoid on a bundle of Hilbert spaces.

Remark 7.3. Unless one is operating in super-categories, such as 2-categories and higher dimensional categories, one needs to distinguish between the *representations of an (algebraic) object* – as defined above – and the *representation of a functor* $\mathcal{S}$ (from $\mathbf{C}$ to the category of sets, **Set**) by an object in an arbitrary category $\mathbf{C}$ as defined next. Thus, in the latter case, a *functor representation* will be defined by a certain *natural equivalence between functors*. Furthermore, one needs consider also the following sequence of functors:

$$\mathcal{R}_{\mathbf{G}} : \mathbf{G} \longrightarrow \mathbf{C}, \qquad \mathcal{R}^*_{\mathbf{C}} : \mathbf{C} \longrightarrow \mathbf{G}, \qquad \mathcal{S} : \mathbf{G} \longrightarrow \mathbf{Set},$$

where $\mathcal{R}_{\mathbf{G}}$ and $\mathcal{R}^*_{\mathbf{C}}$ are adjoint representations as defined above, and $\mathcal{S}$ is the forgetful functor which forgets the group structure; the latter also has a right adjoint $\mathcal{S}^*$. With these notations one obtains the following commutative diagram of adjoint representations and adjoint functors that can be expanded to a square diagram to include either **Top** – the category of topological spaces and homeomorphisms, or **TGrpd**, and/or $\mathbf{C}_{\mathcal{G}} = \mathbf{CM}$ (respectively, the category of topological groupoids, and/or the category of categorical groups and homomorphisms) in a manner analogous to diagrams (9.4) that will be discussed in Section 9 (with the additional, unique adjunction situations to be added in accordingly)

$$\begin{tikzcd}
\mathbf{Set} \arrow[r, shift right, "S^*"'] & \mathbf{G} \arrow[l, shift right, "S"'] \arrow[d, shift left, "\mathcal{R}_C^*"] \\
& \mathbf{C} \arrow[u, shift left, "\mathcal{R}_G"] \arrow[ul, "{F,\ F^*}"]
\end{tikzcd}$$

7.8 Representable functors and their representations

The key notion of *representable functor* was first reported by Grothendieck (also with Dieudonné) during 1960–1962 [110, 107, 108] (see also the earlier publication by Grothendieck [106]). This is a functor $\mathcal{S} : \mathbf{C} \longrightarrow \mathbf{Set}$, from an arbitrary category $\mathbf{C}$ to the category of sets, $\mathbf{Set}$, if it admits a (functor) *representation* defined as follows. A *functor representation* of $\mathcal{S}$ is a pair, (R, ϕ), which consists of an object R of $\mathbf{C}$ and a family ϕ of equivalences $\phi(C) : \mathrm{Hom}_C(R, C) \cong \mathcal{S}(C)$, which is natural in C. When the functor $\mathcal{S}$ has such a representation, it is also said to be *represented by the object* R of $\mathbf{C}$. For each object R of $\mathbf{C}$ one writes $h_R : \mathbf{C} \longrightarrow \mathbf{Set}$ for the covariant Hom-functor $h_R(C) \cong \mathrm{Hom}_{\mathbf{C}}(R, C)$. A *representation* $(\mathcal{R}, \phi)$ of $\mathcal{S}$ is therefore *a natural equivalence of functors*

$$\phi : \ h_R \cong \mathcal{S}.$$

The equivalence classes of such functor representations (defined as natural equivalences) obviously determine an *algebraic groupoid* structure. As a simple example of an *algebraic* functor representation, let us also consider (cf. [152]) the functor $N : \mathbf{Gr} \longrightarrow \mathbf{Set}$ which assigns to each group G its underlying set and to each group homomorphism f the same morphism but regarded just as a function on the underlying sets; such a functor N is called a *forgetful* functor because it "forgets" the group structure. N is a representable functor as it is represented by the additive group $\mathbb{Z}$ of integers and one has the well-known bijection $\mathrm{Hom}_{Gx}(Z, G) \cong \mathcal{S}(G)$ which assigns to each homomorphism $f : \mathbb{Z} \longrightarrow G$ the image $f(1)$ of the generator 1 of $\mathbb{Z}$. In the case of groupoids there is also a natural forgetful functor $F : \mathbf{Grpd} \longrightarrow \mathbf{DirectedGraphs}$ whose left adjoint is the free groupoid on a directed graph, i.e. the groupoid of all paths in the graph.

Is F representable, and if so, what is the object that represents F?

One can also describe (viz. [152]) representable functors in terms of certain universal elements called *universal points*. Thus, consider $\mathcal{S} : \mathbf{C} \to \mathbf{Set}$ and let $\mathbf{C}_{s*}$ be the category whose objects are those pairs (A, x) for which $x \in \mathcal{S}(A)$ and with morphisms $f : (A, x) \longrightarrow (B, y)$ specified as those morphisms $f : A \to B$ of $\mathbf{C}$ such that $\mathcal{S}(f)x = y$; this category $\mathbf{C}_{s*}$ will be called the *category of $\mathcal{S}$-pointed objects* of $\mathbf{C}$. Then one defines a *universal point* for a functor $\mathcal{S} : \mathbf{C} \longrightarrow \mathbf{Set}$ to be an initial object (R, u) in the category $\mathbf{C}_{s*}$. At this point, a general connection between representable functors/functor representations and universal properties is established by the following, fundamental functor representation theorem [152].

Theorem 7.2 (Theorem 7.1 of MacLane [152]). *For each functor $\mathcal{S} : \mathbf{C} \longrightarrow \mathbf{Set}$, the formulas $u = (\phi R)1_R$, and $(\phi c)h = (\mathcal{S}h)u$, (with the latter holding for any morphism $h : R \longrightarrow C$), establish a one-to-one correspondence between the functor representations (R, ϕ) of $\mathcal{S}$ and the universal points (R, u) for $\mathcal{S}$.*

8 Algebraic categories and their representations in the category of Hilbert spaces. Generalization of tensor products

Quantum theories of quasi-particle, or multi-particle, systems are well known to require not just products of Hilbert spaces but instead their tensor products. On the other hand, symmetries are usually built through representations of products of groups such as $\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3)$ in the current 'standard model'; the corresponding Lie algebras are of course $\mathfrak{u}(1)$, $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$. To represent the more complex symmetries involving quantum groups that have underlying Hopf algebras, or in general Grassman–Hopf algebras, associated with many-particle or quasi-particle systems, one is therefore in need of considering new notions of generalized tensor products. We have discussed in Sections 6 and 7 alternative approaches to extended quantum symmetries involving graded Lie agebroids (in quantum gravity), quantum algebroids, convolution products and quantum algebroid representations. The latter approaches can be naturally combined with 'tensor products of quantum algebroids' if a suitable canonical extension of the tensor product notion is selected from several possible alternatives that will be discussed next.

8.1 Introducing tensor products of algebroids and categories

Firstly, we note that tensor products of cubical ω-groupoids have been constructed by Brown and Higgins [45], thus giving rise to a tensor product of crossed complexes, which has been used by Baues and Conduché to define the *'tensor algebra'* of a non-Abelian group [30]. Subsequently, Day and Street in [76] have also considered *Hopf algebras with many objects* in tensor categories [77]. Further work is however needed to explore possible links of these ideas with the functional analysis and operator algebras considered earlier. Thus, in attempting to generalize the notion of Hopf algebra to the many object case, one also needs to consider what could be the notion of tensor product of two R–algebroids C and D. If this can be properly defined one can then expect to see the *composition* in C as some partial functor $m : C\otimes C\longrightarrow C$ and a *diagonal* as some partial functor $\Delta : C\longrightarrow C\otimes C$. The definition of $C\otimes D$ is readily obtained for categories C, D by modifying slightly the definition of the tensor product of groupoids, regarded as crossed complexes in Brown and Higgins [45]. So we define $C\otimes D$ as the pushout of categories

$$\begin{tikzcd} C_0\times D_0 \arrow[r] \arrow[d] & C_1\times D_0 \arrow[d] \\ C_0\times D_1 \arrow[r] & C\otimes D \end{tikzcd}$$

This category may be seen also as generated by the symbols

$$\{c\otimes y \mid c\in C_1\}\cup\{x\otimes d \mid d\in D_1\},$$

for all $x\in C_0$ and $y\in D_0$ subject to the relations given by the compositions in C_1 and on D_1.

The category $G\,\#\,H$ is generated by all elements $(1_x,h)$, $(g,1_y)$ where $g\in G, h\in H$, $x\in G_0$, $y\in H_0$. We will sometimes write g for $(g,1_y)$ and h for $(1_x,h)$. This may seem to be willful ambiguity, but when composites are specified in $G\,\#\,H$, the ambiguity is resolved; for example, if gh is defined in $G\,\#\,H$, then g must refer to $(g,1_y)$, where $y=sh$, and h must refer to $(1_x,h)$, where $x=tg$. This convention simplifies the notation and there is an easily stated solution to the word problem for $G\,\#\,H$. Every element of $G\,\#\,H$ is uniquely expressible in one of the following forms:

i) an identity element $(1_x,1_y)$;

ii) a generating element $(g,1_y)$ or $(1_x,h)$, where $x\in G_0$, $y\in H_0$, $g\in G$, $h\in H$ and g, h are not identities;

iii) a composite $k_1k_2\cdots k_n$ $(n \geq 2)$ of non-identity elements of G or H in which the k_i lie alternately in G and H, and the odd and even products $k_1k_3k_5\cdots$ and $k_2k_4k_6\cdots$ are defined in G or H.

For example, if $g_1 : x \longrightarrow y$, $g_2 : y \longrightarrow z$, in G, g_2 is invertible, and $h_1 : u \longrightarrow v$, $h_2 : v \longrightarrow w$ in H, then the word $g_1h_1g_2h_2g_2^{-1}$ represents an element of $G \# H$ from (x, u) to (y, w). Note that the two occurrences of g_2 refer to different elements of $G \# H$, namely $(g_2, 1_v)$ and $(g_2, 1_w)$. This can be represented as a path in a 2-dimensional grid as follows

$$
\begin{tikzcd}
(x,u) \arrow[d, "g_1"] & (x,v) & (x,w) \\
(y,u) \arrow[r, "h_1"] & (y,v) \arrow[d, "g_2"] & (y,w) \\
(z,u) & (z,v) \arrow[r, "h_2"] & (z,w) \arrow[u, "g_2^{-1}"']
\end{tikzcd}
$$

The similarity with the free product of monoids is obvious and the normal form can be verified in the same way; for example, one can use ‘van der Waerden’s trick’. In the case when C and D are R-algebroids one may consider the pushout in the category of R-algebroids.

Now if $\mathbf{C}$ is a category, we can consider the possibility of a diagonal morphism

$$\Delta : \ \mathbf{C} \longrightarrow \mathbf{C} \# \mathbf{C}.$$

We may also include the possibility of a morphism

$$\mu : \ \mathbf{C} \# C \longrightarrow \mathbf{C}.$$

This seems possible in the algebroid case, namely the sum of the odd and even products. Or at least, μ could be defined on $\mathbf{C}\ \#\mathbf{C}((x, x), (y, y))$

It can be argued that a most significant effect of the use of categories as algebraic structures is to allow for algebraic structures with operations that are partially defined. These were early considered by Higgins in ‘Algebras with a scheme of operators’ [118, 119]. In general, *‘higher dimensional algebra’* (HDA) may be defined as the study of algebraic structures with operations whose domains of definitions are defined by geometric considerations. This allows for a splendid interplay of algebra and geometry, which early appeared in category theory with the use of complex commutative diagrams (see, e.g., [109, 158, 182, 153]). What is needed next is a corresponding interplay with analysis and functional analysis (see, e.g., [178]) that would extend also to quantum operator algebras, their representations and symmetries.

8.2 Construction of weak Hopf algebras via tensor category classification (according to Ostrik [177])

If $\mathbf{k}$ denotes an algebraically closed field, let $\mathbf{C}$ be a tensor category over $\mathbf{k}$. The classification of all semisimple module categories over $\mathbf{C}$ would then allow in principle the construction of all weak Hopf algebras H so that the category of comodules over H is tensor equivalent to $\mathbf{C}$, that is, as realizations of $\mathbf{C}$. There are at least three published cases where such a classification is possible:

(1) when $\mathbf{C}$ is a group theoretical fusion category (as an example when $\mathbf{C}_\gamma$ is the category of representations of a finite group γ, or a Drinfel’d quantum double of a finite group) (see [177]);

(2) when $\mathbf{k}$ is a fusion category attached to quantum SL(2) (see [177, 86, 33, 131, 175, 176, 115]);

(3) when $k = \mathbf{C}_q$ is the category of representations of quantum $\mathrm{SL}_q(2)$ Hopf algebras and q is not a root of unity (see [86]).

This approach was further developed recently for module categories over quantum SL(2) representations in the non-simple case (see also Example 2.4 regarding the quantum $\mathrm{SL}_q(2)$ Hopf algebras for further details), thus establishing a link between weak Hopf algebras and module categories over quantum SL(2) representations (viz. [177]).

Remark 8.1. One notes the condition imposed here of an *algebraically closed* field which is essential for remaining within the bounds of algebraic structures, as fields – in general – are not algebraic.

8.3 Construction of weak Hopf algebras from a matched pair $(\mathbf{V}, \mathbf{H})$ of finite groupoids (following Aguiar and Andruskiewitsch in [2])

As shown in [2], the matched pair of groupoids $(\mathbf{V}, \mathbf{H})$ can be represented by the factorisation of its elements in the following diagram:

$$\begin{tikzcd} P \arrow[r, "x"] \arrow[d, "x\triangleright g"'] & Q \arrow[d, "g"] \\ R \arrow[r, "x\triangleleft g"] & S \end{tikzcd}$$

where g is a morphism or arrow of the vertical groupoid $\mathbf{V}$ and x is a morphism of the matched horizontal groupoid $\mathbf{H}$, can be employed to construct a weak Hopf algebra, or quantum groupoid $\mathbf{k}(\mathbf{V}, \mathbf{H})$. In the above diagram, P is the common base set of objects for both $\mathbf{V}$ and $\mathbf{H}$, and $\triangleright$ and $\triangleleft$ are respectively the mutual *actions* of $\mathbf{V}$ and $\mathbf{H}$ on each other satisfying certain simple axioms (equation (1.1) on page 3 of [2] in *loc. cit*). Furthermore, a matched pair of rotations (η, ξ) for $(\mathbf{V}, \mathbf{H})$ gives rise to a *quasi-triangular weak Hopf* structure Q for $\mathbf{k}(\mathbf{V}, \mathbf{H})$. One can also write explicitly this structure as a tensor product:

$$Q = \sum \xi\big(f^{-1} \triangleleft g^{-1}, g\big) \otimes (\eta(g), f).$$

Remark 8.2. The representation of matched pairs of groupoids introduced in *loc. cit.* is specialized to yield a monoidal structure on the category $\mathbf{Rep}(\mathbf{V}, \mathbf{H})$ and a monoidal functor between such restrictive 'monoidal' representations, and is thus not consistent with the generalized notions of groupoid and functor representations considered, respectively, in Sections 7.3, 7.4, and 7.6–7.8. Nevertheless, the constructions of both weak Hopf algebras and quasi-triangular Hopf by means of matched pairs of groupoids is important for extended symmetry considerations as it suggests the possibility of double groupoid construction of weak Hopf algebroids (and also bialgebroids and double algebroids).

The representations of such higher dimensional algebraic structures will be further discussed in the following Section 9.

9 Double algebroids and double groupoids

There is a body of recent non-Abelian algebraic topology results giving a form of "higher dimensional group (HDG) theory" which is based on intuitive ideas of composing squares or n-cubes rather than just paths as in the case of groups [46].

Such an HDG theory yielded important results in homotopy theory and the homology of discrete groups, and seems also to be connected to a generalized categorical Galois theory introduced by Janelidze and Brown [47]. The HDG approach has also suggested other new constructions in group theory, for example a non-Abelian tensor product of groups. One of the aims of our paper is to proceed towards a corresponding theory for associative algebras and algebroids rather than groups. Then, one also finds that there are many more results and methods in HDG theories that are analogous to those in the lower dimensional group theory, but with a corresponding increase in technical sophistication for the former. Such complications occur mainly at the step of increasing dimension from one to dimension two; thus, we shall deal in this section only with the latter case. The general, n-dimensional case of such results presents significant technical difficulties but is of great potential, and will be considered in subsequent publications.

Thus, in developing a corresponding theory for algebras we expect that in order to obtain a non-trivial theory we shall have to replace, for example, R-algebras by R-algebroids, by which is meant just an R-category for a commutative ring R; in the case when R is the ring of integers, an R-algebroid is just a *'ring with many objects'* in the sense of Mitchell [159, 160]. (for further details see for example Section 4 and other references cited therein). The necessary algebroid concepts were already presented in Section 4. In the following subsections we shall briefly introduce other key concepts needed for such HGD developments. Thus, we begin by considering the simpler structure of double algebras and then proceed to their natural extension to double algebroids.

9.1 Double algebras

Here we approach convolution and the various Hopf structures that we have already discussed from the point of view of 'double structures'. With this purpose in mind, let A be taken to denote one of the following structures: a Hopf, a weak Hopf algebra or a Hopf algebroid (whose base rings need not be commutative). Starting with a Frobenius homomorphism $i : A \longrightarrow A^*$, we consider as in [204] the horizontal (H) and vertical (V) components of the algebra along with a convolution product $(*)$. Specifically, we take unital algebra structures $V = \langle A, \circ, e\rangle$ and $H = \langle A, *, i\rangle$ as leading to a double algebra structure with axioms as given in [204]. Thus the basic framework starts with a quadruple $(V, H, *, i)$. With respect to k–linear maps $\varphi : A \longrightarrow A$, we consider sublagebras $L, R \subset V$ and $B, T \subset H$ in accordance with the Frobenius homorphisms (for $a \in A$):

$$\begin{aligned} &\varphi_L(a) := a * e, \qquad &&\varphi_R(a) := e * a, \\ &\varphi_B(a) := a \circ i, \qquad &&\varphi_T(a) := i \circ a. \end{aligned}$$

Comultiplication of the 'quantum groupoid' arises from the dual bases of φ_B and φ_T with a D_4-symmetry:

$$\begin{tikzcd} {} \arrow[r, "T"] & {} \\ {} \arrow[u, "L"] \arrow[r, "B"'] \arrow[ur, phantom, "A"] & {} \arrow[u, "R"'] \end{tikzcd}$$

9.2 Double algebroids and crossed modules

In [51] Brown and Mosa introduced the notion of *double algebroid*, and its relationship to crossed modules of algebroids was investigated. Here we summarize the main results reported so far, but without providing the proofs that can be found in [162] and [51].

9.2.1 Crossed modules

Let A be an R-algebroid over A_0 and let M be a pre-algebroid over A_0. *Actions* of A on M are defined as follows:

Definition 9.1. A *left action* of A on M assigns to each $m \in M(x,y)$ and $a \in A(w,x)$ an element ${}^a m \in M(w,y)$, satisfying the axioms:

i) ${}^c({}^a m) = {}^{(ca)}m$, ${}^1 m = m$,

ii) ${}^a(mn) = {}^a mn$,

iii) ${}^a(m+m_1) = {}^a m + {}^a m_1$,

iv) ${}^{a+b}(m) = {}^a m + {}^b m$,

v) ${}^a(rm) = r({}^a m) = {}^{ra}(m)$,

for all $m, m_1 \in M(x,y)$, $n \in M(y,z)$, $a, b \in A(w,x)$, $c \in A(u,w)$ and $r \in R$.

Definition 9.2. A *right action* of A on M assigns to each $m \in M(x,y)$, $a \in A(y,z)$ an element $m^a \in M(x,z)$ satisfying the axioms:

i) $(m^a)^c = m^{(ac)}$, $m^1 = m$,

ii) $(mn)^a = mn^a$,

iii) $(m+m_1)^a = m^a + m_1^a$,

iv) $m^{(a+b)} = m^a + m^b$,

v) $(rm)^a = rm^a = m^{ra}$

for all $m, m_1 \in M(x,y)$, $n \in M(y,z)$, $a, b \in A(y,u)$, $c \in A(u,v)$ and $r \in R$.

Left and right actions of A on M *commute* if $({}^a m)^b = {}^a(m^b)$, for all $m \in M(x,y)$, $a \in A(w,x)$, $b \in A(y,u)$.

A *crossed module of algebroids* consists of an R-algebroid A, a pre-algebroid M, both over the same set of objects, and commuting left and right actions of A on M, together with a pre-algebroid morphism $\mu : M \to A$ over the identity on A_0. These must also satisfy the following axioms:

i) $\mu({}^a m) = a(\mu m)$, $\mu(m^b) = (\mu m)b$;

ii) $mn = m^{(\mu n)} = {}^{(\mu m)}n$,

and for all $m \in M(x,y)$, $n \in M(y,z)$, $a \in A(w,x)$, $b \in A(y,u)$.

A *morphism* $(\alpha, \beta) : (A, M, \mu) \longrightarrow (A', M', \mu')$ *of crossed modules* all over the same set of objects is an algebroid morphism $\alpha : A \longrightarrow A'$ and a pre-algebroid morphism $\beta : M \longrightarrow M'$ such that $\alpha\mu = \mu'\beta$ and $\beta({}^a m) = {}^{\alpha a}(\beta m)$, $\beta(m^b) = (\beta m)^{\alpha b}$ for all $a, b \in A$, $m \in M$. Thus one constructs a *category* **CM** *of crossed modules of algebroids*.

Two basic examples of crossed modules are as follows.

(1) Let A be an R-algebroid over A_0 and suppose I is a two-sided ideal in A. Let $i : I \longrightarrow A$ be the inclusion morphism and let A operate on I by $a^c = ac$, ${}^b a = ba$ for all $a \in I$ and $b, c \in A$ such that these products ac, ba are defined. Then $i : I \to A$ is a crossed module.

(2) A two-sided module over the algebroid A is defined to be a crossed module $\mu : M \to A$ in which $\mu m = 0_{xy}$ for all $m \in M(x,y)$, $x, y \in A_0$.

Similar to the case of categorical groups discussed above, a key feature of double groupoids is their relation to crossed modules "of groupoids" [52]. One can thus establish relations between double algebroids with thin structure and crossed modules "of algebroids" analogous to those already found for double groupoids, and also for categorical groups. Thus, it was recently reported that the *category of double algebroids with connections* is equivalent to the *category of crossed modules over algebroids* [51].

9.2.2 Double algebroids

n this subsection we recall the definition of a double algebroid introduced by Brown and Mosa in [51]. Two functors are then constructed, one from the category of double algebroids to the category of crossed modules of algebroids, whereas the other is its unique adjoint functor.

A *double R-algebroid* consists of a double category D such that each category structure has the additional structure of an R-algebroid. More precisely, a double R-algebroid D involves four related R-algebroids:

$$\big(D, D_1, \partial_1^0, \partial_1^1, \varepsilon_1, +_1, \circ_1, \cdot_1\big), \qquad \big(D, D_2, \partial_2^0, \partial_2^1, \varepsilon_2, +_2, \circ_2, \cdot_2\big),$$
$$\big(D_1, D_0, \delta_1^0, \delta_1^1, \varepsilon, +, \circ, .\big), \qquad \big(D_2, D_0, \delta_2^0, \delta_2^1, \varepsilon, +, \circ, .\big)$$

that satisfy the following rules:

i) $\delta_2^i \partial_2^j = \delta_1^j \partial_1^i$ for $i, j \in \{0, 1\}$;

ii)

$$\partial_2^i(\alpha +_1 \beta) = \partial_2^i \alpha + \partial_2^i \beta, \qquad \partial_1^i(\alpha +_2 \beta) = \partial_1^i \alpha + \partial_1^i \beta,$$
$$\partial_2^i(\alpha \circ_1 \beta) = \partial_2^i \alpha \circ \partial_2^i \beta, \qquad \partial_1^i(\alpha \circ_2 \beta) = \partial_1^i \alpha \circ \partial_1^i \beta$$

for $i = 0, 1$, $\alpha, \beta \in D$ and both sides are defined;

iii)

$$r \cdot_1 (\alpha +_2 \beta) = (r \cdot_1 \alpha) +_2 (r \cdot_1 \beta), \qquad r \cdot_2 (\alpha +_1 \beta) = (r \cdot_2 \alpha) +_1 (r \cdot_2 \beta),$$
$$r \cdot_1 (\alpha \circ_2 \beta) = (r \cdot_1 \alpha) \circ_2 (r \cdot_1 \beta), \qquad r \cdot_2 (\alpha \circ_1 \beta) = (r \cdot_2 \alpha) \circ_1 (r \cdot_2 \beta),$$
$$r \cdot_1 (s \cdot_2 \beta) = s \cdot_2 (r \cdot_1 \beta)$$

for all $\alpha, \beta \in D$, $r, s \in R$ and both sides are defined;

iv)

$$(\alpha +_1 \beta) +_2 (\gamma +_1 \lambda) = (\alpha +_2 \gamma) +_1 (\beta +_2 \lambda),$$
$$(\alpha \circ_1 \beta) \circ_2 (\gamma \circ_1 \lambda) = (\alpha \circ_2 \gamma) \circ_1 (\beta \circ_2 \lambda),$$
$$(\alpha +_i \beta) \circ_j (\gamma +_i \lambda) = (\alpha \circ_j \gamma) +_i (\beta \circ_j \lambda)$$

for $i \neq j$, whenever both sides are defined.

A *morphism* $f : \mathsf{D} \to \mathcal{E}$ *of double algebroids* is then defined as a morphism of truncated cubical sets which commutes with all the algebroid structures. Thus, one can construct a category **DA** of double algebroids and their morphisms. The main construction in this subsection is that of two functors η, η' from this category **DA** to the category **CM** of crossed modules of algebroids.

Let D be a double algebroid. One can associate to D a crossed module $\mu : M \longrightarrow D_1$. Here $M(x, y)$ will consist of elements m of D with boundary of the form:

$$\partial m = \begin{pmatrix} & a & \\ 1_y & & 1_x \\ & 0_{xy} & \end{pmatrix},$$

that is $M(x, y) = \{m \in D : \partial_1^1 m = 0_{xy}, \partial_2^0 m = 1_x, \partial_2^1 m = 1_y\}$.

The pre-algebroid structure on M is then induced by the second algebroid structure on D. We abbreviate $\circ_2$ on M and $\circ_1$ on D_1 to juxtaposition. The morphism μ is defined as the restriction of ∂_1^0.

Actions of D_1 on M are defined by

$$^{a}m = (\varepsilon_1 a)m, \qquad m^{b} = m(\varepsilon_1 b).$$

The only non trivial verification of the axioms is that $mn = m^{\mu n} = {}^{\mu m}n$. For this, let m, n have boundaries $\begin{pmatrix} & a & \\ 1 & & 1 \\ & 0 & \end{pmatrix}$, $\begin{pmatrix} & b & \\ 1 & & 1 \\ & 0 & \end{pmatrix}$. Reading in two ways the following diagram (in which unmarked edges are 1's) yields $mn = {}^{a}n$:

$$\begin{array}{|c|c|} \multicolumn{1}{c}{a} & \multicolumn{1}{c}{b} \\ \hline \varepsilon_1 a & n \\ a & 0 \\ \hline m & \\ & \\ \hline \multicolumn{1}{c}{0} & \multicolumn{1}{c}{0} \end{array}$$

Similarly one obtains that $mn = m^{b}$. This shows that $\mu : M \longrightarrow D_1$ is indeed a crossed module. The construction also defines η which readily extends to a functor from the *category of double algebroids* **DA** to **CM**. The second crossed module $\nu : N \longrightarrow D_2$ has D_2 as above, but N consists of elements with boundary of the form $\begin{pmatrix} & 1 & \\ a & & 0 \\ & 1 & \end{pmatrix}$. The actions are defined in a similar manner to that above for M and one constructs a crossed module in the manner suggested above. Therefore, an object in the category of algebroids yields also an *associated crossed module*. In general, the two crossed modules constructed above are not isomorphic. However, if the double algebroid has a *connection pair* (Γ, Γ'), then its two associated crossed modules are isomorphic (cf. [51]). Furthermore, there is also an associated thin structure $\theta : \mathsf{D}_1 \to \mathsf{D}$ which is a morphism of double categories because D_1 also has an associated double algebroid structure derived from that of D_1.

Next one can construct a functor ζ from **CM** to double algebroids. Thus, let $\mu : M \longrightarrow A$ be a crossed module. The double algebroid $D = \zeta(\mu : M \longrightarrow A)$ will coincide with A in dimensions 0 and 1. The set D consists of pairs (m, a) such that $m \in M$, $a = \begin{pmatrix} & a_3 & \\ a_1 & & a_4 \\ & a_2 & \end{pmatrix}$ and $a_3 a_4 - a_1 a_2 = \mu m$. One defines two algebroid structures on D which will turn D into a double algebroid. Its additions and scalar multiplications are defined by:

$$(m, a) +_i (n, b) = (m + n, a +_i b), \qquad r._i\,(m, a) = (m, r._i\, a).$$

The two compositions are defined by:

$$(m, a) \circ_i (n, b) = \begin{cases} (m^{b_4} + {}^{a_2}n, a \circ_i b) & \text{if } i = 1, \\ (m^{b_2} + {}^{a_1}n, a \circ_i b) & \text{if } i = 2. \end{cases}$$

Furthermore, there exists a connection pair for the underlying double category of D given by $\Gamma a = (0, A)$, $\Gamma' a = (0, A')$, where

$$A = \begin{pmatrix} & a & \\ a & & 1 \\ & 1 & \end{pmatrix}, \qquad A' = \begin{pmatrix} & 1 & \\ 1 & & a \\ & a & \end{pmatrix}.$$

Proposition 9.1. *The above construction defines a functor ζ from the category* **CM** *of crossed modules to the category of double algebroids with connection pair $\Gamma a = (0, A)$, $\Gamma' a = (0, A')$*

on the underlying double category, where the connection pair satisfies the following additional properties.

Suppose $u, v, w \in D$ have boundaries

$$\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix}, \qquad \begin{pmatrix} & c & \\ e & & f \\ & b & \end{pmatrix}, \qquad \begin{pmatrix} & g & \\ a & & d \\ & h & \end{pmatrix}$$

respectively , and $r \in R$. Then, the following equations must hold:

i) $\Gamma(a+e) \circ_2 (u +_1 v) \circ_2 \Gamma'(d+f) = (\Gamma' a \circ_2 u \circ_2 \Gamma d) +_2 (\Gamma' e \circ_2 v \circ_2 \Gamma f);$

ii) $\Gamma'(c+g) \circ_1 (u +_2 w) \circ_1 \Gamma(d+f) = (\Gamma' \circ_2 u \circ_2 \Gamma b) +_1 (\Gamma' g \circ_2 w \circ_2 \Gamma h);$

iii) $\Gamma'(ra) \circ_2 (r._1 u) \circ_2 \Gamma(rd) = r._2(\Gamma' a \circ_2 u \circ_2 \Gamma d);$

iv) $\Gamma'(rc) \circ_1 (r._2 u) \circ_1 \Gamma(rd) = r._2(\Gamma' c \circ_1 u \circ_1 \Gamma b).$

Remark 9.1. Edge symmetric double algebroids were also defined in [51], and it was shown that there exist two functors which respectively associate to a double algebroid its corresponding horizontal and vertical crossed modules.

Then, one obtains the result that the categories of crossed modules "of algebroids" and of edge symmetric double algebroids with connection are *equivalent*. As a corollary, one can also show that the two algebroids structures in dimension 2 of a special type of double algebroids are isomorphic, as are their associated crossed modules [51]. This result will be precisely shown in the next subsection.

9.3 The equivalence between the category of crossed modules of algebroids and the category of algebroids with a connection pair. Quantum algebroid symmetries

In order to obtain an equivalence of categories one needs to add in the extra structure of a connection pair to a double algebroid.

Let D be a double algebroid. A *connection pair* for D is a pair of functions $(\Gamma, \Gamma') : D_1 \to D$ which is a connection pair for the underlying double category of $\mathbf{D}$ as in [51, 162]. Thus one has the following properties:

i) If $a : x \longrightarrow y$ in D_1, then $(\Gamma a, \Gamma' a)$ have boundaries given respectively by

$$\begin{pmatrix} & a & \\ a & & 1_y \\ & 1_y & \end{pmatrix}, \qquad \begin{pmatrix} & 1_x & \\ 1_x & & a \\ & a & \end{pmatrix}$$

and

$$\Gamma' a \circ_2 \Gamma a = \varepsilon_1 a, \qquad \Gamma' a \circ_1 \Gamma a = \varepsilon_2 a.$$

ii) If $x \in D_0$, then $\Gamma 1_x = \Gamma' 1_x = \varepsilon_1^2 x$.

iii) The transport laws: if $a \circ b$ is defined in D_1 then

$$\Gamma(a \circ b) = \begin{bmatrix} \Gamma & \varepsilon_1 b \\ \varepsilon_2 b & \Gamma b \end{bmatrix}, \qquad \Gamma'(a \circ b) = \begin{bmatrix} \Gamma' a & \varepsilon_2 a \\ \varepsilon_1 a & \Gamma' b \end{bmatrix}.$$

One then defines 'folding' operations on elements of D for a double algebroid with connection. Let $u \in D$ have boundary $\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix}$. One first sets

$$\psi u = \Gamma' a \circ_2 u \circ_2 \Gamma d$$

as in [51, 162].

Proposition 9.2.

i) Let u, v, w be such that $u \circ_1 v$, $u \circ_2 w$ are defined. Then

$$\psi(u \circ_1 v) = (\psi u \circ_2 \varepsilon_1 \partial_2^1 v) \circ_1 (\varepsilon_1 \partial_2^0 u \circ_2 \psi v),$$
$$\psi(u \circ_2 w) = (\varepsilon_1 \partial_1^0 u \circ_2 \psi w) \circ_1 (\psi u \circ_2 \varepsilon_1 \partial_1^0 w).$$

ii) If $a \in D_1$ then $\psi a = \psi' a = \varepsilon_1 a$.

iii) $\psi u = u$ if and only if $\partial_2^0 u$, $\partial_2^1 u$ are identities.

iv) Let u, v, w be such that $u +_1 v$, $u +_2 w$ are defined. Then

$$\psi(u +_1 v) = \psi u +_2 \psi v, \qquad \psi(u +_2 w) = \psi u +_2 \psi w.$$

v) If $r \in R$ and $u \in D_2$ then $\psi(r._i\, u) = r._i\, \psi u$ for $i = 1, 2$.

One defines an operation $\phi : D \to D$ by

$$\phi u = \psi u -_2 \varepsilon_1 \partial_1^1 \psi u.$$

Let M be the set of elements $u \in D$ such that the boundary ∂u is of the form $\begin{pmatrix} & a & \\ 1 & & 1 \\ & 0 & \end{pmatrix}$. We write 0 for an element of M of the form $\varepsilon_1 0_{xy}$ where 0_{xy} is the algebroid zero of $D_1(x, y)$.

Proposition 9.3. *The operation ϕ has the following properties:*

i) if $\partial u = \begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix}$ then $\partial \phi u = \begin{pmatrix} & cd - ab & \\ 1 & & 1 \\ & 0 & \end{pmatrix}$;

ii) $\phi u = u$ if and only if $u \in M$; in particular $\phi\phi = \phi$;

iii) If $a \in D_1$ then $\phi\gamma a = \phi\gamma' a = \phi\varepsilon_i a = 0$;

iv) $\phi(u +_i v) = \phi u +_2 \phi v$;

v) $\phi(r._i\, u) = r._2(\phi u)$;

vi) $\phi(u \circ_1 v) = (\phi u)^{\partial_1^1 v} +_2 {}^{\partial_1^0 u}(\phi v)$;

vii) $\phi(u \circ_2 w) = (\phi u)^{\partial_1^1 w} +_2 {}^{\partial_1^0 u}(\phi w)$.

With the above constructions one can then prove the following, major result of [51].

Theorem 9.1. *The categories of crossed modules of algebroids and of double algebroids with a connection pair are equivalent. Let* **DA** *denote the category of double algebroids with connection. One has the functors defined above:*

$$\eta :\ \mathbf{DA} \longrightarrow \mathbf{CM}, \qquad \textit{and} \qquad \xi :\ \mathbf{CM} \longrightarrow \mathbf{DA}.$$

The functors $\eta\xi$ and $\xi\eta$ are then each naturally equivalent to the corresponding identity functor for categories **DA** *and* **CM***, respectively.*

Remark 9.2. A word of caution is here in order about the equivalence of categories in general: the equivalence relation may have more than one meaning, that is however always a *global* property; thus, categories that are equivalent may still exhibit substantially different and significant *local* properties, as for example in the case of equivalent categories of semantically distinct, n-valued logic algebras [99].

Remark 9.3. The above theorem also has a significant impact on physical applications of double algebroid representations with a connection pair to extend quantum symmetries in the presence of intense gravitational fields because it allows one to work out such *higher dimensional representations* in terms of those of crossed modules of (lower-dimension) algebroids, as for example in the cases of: Lie, weak Hopf, Grassman–Hopf algebroids or 'Lie' superalgebroids relevant to Quantum Gravity symmetries of intense gravitational fields and 'singularities' that were introduced and discussed in previous sections.

The main result of Brown and Mosa [51] is now stated as follows.

Theorem 9.2 (Brown and Mosa [51]). *The category of crossed modules of R-algebroids is equivalent to the category of double R-algebroids with thin structure.*

Let $\mu : M \longrightarrow A$ be a crossed module. Applying $\eta\xi$ to this yields a crossed module $\nu : N \to B$ say. Then $B = A$ and N consists of pairs (m, a) where $a = \begin{pmatrix} 1 & \begin{matrix} \mu m \\ 0 \end{matrix} & 1 \end{pmatrix}$ for all $m \in M$. Clearly these two crossed modules are naturally isomorphic.

One can now use the category equivalences in the two theorems above to also prove that *in a double algebroid with connection the two algebroid structures in dimension two are isomorphic.* This is accomplished by defining a *reflection* which is analogous to the rotation employed in [52] for double groupoids and to the reflection utilized for those double categories that arise from 2-categories with invertible 2-cells.

Let D be a double algebroid with a connection pair. One defines $\rho : D \longrightarrow D$ by the formula in which assuming $\partial u = \begin{pmatrix} a & \begin{matrix} c \\ b \end{matrix} & d \end{pmatrix}$

$$\rho u = \theta \begin{pmatrix} 1 & \begin{matrix} a \\ ab \end{matrix} & b \end{pmatrix} \circ_1 (\varepsilon_1(cd) -_2 \phi u) \circ_1 \theta \begin{pmatrix} c & \begin{matrix} cd \\ d \end{matrix} & 1 \end{pmatrix}.$$

Thus, one has that: $\partial \rho u = \begin{pmatrix} c & \begin{matrix} a \\ b \end{matrix} & d \end{pmatrix}$, and also the following result of Brown and Mosa [51]:

Theorem 9.3. *The reflection ρ satisfies:*

i) $\rho(a) = a$, $\rho('a) =' a$, $\rho(\varepsilon_1 a) = \varepsilon_1 a$, $\rho(\varepsilon_2 a) = \varepsilon_2 a$ *for all* $a \in D_1$*;*

ii) $\rho(u +_1 v) = \rho u +_2 \rho v$, $\rho(w +_2 x) = \rho w +_1 \rho x$ *whenever* $u +_1 v, w +_2 x$ *are defined;*

iii) $\rho(u \circ_1 v) = \rho u \circ_2 \rho v$, $\rho(w \circ_2 x) = \rho w \circ_1 \rho x$ $u \circ_1 v$, $w \circ_2 x$ *are defined;*

iv) $\rho(r._1 u) = r._2 \rho u$, $\rho(r._2 u) = r._1 \rho u$ *where* $r \in R$.

Remark 9.4. The reflection concept presented above represents the key *internal symmetries* of double algebroids with connection pair, and there are also similar concepts for other higher dimensional structures such as double groupoids, double categories, and so on. Therefore, one can reasonably expect that such reflection notions may also be applicable to all 'quantum doubles', including quantum double groupoids and higher dimensional quantum symmetries that are expected, or predicted, to occur in quantum chromodynamics, and *via* 'Lie' superalgebroids, also in quantum gravity based on lc-GR theories as proposed in subsequent sections.

9.4 Double groupoids

We can take further advantage of the above procedures by reconsidering the earlier, double groupoid case [52] in relationship to a C^*-convolution algebroid that links both 'horizontal' and 'vertical' structures in an internally consistent manner. The geometry of squares and their compositions leads to a common representation of a *double groupoid* in the following form:

$$\mathsf{D} = \begin{tikzcd} S \arrow[r, shift left=2, "s^1"] \arrow[r, shift right=2, "t^1"'] \arrow[d, shift right=2, "s_2"'] \arrow[d, shift left=2, "t_2"] & H \arrow[l] \arrow[d, shift right=2, "s"'] \arrow[d, shift left=2, "t"] \\ V \arrow[u] \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & M \arrow[l] \arrow[u] \end{tikzcd}$$

where M is a set of 'points', H, V are 'horizontal' and 'vertical' groupoids, and S is a set of 'squares' with two compositions. The laws for a double groupoid make it also describable as a groupoid internal to the category of groupoids. Furthermore, because in a groupoid, any composition of commutative squares is also commutative, several groupoid square diagrams of the type shown above can be composed to yield larger square diagrams that are naturally commutative.

Given two groupoids H, V over a set M, there is a double groupoid $\Box(H, V)$ with H, V as horizontal and vertical edge groupoids, and squares given by quadruples $\begin{pmatrix} & h & \\ v & & v' \\ & h' & \end{pmatrix}$ for which we assume always that $h, h' \in H$, $v, v' \in V$ and that the initial and final points of these edges match in M as suggested by the notation, that is for example $sh = sv$, $th = sv'$, ..., etc. The compositions are to be inherited from those of H, V, that is:

$$\begin{pmatrix} & h & \\ v & & v' \\ & h' & \end{pmatrix} \circ_1 \begin{pmatrix} & h' & \\ w & & w' \\ & h'' & \end{pmatrix} = \begin{pmatrix} & h & \\ vw & & v'w' \\ & h'' & \end{pmatrix},$$

$$\begin{pmatrix} & h & \\ v & & v' \\ & h' & \end{pmatrix} \circ_2 \begin{pmatrix} & k & \\ v' & & v'' \\ & k' & \end{pmatrix} = \begin{pmatrix} & hk & \\ v & & v'' \\ & h'k' & \end{pmatrix}.$$

This construction is defined by the right adjoint R to the forgetful functor L which takes the double groupoid as above, to the pair of groupoids (H, V) over M. Furthermore, this right adjoint functor can be utilized to relate *double groupoid representations* to the corresponding pairs of groupoid representations induced by L. Thus, one can obtain a functorial construction of certain double groupoid representations from those of the groupoid pairs (H, V) over M. Further uses of adjointness to classifying groupoid representations related to extended quantum symmetries can also be made through the generalized Galois theory presented in the next subsection; therefore, Galois groupoids constructed with a pair of adjoint functors and their representations may play a central role in such future developments of the mathematical theory of groupoid representations and their applications in quantum physics.

Given a general double groupoid as above, one can define $S\begin{pmatrix} & h & \\ v & & v' \\ & h' & \end{pmatrix}$ to be the set of squares with these as horizontal and vertical edges

$$\mathsf{AD} = \begin{tikzcd} AS \arrow[r, shift left=2, "s^1"] \arrow[r, shift right=2, "t^1"'] \arrow[d, shift right=2, "s_2"'] \arrow[d, shift left=2, "t_2"] & AH \arrow[l] \arrow[d, shift right=2, "s"'] \arrow[d, shift left=2, "t"] \\ AV \arrow[u] \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & M \arrow[l] \arrow[u] \end{tikzcd}$$

for which:

$$AS\begin{pmatrix} v & \begin{matrix} h \\ h' \end{matrix} & v' \end{pmatrix}$$

is the free A-module on the set of squares with the given boundary. The two compositions are then bilinear in the obvious sense.

Alternatively, one can use the convolution construction $\bar{A}\mathsf{D}$ induced by the convolution C^*-algebra over H and V. This allows us to construct for at least a commutative C^*-algebra A a double algebroid (i.e., a set with two algebroid structures), as discussed in the previous subsection. These novel ideas need further development in the light of the algebra of crossed modules of algebroids, developed in [162] and [51], crossed cubes of C^*-algebras following [85], as well as crossed complexes of groupoids [43].

The next, natural extension of this *quantum algebroid* approach to QFT generalized symmetries can now be formulated in terms of *graded Lie algebroids*, or supersymmetry algebroids, for the supersymmetry-based theories of *quantum gravity/supergravity* that were discussed in Section 6.

We shall discuss first in the next subsection an interesting categorical construction of a homotopy double groupoid.

9.5 The generalized Galois theory construction of a homotopy double groupoid [47] and Galois groupoid representations

In two related papers Janelidze [125, 124] outlined a categorical approach to the Galois theory. In a more recent paper in 2004, Brown and Janelidze [47] reported a *homotopy double groupoid* construction of a surjective fibration of Kan simplicial sets based on a generalized, *categorical Galois (GCG) theory* which under certain, well-defined conditions gives a *Galois groupoid* from a pair of adjoint functors. As an example, the standard fundamental group arises in GCG from an adjoint pair between topological spaces and sets. Such a homotopy double groupoid (HDG, explicitly given in diagram 1 of [47]) was also shown to contain the 2-groupoid associated to a map defined by Kamps and Porter [128]; this HDG includes therefore the 2-groupoid of a pair defined by Moerdijk and Svenson [161], the cat^1-group of a fibration defined by Loday [146], and also the classical fundamental crossed module of a pair of pointed spaces introduced by J.H.C. Whitehead. Related aspects concerning homotopical excision, Hurewicz theorems for n-cubes of spaces and van Kampen theorems [209] for diagrams of spaces were subsequently developed in [49, 50].

Two major advantages of this generalized Galois theory construction of HDG that were already pointed out are:

(i) the construction includes information on the map $q : M \to B$ of topological spaces, and

(ii) one obtains different results if the topology of M is varied to a finer topology.

Another advantage of such a categorical construction is the possibility of investigating the global relationships among the category of simplicial sets, $\mathbf{C}_S = \mathbf{Set}^{\Delta^{\mathrm{op}}}$, the category of topological spaces, $\mathbf{Top}$, and the category of groupoids, $\mathbf{Grpd}$. Let I be the fundamental groupoid functor $I = \pi_1 : \mathbf{C}_S \to \mathbf{X}$ from the category $\mathbf{C}_S$ to the category $\mathbf{X} = \mathbf{Grpd}$ of (small) groupoids. Consider next diagram 11 on page 67 of Brown and Janelidze [47]:

$$\begin{tikzcd} \mathbf{Top} \arrow[r, shift right, "S"'] & \mathbf{Set}^{\Delta^{\mathrm{op}}} \arrow[l, shift right, "R"'] \arrow[r, shift left, "I"] & \mathbf{Grpd} \arrow[l, shift left, "H"] \\ & \Delta \arrow[ul, "r"] \arrow[u, "y"] \arrow[ur, "i"'] & \end{tikzcd} \qquad (9.1)$$

where:

- **Top** is the category of topological spaces, S is the singular complex functor and R is its left-adjoint, called the geometric realisation functor;
- $I \vdash H$ is the adjoint pair introduced in Borceux and Janelidze [36], with I being the fundamental groupoid functor, and H being its unique right-adjoint *nerve functor*;
- y is the Yoneda embedding, with r and i being, respectively, the restrictions of R and I respectively along y; thus, r is the singular simplex functor and i carries finite ordinals to codiscrete groupoids on the same sets of objects.

The adjoint functors in the top row of the above diagram are uniquely determined by r and i – up to isomorphisms – as a result of the universal property of y, the Yoneda embedding construction. Furthermore, one notes that there is a natural completion to a square, commutative diagram of the double triangle diagram (9.1) reproduced above by three adjoint functors of the corresponding forgetful functors related to the Yoneda embedding. This natural diagram completion, that may appear trivial at first, leads however to the following Lemma and related Propositions.

Lemma 9.1. *The following diagram* (9.2) *is commutative and there exist canonical natural equivalences between the compositions of the adjoint functor pairs and their corresponding identity functors of the four categories presented in diagram* (9.2)*:*

$$\begin{tikzcd}
\mathbf{Top} \arrow[d, shift right, "f"'] & \mathbf{Set}^{\Delta^{\mathrm{op}}} \arrow[l, shift left, "S"] \arrow[d, shift left, "I"] \\
\mathbf{Set} \arrow[u, shift right, "g"'] \arrow[r, shift left, "G"] & \mathbf{Grpd} \arrow[l, shift left, "F"] \arrow[u, shift left, "H"]
\end{tikzcd} \tag{9.2}$$

with $\mathbf{Top} \xrightarrow{R} \mathbf{Set}^{\Delta^{\mathrm{op}}}$.

The forgetful functors $f : \mathbf{Top} \longrightarrow \mathbf{Set}$, $F : \mathbf{Grpd} \longrightarrow \mathbf{Set}$ *and* $\Phi : \mathbf{Set}^{\Delta^{\mathrm{op}}} \longrightarrow \mathbf{Set}$ *complete this commutative diagram of adjoint functor pairs. The right adjoint of* Φ *is denoted by* $\Phi*$*, and the adjunction pair* $[\Phi, \Phi*]$ *has a mirror-like pair of adjoint functors between* **Top** *and* **Grpd** *when the latter is restricted to its subcategory* **TGrpd** *of topological groupoids, and also when* $\phi : \mathbf{TGrpd} \longrightarrow \mathbf{Top}$ *is a functor that forgets the algebraic structure – but not the underlying topological structure of topological groupoids, which is fully and faithfully carried over to* **Top** *by* ϕ.

Remark 9.5. Diagram (9.2) of adjoint functor pairs can be further expanded by adding to it the category of groups, **Gr**, and by defining a 'forgetful' functor $\psi : \mathbf{Grpd} \longrightarrow \mathbf{Gr}$ that assigns to each groupoid the product of its 'component' groups, thus ignoring the connecting, internal groupoid morphisms. The categorical generalization of the Galois theory for groups can be then related to the adjoint functor $\psi*$ and its pair. As a simple example of the groupoid forgetful functor consider the mapping of an extended symmetry groupoid, $\mathcal{G}_S$, onto the group product $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ that 'forgets' the global symmetry of $\mathcal{G}_S$ and retains only the U(1), SU(2) and SU(3) symmetries of the 'standard model' in physics; here both the groups and $\mathcal{G}_S$ are considered, respectively as special, small categories with one or many objects, and isomorphisms.

Proposition 9.4. *If* $T : \mathbf{C} \longrightarrow \mathbf{Grpd}$ *is any groupoid valued functor then* T *is naturally equivalent to a functor* $\Theta : \mathbf{C} \longrightarrow \mathbf{Grpd}$ *which is univalent with respect to objects.*

The proof is immediate by taking first into account the Lemma 9.1 and diagram (9.2), and then by following the logical proof sequence for the corresponding group category Proposition 10.4 of Mitchell [158]. Note that 'univalent' is also here employed in the sense of Mitchell [158].

This new proposition for groupoid valued functors can be thus considered as a natural extension of the corresponding theorem for group valued functors.

Remark 9.6. The class of natural equivalences of the type $T \to \Theta$ satisfying the conditions in Proposition 9.4 is itself a (large) 2-groupoid whose objects are groupoid valued functors. In particular, when $\mathbf{C} = \mathbf{Top}$ and T is the composite functor $I \circ S : \mathbf{Top} \longrightarrow \mathbf{Set}^{\Delta^{\mathrm{op}}} \longrightarrow \mathbf{Grpd}$, then one obtains the interesting result (a) that the class of functors naturally equivalent with T becomes the (large) 2-groupoid of classical (geometric) *fundamental groupoid functors* π_1 [47]. Moreover, according also to Proposition 2.1 of [47], one has that:

(a) *for every topological space X, $S(X)$ is a Kan complex, and*

(b) *the S-image of a morphism p in* **Top** *is a Kan fibration if and only if p itself is a Serre fibration.*

(For further details the reader is referred to [47].)

9.6 Functor representations of topological groupoids

A representable functor $S : \mathbf{C} \longrightarrow \mathbf{Set}$ as defined in Section 7.8 is also determined by the equivalent condition that there exists an object X in $\mathbf{C}$ so that S is isomorphic to the Hom-functor h^X. In the dual, categorical representation, the Hom-functor h^X is simply replaced by h_X. As an immediate consequence of the Yoneda–Grothendieck lemma the set of natural equivalences between S and h^X (or alternatively h_X) – which has in fact a groupoid structure – is isomorphic with the object $S(X)$. Thus, one may say that if S is a representable functor then $S(X)$ is its (isomorphic) *representation object*, which is also unique up to an isomorphism [158, p. 99]. As an especially relevant example we consider here the *topological groupoid representation* as a functor $\gamma : \mathbf{TGrpd} \longrightarrow \mathbf{Set}$, and related to it, the more restrictive definition of $\gamma : \mathbf{TGrpd} \longrightarrow \mathbf{BHilb}$, where **BHilb** can be selected either as the category of Hilbert bundles or as the category of rigged Hilbert spaces generated through the GNS construction as specified in Definition 5.1 and related equations (5.1) and (5.2).

$$\begin{tikzcd}
\mathbf{Top} \arrow[r, shift right, "M"'] & \mathbf{BHilb} \arrow[l, shift right, "L"'] \arrow[r, shift left, "J"] & \mathbf{TGrpd} \arrow[l, shift left, "K"] \\
& \mathbf{Set} \arrow[ul, "g"] \arrow[u, "m"'] \arrow[ur, "n"'] &
\end{tikzcd} \tag{9.3}$$

Considering the forgetful functors f and F as defined above, one has their respective adjoint functors defined by g and n in diagram (9.3); this construction also leads to a diagram of adjoint functor pairs similar to the ones shown in diagram (9.2). The functor and natural equivalence properties stated in Lemma 9.1 also apply to diagram (9.3) with the exception of those related to the adjoint pair $[\Phi, \Phi*]$ that are replaced by an adjoint pair $[\Psi, \Psi*]$, with $\Psi : \mathbf{BHilb} \longrightarrow \mathbf{Set}$ being the forgetful functor and $\Psi*$ its left adjoint functor. With this construction one obtains the following proposition as a specific realization of Proposition 9.4 adapted to topological groupoids and rigged Hilbert spaces:

Proposition 9.5. *If $\mathcal{R}_o : \mathbf{BHilb} \longrightarrow \mathbf{TGrpd}$ is any topological groupoid valued functor then $\mathcal{R}_o$ is naturally equivalent to a functor $\rho : \mathbf{BHilb} \longrightarrow \mathbf{TGrpd}$ which is univalent with respect to objects.*

Remark 9.7. $\mathcal{R}_o$ and ρ can be considered, respectively, as adjoint Hilbert-functor representations to groupoid, and respectively, topological groupoid functor representations $\mathcal{R}_o^*$ and ρ^* in the category **BHilb** of rigged Hilbert spaces.

Remark 9.8. The connections of the latter result for groupoid representations on rigged Hilbert spaces to the weak C^*-Hopf symmetry associated with quantum groupoids and to the generalized categorical Galois theory warrant further investigation in relation to quantum systems with extended symmetry. Thus, the following corollary and the previous Proposition 9.4 suggest several possible applications of GCG theory to extended quantum symmetries *via* Galois groupoid representations in the category of rigged Hilbert families of quantum spaces that involve interesting adjoint situations and also natural equivalences between such *functor* representations. Then, considering the definition of quantum groupoids as *locally compact* (topological) groupoids with certain extended (quantum) symmetries, their functor representations also have the unique properties specified in Proposition 9.4 and Corollary 9.1, as well as the unique adjointness and natural properties illustrated in diagram (9.3).

Corollary 9.1. *The composite functor* $\Psi \circ \mathcal{R}_o : \mathbf{TGrpd} \longrightarrow \mathbf{BHilb} \longrightarrow \mathbf{Set}$*, has the left adjoint* n *which completes naturally diagram* (9.3)*, with both* $\Psi : \mathbf{BHilb} \longrightarrow \mathbf{Set}$ *and* $\Psi \circ \mathcal{R}_o$ *being forgetful functors.* Ψ *also has a left adjoint* $\Psi*$*, and* $\mathcal{R}_o$ *has a defined inverse, or duality functor* $\Im$ *which assigns in an univalent manner a topological groupoid to a family of rigged Hilbert spaces in* **BHilb** *that are specified via the GNS construction.*

Remark 9.9. The adjoint of the duality functor – which assigns in an univalent manner a family of rigged Hilbert spaces in the category **BHilb** (that are specified *via* the GNS construction) to a topological groupoid – defines a *Hilbert-functor adjoint representation* of topological groupoids; the latter generalizes to dimension **2** the 'standard' notion of (object) groupoid representations. A similar generalization to higher dimensions is also possible for algebroid representations, for example by considering functor representations from the category of double algebroids **DA** (or equivalently from **CM**) to the category **BHilb** of 'rigged' Hilbert spaces.

Remark 9.10. For quantum state spaces and quantum operators the duality functor $J = \Im$: $\mathbf{BHilb} \longrightarrow \mathbf{TGrpd}$ is the *quantum fundamental groupoid (QFG) functor*, that plays a similar role for a quantum state space bundle in the category **BHilb** to that of the fundamental groupoid functor $I = \pi_1 : \mathbf{C}_s \longrightarrow \mathbf{X}$ in diagram (9.1) of the generalized categorical Galois theory of [47] (in the original, this is diagram (11) on page 67). The right adjoint $\mathcal{R}_o$ of the QFG functor $\Im$ thus provides a functor (or 'categorical') representation of *topological* (in fact, locally compact) quantum groupoids by rigged Hilbert spaces (or quantum Hilbert space bundles) in a natural manner. Such rigged quantum Hilbert spaces are the ones actually realized in quantum systems with extended quantum symmetries described by quantum topological groupoids and also represented by the QFG functor in the manner prescribed by the functor π_1 (or I) in the generalized (categorical) Galois theory of Brown and Janelidze as shown in diagram (9.1).

Let us consider next two diagrams that include, respectively, two adjoint situation quintets:

$$(\eta; J \circ M, L \circ K; \mathbf{Top}, \mathbf{TGrpd}), \qquad \text{and} \qquad (\mu; J \circ M, L \circ K; \mathbf{Top}, \mathbf{TGrpd}),$$

$$\begin{tikzcd}
\mathbf{Top} \arrow[r, shift right, "J\circ M"'] & \mathbf{TGrpd} \arrow[l, shift right, "L\circ K"'] \\
& \mathbf{Set} \arrow[ul, "g"] \arrow[u, "n"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Top} \arrow[r, shift right, "J\circ M"'] & \mathbf{TGrpd} \arrow[l, shift right, "L\circ K"'] \\
& \mathbf{BHilb} \arrow[ul, "L"] \arrow[u, "J"']
\end{tikzcd}
\tag{9.4}$$

as well as their complete diagram of adjoint pairs:

$$\begin{tikzcd}
\mathbf{Top} \arrow[r, shift left, "M"] \arrow[d, shift right, "f"'] & \mathbf{BHilb} \arrow[l, shift left, "L"] \arrow[d, shift left, "J"] \\
\mathbf{Set} \arrow[r, shift left, "n"] \arrow[u, shift right, "g"'] & \mathbf{TGrpd} \arrow[l, shift left, "t"] \arrow[u, shift left, "K"]
\end{tikzcd}
\tag{9.5}$$

where the two natural transformations (in fact, not necessarily unique natural equivalences) involved in the adjoint situations are defined between the set-valued bifunctors *via* the families of mappings:

$$\eta_{B,A} : [(L \circ K)(B), A] \longrightarrow [B, (J \circ M)(A)]$$

and

$$\mu_{D,C} : [(L \circ K)(D), C] \longrightarrow [D, (J \circ M)(C)]$$

with A, C in Obj(**Top**) and B, D in Obj(**TGrpd**); further details and the notation employed here are consistent with Chapter V of [158]. Then, one obtains the following proposition as a direct consequence of the above constructions and Proposition 4.1 on page 126 of [158]:

Proposition 9.6. *In the adjoint situations* $(\eta; L \circ K, J \circ M; \mathbf{Top}, \mathbf{TGrpd})$ *and* $(\mu; L \circ K, J \circ M; \mathbf{Top}, \mathbf{TGrpd})$ *of categories and covariant functors defined in diagrams* (9.4) *and* (9.5) *there are respectively one-to-one correspondences* $\eta* : [L \circ K \circ n, g] \longrightarrow [J, J \circ M \circ g]$ *(which is natural in both* g *and* n*), and* $\mu* : [L \circ K \circ J, L] \longrightarrow [J, J \circ M \circ L]$ *(which is natural in both* J *and* L*).*

Remark 9.11. One readily notes that a similar adjointness result holds for **TGrpd** and **BHilb** that involves naturality in $\Im = J$, L and K, t, respectively. Moreover, the functor representations in diagram (9.5) have adjoint functors that in the case of quantum systems link extended quantum symmetries to quantum operator algebra on rigged Hilbert spaces and the locally compact topology of quantum groupoids, assumed here to be endowed with suitable Haar measure systems. In the case of quantum double groupoids a suitable, but rather elaborate, definition of a *double system of Haar measures* can also be introduced (private communication to the first author from Professor M.R. Buneci).

10 Conclusions and discussion

Extended quantum symmetries, recent quantum operator algebra (QOA) developments and also non-Abelian algebraic topology (NAAT) [46] results were here discussed with a view to physical applications in quantum field theories, general molecular and nuclear scattering theories, symmetry breaking, as well as supergravity/supersymmetry based on a *locally covariant* approach to general relativity theories in quantum gravity. Fundamental concepts of QOA and quantum algebraic topology (QAT), such as C^*-algebras, quantum groups, von Neumann/Hopf algebras, quantum supergroups, quantum groupoids, quantum groupoid/algebroid representations and so on, were here considered primarily with a view to their possible extensions and future applications in quantum field theories and beyond.

Recently published mathematical generalizations that represent extended quantum symmetries range from quantum group algebras and quantum superalgebras/quantum supergroups to quantum groupoids, and then further, to quantum topological/Lie groupoids/Lie algebroids [31, 38, 39] in dipole-dipole coupled quasi-particles/bosons in condensed matter (such as: paracrystals/noncrystalline materials/glasses/topologically ordered systems) and nuclear physics, as well as Hamiltonian algebroids and double algebroid/double groupoid/categorical representations in W_N-gravity and more general supergravity theories. We note that supersymmetry was also discussed previously within a different mathematical framework [78]. Several, algebraically simpler, representations of quantum spacetime than QAT have thus been proposed in terms of causal sets, quantized causal sets, and quantum toposes [173, 186, 65, 123, 117]. However, the consistency of such ‘quantum toposes’ with the real quantum logic is yet to be validated; the ‘quantum toposes’ that have been proposed so far are all clearly inconsistent with the Birkhoff–von Neumann quantum logic (see for example, [117]). An alternative, generalized Łukasiewicz topos (GLT) that may allow us avoid such major logical inconsistences with quantum logics has also been developed [20, 21, 22, 24, 101, 99]. We have suggested here several new applications of Grassmann–Hopf algebras/algebroids, graded 'Lie' algebroids, weak Hopf C^*-algebroids, quantum locally compact groupoids to interacting quasi-particle and many-particle quantum systems. These concepts lead to higher dimensional symmetries represented by double groupoids, as well as other higher dimensional algebraic topology structures [51, 162]; they also have potential applications to spacetime structure determination using higher dimensional algebra (HDA) tools and its powerful results to uncover universal, topological invariants of 'hidden' quantum symmetries. New, non-Abelian results may thus be obtained through *higher homotopy, generalized van Kampen theorems* [41, 48], Lie groupoids/algebroids and groupoid atlases, possibly with novel applications to quantum dynamics and local-to-global problems, as well as quantum logic algebras (QLA). Novel mathematical representations in the form of higher homotopy quantum field (HHQFT) and quantum non-Abelian algebraic topology (QNAT) theories have the potential to develop a self-consistent quantum-general relativity theory (QGRT) in the context of supersymmetry algebroids/supersymmetry/supergravity and metric superfields in the Planck limit of spacetime [24, 44]. Especially interesting in QGRT are global representations of fluctuating spacetime structures in the presence of intense, fluctuating quantum gravitational fields. The development of such mathematical representations of extended quantum symmetries and supersymmetry appears as a logical requirement for the unification of quantum field (and especially AQFT) with general relativity theories in QGRT *via* quantum supergravity and NAAT approaches to determining supersymmetry invariants of quantum spacetime geometry.

QNAT is also being applied to develop studies of *non-Abelian quantum Hall* liquids and other many-body quantum systems with topological order [213, 32, 214, 216].

In a subsequent report [25], we shall further consider the development of physical applications of NAAT [46] towards a quantum non-Abelian algebraic topology (QNAT) from the standpoints of the theory of categories-functors-natural equivalences, higher dimensional algebra, as well as quantum logics. This approach can also be further extended and applied to both quantum statistical mechanics and complex systems that exhibit broken symmetry and/or various degrees of topological order in both lower and higher dimensions.

A Appendix

A.1 Von Neumann algebras

Let $\mathcal{H}$ denote a complex (separable) Hilbert space. A *von Neumann algebra* $\mathcal{A}$ acting on $\mathcal{H}$ is a subset of the algebra of all bounded operators $\mathcal{L}(\mathcal{H})$ such that:

(1) $\mathcal{A}$ is closed under the adjoint operation (with the adjoint of an element T denoted by T^*).

(2) $\mathcal{A}$ equals its bicommutant, namely:

$$\mathcal{A} = \{A \in \mathcal{L}(\mathcal{H}) : \forall\, B \in \mathcal{L}(\mathcal{H}), \forall\, C \in \mathcal{A},\ (BC = CB) \Rightarrow (AB = BA)\}.$$

If one calls a *commutant* of a set $\mathcal{A}$ the special set of bounded operators on $\mathcal{L}(\mathcal{H})$ which commute with all elements in $\mathcal{A}$, then this second condition implies that the commutant of the commutant of $\mathcal{A}$ is again the set $\mathcal{A}$.

On the other hand, a von Neumann algebra $\mathcal{A}$ inherits a *unital* subalgebra from $\mathcal{L}(\mathcal{H})$, and according to the first condition in its definition $\mathcal{A}$ does indeed inherit a $*$-*subalgebra* structure, as further explained in the next section on C^*-algebras. Furthermore, one has a notable *Bicommutant Theorem* which states that *$\mathcal{A}$ is a von Neumann algebra if and only if $\mathcal{A}$ is a $*$-subalgebra of $\mathcal{L}(\mathcal{H})$, closed for the smallest topology defined by continuous maps* $(\xi, \eta) \longmapsto \langle A\xi, \eta\rangle$

for all $\langle A\xi, \eta\rangle$*, where* $\langle\cdot,\cdot\rangle$ *denotes the inner product defined on* $\mathcal{H}$. For a well-presented treatment of the geometry of the state spaces of quantum operator algebras, see e.g. [4]; the ring structure of operators in Hilbert spaces was considered in an early, classic paper by Gel'fand and Naimark [98].

A.2 Groupoids

Recall that a *groupoid* $\mathcal{G}$ is a small category in which all morphisms are invertible, and that has a set of objects $X = \mathrm{Ob}(\mathcal{G})$. Thus, a groupoid is a generalisation of a group, in the sense that it is a *generalized 'group with many identities'*, this being possible because its morphism composition – unlike that of a group – is, in general, only *partially defined* (as it is too in the case of abstract categories). One often writes $\mathcal{G}_x^y$ for the set of morphisms in $\mathcal{G}$ from x to y.

A.2.1 Topological groupoid: definition

As is well kown, a *topological groupoid* is just a groupoid *internal* to the category of topological spaces and continuous maps. Thus, a *topological groupoid* consists of a space G, a distinguished subspace $\mathsf{G}^{(0)} = \mathrm{Ob}(\mathsf{G}) \subset \mathsf{G}$, called *the space of objects* of G, together with maps

$$r, s: \ \mathsf{G} \underset{s}{\overset{r}{\rightrightarrows}} \mathsf{G}^{(0)}$$

called the *range* and *source maps* respectively, together with a law of composition

$$\circ: \ \mathsf{G}^{(2)} := \mathsf{G} \times_{\mathsf{G}^{(0)}} \mathsf{G} = \{(\gamma_1, \gamma_2) \in \mathsf{G} \times \mathsf{G}: \ s(\gamma_1) = r(\gamma_2)\} \longrightarrow \mathsf{G},$$

such that the following hold:

(1) $s(\gamma_1 \circ \gamma_2) = r(\gamma_2)$, $r(\gamma_1 \circ \gamma_2) = r(\gamma_1)$, for all $(\gamma_1, \gamma_2) \in \mathsf{G}^{(2)}$.

(2) $s(x) = r(x) = x$, for all $x \in \mathsf{G}^{(0)}$.

(3) $\gamma \circ s(\gamma) = \gamma$, $r(\gamma) \circ \gamma = \gamma$, for all $\gamma \in \mathsf{G}$.

(4) $(\gamma_1 \circ \gamma_2) \circ \gamma_3 = \gamma_1 \circ (\gamma_2 \circ \gamma_3)$.

(5) Each γ has a two-sided inverse γ^{-1} with $\gamma\gamma^{-1} = r(\gamma)$, $\gamma^{-1}\gamma = s(\gamma)$. Furthermore, only for topological groupoids the inverse map needs be continuous.

It is usual to call $\mathsf{G}^{(0)} = \mathrm{Ob}(\mathsf{G})$ *the set of objects* of G. For $u \in \mathrm{Ob}(\mathsf{G})$, the set of arrows $u \longrightarrow u$ forms a group G_u, called the *isotropy group of* G *at* u.

The notion of *internal groupoid* has proved significant in a number of fields, since groupoids generalise bundles of groups, group actions, and equivalence relations. For a further, detailed study of groupoids and topology we refer the reader to the recent textbook by Brown [42].

Examples of groupoids are often encountered; the following are just a few specialized groupoid structures:

(a) locally compact groups, transformation groups , and any group in general,

(b) equivalence relations,

(c) tangent bundles,

(d) the tangent groupoid,

(e) holonomy groupoids for foliations,

(f) Poisson groupoids, and

(g) graph groupoids.

As a simple, helpful example of a groupoid, consider the case (b) above of a groupoid whose morphisms are defined by the equivalence relation in an equivalence class or set. Thus, let R be an *equivalence relation* on a set X. Then R is a groupoid under the following operations: $(x, y)(y, z) = (x, z)$, $(x, y)^{-1} = (y, x)$. Here, $\mathcal{G}^0 = X$, (the diagonal of $X \times X$) and $r((x, y)) = x$, $s((x, y)) = y$.

Thus, $R^2 = \{((x, y), (y, z)) : (x, y), (y, z) \in R\}$. When $R = X \times X$, R is called a *trivial* groupoid. A special case of a trivial groupoid is $R = R_n = \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$. (So every i is equivalent to every j.) Identify $(i, j) \in R_n$ with the matrix unit e_{ij}. Then the groupoid R_n is just matrix multiplication except that we only multiply e_{ij}, e_{kl} when $k = j$, and $(e_{ij})^{-1} = e_{ji}$. We do not really lose anything by restricting the multiplication, since the pairs e_{ij}, e_{kl} excluded from groupoid multiplication just give the 0 product in normal algebra anyway. For a groupoid G_{lc} to be a *locally compact groupoid* means that G_{lc} is required to be a (second countable) *locally compact Hausdorff* space, and the product and also inversion maps are required to be *continuous*. Each G^u_{lc} as well as the unit space G^0_{lc} is closed in G_{lc}. What replaces the left Haar measure on G_{lc} is a system of measures λ^u ($u \in \mathsf{G}^0_{lc}$), where λ^u is a positive regular Borel measure on G^u_{lc} with dense support. In addition, the λ^u s are required to vary continuously (when integrated against $f \in C_c(\mathsf{G}_{lc})$ and to form an invariant family in the sense that for each x, the map $y \mapsto xy$ is a measure preserving homeomorphism from $\mathsf{G}^s_{lc}(x)$ onto $\mathsf{G}^r_{lc}(x)$. Such a system $\{\lambda^u\}$ is called a *left Haar system* for the locally compact groupoid G_{lc}.

This is defined more precisely next.

A.3 Haar systems for locally compact topological groupoids

Let

$$\mathsf{G}_{lc} \overset{r}{\underset{s}{\rightrightarrows}} \mathsf{G}^{(0)}_{lc} = X$$

be a locally compact, locally trivial topological groupoid with its transposition into transitive (connected) components. Recall that for $x \in X$, the *costar of* x denoted CO*(x) is defined as the closed set $\bigcup\{\mathsf{G}_{lc}(y, x) : y \in \mathsf{G}_{lc}\}$, whereby

$$\mathsf{G}_{lc}(x_0, y_0) \hookrightarrow \mathrm{CO}^*(\mathrm{x}) \longrightarrow \mathrm{X},$$

is a principal $\mathsf{G}_{lc}(x_0, y_0)$-bundle relative to fixed base points (x_0, y_0). Assuming all relevant sets are locally compact, then following [199], a *(left) Haar system on* G_{lc} denoted (G_{lc}, τ) (for later purposes), is defined to comprise of i) a measure κ on G_{lc}, ii) a measure μ on X and iii) a measure μ_x on $\mathrm{CO}^*(x)$ such that for every Baire set E of G_{lc}, the following hold on setting $E_x = E \cap \mathrm{CO}^*(x)$:

(1) $x \mapsto \mu_x(E_x)$ is measurable;

(2) $\kappa(E) = \int_x \mu_x(E_x)\, d\mu_x$;

(3) $\mu_z(tE_x) = \mu_x(E_x)$, for all $t \in \mathsf{G}_{lc}(x, z)$ and $x, z \in \mathsf{G}_{lc}$.

The presence of a left Haar system on G_{lc} has important topological implications: it requires that the range map $r : \mathsf{G}_{lc} \to \mathsf{G}^0_{lc}$ is open. For such a G_{lc} with a left Haar system, the vector space $C_c(\mathsf{G}_{lc})$ is a *convolution* $*$*-algebra*, where for $f, g \in C_c(\mathsf{G}_{lc})$:

$$f * g(x) = \int f(t)g(t^{-1}x)d\lambda^{r(x)}(t), \qquad \text{with} \quad f * (x) = \overline{f(x^{-1})}.$$

One has $C^*(\mathsf{G}_{lc})$ to be the *enveloping* C^**-algebra* of $C_c(\mathsf{G}_{lc})$ (and also representations are required to be continuous in the inductive limit topology). Equivalently, it is the completion of

$\pi_{\rm univ}(C_c(\mathsf{G}_{lc}))$ where $\pi_{\rm univ}$ is the *universal representation* of G_{lc}. For example, if $\mathsf{G}_{lc} = R_n$, then $C^*(\mathsf{G}_{lc})$ is just the finite dimensional algebra $C_c(\mathsf{G}_{lc}) = M_n$, the span of the e_{ij}'s.

There exists (viz. [179, pp. 91–92]) a *measurable Hilbert bundle* $(\mathsf{G}^0_{lc}, \mathcal{H}, \mu)$ with $\mathcal{H} = \left\{\mathcal{H}^u_{u\in \mathsf{G}^0_{lc}}\right\}$ and a G-representation L on $\mathcal{H}$ (see also [180, 181]). Then, for every pair ξ, η of square integrable sections of $\mathcal{H}$, it is required that the function $x \mapsto (L(x)\xi(s(x)), \eta(r(x)))$ be ν-measurable. The representation Φ of $C_c(\mathsf{G}_{lc})$ is then given by:

$$\langle \Phi(f)\xi|, \eta\rangle = \int f(x)(L(x)\xi(s(x)), \eta(r(x)))d\nu_0(x).$$

The triple $(\mu, \mathcal{H}, L)$ is called a *measurable* G_{lc}*-Hilbert bundle*.

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