Hypergeometric τ-Functions of the q-Painlevé System of Type $E_7^{(1)\star}$

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Abstract. We present the τ-functions for the hypergeometric solutions to the q-Painlevé system of type $E_7^{(1)}$ in a determinant formula whose entries are given by the basic hypergeometric function ${}_8W_7$. By using the $W(D_5)$ symmetry of the function ${}_8W_7$, we construct a set of twelve solutions and describe the action of $\widetilde{W}(D_6^{(1)})$ on the set.

Key words: q-Painlevé system; q-hypergeometric function; Weyl group; τ-function

2000 Mathematics Subject Classification: 33D15; 33D05; 33D60; 33E17

1 Introduction

A natural framework for discrete Painlevé equations by means of the geometry of rational surfaces has been proposed by Sakai [17]. Each equation is defined by the group of Cremona transformations on a family of surfaces obtained by blowing-up at nine points on $\mathbb{P}^2$. According to the types of rational surfaces, the discrete Painlevé equations are classified in terms of affine root systems. Also, their symmetries are described by means of affine Weyl groups, the lattice part of which gives rise to difference equations. For instance, the q-Painlevé system of type $E_7^{(1)}$, which is the main object of this paper, is a discrete dynamical system defined on a family of rational surfaces parameterized by nine-point configurations on $\mathbb{P}^2$ such that six points among them are on a conic and other three are on a line [17]. An explicit expression for the system of q-difference equations is given by [15]

$$\begin{aligned}
\frac{(f\overline{g}-\overline{t}t)(fg-t^2)}{(f\overline{g}-1)(fg-1)} &= \frac{(f-b_1t)(f-b_2t)(f-b_3t)(f-b_4t)}{(f-b_5)(f-b_6)(f-b_7)(f-b_8)},\\
\frac{(fg-t^2)(\underline{f}g-t\underline{t})}{(fg-1)(\underline{f}g-1)} &= \frac{\left(g-\frac{t}{b_1}\right)\left(g-\frac{t}{b_2}\right)\left(g-\frac{t}{b_3}\right)\left(g-\frac{t}{b_4}\right)}{\left(g-\frac{1}{b_5}\right)\left(g-\frac{1}{b_6}\right)\left(g-\frac{1}{b_7}\right)\left(g-\frac{1}{b_8}\right)},
\end{aligned}\tag{1.1}$$

where t is the independent variable and the time evolution of the dependent variables is given by $\overline{g}=g(qt)$ and $\underline{f}=f(t/q)$. The parameters b_i $(i=1,2,\dots,8)$ satisfy $b_1b_2b_3b_4=q$ and $b_5b_6b_7b_8=1$.

Similarly to the Painlevé differential equations, the discrete Painlevé equations admit particular solutions expressible in terms of various hypergeometric functions. Regarding the q-difference Painlevé equations, the hypergeometric solutions to those equations have been constructed by means of a geometric approach and direct linearization of the q-difference Riccati

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equations [9, 10]. In particular, the Riccati solution to the system of q-difference equations (1.1) is expressed in terms of the q-hypergeometric series

$$
\begin{aligned}
{}_8W_7(a_0;a_1,\ldots,a_5;q,z) &= {}_8\varphi_7\left(\begin{matrix} a_0, qa_0^{1/2}, -qa_0^{1/2}, a_1\cdots, a_5 \\ a_0^{1/2}, -a_0^{1/2}, qa_0/a_1,\cdots, qa_0/a_5 \end{matrix}\, ;q,z\right) \\
&= \sum_{k=0}^{\infty} \frac{(1-a_0q^{2k})}{(1-a_0)} \frac{(a_0;q)_k}{(q;q)_k} \prod_{i=1}^{5} \frac{(a_i;q)_k}{(qa_0/a_i;q)_k} z^k, \qquad z = \frac{q^2a_0^2}{a_1a_2a_3a_4a_5},
\end{aligned}
\tag{1.2}
$$

where $(a;q)_k = \prod_{i=0}^{k-1}(1-aq^i)$.

The purposes of this paper are to propose a formulation for the q-Painlevé system of type $E_7^{(1)}$ by means of the lattice τ-functions and to completely determine the τ-functions for the hypergeometric solutions (hypergeometric τ-functions for short) of the system.

This paper is organized as follows. In Section 2, we give a formulation for the q-Painlvé system of type $E_7^{(1)}$ in terms of the lattice τ-functions. Section 3 is devoted to a preparation for constructing the hypergeometric τ-functions. We decompose the lattice, each of whose elements indicates the τ-function, into a family of six-dimensional lattices.

In Sections 4–6, we construct the hypergeometric τ-functions. We find that a q-analogue of the double gamma function appears as a normalization factor of the hypergeometric τ-functions in Section 4. In Section 5, we find that a class of bilinear equations for the lattice τ-functions yields the contiguity relations for the q-hypergeometric function ${}_8W_7$. As is well-known, the q-hypergeometric function ${}_8W_7$ possesses the $W(D_5)$-symmetry [13]. From that, we can construct a set of twelve solutions corresponding to the coset $W(D_6)/W(D_5)$, and describe the action of $\widetilde{W}\big(D_6^{(1)}\big)$ on the set of solutions.

One of the important features of the hypergeometric solutions to the continuous and discrete Painlevé equations is that they can be expressed in terms of Wronskians or Casorati determinants [4, 12, 7, 3, 16]. In Section 6, we show that the hypergeometric τ-functions of the q-Painlevé system of type $E_7^{(1)}$ are expressed by "two-directional Casorati determinants". As a consequence, we get an explicit expression for the hypergeometric solutions to the q-difference Painlevé equation (1.1), which is proposed in Corollary 6.1.

2 The q-Painlevé system of type $E_7^{(1)}$

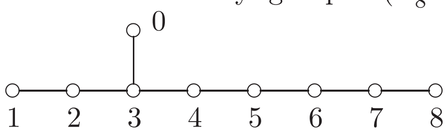
2.1 The discrete Painlevé system of type $E_8^{(1)}$

At first, we give a brief review of the formulation for the discrete Painlevé system of type $E_8^{(1)}$ in terms of the lattice τ-functions [8, 11].

Let $\mathcal{L} = \bigoplus_{i=0}^{9} \mathbb{Z}\mathrm{e}_i$ be a lattice with a basis $\{\mathrm{e}_0, \mathrm{e}_1, \ldots, \mathrm{e}_9\}$, and define a symmetric bilinear form $\langle\ ,\ \rangle : \mathcal{L}\times\mathcal{L} \to \mathbb{Z}$ by

$$
\langle \mathrm{e}_0, \mathrm{e}_0\rangle = -1, \qquad \langle \mathrm{e}_i, \mathrm{e}_i\rangle = 1 \quad (i=1,2,\ldots,9), \qquad \langle \mathrm{e}_i, \mathrm{e}_j\rangle = 0 \quad (i,j=0,1,\ldots,9;\ i\neq j).
$$

Consider the affine Weyl group $W(E_8^{(1)}) = \langle s_0, s_1, \ldots, s_8\rangle$ associated with the Dynkin diagram



The lattice $\mathcal{L}$ admits a natural linear action of $W(E_8^{(1)})$ defined by $s_i\,.\,\Lambda = \Lambda - \langle h_i, \Lambda\rangle h_i$ for $\Lambda \in \mathcal{L}$, where h_i $(i=0,1,\ldots,8)$ are the simple coroots defined by $h_0 = \mathrm{e}_0 - \mathrm{e}_1 - \mathrm{e}_2 - \mathrm{e}_3$ and

$h_i = \mathrm{e}_i - \mathrm{e}_{i+1}$ $(i = 1, \dots, 8)$. The canonical central element $c = 3\mathrm{e}_0 - \mathrm{e}_1 - \cdots - \mathrm{e}_9$ is orthogonal to all the simple coroots h_i, and hence $W(E_8^{(1)})$-invariant.

The parameter space for the discrete Painlevé system of type $E_8^{(1)}$ is the ten-dimensional vector space $\bigoplus_{i=0}^{9} \mathbb{C}\mathrm{e}_i$, whose coordinates are denoted by $\varepsilon_i = \langle \mathrm{e}_i, \cdot \rangle$ $(i = 0, 1, \dots, 9)$. The root lattice $Q(E_8^{(1)}) = \bigoplus_{i=0}^{8} \mathbb{Z}\alpha_i$ is generated by the simple roots $\alpha_0 = \varepsilon_0 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3$ and $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(i = 1, \dots, 8)$. The affine Weyl group $W(E_8^{(1)})$ acts on the coordinate function ε_i in a similar way to on the basis e_i. The $W(E_8^{(1)})$-invariant element corresponding to c is given by $\delta = \langle c, \cdot \rangle = 3\varepsilon_0 - \varepsilon_1 - \cdots - \varepsilon_9$, which is called the null root and plays the role of the scaling constant for difference equations in the context of the discrete Painlevé equations. For simplicity, we denote the reflection s_α with respect to the root $\alpha = \varepsilon_{ij} = \varepsilon_i - \varepsilon_j$ or $\alpha = \varepsilon_{ijk} = \varepsilon_0 - \varepsilon_i - \varepsilon_j - \varepsilon_k$ for $i, j, k \in \{1, 2, \dots, 9\}$ by s_{ij} or s_{ijk}, respectively. Also, we often use the notation $\mathrm{e}_{ij} = \mathrm{e}_i - \mathrm{e}_j$ and $\mathrm{e}_{ijk} = \mathrm{e}_0 - \mathrm{e}_i - \mathrm{e}_j - \mathrm{e}_k$.

For each $\alpha \in Q\big(E_8^{(1)}\big)$, the action of the translation operator $T_\alpha \in W\big(E_8^{(1)}\big)$ is given by [6]

$$T_\alpha(\Lambda) = \Lambda + \langle c, \Lambda \rangle\, h - \left(\frac{1}{2} \langle h, h \rangle \langle c, \Lambda \rangle + \langle h, \Lambda \rangle \right) c \qquad (\Lambda \in \mathcal{L}) \tag{2.1}$$

by using the element $h \in \mathcal{L}$ such that $\alpha = \langle h, \cdot \rangle$. Note that we have $T_\alpha T_\beta = T_\beta T_\alpha$ and $w T_\alpha w^{-1} = T_{w.\alpha}$ for any $w \in W\big(E_8^{(1)}\big)$. When $\alpha = \varepsilon_{ij}$ or ε_{ijk}, we also denote the translation T_α simply by T_{ij} or T_{ijk}, respectively. They can be expressed by

$$\begin{aligned} T_{ij} &= s_{il_1l_2} s_{il_3l_4} s_{l_5l_6l_7} s_{il_3l_4} s_{il_1l_2} s_{ij}, \qquad && \{i, j, l_1, \dots, l_7\} = \{1, 2, \dots, 9\}, \\ T_{ijk} &= s_{l_1l_2l_3} s_{l_4l_5l_6} s_{l_1l_2l_3} s_{ijk}, && \{i, j, k, l_1, \dots, l_6\} = \{1, 2, \dots, 9\}. \end{aligned}$$

Let us introduce a family of dependent variables $\tau_\Lambda = \tau_\Lambda(\varepsilon)$, $\varepsilon = (\varepsilon_0, \dots, \varepsilon_9)$, indexed by $\Lambda \in M$, where M is the $W(E_8^{(1)})$-orbit defined by

$$M = W\big(E_8^{(1)}\big) . \mathrm{e}_1 = \{\Lambda \in \mathcal{L} \,|\, \langle c, \Lambda \rangle = -1, \langle \Lambda, \Lambda \rangle = 1\} \subset \mathcal{L}.$$

The action of $W\big(E_8^{(1)}\big)$ on the lattice τ-functions τ_Λ is defined by $w(\tau_\Lambda) = \tau_{w.\Lambda}$ for any $w \in W\big(E_8^{(1)}\big)$. The discrete Painlevé system of type $E_8^{(1)}$ is equivalent to the overdetermined system defined by the bilinear equations

$$[\varepsilon_{jk}][\varepsilon_{jkl}]\tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_l} + [\varepsilon_{ki}][\varepsilon_{kil}]\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_l} + [\varepsilon_{ij}][\varepsilon_{ijl}]\tau_{\mathrm{e}_k}\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_l} = 0$$

for any mutually distinct indices $i, j, k, l \in \{1, 2, \dots, 9\}$, as well as their $W\big(E_8^{(1)}\big)$-transforms

$$[w(\varepsilon_{jk})][w(\varepsilon_{jkl})]\tau_{w.\mathrm{e}_i}\tau_{w.(\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_l)} + (i, j, k)\text{-cyclic} = 0$$

for any $w \in W\big(E_8^{(1)}\big)$. Here, $[x]$ is a nonzero odd holomorphic function on $\mathbb{C}$ satisfying the Riemann relation

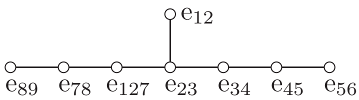
$$[x + y][x - y][u + v][u - v] = [x + u][x - u][y + v][y - v] - [x + v][x - v][y + u][y - u]$$

for any $x, y, u, v \in \mathbb{C}$. There are three classes of such functions; elliptic, trigonometric and rational. These three cases correspond to the three types of difference equations, namely, elliptic difference, q-difference and ordinal difference, respectively. The lattice part of $W\big(E_8^{(1)}\big)$ gives rise to the difference Painlevé equation.

2.2 The q-Painlevé system of type $E_7^{(1)}$

Let us propose a formulation for the q-Painlevé system of type $E_7^{(1)}$ by means of the lattice τ-functions, using by the notation introduced in the previous subsection. A derivation of the formulation is discussed in Appendices.

The q-Painlevé system of type $E_7^{(1)}$ is a discrete dynamical system defined on a family of rational surfaces parameterized by nine-point configurations on $\mathbb{P}^2$ such that six points among them are on a conic C and other three are on a line L [17]. Here, we set $p_1, p_2, p_3, p_4, p_5, p_6 \in C$ and $p_7, p_8, p_9 \in L$. In what follows, the symbols C and L also mean the index sets $C = \{1, 2, 3, 4, 5, 6\}$ and $L = \{7, 8, 9\}$, respectively. And we often use $i, j, k, \ldots$ and r, s as the elements of C and L, respectively. In this setting, the symmetric groups $\mathfrak{S}_6^C = \langle s_{12}, \ldots, s_{56}\rangle$ and $\mathfrak{S}_3^L = \langle s_{78}, s_{89}\rangle$ naturally act on the configuration space as the permutation of the points on C and L, respectively. Also, the standard Cremona transformation with respect to (p_1, p_2, p_7) is well-defined as a birational action on the space. They generate the affine Weyl group $W(E_7^{(1)}) = \langle s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{78}, s_{89}, s_{127}\rangle$. The associated Dynkin diagram and its automorphism are realized by



and $\pi = s_{123}s_{47}s_{58}s_{69}$, respectively. Thus we find that the extended affine Weyl group $\widetilde{W}(E_7^{(1)}) = \langle s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{78}, s_{89}, s_{127}, \pi\rangle$ acts on the configuration space.

The lattice τ-functions $\tau_\Lambda = \tau_\Lambda(\varepsilon)$ for the q-Painlevé system of type $E_7^{(1)}$ are indexed by

$$\Lambda \in M^{E_7} = \widetilde{W}(E_7^{(1)}) \,.\, \mathrm{e}_1 = M^C \coprod M^L,$$

where

$$M^C = \{\Lambda \in M \,|\, \langle \mathrm{e}_{789}, \Lambda\rangle = 0\} = W(E_7^{(1)}) \,.\, \mathrm{e}_1,$$
$$M^L = \{\Lambda \in M \,|\, \langle \mathrm{e}_{789}, \Lambda\rangle = -1\} = W(E_7^{(1)}) \,.\, \mathrm{e}_7.$$

The action of $\widetilde{W}(E_7^{(1)})$ on the lattice τ-functions τ_Λ is defined by $w(\tau_\Lambda) = \tau_{w.\Lambda}$ for any $w \in \widetilde{W}(E_7^{(1)})$. The q-Painlevé system of type $E_7^{(1)}$ is equivalent to the overdetermined system defined by the bilinear equations

$$\begin{aligned}
&[\varepsilon_{rs}]\tau^C_{\mathrm{e}_j}\tau^L_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j} = [\varepsilon_{ijs}]\tau^L_{\mathrm{e}_r}\tau^C_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_r} - [\varepsilon_{ijr}]\tau^L_{\mathrm{e}_s}\tau^C_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_s},\\
&[\varepsilon_{jk}]\tau^L_{\mathrm{e}_r}\tau^C_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_r} = [\varepsilon_{ikr}]\tau^C_{\mathrm{e}_j}\tau^L_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j} - [\varepsilon_{ijr}]\tau^C_{\mathrm{e}_k}\tau^L_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_k},
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
&[\varepsilon_{ij}][\varepsilon_{ijr}]\tau^C_{\mathrm{e}_k}\tau^C_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_r} + (i,j,k)\text{-cyclic} = 0,\\
&[\varepsilon_{ij}][\varepsilon_{kl}]\tau^L_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j}\tau^L_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_l} + (i,j,k)\text{-cyclic} = 0,
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
&\tau^C_{\mathrm{e}_i}\tau^C_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_9} - \tau^C_{\mathrm{e}_j}\tau^C_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_9} + [\varepsilon_{ij}][\varepsilon_{ij9}]\, d_L \tau^L_{\mathrm{e}_7}\tau^L_{\mathrm{e}_8} = 0,\\
&\tau^L_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_4}\tau^L_{\mathrm{e}_0-\mathrm{e}_2-\mathrm{e}_3} - \tau^L_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_3}\tau^L_{\mathrm{e}_0-\mathrm{e}_2-\mathrm{e}_4} + [\varepsilon_{12}][\varepsilon_{34}]\, d_C \tau^C_{\mathrm{e}_5}\tau^C_{\mathrm{e}_6} = 0
\end{aligned} \tag{2.4}$$

for mutually distinct indices $i, j, k, l \in C$ and $r, s \in L$, as well as their $\widetilde{W}(E_7^{(1)})$-transforms. The superscript C (resp. L) denotes that the τ-function is indexed by $\Lambda \in M^C$ (resp. $\Lambda \in M^L$), and we leave it out when it is unnecessarily. It is possible to fix the function $[x]$ as $[x] = e(\frac{1}{2}x) - e(-\frac{1}{2}x)$, $e(x) = e^{\pi\sqrt{-1}x}$, without loss of generality. The factors d_L and d_C in (2.4) correspond to the irreducible components of the anti-canonical devisor $\mathcal{D}_L = \mathrm{e}_0 - \mathrm{e}_7 - \mathrm{e}_8 - \mathrm{e}_9$ and $\mathcal{D}_C = 2\mathrm{e}_0 - \mathrm{e}_1 - \cdots - \mathrm{e}_6$, respectively. These factors are $W(E_7^{(1)})$-invariant and the action of π is given by $\pi : d_L \leftrightarrow d_C$.

The translation operators with respect to the root vectors $\varepsilon_{ij}, \varepsilon_{rs}, \varepsilon_{ijr} \in Q\big(E_7^{(1)}\big)$ are denoted by T_{ij}, T_{rs} and T_{ijr}, respectively. Also, there exist fifty six translation operators that move a lattice point $\Lambda \in M^{E_7}$ to its nearest ones. Let us denote such operators by $\widetilde{T}_{ir}$, $\widetilde{T}_{ijk}$ and $\widetilde{T}_{irs}$ according to the action on $Q\big(E_7^{(1)}\big)$;

$$\begin{aligned} \widetilde{T}_{17}: &\quad \varepsilon_{78} \mapsto \varepsilon_{78} + \delta, &\quad \varepsilon_{12} \mapsto \varepsilon_{12} - \delta, \\ \widetilde{T}_{123}: &\quad \varepsilon_{127} \mapsto \varepsilon_{127} - \delta, &\quad \varepsilon_{34} \mapsto \varepsilon_{34} + \delta, \end{aligned}$$

for instance. We find that these operators can be realized as $\widetilde{T}_{ir} = T_{ir} s_{789}$, $\widetilde{T}_{ijk} = s_{789} T_{ijk}$ and $\widetilde{T}_{irs} = T_{irs} s_{789}$, respectively, in terms of the Weyl group $W\big(E_8^{(1)}\big)$. Then, from the formula (2.1), the action on a lattice point can be calculated as $\widetilde{T}_{17}(\mathrm{e}_9) = T_{17}(\mathrm{e}_0 - \mathrm{e}_7 - \mathrm{e}_8) = \mathrm{e}_0 - \mathrm{e}_1 - \mathrm{e}_8$, for example. Note that we have the relations such as $\widetilde{T}_{19}\widetilde{T}_{178} = 1$ and $\widetilde{T}_{123}\widetilde{T}_{456} = 1$. The translation operators with respect to the root vectors can be expressed by $T_{ij} = \widetilde{T}_{ir}\widetilde{T}_{jr}^{-1}$, $T_{rs} = \widetilde{T}_{ir}^{-1}\widetilde{T}_{is}$ and $T_{ijr} = \widetilde{T}_{ijk}\widetilde{T}_{kr}$.

Proposition 2.1. *If the lattice τ-functions τ_Λ $(\Lambda \in M^{E_7})$ satisfy the bilinear equations* (2.2) *and their $\widetilde{W}\big(E_7^{(1)}\big)$-transforms, then they also satisfy* (2.3) *and their $\widetilde{W}\big(E_7^{(1)}\big)$-transforms.*

This is easily verified by a direct calculation. From this proposition, we see that it is not necessary to consider the bilinear equations (2.3) for constructing a solution to the q-Painlevé system of type $E_7^{(1)}$. However, as we will see Section 6, we use the bilinear equations of type (2.3) in order to get a nicer determinant formula for the hypergeometric τ-functions. Then, we will treat all types of bilinear equations below, although the discussion becomes technically complicated as a consequence.

Let us introduce the dependent variables f and g by

$$\begin{aligned} f &= e\left(\tfrac{1}{8}\alpha_l - \tfrac{1}{8}\alpha_r + \tfrac{1}{4}\varepsilon_{12} + \tfrac{1}{8}\delta\right) \frac{e(\frac{1}{4}\varepsilon_{13})\tau_{\mathrm{e}_1}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2} - e(-\frac{1}{4}\varepsilon_{13})\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_0-\mathrm{e}_2-\mathrm{e}_3}}{e(\frac{1}{4}\varepsilon_{13})\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_0-\mathrm{e}_2-\mathrm{e}_3} - e(-\frac{1}{4}\varepsilon_{13})\tau_{\mathrm{e}_1}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}}, \\ g &= e\left(\tfrac{1}{8}\alpha_r - \tfrac{1}{8}\alpha_l + \tfrac{1}{4}\varepsilon_{12} - \tfrac{1}{8}\delta\right) \frac{e(\frac{1}{4}\varepsilon_{23})\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_3} - e(-\frac{1}{4}\varepsilon_{23})\tau_{\mathrm{e}_2}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}}{e(\frac{1}{4}\varepsilon_{23})\tau_{\mathrm{e}_2}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2} - e(-\frac{1}{4}\varepsilon_{23})\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_3}} \end{aligned}$$

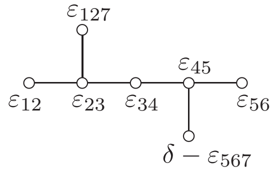
with $\alpha_l = 3\varepsilon_{127} + 2\varepsilon_{78} + \varepsilon_{89}$ and $\alpha_r = 3\varepsilon_{34} + 2\varepsilon_{45} + \varepsilon_{56}$. Then, one get the explicit expression for the q-difference equations (1.1), a derivation of which is discussed in Appendix C.

3 A family of six-dimensional lattices and the bilinear equations

As a preparation for constructing the hypergeometric τ-functions, we decompose the lattice M^{E_7} into a family of six-dimensional lattices according to the value of the symmetric bilinear form with the coroot vector $\mathrm{e}_{89} = \mathrm{e}_8 - \mathrm{e}_9$;

$$M^{E_7} = \coprod_{n\in\mathbb{Z}} M_n, \qquad M_n = \big\{\Lambda \in M^{E_7} \,|\, \langle \Lambda, \mathrm{e}_{89}\rangle = n\big\}.$$

Parallel to this decomposition, let us consider the orthogonal complement of ε_{89} in the root lattice $Q\big(E_7^{(1)}\big)$. Then we get the root lattice $Q\big(D_6^{(1)}\big)$ corresponding to the Dynkin diagram



Since we have $\varepsilon_{127}+\varepsilon_{12}+2\varepsilon_{23}+2\varepsilon_{34}+2\varepsilon_{45}+\varepsilon_{56}+(\delta-\varepsilon_{567})=\delta$, the same δ denotes the null root of $Q\big(D_6^{(1)}\big)$. The corresponding simple reflections generate the affine Weyl group $W\big(D_6^{(1)}\big)=\langle s_{127},s_{12},s_{23},s_{34},s_{45},s_{56},s_{\delta-\varepsilon_{567}}\rangle$. Note that the finite Weyl group $W(D_6)=\langle s_{127},s_{12},s_{23},s_{34},s_{45},s_{56}\rangle$ includes the symmetric group $\mathfrak{S}_6=\langle s_{12},s_{23},s_{34},s_{45},s_{56}\rangle$ as a subgroup. In this realization, an automorphism of the above Dynkin diagram can be expressed by $\rho=\pi s_{157}s_{168}s_{24}s_{26}s_{35}s_{79}$ whose action on the simple roots of $Q\big(D_6^{(1)}\big)$ is given by

$$\rho:\quad \varepsilon_{12}\leftrightarrow\delta-\varepsilon_{567},\qquad \varepsilon_{127}\leftrightarrow\varepsilon_{56},\qquad \varepsilon_{23}\leftrightarrow\varepsilon_{45}.$$

The extended affine Weyl group $\widetilde{W}\big(D_6^{(1)}\big)=\langle s_{127},s_{12},s_{23},s_{34},s_{45},s_{56},s_{\delta-\varepsilon_{567}},\rho\rangle$ acts transitively on each M_n. Regarding the translation operators, we have $\widetilde{T}_{i7},\widetilde{T}_{ijk}\in\widetilde{W}\big(D_6^{(1)}\big)$ for $i,j,k\in C=\{1,2,\ldots,6\}$, which can be expressed in the form $\widetilde{T}_\alpha=\rho w$, $w\in W\big(D_6^{(1)}\big)$.

According to the location of the lattice τ-functions, one can classify the bilinear equations (2.2) into the following four types:

$(\mathrm{A})_n$: Two on each of M_{n-1}, M_n and M_{n+1}, respectively.
$(\mathrm{B})_n$: Four on M_n, and one on M_{n+1} and M_{n-1}, respectively.
$(\mathrm{C})_n$: Three on M_{n+1} and M_n, respectively.
$(\mathrm{D})_n$: Six on M_n.

The bilinear equations of type $(\mathrm{C})_n$ are further classified into two types. The first one is that all of three τ-functions on M_{n+1} belong to M^C (or M^L), which is denoted by $(\mathrm{C})_n^{\mathrm{r}}$. The second is that one of three τ-functions on M_{n+1} belongs to M^C (or M^L), denoted by $(\mathrm{C})_n^{\mathrm{i}}$. Typical bilinear equations are given by

$$\begin{aligned}
&(\mathrm{A})_0 && [\varepsilon_{89}]\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j}=[\varepsilon_{ij9}]\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_8}-[\varepsilon_{ij8}]\tau_{\mathrm{e}_9}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_9},\\
&(\mathrm{B})_0 && [\varepsilon_{78}]\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j}=[\varepsilon_{ij8}]\tau_{\mathrm{e}_7}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_7}-[\varepsilon_{ij7}]\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_8},\\
&(\mathrm{C})_0^{\mathrm{i}} && [\varepsilon_{jk}]\tau_{\mathrm{e}_8}\tau_{2\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j-\mathrm{e}_k-\mathrm{e}_8-\mathrm{e}_9}=[\varepsilon_{ik8}]\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_9}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j}-[\varepsilon_{ij8}]\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_9}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_k},\\
&(\mathrm{C})_0^{\mathrm{r}} && [\varepsilon_{ij}]\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_9}+(i,j,k)\text{-cyclic}=0,\\
&(\mathrm{D})_0 && [\varepsilon_{jk}]\tau_{\mathrm{e}_7}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_7}=[\varepsilon_{ik7}]\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j}-[\varepsilon_{ij7}]\tau_{\mathrm{e}_k}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_k}
\end{aligned}\tag{3.1}$$

for mutually distinct indices $i,j,k\in C$. The bilinear equations (2.3) are also classified in a similar way into four types, each of which we denote by $(\mathrm{A})'_n$, $(\mathrm{B})'_n$, $(\mathrm{C})'_n$ and $(\mathrm{D})'_n$ to distinguish them from the bilinear equations (2.2). Typical equations are given by

$$\begin{aligned}
&(\mathrm{A})'_0 && [\varepsilon_{78}][\delta-\varepsilon_{569}]\tau_{\mathrm{e}_9}\tau_{2\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2-\mathrm{e}_3-\mathrm{e}_4-\mathrm{e}_9}+(7,8,9)\text{-cyclic}=0,\\
&(\mathrm{B})'_0 && [\varepsilon_{ij}][\varepsilon_{kl}]\tau_{\mathrm{e}_8}\tau_{2\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j-\mathrm{e}_k-\mathrm{e}_l-\mathrm{e}_8}\\
& && \qquad=[\varepsilon_{il8}][\varepsilon_{jk8}]\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_k}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_l}-[\varepsilon_{jl8}][\varepsilon_{ik8}]\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_k}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_l},\\
&(\mathrm{C})'_0 && [\varepsilon_{ij}][\varepsilon_{ij9}]\tau_{\mathrm{e}_k}\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_9}+(i,j,k)\text{-cyclic}=0,\\
&(\mathrm{D})'_0 && [\varepsilon_{ij}][\varepsilon_{kl}]\tau_{ij}\tau_{kl}+(i,j,k)\text{-cyclic}=0,\qquad [\varepsilon_{ij}][\varepsilon_{ij7}]\tau_k\tau_{k7}+(i,j,k)\text{-cyclic}=0
\end{aligned}\tag{3.2}$$

for mutually distinct indices $i,j,k,l\in C$. The bilinear equations (2.4) are also classified into the type $(\mathrm{A})_n^{\mathrm{d}}$, $(\mathrm{B})_n^{\mathrm{d}}$, $(\mathrm{C})_n^{\mathrm{d}}$ and $(\mathrm{D})_n^{\mathrm{d}}$. Typical equations are given by

$$\begin{aligned}
&(\mathrm{A})_0^{\mathrm{d}} && \tau_{\mathrm{e}_8}\tau_{2\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2-\mathrm{e}_3-\mathrm{e}_4-\mathrm{e}_8}-\tau_{\mathrm{e}_9}\tau_{2\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2-\mathrm{e}_3-\mathrm{e}_4-\mathrm{e}_9}+[\delta-\varepsilon_{567}][\varepsilon_{89}]\,d_C\tau_{\mathrm{e}_5}\tau_{\mathrm{e}_6}=0,\\
&(\mathrm{B})_0^{\mathrm{d}} && \tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_7}-\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_7}+[\varepsilon_{ij}][\varepsilon_{ij7}]\,d_L\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_9}=0,\\
& && \tau_{\mathrm{e}_8}\tau_{2\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2-\mathrm{e}_3-\mathrm{e}_4-\mathrm{e}_8}-\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}\tau_{\mathrm{e}_0-\mathrm{e}_3-\mathrm{e}_4}-[\varepsilon_{128}][\varepsilon_{348}]\,d_C\tau_{\mathrm{e}_5}\tau_{\mathrm{e}_6}=0,\\
&(\mathrm{C})_0^{\mathrm{d}} && \tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_9}-\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_9}+[\varepsilon_{ij}][\varepsilon_{ij9}]\,d_L\tau_{\mathrm{e}_7}\tau_{\mathrm{e}_8}=0,\\
&(\mathrm{D})_0^{\mathrm{d}} && \tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_4}\tau_{\mathrm{e}_0-\mathrm{e}_2-\mathrm{e}_3}-\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_3}\tau_{\mathrm{e}_0-\mathrm{e}_2-\mathrm{e}_4}+[\varepsilon_{12}][\varepsilon_{34}]\,d_C\tau_{\mathrm{e}_5}\tau_{\mathrm{e}_6}=0
\end{aligned}\tag{3.3}$$

for mutually distinct indices $i,j\in C$.

Lemma 3.1. *Any bilinear equation of type* $(\mathrm{A})_0$ *can be obtained by an action of* $\widetilde{W}\big(D_6^{(1)}\big)$ *on the first equation of* (3.1). *Also, we have a similar situation regarding each case of type* $(\mathrm{B})_0$, $(\mathrm{C})_0^{\mathrm{r}}$, $(\mathrm{C})_0^{\mathrm{i}}$ *and* $(\mathrm{D})_0$, *respectively.*

Proof. Any lattice τ-function on M_1 can be transformed to τ_{e_8} by an action of $\widetilde{W}\big(D_6^{(1)}\big)$. Searching for $\Lambda \in M_{-1}$ such that $\langle \Lambda + \mathrm{e}_8, \Lambda + \mathrm{e}_8 \rangle = 0$, we find that the lattice τ-functions on M_{-1} which can pair with τ_{e_8} are $\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_8}, \tau_{2\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j-\mathrm{e}_k-\mathrm{e}_7-\mathrm{e}_8}$ and $\tau_{c+\mathrm{e}_i+\mathrm{e}_9-\mathrm{e}_7}$ for mutually distinct indices $i, j, k \in C$. Any of them can be transformed to $\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_8}$ by an action of $W(D_6)$. Since τ_{e_8} is invariant under the action of $W(D_6)$, we find that one of the pairs of the lattice τ-functions in a bilinear equation of type $(\mathrm{A})_0$ can be transformed to $\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_8}$ by an action of $\widetilde{W}\big(D_6^{(1)}\big)$. Note that three pairs of the lattice τ-functions in a bilinear equation have a common barycenter. Therefore, the bilinear equations of type $(\mathrm{A})_0$ including the term $\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_8}$ are reduced to

$$[\varepsilon_{89}]\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j} = [\varepsilon_{ij9}]\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_8} - [\varepsilon_{ij8}]\tau_{\mathrm{e}_9}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_9},$$
$$[\varepsilon_{89}]\tau_{\mathrm{e}_7}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_7} = [\varepsilon_{79}]\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_8} - [\varepsilon_{78}]\tau_{\mathrm{e}_9}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_9},$$

which are transformed by the action of the Dynkin diagram automorphism $\rho \in \widetilde{W}\big(D_6^{(1)}\big)$ to each other. The proof for the other types of bilinear equations is given in a similar way. ■

From this lemma and similar consideration for the bilinear equations (3.2) and (3.3), we immediately get the following proposition.

Proposition 3.1. *Fix* $n \in \mathbb{Z}$.

1. *All the bilinear equations of type* $(\mathrm{A})_n$ *can be transformed by the action of* $\widetilde{W}\big(D_6^{(1)}\big)$ *to one another. Also, we have a similar situation regarding each case of type* $(\mathrm{B})_n$, $(\mathrm{C})_n^{\mathrm{i}}$, $(\mathrm{C})_n^{\mathrm{r}}$ *and* $(\mathrm{D})_n$, *respectively.*
2. *All the bilinear equations of type* $(\mathrm{A})_n'$ *can be transformed by the action of* $\widetilde{W}\big(D_6^{(1)}\big)$ *to one another. Also, we have a similar situation regarding each case of type* $(\mathrm{B})_n'$ *and* $(\mathrm{C})_n'$, *respectively. The set of all the bilinear equations of type* $(\mathrm{D})_n'$ *is decomposed to two orbits by the action of* $\widetilde{W}\big(D_6^{(1)}\big)$.
3. *All the bilinear equations of type* $(\mathrm{A})_n^{\mathrm{d}}$ *can be transformed by the action of* $\widetilde{W}\big(D_6^{(1)}\big)$ *to one another. Also, we have a similar situation regarding each case of type* $(\mathrm{C})_n^{\mathrm{d}}$ *and* $(\mathrm{D})_n^{\mathrm{d}}$, *respectively. The set of all the bilinear equations of type* $(\mathrm{B})_n^{\mathrm{d}}$ *is decomposed to two orbits by the action of* $\widetilde{W}\big(D_6^{(1)}\big)$.

Let us discuss the relationships among the above types of bilinear equations.

Proposition 3.2. *If the lattice* τ*-functions satisfy all the bilinear equations of type* $(\mathrm{B})_n$, *then they also satisfy those of type* $(\mathrm{A})_n$; *that is,*

1. $(\mathrm{B})_n \;\Rightarrow\; (\mathrm{A})_n$.

Similarly, we have

2. $(\mathrm{C})_n^{\mathrm{i}} \;\Rightarrow\; (\mathrm{C})_n^{\mathrm{r}}$.

Moreover, if $\tau_\Lambda \neq 0$ *for* $\Lambda \in M_{n-1}$, *we have the following:*

3. $(\mathrm{C})_{n-1}^{\mathrm{i}} \;\Rightarrow\; (\mathrm{D})_n$.
4. $(\mathrm{A})_n$, $(\mathrm{C})_{n-1}^{\mathrm{i}} \;\Rightarrow\; (\mathrm{C})_n^{\mathrm{i}}$.

Proof. It is sufficient to verify the statement for a certain $n \in \mathbb{Z}$. The first and second statements are easily verified for the case of $n = 0$.

3. $(\mathrm{C})^{\mathrm{i}}_0 \Rightarrow (\mathrm{D})_1$. Let us consider the following bilinear equation

$$[\varepsilon_{23}]\tau_{\mathrm{e}_4}\tau_{2\mathrm{e}_0-\mathrm{e}_2-\mathrm{e}_3-\mathrm{e}_4-\mathrm{e}_5-\mathrm{e}_9} = [\varepsilon_{349}]\tau_{\mathrm{e}_0-\mathrm{e}_2-\mathrm{e}_9}\tau_{\mathrm{e}_0-\mathrm{e}_3-\mathrm{e}_5} - [\varepsilon_{249}]\tau_{\mathrm{e}_0-\mathrm{e}_3-\mathrm{e}_9}\tau_{\mathrm{e}_0-\mathrm{e}_2-\mathrm{e}_5}$$

of type $(\mathrm{C})^{\mathrm{i}}_0$. Multiplying this equation by $\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_9}$ and summing up its $(1,2,3)$-cyclic permutations, we get a bilinear equation of type $(\mathrm{D})_1$.

4. $(\mathrm{A})_0$ and $(\mathrm{C})^{\mathrm{i}}_{-1} \Rightarrow (\mathrm{C})^{\mathrm{i}}_0$. Let us consider the following bilinear equation of type $(\mathrm{C})^{\mathrm{i}}_{-1}$

$$[\varepsilon_{jk}]\tau_{\mathrm{e}_9}\tau_{2\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j-\mathrm{e}_k-\mathrm{e}_8-\mathrm{e}_9} = [\varepsilon_{ik9}]\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_8}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j} - [\varepsilon_{ij9}]\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_8}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_k}.$$

Multiplying both right and left-hand sides by τ_{e_8} and using the first equation of (3.1), we get

$$\begin{aligned}
&\tau_{\mathrm{e}_9} \times [\varepsilon_{jk}]\tau_{\mathrm{e}_8}\tau_{2\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j-\mathrm{e}_k-\mathrm{e}_8-\mathrm{e}_9}\\
&\qquad = \tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j} \times \big([\varepsilon_{ik8}]\tau_{\mathrm{e}_9}\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_9} + [\varepsilon_{89}]\tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_k}\big) - \{j \leftrightarrow k\}\\
&\qquad = \tau_{\mathrm{e}_9} \times \big([\varepsilon_{ik8}]\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_9}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j} - [\varepsilon_{ij8}]\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_9}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_k}\big),
\end{aligned}$$

which is equivalent to the third equation of (3.1). ∎

Also, by not difficult but tedious procedure, we get the following propositions.

Proposition 3.3. *If the lattice τ-functions satisfy all the bilinear equations of type $(\mathrm{B})^{\mathrm{d}}_n$, then they also satisfy those of type $(\mathrm{A})^{\mathrm{d}}_n$; that is,*

1. $(\mathrm{B})^{\mathrm{d}}_n \Rightarrow (\mathrm{A})^{\mathrm{d}}_n$.

Similarly, if $\tau_\Lambda \neq 0$ for $\Lambda \in M_{n-1}$, we have the following:

2. $(\mathrm{A})_n,\ (\mathrm{C})^{\mathrm{d}}_{n-1} \Rightarrow (\mathrm{C})^{\mathrm{d}}_n$,
3. $(\mathrm{C})^{\mathrm{i}}_{n-1},\ (\mathrm{C})^{\mathrm{d}}_{n-1} \Rightarrow (\mathrm{D})^{\mathrm{d}}_n$,
4. $(\mathrm{C})^{\mathrm{d}}_{n-1},\ (\mathrm{C})^{\mathrm{i}}_{n-1},\ (\mathrm{B})_n \Rightarrow (\mathrm{B})^{\mathrm{d}}_n$.

Proposition 3.4. *If the lattice τ-functions satisfy all the bilinear equations of type $(\mathrm{C})^{\mathrm{i}}_n$, then they also satisfy those of type $(\mathrm{C})'_n$; that is,*

1. $(\mathrm{C})^{\mathrm{i}}_n \Rightarrow (\mathrm{C})'_n$.

Similarly, we have

2. $(\mathrm{B})'_n \Rightarrow (\mathrm{A})'_n$.

Moreover, if $\tau_\Lambda \neq 0$ for $\Lambda \in M_{n-1}$, we have the following:

3. $(\mathrm{C})'_{n-1},\ (\mathrm{D})_n \Rightarrow (\mathrm{D})'_n$,
4. $(\mathrm{B})'_n,\ (\mathrm{C})^{\mathrm{i}}_{n-1} \Rightarrow (\mathrm{B})_n$.

4 The construction of the τ-functions on M_0

Hereafter, we construct the hypergeometric τ-functions for the q-Painlevé system of type $E_7^{(1)}$ by imposing the following boundary condition

$$\tau_{\Lambda_{-1}} = 0 \qquad \text{for any} \quad \Lambda_{-1} \in M_{-1} \tag{4.1}$$

and $\tau_{\Lambda_0} \neq 0$ for any $\Lambda_0 \in M_0$. In this section, we discuss the construction of the τ-functions on the lattice M_0.

First, let us consider the following bilinear equations of type $(\mathrm{A})_0$, $(\mathrm{A})'_0$ and $(\mathrm{A})^{\mathrm{d}}_0$

$$\begin{aligned}
&[\varepsilon_{89}]\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j} = [\varepsilon_{ij9}]\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_8} - [\varepsilon_{ij8}]\tau_{\mathrm{e}_9}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_9} \qquad (i,j \in C),\\
&[\varepsilon_{78}][\delta-\varepsilon_{569}]\tau_{\mathrm{e}_9}\tau_{2\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2-\mathrm{e}_3-\mathrm{e}_4-\mathrm{e}_9} + (7,8,9)\text{-cyclic} = 0,\\
&[\delta-\varepsilon_{567}][\varepsilon_{89}]d_C\tau_{\mathrm{e}_5}\tau_{\mathrm{e}_6} + \tau_{\mathrm{e}_8}\tau_{2\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2-\mathrm{e}_3-\mathrm{e}_4-\mathrm{e}_8} - \tau_{\mathrm{e}_9}\tau_{2\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2-\mathrm{e}_3-\mathrm{e}_4-\mathrm{e}_9} = 0.
\end{aligned} \tag{4.2}$$

The boundary condition (4.1) leads us to

$$[\varepsilon_{89}] = 0 \quad \Leftrightarrow \quad \varepsilon_{89} = \omega \in \mathbb{Z}. \tag{4.3}$$

All the bilinear equations of type $(\mathrm{A})_0$, $(\mathrm{A})'_0$ and $(\mathrm{A})^{\mathrm{d}}_0$ hold under the conditions (4.1) and (4.3), since they can be obtained by the action of $\widetilde{W}\big(D_6^{(1)}\big) = \langle s_{127}, s_{12}, \dots, s_{56}, s_{\delta-\varepsilon_{567}}, \rho\rangle$ on (4.2) and the coefficient $[\varepsilon_{89}]$ is $\widetilde{W}\big(D_6^{(1)}\big)$-invariant.

Under the boundary condition (4.1), the bilinear equations of type $(\mathrm{B})_0$, $(\mathrm{B})'_0$ and $(\mathrm{B})^{\mathrm{d}}_0$ are expressed in terms of the lattice τ-functions on M_0. Typical equations of these types are given by

$$\begin{aligned}
&[\varepsilon_{78}]\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j} = [\varepsilon_{ij8}]\tau_{\mathrm{e}_7}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_7} - [\varepsilon_{ij7}]\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_8},\\
&[\varepsilon_{ij}][\varepsilon_{kl}]\tau_{\mathrm{e}_8}\tau_{2\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j-\mathrm{e}_k-\mathrm{e}_l-\mathrm{e}_8} = [\varepsilon_{il8}][\varepsilon_{jk8}]\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_k}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_l} - [\varepsilon_{jl8}][\varepsilon_{ik8}]\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_k}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_l},\\
&\tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_7} - \tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_7} + [\varepsilon_{ij}][\varepsilon_{ij7}]\, d_L\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_9} = 0,\\
&\tau_{\mathrm{e}_8}\tau_{2\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2-\mathrm{e}_3-\mathrm{e}_4-\mathrm{e}_8} - \tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}\tau_{\mathrm{e}_0-\mathrm{e}_3-\mathrm{e}_4} = [\varepsilon_{128}][\varepsilon_{348}]\, d_C\tau_{\mathrm{e}_5}\tau_{\mathrm{e}_6}
\end{aligned}$$

for mutually distinct indices $i, j, k, l \in C$. These are reduced to

$$\begin{aligned}
&[\varepsilon_{79}]\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j} = [\varepsilon_{ij9}]\tau_{\mathrm{e}_7}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_7}, \qquad \tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_7} = \tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_7},\\
&\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}\tau_{\mathrm{e}_0-\mathrm{e}_3-\mathrm{e}_4} + [\varepsilon_{129}][\varepsilon_{349}]\, d_C\tau_{\mathrm{e}_5}\tau_{\mathrm{e}_6} = 0
\end{aligned} \tag{4.4}$$

and

$$[\varepsilon_{il9}][\varepsilon_{jk9}]\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_k}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_l} = [\varepsilon_{jl9}][\varepsilon_{ik9}]\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_k}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_l} \tag{4.5}$$

due to the conditions (4.1) and (4.3). Obviously, the equation (4.5) can be derived from the third equation of (4.4) and its $\mathfrak{S}_6$-transforms. Also, it is not difficult to see that all the bilinear equations of type $(\mathrm{D})_0$, $(\mathrm{D})'_0$ and $(\mathrm{D})^{\mathrm{d}}_0$ can be derived from the equations (4.4) and their $\widetilde{W}\big(D_6^{(1)}\big)$-transforms. Then, it is sufficient to consider the equations (4.4) and their $\widetilde{W}\big(D_6^{(1)}\big)$-transforms for constructing the hypergeometric τ-functions on M_0.

Let us consider a pair of non-zero meromorphic functions $(G(x), F(x))$ satisfying the difference equations $G(x+\delta) = \epsilon\,[x]\,G(x)$ and $F(x+\delta) = G(x)F(x)$ with a constant $\epsilon \in \mathbb{C}^*$. When $\operatorname{Im}\delta > 0$, a typical choice of such functions is given by

$$G(x) = \frac{e\left(-\frac{\delta}{2}\binom{x/\delta}{2}\right)}{(u;q)_\infty}, \qquad F(x) = e\left(-\frac{\delta}{2}\binom{x/\delta}{3}\right)(u;q,q)_\infty,$$

where $u = e(x)$, $q = e(\delta)$, $(u;q,q) = \prod_{i,j=0}^{\infty}(1-uq^{i+j})$ and $\epsilon = -1$. For other choices of $(G(x),F(x),\epsilon)$, see Appendix in [14]. In what follows, we fix a triplet $(G_+(x),F_+(x),\epsilon_+)$ with a constant factor ϵ_+, namely we have

$$G_+(x+\delta) = \epsilon_+[x]\,G_+(x), \qquad F_+(x+\delta) = G_+(x)F_+(x). \tag{4.6}$$

Also, we introduce a pair of functions $(G_-(x),F_-(x))$ by the relations

$$F_-(x) = F_+(2\delta+\omega-x), \qquad G_-(x)G_+(\delta+\omega-x) = 1. \tag{4.7}$$

Note that these functions satisfy the difference equations

$$G_-(x+\delta) = \epsilon_-[x]\,G_-(x), \qquad F_-(x+\delta) = G_-(x)F_-(x) \tag{4.8}$$

with $\epsilon_- = (-1)^{\omega+1}\epsilon_+$.

Moreover, we consider a triplet of functions $(\mathcal{A}_+(x),\mathcal{B}_+(x),\mathcal{C}_+(x))$ defined by the difference equations

$$\begin{aligned}
&\frac{\mathcal{A}_+(x+\delta)\mathcal{A}_+(x-\delta)}{\mathcal{A}_+(x)\mathcal{A}_+(x)} = e(\alpha x+\mathfrak{a}),\\
&\frac{\mathcal{B}_+(x+\delta)\mathcal{B}_+(x-\delta)}{\mathcal{B}_+(x)\mathcal{B}_+(x)} = e(\alpha x+\mathfrak{b}),\\
&\frac{\mathcal{C}_+(x+\delta)\mathcal{C}_+(x-\delta)}{\mathcal{C}_+(x)\mathcal{C}_+(x)} = e(-\alpha x+\mathfrak{c}),
\end{aligned} \tag{4.9}$$

where $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and α are the complex constants satisfying $e(2\alpha\delta+4\mathfrak{b}+2\mathfrak{c}) = (-1)^{\omega+1}$ and $(-1)^{\omega+1}\epsilon_+^2 e(\alpha\omega+2\mathfrak{a}) + d_C\, e(5\mathfrak{b}+3\mathfrak{c}) = 0$. A typical example of such functions is given by $\mathcal{A}_+(x) = e(\delta\alpha\binom{x/\delta+1}{3} + \mathfrak{a}\binom{x/\delta}{2})$. Also, we introduce the functions $\mathcal{A}_-(x)$, $\mathcal{B}_-(x)$ and $\mathcal{C}_-(x)$ by the relations

$$\mathcal{A}_-(x) = \mathcal{A}_+(2\delta+\omega-x), \qquad \mathcal{B}_-(x) = \mathcal{B}_+(2\delta-x), \qquad \mathcal{C}_-(x) = \mathcal{C}_+(2\delta-x). \tag{4.10}$$

Definition 4.1. For each $\Lambda_0 \in M_0$, we define the twelve functions $\tau^{(a;\pm)}_{\Lambda_0}(\varepsilon)$ $(a\in C)$ by

$$\begin{aligned}
\tau^{(a;\pm)}_{\Lambda_0}(\varepsilon) = {}& F_\pm(\varepsilon_{79}+(\langle \mathrm{e}_{79},\Lambda_0\rangle+1)\delta) \prod_{i,j\in C;\, i<j} F_{\pm\kappa^{(a)}_{ij}}(\varepsilon_{ij9}+(\langle \mathrm{e}_{ij9},\Lambda_0\rangle+1)\delta)\\
&\times \mathcal{A}_\pm(\varepsilon_{79}+(\langle \mathrm{e}_{79},\Lambda_0\rangle+1)\delta) \prod_{i,j\in C;\, i<j} \mathcal{A}_{\pm\kappa^{(a)}_{ij}}(\varepsilon_{ij9}+(\langle \mathrm{e}_{ij9},\Lambda_0\rangle+1)\delta)\\
&\times \prod_{i\in C_a} \mathcal{B}_\pm(\varepsilon_{ia7}+(\langle \mathrm{e}_{ia7},\Lambda_0\rangle+1)\delta)\,\mathcal{B}_\pm(\varepsilon_{ia}+(\langle \mathrm{e}_{ia},\Lambda_0\rangle+1)\delta)\\
&\times \mathcal{C}_\pm(\varepsilon_{aa7}+(\langle \mathrm{e}_{aa7},\Lambda_0\rangle+1)\delta),
\end{aligned} \tag{4.11}$$

where $\kappa^{(a)}_{ij}$ is the sign factor defined by $\kappa^{(a)}_{ij} = (-1)^{\sharp(\{i,j\}\cap\{a\})}$ and $C_a = C\backslash\{a\}$.

Theorem 4.1. *The action of $\widetilde{W}(D_6^{(1)})$ on the functions $\tau^{(a;\pm)}_{\Lambda_0}(\varepsilon)$ is described as follows:*

1. *For any translation operator $T\in\widetilde{W}(D_6^{(1)})$, we have $\tau^{(a;\pm)}_{T.\Lambda_0}(\varepsilon) = \tau^{(a;\pm)}_{\Lambda_0}(T(\varepsilon))$.*
2. *For any permutation $\sigma\in\mathfrak{S}_6$, we have $\tau^{(\sigma(a);\pm)}_{\sigma.\Lambda_0}(\varepsilon) = \tau^{(a;\pm)}_{\Lambda_0}(\sigma(\varepsilon))$.*
3. *Take two mutually distinct indices $i,j\in C$.*

(a) *If* $a \notin \{i,j\}$, *then* $\tau^{(a;\pm)}_{s_{ij7}.\Lambda_0}(\varepsilon) = \tau^{(a;\pm)}_{\Lambda_0}(s_{ij7}(\varepsilon))$.

(b) *If* $a \in \{i,j\}$, *then* $\tau^{(a;\pm)}_{s_{ij7}.\Lambda_0}(\varepsilon) = \tau^{(b;\mp)}_{\Lambda_0}(s_{ij7}(\varepsilon))$, *where* b *is an index such that* $\{a,b\} = \{i,j\}$.

4. *The action of the central element* $w_c \in W(D_6)$ *is given by* $\tau^{(a;\mp)}_{w_c.\Lambda_0}(\varepsilon) = \tau^{(a;\pm)}_{\Lambda_0}(w_c(\varepsilon))$.

Proof. The first and second statements are obvious from the definition of $\tau^{(a;\pm)}_{\Lambda_0}(\varepsilon)$. The third statement is guaranteed by the relations (4.7) and (4.10). Since we have

$$w_c: \ \varepsilon_{79} \mapsto \delta + \omega - \varepsilon_{79}, \qquad \varepsilon_{ij9} \mapsto \delta + \omega - \varepsilon_{ij9} \quad (i,j \in C),$$

one can verify the fourth statement by using the relations (4.7) and (4.10). ■

Remark 4.1. The central element $w_c \in W(D_6)$ can be expressed by $w_c = s_{12}s_{127}s_{34}s_{347}s_{56}s_{567}$. It is easy to see that we have $Tw_c = w_cT^{-1}$ for any translation operator $T \in \widetilde{W}(D_6^{(1)})$.

Let S be a label set defined by $S = \{(a\,;\,\epsilon)\,|\,a \in C,\ \epsilon = \pm 1\}$. By using the difference equations (4.6), (4.8) and (4.9), one can verify that the family of functions $\{\tau^{(\eta)}_{\Lambda_0}(\varepsilon)\}_{\Lambda_0 \in M_0}$ for each label $\eta \in S$ satisfies the bilinear equations (4.4). Also, the set of the functions $\{\tau^{(\eta)}_{\Lambda_0}(\varepsilon)\,|\,\eta \in S, \Lambda_0 \in M_0\}$ is consistent with respect to the action of $\widetilde{W}\big(D_6^{(1)}\big)$ in the sense of Theorem 4.1. Then, we have the following theorem.

Theorem 4.2. *For each label* $\eta \in S$, *the family of functions* $\{\tau^{(\eta)}_{\Lambda_0}(\varepsilon)\}_{\Lambda_0 \in M_0}$ *defined by* (4.11) *satisfies all the bilinear equations of type* $(\mathrm{B})_0$, $(\mathrm{B})'_0$, $(\mathrm{B})^{\mathrm{d}}_0$, $(\mathrm{D})_0$, $(\mathrm{D})'_0$ *and* $(\mathrm{D})^{\mathrm{d}}_0$ *under the conditions* (4.1) *and* (4.3).

Before discussing the construction of the hypergeometric τ-functions on M_n for $n \in \mathbb{Z}_{\geq 1}$, we mention those on M_n for $n \in \mathbb{Z}_{<0}$.

Lemma 4.1. *For any fixed* $n \in \mathbb{Z}_{<0}$, *we have* $\tau_{\Lambda_n}(\varepsilon) = 0$ *for any* $\Lambda_n \in M_n$ *under the conditions* (4.1) *and* (4.3).

5 The construction of the τ-functions on M_1

In this section, we construct the hypergeometric τ-functions on M_1. We find that a class of bilinear equations for the lattice τ-functions yields the contiguity relations for the q-hypergeometric function ${}_8W_7$ [5, 1]. As is well-known, the q-hypergeometric function ${}_8W_7$ possesses the $W(D_5)$-symmetry [13]. From that, we can construct a set of twelve solutions corresponding to the coset $W(D_6)/W(D_5)$, and describe the action of $\widetilde{W}(D_6^{(1)})$ on the set of solutions.

5.1 The q-hypergeometric function ${}_8W_7$ and its transformation formula

Fix a complex number q with $0 < |q| < 1$. Let us consider the basic hypergeometric function ${}_8W_7 = {}_8W_7(a_0; a_1, \ldots, a_5; q, z)$ defined by (1.2). It is well-known that this function admits the transformation formula [5, 1]

$$\begin{aligned}{}_8W_7(a_0; a_1, a_2, a_3, a_4, a_5; q, z) &= \frac{\left(qa_0, \frac{qa_0}{a_4a_5}, \frac{q^2a_0^2}{a_1a_2a_3a_4}, \frac{q^2a_0^2}{a_1a_2a_3a_5}; q\right)_\infty}{\left(\frac{qa_0}{a_4}, \frac{qa_0}{a_5}, \frac{q^2a_0^2}{a_1a_2a_3}, \frac{q^2a_0^2}{a_1a_2a_3a_4a_5}; q\right)_\infty} \\ &\quad \times {}_8W_7\left(\frac{qa_0^2}{a_1a_2a_3}; \frac{qa_0}{a_2a_3}, \frac{qa_0}{a_1a_3}, \frac{qa_0}{a_1a_2}, a_4, a_5; q, \frac{qa_0}{a_4a_5}\right),\end{aligned}$$

which can be expressed by the following identity

$$\frac{(q^2a_0^2/a_1a_2a_3a_4a_5;q)_\infty \prod_{k=1}^{5}(qa_0/a_k;q)_\infty}{(qa_0;q)_\infty}\,{}_8W_7(a_0;a_1,\dots,a_5;q,z)$$
$$=\frac{(q^2\widetilde{a}_0^2/\widetilde{a}_1\widetilde{a}_2\widetilde{a}_3\widetilde{a}_4\widetilde{a}_5;q)_\infty \prod_{k=1}^{5}(q\widetilde{a}_0/\widetilde{a}_k;q)_\infty}{(q\widetilde{a}_0;q)_\infty}\,{}_8W_7(\widetilde{a}_0;\widetilde{a}_1,\dots,\widetilde{a}_5;q,\widetilde{z})$$

with respect to the coordinate transformation

$$\widetilde{a}_0=qa_0^2/a_1a_2a_3,$$
$$\widetilde{a}_1=qa_0/a_2a_3,\qquad \widetilde{a}_2=qa_0/a_1a_3,\qquad \widetilde{a}_3=qa_0/a_1a_2,$$
$$\widetilde{a}_4=a_4,\quad \widetilde{a}_5=a_5.$$

In this form, the function is manifestly invariant under the permutation of the parameters $a_1,\dots,a_5$.

Assume that $\operatorname{Im}\delta>0$. We relate the variables a_i to ε_j by

$$a_0=e(\delta-\varepsilon_{669}),\qquad a_i=e(\delta-\varepsilon_{i69})\quad (i=1,2,\dots,5),\qquad q=e(\delta). \tag{5.1}$$

Since the action of $s_{457}\in W(D_6)=\langle s_{12},s_{23},s_{34},s_{45},s_{56},s_{127}\rangle$ on the variables a_i is given by

$$s_{457}:\quad a_0\mapsto qa_0^2/a_1a_2a_3,$$
$$a_1\mapsto qa_0/a_2a_3,\qquad a_2\mapsto qa_0/a_1a_3,\qquad a_3\mapsto qa_0/a_1a_2,$$
$$a_4\mapsto a_4,\quad a_5\mapsto a_5,$$

we see that this action leads us to the above transformation formula for ${}_8W_7$.

Let us introduce the function $\mu^{(6)}(\varepsilon)$ that is invariant under the action of the symmetric group $\mathfrak{S}_5=\langle s_{12},s_{23},s_{34},s_{45}\rangle\subset\mathfrak{S}_6$ and satisfies

$$\frac{\mu^{(6)}(s_{457}(\varepsilon))}{\mu^{(6)}(\varepsilon)}=\frac{g_+(\varepsilon_{459})g_+(2\delta-\varepsilon_{669})\prod_{i=4,5}g_+(\delta+\omega-\varepsilon_{i67})}{g_+(\varepsilon_{79})g_+(2\delta-\varepsilon_{669}-\varepsilon_{457})\prod_{i=4,5}g_+(\delta+\omega-\varepsilon_{i6})}, \tag{5.2}$$

where $g_+(x)$ is given by $G_+(x)=\dfrac{g_+(x)}{(u;q)_\infty}$ with $u=e(x)$ and $q=e(\delta)$. The relation (5.2) means that the function $\dfrac{g_+(2\delta-\varepsilon_{669})}{g_+(\varepsilon_{79})}\prod_{i\in C_6}\dfrac{1}{g_+(\delta+\omega-\varepsilon_{i6})}\mu^{(6)}(\varepsilon)$ is invariant under the action of s_{457}. Then, we see that the function

$$\mu^{(6)}(\varepsilon)\frac{G_+(2\delta-\varepsilon_{669})}{G_+(\varepsilon_{79})}\prod_{i\in C_6}G_-(\varepsilon_{i6})\Phi^{(6)}(\varepsilon), \tag{5.3}$$

where $\Phi^{(6)}(\varepsilon)={}_8W_7(a_0;a_1,a_2,a_3,a_4,a_5;q,z)$, is invariant under the action of the finite Weyl group $W(D_5)=\langle s_{12},s_{23},s_{34},s_{45},s_{127}\rangle\subset W(D_6)$.

5.2 The contiguity relations for ${}_8W_7$

It is also known that the q-hypergeometric function $\Phi^{(6)}={}_8W_7$ satisfies the following contiguity relations [5, 1]

$$(a_1-a_2)(1-z)\Phi^{(6)}=a_1\frac{\prod_{i=3}^{5}(1-qa_0/a_1a_i)}{1-qa_0/a_1}\Phi^{(6)}|_{a_1\mapsto a_1/q}$$

$$- a_2\frac{\prod\limits_{i=3}^{5}(1-qa_0/a_2a_i)}{1-qa_0/a_2}\Phi^{(6)}|_{a_2\mapsto a_2/q}, \tag{5.4}$$

$$\begin{aligned}(a_2-a_1)(1-a_0/a_1a_2)\Phi^{(6)} &= (1-a_1)(1-a_0/a_1)\Phi^{(6)}|_{a_1\mapsto qa_1}\\ &\quad - (1-a_2)(1-a_0/a_2)\Phi^{(6)}|_{a_2\mapsto qa_2},\end{aligned} \tag{5.5}$$

$$\begin{aligned}(1-a_0/a_1)(1-z)\Phi^{(6)} &= \frac{\prod\limits_{i=1}^{5}(1-a_0/a_i)}{(1-q^{-1}a_0)(1-a_0)(1-q^{-1}a_1)}\Phi^{(6)}(-)\\ &\quad - q^{-1}a_1\frac{\prod\limits_{i=2}^{5}(1-qa_0/a_1a_i)}{(1-q^{-1}a_1)(1-qa_0/a_1)}\Phi^{(6)}|_{a_1\mapsto a_1/q},\end{aligned} \tag{5.6}$$

$$\Phi^{(6)}|_{a_1\mapsto qa_1} - \Phi^{(6)} = q^{-1}z\frac{(1-qa_0)(1-q^2a_0)\prod\limits_{i=2}^{5}(1-a_i)}{(1-a_0/a_1)\prod\limits_{i=1}^{5}(1-qa_0/a_i)}\Phi^{(6)}(+), \tag{5.7}$$

where $\Phi^{(6)}(\pm) = \Phi^{(6)}|_{a_0\mapsto q^{\pm 2}a_0, a_1\mapsto q^{\pm 1}a_1,\dots,a_5\mapsto q^{\pm 1}a_5}$.

Noticing that the action of translation operators $\widetilde{T}_{i7} \in \widetilde{W}\big(D_6^{(1)}\big)$ $(i \in C)$ on the variables a_i $(i = 0, 1, \dots, 5)$ is given by

$$\widetilde{T}_{i7}:\ a_i \mapsto q^{-1}a_i, \qquad \widetilde{T}_{67}:\ a_0 \mapsto q^{-2}a_0, \qquad a_i \mapsto q^{-1}a_i \qquad (i \in C_6),$$

we see that the contiguity relations (5.4) and (5.5) can be rewritten as

$$(-1)^{\omega}[\varepsilon_{jk}][\varepsilon_{79}]\,\Phi^{(6)}(\varepsilon) = \frac{\prod\limits_{l\in C_6\setminus\{j,k\}}[\varepsilon_{jl9}]}{[\varepsilon_{j6}-\delta]}\Phi^{(6)}\big(\widetilde{T}_{j7}(\varepsilon)\big) - \frac{\prod\limits_{l\in C_6\setminus\{j,k\}}[\varepsilon_{kl9}]}{[\varepsilon_{k6}-\delta]}\Phi^{(6)}\big(\widetilde{T}_{k7}(\varepsilon)\big) \tag{5.8}$$

and

$$[\varepsilon_{jk}][\varepsilon_{jk9}-\delta]\,\Phi^{(6)}(\varepsilon) = [\varepsilon_{j69}-\delta][\varepsilon_{j6}]\Phi^{(6)}\big(\widetilde{T}_{j7}^{-1}(\varepsilon)\big) - [\varepsilon_{k69}-\delta][\varepsilon_{k6}]\Phi^{(6)}\big(\widetilde{T}_{k7}^{-1}(\varepsilon)\big), \tag{5.9}$$

respectively, for $j, k \in C_6$. Similarly, the contiguity relations (5.6) and (5.7) are expressed by

$$\begin{aligned}(-1)^{\omega}[\varepsilon_{k6}][\varepsilon_{k69}][\varepsilon_{79}]\Phi^{(6)}(\varepsilon) &= \frac{\prod\limits_{l\in C_6\setminus\{k\}}[\varepsilon_{kl9}]}{[\varepsilon_{k6}-\delta]}\Phi^{(6)}\big(\widetilde{T}_{k7}(\varepsilon)\big)\\ &\quad - \frac{\prod\limits_{l\in C_6}[\varepsilon_{l6}]}{[\delta-\varepsilon_{669}][-\varepsilon_{669}]}\Phi^{(6)}\big(\widetilde{T}_{67}(\varepsilon)\big)\end{aligned} \tag{5.10}$$

and

$$\Phi^{(6)}(\varepsilon) = \Phi^{(6)}(\widetilde{T}_{k7}^{-1}(\varepsilon)) - \frac{[3\delta-\varepsilon_{669}][2\delta-\varepsilon_{669}]\prod\limits_{l\in C_6\setminus\{k\}}[\varepsilon_{l69}-\delta]}{[\varepsilon_{k6}]\prod\limits_{l\in C_6}[\varepsilon_{l6}-\delta]}\Phi^{(6)}\big(\widetilde{T}_{67}^{-1}(\varepsilon)\big), \tag{5.11}$$

respectively, for $k \in C_6$.

Let us introduce the function $\Psi^{(6)}(\varepsilon)$ by

$$\prod_{i,j\in C_6;\, i<j}\frac{1}{G_+(\varepsilon_{ij9})}\,\Psi^{(6)}(\varepsilon) = \mu^{(6)}(\varepsilon)\,\frac{G_+(2\delta-\varepsilon_{669})}{G_+(\varepsilon_{79})}\prod_{i\in C_6}G_-(\varepsilon_{i6})\,\Phi^{(6)}(\varepsilon),$$

where the right-hand side is the $W(D_5)$-invariant function (5.3). We see that the function $\Psi^{(6)}(\varepsilon)$ is $\mathfrak{S}_5$-invariant and satisfies the relation

$$\prod_{i,j\in\{1,2,3\};\, i<j} G_+(\varepsilon_{ij9}) \prod_{i=1,2,3} G_-(\varepsilon_{i69})\, G_+(\varepsilon_{459})\, \Psi^{(6)}(s_{457}(\varepsilon)) = G_+(\varepsilon_{79})\Psi^{(6)}(\varepsilon).$$

Suppose that the correction factor $\mu^{(6)}(\varepsilon)$, introduced in the previous subsection, satisfies the difference equation $\mu^{(6)}(\widetilde{T}_{i7}(\varepsilon)) = (-1)^{\omega}\mu^{(6)}(\varepsilon)\,(i\in C)$. Then both of the contiguity relations (5.8) and (5.10) yield

$$(-1)^{\omega+1}\epsilon_+^2[\varepsilon_{jk}][\varepsilon_{jk9}]\Psi^{(6)}(\varepsilon) = \Psi^{(6)}\big(\widetilde{T}_{j7}(\varepsilon)\big) - \Psi^{(6)}\big(\widetilde{T}_{k7}(\varepsilon)\big)$$

for $j,k\in C$. Similarly, we see that (5.9) and (5.11) are reduced to

$$\epsilon_+^{-2}[\varepsilon_{jk}][\varepsilon_{79}-\delta]\Psi^{(6)}(\varepsilon) = \prod_{l\in C\setminus\{j,k\}}[\varepsilon_{kl9}-\delta]\Psi^{(6)}\big(\widetilde{T}_{k7}^{-1}(\varepsilon)\big) - \prod_{l\in C\setminus\{j,k\}}[\varepsilon_{jl9}-\delta]\Psi^{(6)}\big(\widetilde{T}_{j7}^{-1}(\varepsilon)\big)$$

for $j,k\in C$. It is easy to see that the function $\Psi^{(a)}(\varepsilon)$ $(a\in C_6)$ defined by $\Psi^{(a)}(\varepsilon) = \Psi^{(6)}(s_{a6}(\varepsilon))$ satisfies the same contiguity relations as those for $\Psi^{(6)}(\varepsilon)$.

Proposition 5.1. *Each of the functions $\Psi^{(a)}(\varepsilon)$ $(a\in C)$ satisfies the contiguity relations*

$$(-1)^{\omega+1}\epsilon_+^2[\varepsilon_{jk}][\varepsilon_{jk9}]\Psi^{(a)}(\varepsilon) = \Psi^{(a)}\big(\widetilde{T}_{j7}(\varepsilon)\big) - \Psi^{(a)}\big(\widetilde{T}_{k7}(\varepsilon)\big), \tag{5.12}$$

$$\begin{aligned}&\epsilon_+^{-2}[\varepsilon_{jk}][\varepsilon_{79}-\delta]\Psi^{(a)}(\varepsilon)\\ &\quad = \prod_{l\in C\setminus\{j,k\}}[\varepsilon_{kl9}-\delta]\Psi^{(a)}\big(\widetilde{T}_{k7}^{-1}(\varepsilon)\big) - \prod_{l\in C\setminus\{j,k\}}[\varepsilon_{jl9}-\delta]\Psi^{(a)}\big(\widetilde{T}_{j7}^{-1}(\varepsilon)\big)\end{aligned} \tag{5.13}$$

for mutually distinct indices $j,k\in C$.

Here, we give a remark on choice of the correction factor $\mu^{(6)}(\varepsilon)$. The function $\mu^{(6)}(\varepsilon)$ in the form

$$\mu^{(6)}(\varepsilon) = \nu^{(6)}(\varepsilon)\frac{g_+(\varepsilon_{79})\prod_{i\in C_6} g_+(\delta+\omega-\varepsilon_{i6})}{g_+(2\delta-\varepsilon_{669})},$$

where $\nu^{(6)}(\varepsilon)$ is a $W(D_5)$-invariant function, is manifestly $\mathfrak{S}_5$-invariant and satisfies the relation (5.2). Due to $g_+(x+\delta) = -\epsilon_+ e(-\frac{1}{2}x)g_+(x)$, what we have to do is to find a $W(D_5)$-invariant function $\nu^{(6)}(\varepsilon)$ satisfying the difference equations

$$\begin{aligned}&\nu^{(6)}(\widetilde{T}_{i7}(\varepsilon)) = (-1)^{\omega}\epsilon_+^{-2}e\left(\tfrac{1}{2}(\varepsilon_{79}-\varepsilon_{i6}+\delta+\omega)\right)\nu^{(6)}(\varepsilon) \qquad (i\in C_6),\\ &\nu^{(6)}(\widetilde{T}_{67}(\varepsilon)) = (-1)^{\omega}\epsilon_+^{2}e\left(\tfrac{1}{2}\varepsilon_{669}\right)\nu^{(6)}(\varepsilon).\end{aligned} \tag{5.14}$$

It is easy to see that the function $\nu^{(6)}(\varepsilon)$ in the form

$$\nu^{(6)}(\varepsilon) = \varphi_1(\varepsilon_{79})\prod_{i,j\in C_6\,;\, i<j}\varphi_1(\varepsilon_{ij9})\prod_{i\in C_6}\varphi_1(\delta+\omega-\varepsilon_{i69})\prod_{i\in C_6}\varphi_2(\varepsilon_{i67})\varphi_2(\varepsilon_{i6})\varphi_3(\varepsilon_{667}),$$

where $\varphi_i(x)$ $(i=1,2,3)$ are arbitrary functions, is $W(D_5)$-invariant. When $\varphi_i(x)$ $(i=1,2,3)$ satisfy $\varphi_i(x+\delta) = e(\alpha_i x+\beta_i)\varphi_i(x)$ with $\alpha_3 = 2\alpha_1-\alpha_2$, $8\alpha_1+4\alpha_2=1$ and $\epsilon_+^{-2}e((\alpha_1-\alpha_2)\delta + \alpha_2\omega + (-4\beta_1+5\beta_2+\beta_3)) = 1$, the function $\nu^{(6)}(\varepsilon)$ satisfies the difference equations (5.14). A typical choice of them is given by $\varphi_i(x) = e(\alpha_i\delta\binom{x/\delta}{2} + \beta_i x/\delta)$. It is possible to determine $\mu^{(6)}(\varepsilon)$ according to the choice of the functions $\varphi_i(x)$ $(i=1,2,3)$ and $G_+(x)$. We have proposed some examples of the functions $G_+(x)$ and $F_+(x)$ in Appendix of [14].

5.3 Twelve solutions

Hereafter, we denote $\Psi^{(a)}(\varepsilon)$ by $\Psi^{(a;+)}(\varepsilon)$. Since the action of the central element $w_c \in W(D_6)$ on the variables a_i $(i = 0,1,\dots,5)$ is given by $w_c(a_i) = q/a_i$, the application of w_c to the contiguity relations (5.12) and (5.13) leads us to

$$\epsilon_+^2[\varepsilon_{jk}][\varepsilon_{jk9}-\delta]\check{\Psi}^{(a;+)}(\varepsilon) = \check{\Psi}^{(a;+)}\big(\widetilde{T}_{k7}^{-1}(\varepsilon)\big) - \check{\Psi}^{(a;+)}\big(\widetilde{T}_{j7}^{-1}(\varepsilon)\big),$$
$$(-1)^{\omega}\epsilon_+^{-2}[\varepsilon_{jk}][\varepsilon_{79}]\check{\Psi}^{(a;+)}(\varepsilon) = \prod_{l\in C\setminus\{j,k\}}[\varepsilon_{kl9}]\check{\Psi}^{(a;+)}\big(\widetilde{T}_{k7}(\varepsilon)\big) - \prod_{l\in C\setminus\{j,k\}}[\varepsilon_{jl9}]\check{\Psi}^{(a;+)}\big(\widetilde{T}_{j7}(\varepsilon)\big),$$

where $\check{\Psi}^{(a;+)}(\varepsilon) = \Psi^{(a;+)}(w_c(\varepsilon))$. Let us introduce the function $\Psi^{(a;-)}(\varepsilon)$ by

$$\Psi^{(a;-)}(\varepsilon) = \mathcal{E}^{(a;+)}(\varepsilon)\mathcal{G}^{(a;+)}(\varepsilon)\check{\Psi}^{(a;+)}(\varepsilon),$$
$$\mathcal{E}^{(a;+)}(\varepsilon) = \mathcal{A}'_+(\varepsilon_{79})\prod_{i,j\in C_a;\, i<j}\mathcal{A}'_+(\varepsilon_{ij9})\prod_{i\in C_a}\mathcal{A}'_+(\delta+\omega-\varepsilon_{ia9})\prod_{i\in C_a}\mathcal{B}'_+(\varepsilon_{ia7})\mathcal{B}'_+(\varepsilon_{ia})\mathcal{C}'_+(\varepsilon_{aa7}),$$
$$\mathcal{G}^{(a;+)}(\varepsilon) = \frac{\prod\limits_{i\in C_a}G_+(\varepsilon_{ia9})\prod\limits_{i,j\in C_a;\, i<j}G_-(\varepsilon_{ij9})}{G_-(\varepsilon_{79})},$$

where the functions $\mathcal{A}'_+(x)$, $\mathcal{B}'_+(x)$ and $\mathcal{C}'_+(x)$ are expressed in terms of $\mathcal{A}_+(x)$, $\mathcal{B}_+(x)$ and $\mathcal{C}_+(x)$, introduced in the previous section, by

$$\mathcal{A}'_+(x) = \frac{\mathcal{A}_+(2\delta+\omega-x)}{\mathcal{A}_+(\delta+\omega-x)}, \qquad \mathcal{B}'_+(x) = \frac{\mathcal{B}_+(-x)}{\mathcal{B}_+(2\delta-x)}, \qquad \mathcal{C}'_+(x) = \frac{\mathcal{C}_+(-\delta-x)}{\mathcal{C}_+(3\delta-x)}.$$

When we set $d_C = d_L = (-1)^{\omega}\epsilon_+^2$, the factors $\mathcal{E}^{(a;+)}(\varepsilon)$ and $\mathcal{G}^{(a;+)}(\varepsilon)$ satisfy the difference equations

$$\mathcal{E}^{(a;+)}\big(\widetilde{T}_{i7}(\varepsilon)\big) = (-1)^{\omega+1}\mathcal{E}^{(a:+)}(\varepsilon), \qquad \mathcal{G}^{(a:+)}\big(\widetilde{T}_{i7}(\varepsilon)\big) = (-1)^{\omega+1}\epsilon_+^4\frac{\prod\limits_{l\in C_i}[\varepsilon_{il9}]}{[\varepsilon_{79}]}\mathcal{G}^{(a:+)}(\varepsilon)$$

for $i \in C$, and we get $e(2\alpha\delta + 4\mathfrak{b} + 2\mathfrak{c}) = (-1)^{\omega+1}$ and $e(\alpha\omega + 2\mathfrak{a}) = e(5\mathfrak{b} + 3\mathfrak{c})$. Thus, we find that each of the functions $\Psi^{(a;-)}(\varepsilon)$ satisfies the contiguity relations

$$\epsilon_+^{-2}[\varepsilon_{jk}][\varepsilon_{79}-\delta]\Psi^{(a;-)}(\varepsilon) = \prod_{l\in C\setminus\{j,k\}}[\varepsilon_{kl9}-\delta]\Psi^{(a;-)}\big(\widetilde{T}_{k7}^{-1}(\varepsilon)\big) - \prod_{l\in C\setminus\{j,k\}}[\varepsilon_{jl9}-\delta]\Psi^{(a;-)}\big(\widetilde{T}_{j7}^{-1}(\varepsilon)\big),$$
$$(-1)^{\omega+1}\epsilon_+^2[\varepsilon_{jk}][\varepsilon_{jk9}]\Psi^{(a;-)}(\varepsilon) = \Psi^{(a;-)}\big(\widetilde{T}_{j7}(\varepsilon)\big) - \Psi^{(a;-)}\big(\widetilde{T}_{k7}(\varepsilon)\big),$$

which are the same as those for $\Psi^{(a;+)}(\varepsilon)$.

Theorem 5.1. *Each of the twelve functions $\Psi(\varepsilon) = \Psi^{(a;\pm)}(\varepsilon)$ gives rise to the solution of the contiguity relations*

$$(-1)^{\omega+1}\epsilon_+^2[\varepsilon_{jk}][\varepsilon_{jk9}]\Psi(\varepsilon) = \Psi\big(\widetilde{T}_{j7}(\varepsilon)\big) - \Psi\big(\widetilde{T}_{k7}(\varepsilon)\big),$$
$$\epsilon_+^{-2}[\varepsilon_{jk}][\varepsilon_{79}-\delta]\Psi(\varepsilon) = \prod_{l\in C\setminus\{j,k\}}[\varepsilon_{kl9}-\delta]\Psi\big(\widetilde{T}_{k7}^{-1}(\varepsilon)\big) - \prod_{l\in C\setminus\{j,k\}}[\varepsilon_{jl9}-\delta]\Psi\big(\widetilde{T}_{j7}^{-1}(\varepsilon)\big)$$

for mutually distinct indices $j,k \in C$.

From these contiguity relations, one can get the q-hypergeometric equation of the second order. The functions $\Psi^{(a;\pm)}(\varepsilon)$ coincide with the twelve pairwise linearly independent solutions to the q-hypergeometric equation constructed by Gupta and Masson [2].

Furthermore, we introduce the function $\mathcal{E}^{(a;-)}(\varepsilon)$ by

$$\mathcal{E}^{(a;-)}(\varepsilon) = \mathcal{A}'_-(\varepsilon_{79}) \prod_{i,j\in C_a;\, i<j} \mathcal{A}'_-(\varepsilon_{ij9}) \prod_{i\in C_a} \mathcal{A}'_-(\delta+\omega-\varepsilon_{ia9}) \prod_{i\in C_a} \mathcal{B}'_-(\varepsilon_{ia7})\mathcal{B}'_-(\varepsilon_{ia})\mathcal{C}'_-(\varepsilon_{aa7}),$$

where $\mathcal{A}'_-(x)$, $\mathcal{B}'_-(x)$ and $\mathcal{C}'_-(x)$ are defined by $\mathcal{A}'_-(x)\mathcal{A}'_+(\delta+\omega-x) = 1$, $\mathcal{B}'_-(x)\mathcal{B}'_+(-x) = 1$ and $\mathcal{C}'_-(x)\mathcal{C}'_+(-x) = 1$, respectively. By construction, we have the following proposition.

Proposition 5.2. *The action of $W(D_6)$ on the functions $\Psi^{(a;\pm)}(\varepsilon)$ is described as follows:*

1. *For any permutation $\sigma \in \mathfrak{S}_6$, we have $\Psi^{(a;\pm)}(\sigma(\varepsilon)) = \Psi^{(\sigma(a);\pm)}(\varepsilon)$.*
2. *Take two mutually distinct indices $i, j \in C$.*

   (a) *If $a \notin \{i,j\}$, then*

$$\Psi^{(a;\pm)}(s_{ij7}(\varepsilon)) = \frac{G_\pm(\varepsilon_{79})}{G_\pm(\varepsilon_{ij9}) \prod\limits_{\substack{k,l\in C\setminus\{i,j,a\};\\ k<l}} G_\pm(\varepsilon_{kl9}) \prod\limits_{k\in C\setminus\{i,j,a\}} G_\mp(\varepsilon_{ka9})} \Psi^{(a;\pm)}(\varepsilon).$$

   (b) *If $a \in \{i,j\}$, then*

$$\Psi^{(a;\pm)}(s_{ij7}(\varepsilon)) = \frac{1}{\mathcal{E}^{(b;\pm)}(\varepsilon)} \frac{G_\mp(\varepsilon_{79})}{G_\pm(\varepsilon_{ij9}) \prod\limits_{k,l\in C\setminus\{i,j\};\, k<l} G_\mp(\varepsilon_{kl9})} \Psi^{(b;\mp)}(\varepsilon),$$

   where b is an index such that $\{a,b\} = \{i,j\}$.

3. *The action of the central element $w_c \in W(D_6)$ is given by*

$$\Psi^{(a;\mp)}(\varepsilon) = \mathcal{E}^{(a;\pm)}(\varepsilon) \frac{\prod\limits_{i\in C_a} G_\pm(\varepsilon_{ia9}) \prod\limits_{i,j\in C_a;\, i<j} G_\mp(\varepsilon_{ij9})}{G_\mp(\varepsilon_{79})} \Psi^{(a;\pm)}(w_c(\varepsilon)).$$

The set of twelve functions $\Psi^{(a;\pm)}(\varepsilon)$ corresponds to the coset $W(D_6)/W(D_5)$, as we will see below. Note that $|W(D_6)/W(D_5)| = 12$.

5.4 The τ-functions on M_1

Here, we construct the functions $\tau_{\Lambda_1}(\varepsilon)$ $(\Lambda_1 \in M_1)$ on the basis of the discussion in the previous subsections. The bilinear equations to be considered are of type $(\mathrm{C})_0$, $(\mathrm{C})'_0$, $(\mathrm{C})^{\mathrm{d}}_0$, $(\mathrm{D})_1$, $(\mathrm{D})'_1$ and $(\mathrm{D})^{\mathrm{d}}_1$, since the functions $\tau_{\Lambda_0}(\varepsilon)$ $(\Lambda_0 \in M_0)$ are already known.

It is easy to get the following lemma.

Lemma 5.1. *If the lattice τ-functions satisfy all the bilinear equations of type $(\mathrm{B})^{\mathrm{d}}_0$ and $(\mathrm{C})^{\mathrm{d}}_0$ under the boundary condition (4.1), then they also satisfy those of type $(\mathrm{C})^{\mathrm{i}}_0$.*

From this lemma, we see that it is sufficient for constructing the hypergeometric τ-functions on M_1 to consider the bilinear equations of type $(\mathrm{C})^{\mathrm{d}}_0$.

Definition 5.1. For each $\Lambda_1 \in M_1$, we define the twelve functions $\tau_{\Lambda_1}^{(a;\pm)}(\varepsilon)$ by

$$\tau_{\Lambda_1}^{(a;\pm)}(\varepsilon) = \mathcal{N}_{\Lambda_1}^{(a;\pm)}(\varepsilon)\, \Psi_{\Lambda_1}^{(a;\pm)}(\varepsilon), \tag{5.15}$$

where $\mathcal{N}_{\Lambda_1}^{(a;\pm)}(\varepsilon)$ is given by

$$\begin{aligned}
\mathcal{N}_{\Lambda_1}^{(a;\pm)}(\varepsilon) = {} & F_\pm(\varepsilon_{79} + (\langle \mathrm{e}_{79}, \Lambda_1\rangle + 1)\delta) \prod_{i,j\in C;\, i<j} F_{\pm\kappa_{ij}^{(a)}}(\varepsilon_{ij9} + \langle \mathrm{e}_{ij9}, \Lambda_1\rangle\delta) \\
& \times \mathcal{A}_\pm(\varepsilon_{79} + (\langle \mathrm{e}_{79}, \Lambda_1\rangle + 1)\delta) \\
& \times \prod_{i,j\in C_a;\, i<j} \mathcal{A}_\pm(\varepsilon_{ij9} + (\langle \mathrm{e}_{ij9}, \Lambda_1\rangle + 1)\delta) \prod_{i\in C_a} \mathcal{A}_\mp(\varepsilon_{ia9} + \langle \mathrm{e}_{ia9}, \Lambda_1\rangle\delta) \\
& \times \prod_{i\in C_a} \mathcal{B}_\pm(\varepsilon_{ia7} + \langle \mathrm{e}_{ia7}, \Lambda_1\rangle\delta)\mathcal{B}_\pm(\varepsilon_{ia} + \langle \mathrm{e}_{ia}, \Lambda_1\rangle\delta) \\
& \times \mathcal{C}_\pm(\varepsilon_{aa7} + (\langle \mathrm{e}_{aa7}, \Lambda_1\rangle - 1)\delta),
\end{aligned}$$

and $\Psi_{\Lambda_1}^{(a;\pm)}(\varepsilon) = \Psi^{(a;\pm)}(\varepsilon + \langle \mathrm{e}, \Lambda_1\rangle\delta)$.

Theorem 5.2. *The action of $\widetilde{W}\big(D_6^{(1)}\big)$ on the functions $\tau_{\Lambda_1}^{(a;\pm)}(\varepsilon)$ is described as follows:*

1. *For any translation operator $T \in \widetilde{W}\big(D_6^{(1)}\big)$, we have $\tau_{T.\Lambda_1}^{(a;\pm)}(\varepsilon) = \tau_{\Lambda_1}^{(a;\pm)}(T(\varepsilon))$.*
2. *For any permutation $\sigma \in \mathfrak{S}_6$, we have $\tau_{\sigma.\Lambda_1}^{(\sigma(a);\pm)}(\varepsilon) = \tau_{\Lambda_1}^{(a;\pm)}(\sigma(\varepsilon))$.*
3. *Take two mutually distinct indices $i, j \in C$.*
   - (a) *If $a \notin \{i, j\}$, then $\tau_{s_{ij7}.\Lambda_1}^{(a;\pm)}(\varepsilon) = \tau_{\Lambda_1}^{(a;\pm)}(s_{ij7}(\varepsilon))$.*
   - (b) *If $a \in \{i, j\}$, then $\tau_{s_{ij7}.\Lambda_1}^{(a;\pm)}(\varepsilon) = \tau_{\Lambda_1}^{(b;\mp)}(s_{ij7}(\varepsilon))$, where b is an index such that $\{a, b\} = \{i, j\}$.*
4. *The action of the central element $w_c \in W(D_6)$ is given by $\tau_{w_c.\Lambda_1}^{(a;\mp)}(\varepsilon) = \tau_{\Lambda_1}^{(a;\pm)}(w_c(\varepsilon))$.*

Proof. The first and second statements are obvious from the definition of $\tau_{\Lambda_1}^{(a;\pm)}(\varepsilon)$. The third and fourth statements are guaranteed by Proposition 5.2 and (5.15). ∎

Corollary 5.1. *For the particular element $\mathrm{e}_8 \in M_1$, the set of twelve functions*

$$\begin{aligned}
\tau_{\mathrm{e}_8}^{(a;\pm)}(\varepsilon) = {} & F_\pm(\varepsilon_{79} + \delta) \prod_{i,j\in C;\, i<j} F_{\pm\kappa_{ij}^{(a)}}(\varepsilon_{ij9}) \\
& \times \mathcal{A}_\pm(\varepsilon_{79} + \delta) \prod_{i,j\in C_a;\, i<j} \mathcal{A}_\pm(\varepsilon_{ij9} + \delta) \prod_{i\in C_a} \mathcal{A}_\mp(\varepsilon_{ia9}) \\
& \times \prod_{i\in C_a} \mathcal{B}_\pm(\varepsilon_{ia7})\, \mathcal{B}_\pm(\varepsilon_{ia}) \times \mathcal{C}_\pm(\varepsilon_{aa7} - \delta)\Psi^{(a;\pm)}(\varepsilon)
\end{aligned}$$

is stabilized by $W(D_6)$[1]*. For each label $(a;\pm) \in S$, the isotropy subgroup of $\tau_{\mathrm{e}_8}^{(a;\pm)}(\varepsilon)$ is isomorphic to $W(D_5)$;*

$$\tau_{\mathrm{e}_8}^{(6;\pm)}(w(\varepsilon)) = \tau_{\mathrm{e}_8}^{(6;\pm)}(\varepsilon), \qquad w \in W(D_5) = \langle s_{12}, s_{23}, s_{34}, s_{45}, s_{127}\rangle$$

for instance.

[1]Note that $\mathrm{e}_8 \in M_1$ is $W(D_6)$-invariant.

Let us consider the bilinear equation of type $(\mathrm{C})_0^{\mathrm{d}}$

$$\tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_9} - \tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_9} + [\varepsilon_{ij}][\varepsilon_{ij9}]d_L\tau_{\mathrm{e}_7}\tau_{\mathrm{e}_8} = 0 \tag{5.16}$$

for mutually distinct indices $i, j \in C$. Substituting (4.11) and (5.15) into (5.16), we get for $\Psi^{(a;\pm)}(\varepsilon)$ the linear relation

$$(-1)^{\omega+1}\epsilon_+^2[\varepsilon_{ij}][\varepsilon_{ij9}]\Psi^{(a;\pm)}(\varepsilon) = \Psi^{(a;\pm)}(\widetilde{T}_{i7}(\varepsilon)) - \Psi^{(a;\pm)}(\widetilde{T}_{j7}(\varepsilon)).$$

Similarly, the application of the central element $w_c \in W(D_6)$ to the bilinear equation (5.16) leads us to

$$\begin{aligned}\epsilon_+^{-2}[\varepsilon_{jk}][\varepsilon_{79}-\delta]\Psi^{(a;\pm)}(\varepsilon) = &\prod_{l\in C\setminus\{j,k\}}[\varepsilon_{kl9}-\delta]\Psi^{(a;\pm)}\big(\widetilde{T}_{k7}^{-1}(\varepsilon)\big)\\ &- \prod_{l\in C\setminus\{j,k\}}[\varepsilon_{jl9}-\delta]\Psi^{(a;\pm)}\big(\widetilde{T}_{j7}^{-1}(\varepsilon)\big)\end{aligned}$$

for mutually distinct indices $i, j \in C$. These are precisely the contiguity relations in Theorem 5.1.

Also, the set of functions $\{\tau_{\Lambda_1}^{(\eta)}(\varepsilon)\,|\,\eta \in S, \Lambda_1 \in M_1\}$ is consistent with respect to the action of $\widetilde{W}\big(D_6^{(1)}\big)$ in the sense of Theorem 5.2. Therefore, we have the following theorem due to Propositions 3.2, 3.3, 3.4 and Lemma 5.1.

Theorem 5.3. *For each label $\eta \in S$, the family of functions $\{\tau_\Lambda^{(\eta)}(x)\}_{\Lambda\in M_0\coprod M_1}$ defined by* (4.11) *and* (5.15) *satisfies all the bilinear equations of type* $(\mathrm{C})_0^{\mathrm{d}}$, $(\mathrm{C})_0^{\mathrm{i}}$, $(\mathrm{C})_0^{\mathrm{r}}$, $(\mathrm{C})_0'$, $(\mathrm{D})_1^{\mathrm{d}}$, $(\mathrm{D})_1$ *and* $(\mathrm{D})_1'$ *under the conditions* (4.1) *and* (4.3).

Remark 5.1. From this theorem, we see that the bilinear equations of type $(\mathrm{C})_0^{\mathrm{d}}$, $(\mathrm{C})_0^{\mathrm{i}}$, $(\mathrm{C})_0^{\mathrm{r}}$ and $(\mathrm{C})_0'$ imply the contiguity relations for the q-hypergeometric function ${}_8W_7$. Also, we get the quadratic relations for ${}_8W_7$ from the bilinear equations of type $(\mathrm{D})_1^{\mathrm{d}}$, $(\mathrm{D})_1$ and $(\mathrm{D})_1'$.

Remark 5.2. From the result in this section, one can get an explicit expression for the so-called Riccati solution to the system of q-difference equations (1.1), in terms of the functions $\tau_{\Lambda_1}^{(\eta)}(\varepsilon)$. When the label $\eta \in S$ is fixed, one can express b_i $(i = 1, 2, \dots, 8)$ and t in terms of the parameters of the q-hypergeometric function ${}_8W_7$. On the other hand, another expression for the Riccati solution has been proposed in [10], which is constructed under the condition $b_1b_3 = b_5b_7$; that is, $\varepsilon_{358} \in \mathbb{Z}$. Comparing this with the condition (4.3), we find that these two expressions can be transformed to each other by a Bäcklund transformation.

6 A determinant formula for the hypergeometric τ-functions

One of the important features of the hypergeometric solutions to the continuous and discrete Painlevé equations is that they can be expressed in terms of Wronskians or Casorati determinants [4, 12, 7, 3, 16]. In this section, we show that the hypergeometric τ-functions on M_n $(n \in \mathbb{Z}_{\geq 2})$ are expressed by "two-directional Casorati determinants" of order n.

Let us introduce the auxiliary variables x_i $(i = 0, 1, \dots, 6)$ by $x_0 = \delta - \varepsilon_{78}$ and $x_i = \frac{1}{2}\varepsilon_{ii9}$ $(i \in C)$, where we have $x_0 + x_1 + \cdots + x_6 = 2\delta + 2\omega$. Under the conditions (4.1) and (4.3), the functions $\tau_\Lambda(\varepsilon)$ depend on x_i (and ω). In what follows, we denote the hypergeometric τ-functions by $\tau_\Lambda(x)$ instead of by $\tau_\Lambda(\varepsilon)$ for convenience. Also, we denote a function $f^{(\eta)}(x)$ $(\eta \in S)$ by $f(\eta; x)$.

For each $n \in \mathbb{Z}_{\geq 0}$, we define the twelve functions $K_n(\eta; x) = K_n^{(a;\pm)}(x)$ by the following "two-directional Casorati determinants"

$$K_{2m}\left(\eta; x_0 + \tfrac{2m-1}{2}\delta, x_i + \tfrac{2m-1}{4}\delta\right)\Big|_{x_i\mapsto x_i-(m-1)\delta\ (i=1,2,3,4)}$$

$$= \det\left(\Psi(b-m, m+1-b, a-m, m+1-a)\right)_{a,b=1}^{2m},$$

$$K_{2m+1}\left(\eta; x_0+m\delta, x_i+\tfrac{m}{2}\delta\right)\Big|_{x_i\mapsto x_i-m\delta\,(i=1,2,3,4)}$$
$$= \det\left(\Psi(b-m-1, m+1-b, a-m-1, m+1-a)\right)_{a,b=1}^{2m+1},$$

where $\Psi(m_1,m_2,m_3,m_4) = \Psi(\eta;x)|_{x_i\mapsto x_i+m_i\delta\,(i=1,2,3,4)}$, and $\Psi(\eta;x)$ is the hypergeometric function multiplied by some normalization factors, introduced in Section 5.2 and 5.3. The first some members of $K_n(\eta;x)$ are given as follows:

$$K_0(x)=1, \qquad K_1(x)=\Psi(x), \qquad K_2\left(x_0+\tfrac12\delta, x_i+\tfrac14\delta\right) = \begin{vmatrix} \Psi^{24}(x) & \Psi^{14}(x) \\ \Psi^{23}(x) & \Psi^{13}(x) \end{vmatrix},$$

$$K_3\left(x_0+\delta, x_i+\tfrac12\delta\right)\Big|_{x_i\mapsto x_i-\delta\,(i=1,2,3,4)} = \begin{vmatrix} \Psi^{24}_{13}(x) & \Psi^{4}_{3}(x) & \Psi^{14}_{23}(x) \\ \Psi^{2}_{1}(x) & \Psi(x) & \Psi^{1}_{2}(x) \\ \Psi^{23}_{14}(x) & \Psi^{3}_{4}(x) & \Psi^{13}_{24}(x) \end{vmatrix},$$

where we omit the label $\eta\in S$ for simplicity, and $\Psi^{i_1\dots i_r}_{j_1\dots j_r}(x) = \Psi(x)\Big|_{\substack{x_i\mapsto x_i+\delta\,(i=i_1,\dots,i_r)\\ x_j\mapsto x_j-\delta\,(j=j_1,\dots,j_r)}}$. By using Jacobi's identity, one can easily see that each of the functions $K_n(\eta;x)$ satisfies the relation

$$K_{n+1}(\eta;x)K_{n-1}^{(1234)}\left(\eta;x_0-\delta, x_i-\tfrac{\delta}{2}\right)$$
$$= K_n^{(24)}\left(\eta;x_0-\tfrac{\delta}{2}, x_i-\tfrac{\delta}{4}\right)K_n^{(13)}\left(\eta;x_0-\tfrac{\delta}{2}, x_i-\tfrac{\delta}{4}\right)$$
$$- K_n^{(14)}\left(\eta;x_0-\tfrac{\delta}{2}, x_i-\tfrac{\delta}{4}\right)K_n^{(23)}\left(\eta;x_0-\tfrac{\delta}{2}, x_i-\tfrac{\delta}{4}\right),$$

where $K_n^{(i_1\dots i_r)}(\eta;x) = K_n(\eta;x)\,|_{x_i\mapsto x_i+\delta\,(i=i_1,\dots,i_r)}$.

Definition 6.1. For each $n\in\mathbb{Z}_{\geq 0}$, we define the twelve functions $\tau_n(\eta;x)$ by $\tau_n(\eta;x) = \Upsilon_n(\eta;x)K_n(\eta;x)$. The normalization factor $\Upsilon_n(\eta;x) = \Upsilon_n^{(a;\pm)}(x)$ is given by

$$\Upsilon_n^{(a;\pm)}(x) = \frac{1}{c_n(x)}F_\mp\left(x_0+\tfrac{1-n}{2}\delta\right)\prod_{i,j\in C;\,i<j}F_{\pm\kappa_{ij}^{(a)}}\left(x_i+x_j+\tfrac{1-n}{2}\delta\right)$$
$$\times\mathcal{A}_\mp\left(x_0+\tfrac{1-n}{2}\delta\right)\prod_{i,j\in C_a;\,i<j}\mathcal{A}_\pm\left(x_i+x_j+\tfrac{n+1}{2}\delta\right)\prod_{i\in C_a}\mathcal{A}_\mp\left(x_i+x_a+\tfrac{1-n}{2}\delta\right)$$
$$\times\prod_{i\in C_a}\mathcal{B}_\pm(x_0+x_i+x_a-\omega-n\delta)\mathcal{B}_\pm(x_a-x_i+(1-n)\delta)\mathcal{C}_\pm(x_0+2x_a-\omega-2n\delta),$$

where the functions $F_\pm(x)$, $\mathcal{A}_\pm(x)$, $\mathcal{B}_\pm(x)$ and $\mathcal{C}_\pm(x)$ are introduced in Section 4. The factor $c_n(x)$ is defined by

$$c_n(x) = (-1)^{(\omega+1)\binom{n}{2}}\epsilon_+^{4\binom{n}{2}}\prod_{r=1}^{n-1}[x_1-x_2+I_r\delta][x_3-x_4+I_r\delta]$$
$$\times\prod_{r=1}^{n-1}\left[x_1+x_2+\left(r-\tfrac{n+1}{2}\right)\delta\right]^r\left[x_3+x_4+\left(r-\tfrac{n+1}{2}\right)\delta\right]^r,$$

where I_r $(r=1,2,\dots)$ is the subset of $\mathbb{Z}$ given by $I_r=\{-r+1,-r+3,\dots,r-3,r-1\}$ and $[x+I_r\delta] = \prod_{k\in I_r}[x+k\delta]$.

Proposition 6.1. *We have the following bilinear relation*

$$[x_1-x_2][x_3-x_4]\tau_{n+1}(\eta;x)\tau_{n-1}^{(1234)}\left(\eta;x_0-\delta, x_i-\tfrac{\delta}{2}\right)$$

$$= \left[x_1 + x_4 - \tfrac{n}{2}\delta\right] \left[x_2 + x_3 - \tfrac{n}{2}\delta\right] \tau_n^{(24)}\left(\eta; x_0 - \tfrac{\delta}{2}, x_i - \tfrac{\delta}{4}\right) \tau_n^{(13)}\left(\eta; x_0 - \tfrac{\delta}{2}, x_i - \tfrac{\delta}{4}\right)$$
$$- \left[x_2 + x_4 - \tfrac{n}{2}\delta\right] \left[x_1 + x_3 - \tfrac{n}{2}\delta\right] \tau_n^{(14)}\left(\eta; x_0 - \tfrac{\delta}{2}, x_i - \tfrac{\delta}{4}\right) \tau_n^{(23)}\left(\eta; x_0 - \tfrac{\delta}{2}, x_i - \tfrac{\delta}{4}\right). \tag{6.1}$$

This proposition is easily verified by noticing that the normalization factor $\Upsilon_n(\eta; x)$ satisfies the relation

$$[x_1 - x_2][x_3 - x_4]\Upsilon_{n+1}(\eta; x)\Upsilon_{n-1}^{(1234)}\left(\eta; x_0 - \delta, x_i - \tfrac{\delta}{2}\right)$$
$$= \left[x_1 + x_4 - \tfrac{n}{2}\delta\right] \left[x_2 + x_3 - \tfrac{n}{2}\delta\right] \Upsilon_n^{(24)}\left(\eta; x_0 - \tfrac{\delta}{2}, x_i - \tfrac{\delta}{4}\right) \Upsilon_n^{(13)}\left(\eta; x_0 - \tfrac{\delta}{2}, x_i - \tfrac{\delta}{4}\right)$$
$$= \left[x_2 + x_4 - \tfrac{n}{2}\delta\right] \left[x_1 + x_3 - \tfrac{n}{2}\delta\right] \Upsilon_n^{(14)}\left(\eta; x_0 - \tfrac{\delta}{2}, x_i - \tfrac{\delta}{4}\right) \Upsilon_n^{(23)}\left(\eta; x_0 - \tfrac{\delta}{2}, x_i - \tfrac{\delta}{4}\right).$$

Definition 6.2. For each $\Lambda_n \in M_n$ $(n \in \mathbb{Z})$, we define the twelve functions $\tau_{\Lambda_n}(\eta; x)$ by

$$\tau_{\Lambda_n}(\eta; x) = \tau_n\left(\eta; x + l^{(n)}\delta\right),$$
$$l_0^{(n)} = \langle v_0, \Lambda_n\rangle + \tfrac{1-n}{2}, \qquad l_i^{(n)} = \langle v_i, \Lambda_n\rangle + \tfrac{1-n}{4} \qquad (i \in C), \tag{6.2}$$

where the vectors v_i are defined by $v_0 = c - \mathrm{e}_{78}$ and $v_i = \frac{1}{2}\mathrm{e}_{ii9}$ $(i \in C)$ that correspond to the variables x_i.

We show below that the functions $\tau_{\Lambda_n}(\eta; x)$ are precisely the hypergeometric τ-functions on M_n. As a preparation, let us define the action of $\widetilde{W}\left(D_6^{(1)}\right)$ on the label set $S = \{(a, \epsilon) \,|\, a \in C, \epsilon = \pm 1\}$.

Definition 6.3. We define the action of $\widetilde{W}(D_6^{(1)})$ on the label $\eta = (a, \epsilon) \in S$ as follows:

1. The label is invariant under the action of any translation.
2. The action of any permutation $\sigma \in \mathfrak{S}_6$ is defined by $\sigma : (a; \pm) \mapsto (\sigma(a); \pm)$.
3. Take two mutually distinct indices $i, j \in C$. If $a \notin \{i, j\}$, then $s_{ij7} : (a; \pm) \mapsto (a; \pm)$. Otherwise, we have $s_{ij7} : (a; \pm) \mapsto (b; \mp)$, where b is an index such that $\{a, b\} = \{i, j\}$.
4. The action of the central element w_c is defined by $w_c : (a; \pm) \mapsto (a; \mp)$.

Theorem 6.1.

1. *For each $\eta \in S$, the family of functions $\{\tau_\Lambda(\eta; x)\}_{\Lambda \in M^{E_7}}$ satisfies all the bilinear equations for the q-Painlevé system of type $E_7^{(1)}$ under the conditions* (4.1) *and* (4.3)*.*
2. *For each $n \in \mathbb{Z}$, the action of $\widetilde{W}\left(D_6^{(1)}\right)$ on the set of functions $\{\tau_{\Lambda_n}(\eta; x) \,|\, \eta \in S,\, \Lambda_n \in M_n\}$ is described by $\tau_{w.\Lambda_n}(w(\eta); x) = \tau_{\Lambda_n}(\eta; w(x))$ for any $w \in \widetilde{W}\left(D_6^{(1)}\right)$.*

Let us verify the first statement. We consider the bilinear equations

$$[\varepsilon_{12}][\varepsilon_{34}]\tau_{L_{m,1}+\mathrm{e}_8}\tau_{L_{m,-1}+2\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2-\mathrm{e}_3-\mathrm{e}_4-\mathrm{e}_8}$$
$$= [\varepsilon_{148} - m\delta][\varepsilon_{238} - m\delta]\tau_{L_{m,0}+\mathrm{e}_0-\mathrm{e}_2-\mathrm{e}_4}\tau_{L_{m,0}+\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_3}$$
$$- [\varepsilon_{248} - m\delta][\varepsilon_{138} - m\delta]\tau_{L_{m,0}+\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_4}\tau_{L_{m,0}+\mathrm{e}_0-\mathrm{e}_2-\mathrm{e}_3} \tag{6.3}$$

and

$$[\varepsilon_{12}][\varepsilon_{34}]\tau_{L_{m,2}+c+\mathrm{e}_{89}+\mathrm{e}_7}\tau_{L_{m,0}+c+2\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2-\mathrm{e}_3-\mathrm{e}_4-\mathrm{e}_8-\mathrm{e}_9+\mathrm{e}_7}$$
$$= [\varepsilon_{148} - m\delta][\varepsilon_{238} - m\delta]\tau_{L_{m,1}+c+\mathrm{e}_{249}+\mathrm{e}_7}\tau_{L_{m,1}+c+\mathrm{e}_{139}+\mathrm{e}_7}$$
$$- [\varepsilon_{248} - m\delta][\varepsilon_{138} - m\delta]\tau_{L_{m,1}+c+\mathrm{e}_{149}+\mathrm{e}_7}\tau_{L_{m,1}+c+\mathrm{e}_{239}+\mathrm{e}_7}, \tag{6.4}$$

where $L_{m,n} = m(m+n)c + m\mathrm{e}_{89}$ $(m \in \mathbb{Z})$, which are of type $(\mathrm{B})'_{2m}$ and $(\mathrm{B})'_{2m+1}$, respectively. Substituting (6.2), we see that these bilinear equations are satisfied thanks to (6.1).

In order to verify the second statement of Theorem 6.1, we use the following lemma.

Lemma 6.1. *Suppose that the functions $\tau_{\Lambda_{n-1}}(\eta;x)$ and $\tau_{\Lambda_n}(\eta;x)$ satisfy all the bilinear equations of type* $(\mathrm{D})'_n$ *and* $(\mathrm{C})'_{n-1}$, *and that we have the relations $\tau_{w.\Lambda_{n-1}}(w(\eta);x)=\tau_{\Lambda_{n-1}}(\eta;w(x))$ and $\tau_{w.\Lambda_n}(w(\eta);x)=\tau_{\Lambda_n}(\eta;w(x))$ for any $w\in W(D_6)$. Then the function $\tau_{\Lambda_{n+1}}(\eta;x)$ determined by the bilinear equation* (6.3) *or* (6.4) *also satisfies $\tau_{w.\Lambda_{n+1}}(w(\eta);x)=\tau_{\Lambda_{n+1}}(\eta;w(x))$ for any $w\in W(D_6)$.*

Proof. From the assumption, we have the bilinear relations

$$[x_1-x_2][x_3-x_4]\tau_n^{(12)}(\eta;x)\tau_n^{(34)}(\eta;x)+(1,2,3)\text{-cyclic}=0, \tag{6.5}$$

$$[x_3-x_4]\left[x_1+x_5-\tfrac{n-1}{2}\delta\right]\tau_{n-1}^{(1234)}\left(\eta;x_0-\tfrac{\delta}{2},x_i-\tfrac{\delta}{4}\right)\tau_n^{(25)}(\eta;x)+(3,4,5)\text{-cyclic}=0,$$

$$[x_3-x_4]\left[x_2+x_5-\tfrac{n-1}{2}\delta\right]\tau_{n-1}^{(1234)}\left(\eta;x_0-\tfrac{\delta}{2},x_i-\tfrac{\delta}{4}\right)\tau_n^{(15)}(\eta;x)+(3,4,5)\text{-cyclic}=0, \tag{6.6}$$

and the relations $\tau_{n-1}(w(\eta);x)=\tau_{n-1}(\eta;w(x))$ and $\tau_n(w(\eta);x)=\tau_n(\eta;w(x))$ for any $w\in W(D_6)=\langle s_{12},s_{23},s_{34},s_{45},s_{56},s_{127}\rangle$. What we have to do is to show that the function $\tau_{n+1}(\eta;x)$ determined by the recurrence relation (6.1) also satisfies

$$\tau_{n+1}(w(\eta);x)=\tau_{n+1}(\eta;w(x)) \tag{6.7}$$

for any $w\in W(D_6)$. It is obvious that we have (6.7) for $w=s_{12},s_{34},s_{56}$ and s_{127} under the assumption. Then, it is sufficient to verify (6.7) for $w=s_{23}$ and s_{45}. Replacing x by $\widetilde{x}=s_{23}(x)$ in the recurrence relation (6.1), we get

$$\begin{aligned}
&[x_1-x_3][x_2-x_4]\tau_{n+1}(\eta;\widetilde{x})\tau_{n-1}^{(1234)}\left(\widetilde{\eta};x_0-\delta,x_i-\tfrac{\delta}{2}\right)\\
&\quad=\left[x_1+x_4-\tfrac{n}{2}\delta\right]\left[x_2+x_3-\tfrac{n}{2}\delta\right]\tau_n^{(34)}\left(\widetilde{\eta};x_0-\tfrac{\delta}{2},x_i-\tfrac{\delta}{4}\right)\tau_n^{(12)}\left(\widetilde{\eta};x_0-\tfrac{\delta}{2},x_i-\tfrac{\delta}{4}\right)\\
&\quad-\left[x_3+x_4-\tfrac{n}{2}\delta\right]\left[x_1+x_2-\tfrac{n}{2}\delta\right]\tau_n^{(14)}\left(\widetilde{\eta};x_0-\tfrac{\delta}{2},x_i-\tfrac{\delta}{4}\right)\tau_n^{(23)}\left(\widetilde{\eta};x_0-\tfrac{\delta}{2},x_i-\tfrac{\delta}{4}\right),
\end{aligned}$$

where $\widetilde{\eta}=s_{23}(\eta)$. Then, the bilinear equation (6.5) yields $\tau_{n+1}(\widetilde{\eta};x)=\tau_{n+1}(\eta;\widetilde{x})$. Similarly, replacing x by $\widetilde{x}=s_{45}(x)$ in the recurrence relation (6.1), we get

$$\begin{aligned}
&[x_1-x_2][x_3-x_5]\tau_{n+1}(\eta;\widetilde{x})\tau_{n-1}^{(1235)}\left(\widetilde{\eta};x_0-\delta,x_i-\tfrac{\delta}{2}\right)\\
&=\left[x_1+x_5-\tfrac{n}{2}\delta\right]\left[x_2+x_3-\tfrac{n}{2}\delta\right]\tau_n^{(25)}\left(\widetilde{\eta};x_0-\tfrac{\delta}{2},x_i-\tfrac{\delta}{4}\right)\tau_n^{(13)}\left(\widetilde{\eta};x_0-\tfrac{\delta}{2},x_i-\tfrac{\delta}{4}\right)\\
&-\left[x_2+x_5-\tfrac{n}{2}\delta\right]\left[x_1+x_3-\tfrac{n}{2}\delta\right]\tau_n^{(15)}\left(\widetilde{\eta};x_0-\tfrac{\delta}{2},x_i-\tfrac{\delta}{4}\right)\tau_n^{(23)}\left(\widetilde{\eta};x_0-\tfrac{\delta}{2},x_i-\tfrac{\delta}{4}\right),
\end{aligned}$$

where $\widetilde{\eta}=s_{45}(\eta)$. From the bilinear relations (6.6), we get $\tau_{n+1}(\widetilde{\eta};x)=\tau_{n+1}(\eta;\widetilde{x})$. ■

We already have $\tau_{w.\Lambda_0}(w(\eta);x)=\tau_{\Lambda_0}(\eta;w(x))$ and $\tau_{w.\Lambda_1}(w(\eta);x)=\tau_{\Lambda_1}(\eta;w(x))$ for any $w\in W(D_6)$ from Theorems 4.1 and 5.2. Also, these functions satisfy all the bilinear equations of type $(\mathrm{C})'_0$ and $(\mathrm{D})'_1$ from Theorem 5.3. Then we have $\tau_{w.\Lambda_2}(w(\eta);x)=\tau_{\Lambda_2}(\eta;w(x))$ for any $w\in\widetilde{W}\big(D_6^{(1)}\big)$ from Lemma 6.1. Applying Propositions 3.2, 3.3 and 3.4 repeatedly, we can verify the second statement of Theorem 6.1.

With respect to the system of q-difference equations (1.1), one can get the explicit expression for the hypergeometric solutions in terms of the functions $\tau_n(\eta;x)$ introduced in Definition 6.1. When the label $\eta\in S$ is fixed, one can express b_i $(i=1,2,\dots,8)$ and t in terms of the parameters of the q-hypergeometric function ${}_8W_7$. For instance, in the case of $\eta=(6;+)\in S$, we have the following.

Corollary 6.1. *Define the functions $f_n(x)$ and $g_n(x)$ by*

$$f_n(x)=\left(\frac{qa_0a_4a_5}{a_1a_2a_3}\right)^{1/4}t^{1/2}\frac{N_{f,n}(x)}{D_{f,n}(x)},\qquad g_n(x)=\left(\frac{a_1a_2a_3}{qa_0a_4a_5}\right)^{1/4}t^{1/2}\frac{N_{g,n}(x)}{D_{g,n}(x)}$$

with

$$\begin{aligned}
N_{f,n}(x) &= (a_1/a_3)^{1/4}\tau_n^{[1]}\left(x_0-\tfrac{3}{2}\delta, x_i+\tfrac{\delta}{4}\right)\tau_n^{(12)}\left(x_0+\tfrac{3}{2}\delta, x_i-\tfrac{\delta}{4}\right)\\
&\quad -(a_3/a_1)^{1/4}\tau_n^{[3]}\left(x_0-\tfrac{3}{2}\delta, x_i+\tfrac{\delta}{4}\right)\tau_n^{(23)}\left(x_0+\tfrac{3}{2}\delta, x_i-\tfrac{\delta}{4}\right),\\
D_{f,n}(x) &= (a_1/a_3)^{1/4}\tau_n^{[3]}\left(x_0-\tfrac{3}{2}\delta, x_i+\tfrac{\delta}{4}\right)\tau_n^{(23)}\left(x_0+\tfrac{3}{2}\delta, x_i-\tfrac{\delta}{4}\right)\\
&\quad -(a_3/a_1)^{1/4}\tau_n^{[1]}\left(x_0-\tfrac{3}{2}\delta, x_i+\tfrac{\delta}{4}\right)\tau_n^{(12)}\left(x_0+\tfrac{3}{2}\delta, x_i-\tfrac{\delta}{4}\right),\\
N_{g,n}(x) &= (a_2/a_3)^{1/4}\tau_n^{[3]}\left(x_0-\tfrac{3}{2}\delta, x_i+\tfrac{\delta}{4}\right)\tau_n^{(13)}\left(x_0+\tfrac{3}{2}\delta, x_i-\tfrac{\delta}{4}\right)\\
&\quad -(a_3/a_2)^{1/4}\tau_n^{[2]}\left(x_0-\tfrac{3}{2}\delta, x_i+\tfrac{\delta}{4}\right)\tau_n^{(12)}\left(x_0+\tfrac{3}{2}\delta, x_i-\tfrac{\delta}{4}\right),\\
D_{g,n}(x) &= (a_2/a_3)^{1/4}\tau_n^{[2]}\left(x_0-\tfrac{3}{2}\delta, x_i+\tfrac{\delta}{4}\right)\tau_n^{(12)}\left(x_0+\tfrac{3}{2}\delta, x_i-\tfrac{\delta}{4}\right)\\
&\quad -(a_3/a_2)^{1/4}\tau_n^{[3]}\left(x_0-\tfrac{3}{2}\delta, x_i+\tfrac{\delta}{4}\right)\tau_n^{(13)}\left(x_0+\tfrac{3}{2}\delta, x_i-\tfrac{\delta}{4}\right),
\end{aligned}$$

where $a_0, a_1, \ldots, a_5$ are the parameters of the hypergeometric function ${}_8W_7(a_0; a_1, \ldots, a_5; q, z)$ defined by (5.1)*, $\tau_n^{(i_1i_2)}(x) = \tau_n(x)|_{x_i\mapsto x_i+\delta\,(i=i_1,i_2)}$ and $\tau_n^{[i]}(x) = \tau_n(x)|_{x_i\mapsto x_i-\delta}$. Let $b_{i,n}$ $(i = 1,2,\ldots,8)$ be the parameters defined by*

$$\begin{aligned}
b_{1,n} &= -q^{1/4}a_3^{-3/4}(a_0a_4a_5)^{1/4}, & b_{2,n} &= -q^{1/4}a_4^{-3/4}(a_0a_3a_5)^{1/4},\\
b_{3,n} &= -q^{1/4}a_5^{-3/4}(a_0a_3a_4)^{1/4}, & b_{4,n} &= -q^{1/4}a_0^{-3/4}(a_3a_4a_5)^{1/4},\\
b_{5,n} &= -q^{1/4}(a_1a_2)^{-1/2}(a_0a_3a_4a_5)^{1/4}, & b_{6,n} &= -q^{5/4}a_0^{5/4}(a_1a_2)^{-1/2}(a_3a_4a_5)^{-3/4},\\
b_{7,n} &= -q^{n/2-3/4}a_0^{-3/4}(a_1a_2)^{1/2}(a_3a_4a_5)^{1/4}, & b_{8,n} &= -q^{-n/2-3/4}a_0^{-3/4}(a_1a_2)^{1/2}(a_3a_4a_5)^{1/4}
\end{aligned}$$

and $t = (a_1/a_2)^{1/2}$. Then, $f = f_n(x)$ and $g = g_n(x)$ with $b_i = b_{i,n}$ give rise to a solution of the system of q-difference equations (1.1)*.*

A The q-Painlevé system of type $E_7^{(1)}$

A.1 Point configurations and Cremona transformations

As mentioned in Section 2, the q-Painlevé system of type $E_7^{(1)}$ is a discrete dynamical system defined on a family of rational surfaces parameterized by nine-point configurations on $\mathbb{P}^2$ such that six points among them are on a conic C and other three are on a line L [17]. In this section, we set $p_1, p_2, p_4, p_5, p_6, p_7 \in C$ and $p_3, p_8, p_9 \in L$ so that the standard Cremona transformation with respect to (p_1, p_2, p_3) is well-defined as a birational action on the configuration space. One can parameterize the configuration space by [17, 9]

$$X = \begin{bmatrix} 1 & 1 & -u_3 & 1 & 1 & 1 & 1 & -u_8 & -u_9 & x_1\\ u_1 & u_2 & 0 & u_4 & u_5 & u_6 & u_7 & 0 & 0 & x_2\\ u_1^2 & u_2^2 & 1 & u_4^2 & u_5^2 & u_6^2 & u_7^2 & 1 & 1 & x_3 \end{bmatrix},$$

where $u_1, u_2, \ldots, u_9$ are parameters satisfying $u_1u_2\cdots u_9 = q^{-1}$ $(q \in \mathbb{C}^*)$ and the tenth column denotes the coordinates of a general point on $\mathbb{P}^2$. The symmetric group $\mathfrak{S}_6^C \times \mathfrak{S}_3^L$ with $\mathfrak{S}_6^C = \langle s_{12}, s_{24}, s_{45}, s_{56}, s_{67}\rangle$ and $\mathfrak{S}_3^L = \langle s_{38}, s_{89}\rangle$ naturally acts on the space as $\sigma(u_j) = u_{\sigma(j)}$ for any $\sigma \in \mathfrak{S}_6^C \times \mathfrak{S}_3^L$.

Let us normalize X by an action of $GL_3(\mathbb{C})$ as

$$Y = \begin{bmatrix} 1 & 0 & 0 & u_{14} & \ldots & u_{19} & y_1\\ 0 & 1 & 0 & u_{24} & \ldots & u_{29} & y_2\\ 0 & 0 & 1 & u_{34} & \ldots & u_{39} & y_3 \end{bmatrix}.$$

The coordinates u_{ij} can be expressed by

$$u_{ij} = \begin{cases} y_i^C(u_j) & (j = 1, 2, 4, 5, 6, 7), \\ y_i^L(u_j) & (j = 3, 8, 9), \end{cases} \tag{A.1}$$

where $y_i^C(u)$ and $y_i^L(u)$ are defined by

$$y_1^C(u) = \frac{(u_2 - u)(1 - u_2u_3u)}{(u_2 - u_1)(1 - u_1u_2u_3)}, \qquad y_2^C(u) = \frac{(u_1 - u)(1 - u_1u_3u)}{(u_1 - u_2)(1 - u_1u_2u_3)},$$
$$y_3^C(u) = \frac{(u_1 - u)(u_2 - u)}{(1 - u_1u_2u_3)}$$

and

$$y_1^L(u) = \frac{u_2(u_3 - u)}{(u_2 - u_1)(1 - u_1u_2u_3)}, \qquad y_2^L(u) = \frac{u_1(u_3 - u)}{(u_1 - u_2)(1 - u_1u_2u_3)},$$
$$y_3^L(u) = \frac{1 - u_1u_2u}{1 - u_1u_2u_3},$$

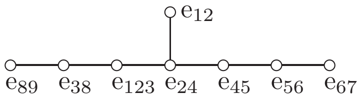
respectively. We further normalize X (or Y) by an action of $PGL_3(\mathbb{C})$ as

$$Z = \begin{bmatrix} 1 & 0 & 0 & 1 & v_{15} & \dots & v_{19} & z_1 \\ 0 & 1 & 0 & 1 & v_{25} & \dots & v_{29} & z_2 \\ 0 & 0 & 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix},$$

where the coordinates v_{ij} and z_i are expressed by

$$v_{ij} = \frac{u_{34}\, u_{ij}}{u_{i4}\, u_{3j}}, \qquad z_i = \frac{u_{34}\, y_i}{u_{i4}\, y_3} \qquad (i = 1, 2;\ j = 5, 6, 7, 8, 9). \tag{A.2}$$

The action of the standard Cremona transformation with respect to (p_1, p_2, p_3), denoted by s_{123}, on these variables is given by $s_{123}(v_{ij}) = 1/v_{ij}$ and $s_{123}(z_i) = 1/z_i$. This transformation together with the symmetric group $\mathfrak{S}_6^C \times \mathfrak{S}_3^L$ generates the affine Weyl group $W(E_7^{(1)}) = \langle s_{12}, s_{24}, s_{45}, s_{56}, s_{67}, s_{38}, s_{89}, s_{123}\rangle$ associated with the Dynkin diagram



In this realization of $W(E_7^{(1)}) \subset W(E_8^{(1)}) = \langle s_{12}, s_{23}, \dots, s_{89}, s_{123}\rangle$, the automorphism of the above Dynkin diagram can be expressed by $\pi = s_{124}\, s_{35}\, s_{68}\, s_{79}$. The action of the extended affine Weyl group $\widetilde{W}(E_7^{(1)}) = \langle s_{12}, s_{24}, s_{45}, s_{56}, s_{67}, s_{38}, s_{89}, s_{123}, \pi\rangle$ on the configuration space is given by

$$\begin{aligned}
&s_{12}(z_1) = z_2, && s_{12}(z_2) = z_1, && s_{38}(z_1) = \frac{z_1 - v_{18}}{1 - v_{18}}, && s_{38}(z_2) = \frac{z_2 - v_{28}}{1 - v_{28}}, \\
&s_{123}(z_1) = \frac{1}{z_1}, && s_{123}(z_2) = \frac{1}{z_2}, && s_{24}(z_1) = \frac{z_2 - z_1}{z_2 - 1}, && s_{24}(z_2) = \frac{z_2}{z_2 - 1}, \\
&s_{45}(z_1) = \frac{z_1}{v_{15}}, && s_{45}(z_2) = \frac{z_2}{v_{25}}
\end{aligned} \tag{A.3}$$

and

$$\pi(v_{ij}) = \frac{v_{ik} - v_{i5}}{v_{ik} - 1} \quad (k = 8, 9, 6, 7 \quad \text{for} \quad j = 6, 7, 8, 9), \qquad \pi(z_i) = \frac{z_i - v_{i5}}{z_i - 1} \tag{A.4}$$

for $i=1,2$. From this representation, we obtain a family of functional equations

$$w(v_{ij}) = S_{ij}^w(v), \qquad w(z_i) = R_i^w(v;z) \tag{A.5}$$

for each $w \in \widetilde{W}(E_7^{(1)})$, where $S_{ij}^w(v)$ and $R_i^w(v;z)$ are some rational functions.

Let us introduce the variables $c_i = e(\varepsilon_i)$ $(i=0,1,\dots,9)$, where ε_i are the coordinate functions introduced in Section 2, and suppose that the parameters $u_1,\dots,u_9$ are expressed by $u_j = c_j/c_0^r$ $(j=1,2,4,5,6,7)$ and $u_j = c_j/c_0^s$ $(j=3,8,9)$ with $2r+s=1$. Then, $y_i^C(u)$ and $y_i^L(u)$ are expressed by

$$\begin{aligned} y_1^C(t/c_0^r) &= \frac{(1-t/c_2)(1-c_2c_3t/c_0)}{(1-c_1/c_2)(1-c_1c_2c_3/c_0)}, \qquad y_2^C(t/c_0^r) = \frac{(1-t/c_1)(1-c_1c_3t/c_0)}{(1-c_2/c_1)(1-c_1c_2c_3/c_0)}, \\ y_3^C(t/c_0^r) &= \frac{c_1c_2(1-t/c_1)(1-t/c_2)}{c_0^{2r}(1-c_1c_2c_3/c_0)} \end{aligned} \tag{A.6}$$

and

$$\begin{aligned} y_1^L(t/c_0^s) &= \frac{c_3(1-t/c_3)}{c_0^s(1-c_1/c_2)(1-c_1c_2c_3/c_0)}, \qquad y_2^L(t/c_0^s) = \frac{c_3(1-t/c_3)}{c_0^s(1-c_2/c_1)(1-c_1c_2c_3/c_0)}, \\ y_3^L(t/c_0^s) &= \frac{1-c_1c_2t/c_0}{1-c_1c_2c_3/c_0}, \end{aligned} \tag{A.7}$$

respectively. These expressions with (A.1) and (A.2) give us

$$\begin{aligned} v_{1j} &= \frac{(u_1-u_4)(1-u_2u_3u_j)}{(1-u_2u_3u_4)(u_1-u_j)} = \frac{[\varepsilon_{14}][\varepsilon_{23j}]}{[\varepsilon_{234}][\varepsilon_{1j}]}, \\ v_{2j} &= \frac{(u_2-u_4)(1-u_1u_3u_j)}{(1-u_1u_3u_4)(u_2-u_j)} = \frac{[\varepsilon_{24}][\varepsilon_{13j}]}{[\varepsilon_{134}][\varepsilon_{2j}]} \end{aligned} \tag{A.8}$$

for $j=5,6,7$ and

$$\begin{aligned} v_{1j} &= \frac{u_2(u_1-u_4)(u_3-u_j)}{(1-u_2u_3u_4)(1-u_1u_2u_j)} = \frac{[\varepsilon_{14}][\varepsilon_{3j}]}{[\varepsilon_{234}][\varepsilon_{12j}]}, \\ v_{2j} &= \frac{u_1(u_2-u_4)(u_3-u_j)}{(1-u_1u_3u_4)(1-u_1u_2u_j)} = \frac{[\varepsilon_{24}][\varepsilon_{3j}]}{[\varepsilon_{134}][\varepsilon_{12j}]} \end{aligned} \tag{A.9}$$

for $j=8,9$. We find that these expressions satisfy $w(v_{ij})(\varepsilon) = v_{ij}(w(\varepsilon))$ for any $w \in \widetilde{W}\big(E_7^{(1)}\big)$, namely (A.8) and (A.9) give a solution to the first equation of (A.5). Also, we see that the functions

$$z_1 = \frac{[\varepsilon_{14}]}{[\varepsilon_{234}]}\frac{[\varepsilon_{123}+\varepsilon_1-t]}{[\varepsilon_1-t]}, \qquad z_2 = \frac{[\varepsilon_{24}]}{[\varepsilon_{134}]}\frac{[\varepsilon_{123}+\varepsilon_2-t]}{[\varepsilon_2-t]} \tag{A.10}$$

and

$$z_1 = \frac{[\varepsilon_{14}]}{[\varepsilon_{234}]}\frac{[\varepsilon_3-t]}{[\varepsilon_{123}+\varepsilon_3-t]}, \qquad z_2 = \frac{[\varepsilon_{24}]}{[\varepsilon_{134}]}\frac{[\varepsilon_3-t]}{[\varepsilon_{123}+\varepsilon_3-t]} \tag{A.11}$$

satisfy $z_i(w(\varepsilon);t) = R_i^w(\varepsilon; z(\varepsilon;t))$ for any $w \in \widetilde{W}\big(E_7^{(1)}\big)$. This means that each of (A.10) and (A.11) provides a one-parameter family of solutions to the second equation of (A.5). These solutions will be called the canonical solution, which correspond to the vertical solution in the context of the differential Painlevé equations.

A.2 The lattice τ-functions and the bilinear equations

Here, we introduce a framework of the lattice τ-functions and show that the action of the extended affine Weyl group $\widetilde{W}(E_7^{(1)})$ is transformed into the bilinear equations for the lattice τ-functions.

Recall that the lattice τ-functions for the discrete Painlevé system of type $E_8^{(1)}$ are indexed by $\Lambda \in M = W\big(E_8^{(1)}\big) \,.\, \mathrm{e}_1 = \{\, \Lambda \in \mathcal{L} \,|\, \langle \Lambda, \Lambda\rangle = 1,\ \langle c, \Lambda\rangle = -1 \,\}$ [8]. Let us decompose the central element $c = 3\mathrm{e}_0 - \mathrm{e}_1 - \mathrm{e}_2 - \cdots - \mathrm{e}_9$ into two irreducible components by [17]

$$c = \mathcal{D}_C + \mathcal{D}_L, \qquad \begin{cases} \mathcal{D}_C = 2\mathrm{e}_0 - \mathrm{e}_1 - \mathrm{e}_2 - \mathrm{e}_4 - \mathrm{e}_5 - \mathrm{e}_6 - \mathrm{e}_7, \\ \mathcal{D}_L = \mathrm{e}_0 - \mathrm{e}_3 - \mathrm{e}_8 - \mathrm{e}_9 \end{cases}$$

corresponding to the conic C and the line L. Then, we have two $W(E_7^{(1)})$-orbits

$$M^C = \{\Lambda \in M \,|\, \langle \mathcal{D}_C, \Lambda\rangle = -1,\ \langle \mathcal{D}_L, \Lambda\rangle = 0\,\} = W\big(E_7^{(1)}\big) \,.\, \mathrm{e}_1,$$
$$M^L = \{\Lambda \in M \,|\, \langle \mathcal{D}_C, \Lambda\rangle = 0,\ \langle \mathcal{D}_L, \Lambda\rangle = -1\,\} = W\big(E_7^{(1)}\big) \,.\, \mathrm{e}_3,$$

which are transformed by the action of the Dynkin diagram automorphism $\pi \in \widetilde{W}\big(E_7^{(1)}\big)$ to each other. Hereafter, we consider the lattice τ-functions τ_Λ for $\Lambda \in M^{E_7} = M^C \coprod M^L = \widetilde{W}\big(E_7^{(1)}\big)\mathrm{e}_1$, on which the action of $w \in \widetilde{W}\big(E_7^{(1)}\big)$ is defined by $w(\tau_\Lambda) = \tau_{w.\Lambda}$.

Suppose that the variables y_i are expressed by

$$y_1 = \frac{\tau_{\mathrm{e}_2}\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_0-\mathrm{e}_2-\mathrm{e}_3}}{N_1}, \qquad y_2 = \frac{\tau_{\mathrm{e}_1}\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_3}}{N_2}, \qquad y_3 = \frac{\tau_{\mathrm{e}_1}\tau_{\mathrm{e}_2}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}}{N_3},$$

where the normalization factors N_1, N_2 and N_3 are certain functions of ε_i $(i = 0, 1, \ldots, 9)$. Denote the τ-functions for the canonical solution on the conic C by $\tau_\Lambda|_C$ $(\Lambda \in M^{E_7})$;

$$y_1^C = \frac{\tau_{\mathrm{e}_2}|_C\,\tau_{\mathrm{e}_3}|_C\,\tau_{\mathrm{e}_0-\mathrm{e}_2-\mathrm{e}_3}|_C}{N_1}, \qquad y_2^C = \frac{\tau_{\mathrm{e}_1}|_C\,\tau_{\mathrm{e}_3}|_C\,\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_3}|_C}{N_2},$$
$$y_3^C = \frac{\tau_{\mathrm{e}_1}|_C\,\tau_{\mathrm{e}_2}|_C\,\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}|_C}{N_3}.$$

Comparing this expression with (A.6) and (A.7), one can assume that the τ-functions for the canonical solutions on C and L are expressed by

$$\tau_{\mathrm{e}_j}|_C = \begin{cases} (1 - e(t - \varepsilon_j))\, e(\alpha\varepsilon_j) & (j = 1, 2, 4, 5, 6, 7), \\ e(\beta\varepsilon_j) & (j = 3, 8, 9), \end{cases}$$

and

$$\tau_{\mathrm{e}_j}|_L = \begin{cases} e(\beta\varepsilon_j) & (j = 1, 2, 4, 5, 6, 7), \\ (1 - e(t - \varepsilon_j))\, e(\alpha\varepsilon_j) & (j = 3, 8, 9), \end{cases}$$

respectively, so that the canonical solutions y_i^C and y_i^L are transformed by the action of the Dynkin diagram automorphism π to each other. These requirements lead us to $r = 1/4$, $s = 1/2$ and $\beta = \alpha - 1/2$, and we get $N_1 = -c_0^{\alpha-1/2}c_1[\varepsilon_{12}][\varepsilon_{123}]$, $N_2 = c_0^{\alpha-1/2}c_2[\varepsilon_{12}][\varepsilon_{123}]$ and $N_3 = c_0^{\alpha-1/2}c_3^{1/2}[\varepsilon_{123}]$.

Let us introduce the variables $f_i\,(i = 1, 2, 3)$ by $f_1 = \dfrac{\tau_{\mathrm{e}_0-\mathrm{e}_2-\mathrm{e}_3}}{\tau_{\mathrm{e}_1}}$, $f_2 = \dfrac{\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_3}}{\tau_{\mathrm{e}_2}}$ and $f_3 = \dfrac{\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}}{\tau_{\mathrm{e}_3}}$. Then, the inhomogeneous coordinates z_1 and z_2 are expressed by

$$z_1 = \frac{[\varepsilon_{14}]}{[\varepsilon_{234}]}\frac{f_1}{f_3}, \qquad z_2 = \frac{[\varepsilon_{24}]}{[\varepsilon_{134}]}\frac{f_2}{f_3}. \tag{A.12}$$

From (A.3), (A.4) and (A.12), one thus obtain a realization of the extended affine Weyl group $\widetilde{W}(E_7^{(1)})$ as a group of birational transformations.

Theorem A.1. *The action of* $\widetilde{W}(E_7^{(1)})$ *on the variables* (f_1, f_2, f_3) *and* $(\tau_{\mathrm{e}_1}, \dots, \tau_{\mathrm{e}_9})$ *is given by*

$$\sigma(\tau_{\mathrm{e}_i}) = \tau_{\mathrm{e}_{\sigma(i)}} \qquad (\sigma \in \mathfrak{S}_6^C \times \mathfrak{S}_3^L),$$
$$s_{123}(\tau_{\mathrm{e}_1}) = \tau_{\mathrm{e}_1} f_1, \qquad s_{123}(\tau_{\mathrm{e}_2}) = \tau_{\mathrm{e}_2} f_2, \qquad s_{123}(\tau_{\mathrm{e}_3}) = \tau_{\mathrm{e}_3} f_3,$$
$$s_{12}(f_1) = f_2, \qquad s_{12}(f_2) = f_1,$$
$$s_{38}(f_1) = \frac{\tau_{\mathrm{e}_3}}{\tau_{\mathrm{e}_8}} \frac{[\varepsilon_{128}]f_1 - [\varepsilon_{38}]f_3}{[\varepsilon_{123}]}, \qquad s_{38}(f_2) = \frac{\tau_{\mathrm{e}_3}}{\tau_{\mathrm{e}_8}} \frac{[\varepsilon_{128}]f_2 - [\varepsilon_{38}]f_3}{[\varepsilon_{123}]}, \qquad s_{38}(f_3) = \frac{\tau_{\mathrm{e}_3}}{\tau_{\mathrm{e}_8}} f_3,$$
$$s_{123}(f_1) = \frac{1}{f_1}, \qquad s_{123}(f_2) = \frac{1}{f_2}, \qquad s_{123}(f_3) = \frac{1}{f_3},$$
$$s_{24}(f_1) = \frac{\tau_{\mathrm{e}_2}}{\tau_{\mathrm{e}_4}} \frac{[\varepsilon_{14}][\varepsilon_{134}]f_1 - [\varepsilon_{24}][\varepsilon_{234}]f_2}{[\varepsilon_{12}][\varepsilon_{123}]}, \qquad s_{24}(f_2) = \frac{\tau_{\mathrm{e}_2}}{\tau_{\mathrm{e}_4}} f_2,$$
$$s_{24}(f_3) = \frac{\tau_{\mathrm{e}_2}}{\tau_{\mathrm{e}_4}} \frac{[\varepsilon_{134}]f_3 - [\varepsilon_{24}]f_2}{[\varepsilon_{123}]},$$

and

$$\pi(\tau_{\mathrm{e}_1}) = \frac{\tau_{\mathrm{e}_1}\tau_{\mathrm{e}_3}}{\tau_{\mathrm{e}_4}} \frac{-[\varepsilon_{14}]f_1 + [\varepsilon_{234}]f_3}{[\varepsilon_{123}]}, \qquad \pi(\tau_{\mathrm{e}_2}) = \frac{\tau_{\mathrm{e}_2}\tau_{\mathrm{e}_3}}{\tau_{\mathrm{e}_4}} \frac{-[\varepsilon_{24}]f_2 + [\varepsilon_{134}]f_3}{[\varepsilon_{123}]},$$
$$\pi(\tau_{\mathrm{e}_3}) = \tau_{\mathrm{e}_5}, \qquad \pi(\tau_{\mathrm{e}_4}) = \tau_{\mathrm{e}_3} f_3, \qquad \pi(\tau_{\mathrm{e}_5}) = \tau_{\mathrm{e}_3},$$
$$\pi(\tau_{\mathrm{e}_6}) = \tau_{\mathrm{e}_8}, \qquad \pi(\tau_{\mathrm{e}_7}) = \tau_{\mathrm{e}_9}, \qquad \pi(\tau_{\mathrm{e}_8}) = \tau_{\mathrm{e}_6}, \qquad \pi(\tau_{\mathrm{e}_9}) = \tau_{\mathrm{e}_7},$$
$$\pi(f_1) = \frac{\tau_{\mathrm{e}_4}}{\tau_{\mathrm{e}_5}} \frac{[\varepsilon_{15}]f_1 - [\varepsilon_{235}]f_3}{[\varepsilon_{14}]f_1 - [\varepsilon_{234}]f_3}, \qquad \pi(f_2) = \frac{\tau_{\mathrm{e}_4}}{\tau_{\mathrm{e}_5}} \frac{[\varepsilon_{25}]f_2 - [\varepsilon_{135}]f_3}{[\varepsilon_{24}]f_2 - [\varepsilon_{134}]f_3}, \qquad \pi(f_3) = \frac{\tau_{\mathrm{e}_4}}{\tau_{\mathrm{e}_5}}.$$

These give rise to a representation of $\widetilde{W}(E_7^{(1)})$.

From this theorem, we immediately obtain the bilinear equations (2.2) and (2.3) for mutually distinct indices $i, j, k, l \in \{1, 2, 4, 5, 6, 7\}$ and $r, s \in \{3, 8, 9\}$.

B Another representation

In this section we again set $C = \{1, 2, 3, 4, 5, 6\}$ and $L = \{7, 8, 9\}$. The lattice τ-functions τ_Λ ($\Lambda \in M^{E_7}$) for the q-Painlevé system of type $E_7^{(1)}$ satisfy the following bilinear equations

$$\begin{aligned} &[\varepsilon_{rs}]\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j} = [\varepsilon_{ijs}]\tau_{\mathrm{e}_r}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_r} - [\varepsilon_{ijr}]\tau_{\mathrm{e}_s}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_s}, \\ &[\varepsilon_{jk}]\tau_{\mathrm{e}_r}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_r} = [\varepsilon_{ikr}]\tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j} - [\varepsilon_{ijr}]\tau_{\mathrm{e}_k}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_k}, \end{aligned} \tag{B.1}$$

and

$$[\varepsilon_{ij}][\varepsilon_{ijr}]\tau_{\mathrm{e}_k}\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_r} + (i,j,k)\text{-cyclic} = 0,$$
$$[\varepsilon_{ij}][\varepsilon_{kl}]\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_k-\mathrm{e}_l} + (i,j,k)\text{-cyclic} = 0,$$

where $i, j, k, l \in C$ and $r, s \in L$.

As discussed in [8, 11], when $\Lambda = d\mathrm{e}_0 - \nu_1\mathrm{e}_1 - \cdots - \nu_9\mathrm{e}_9$, the τ-function τ_Λ is characterized by a homogeneous polynomial of degree d in the homogeneous coordinates of $\mathbb{P}^2$ which has a zero of multiplicity $\geq \nu_j$ at p_j for each $j = 1, \dots, 9$. From this geometric consideration, we find that we have the following bilinear equation

$$\tau_{\mathrm{e}_i}\tau_{\mathrm{e}_0-\mathrm{e}_i-\mathrm{e}_9} - \tau_{\mathrm{e}_j}\tau_{\mathrm{e}_0-\mathrm{e}_j-\mathrm{e}_9} + [\varepsilon_{ij}][\varepsilon_{ij9}]\, d_L \tau_{\mathrm{e}_7}\tau_{\mathrm{e}_8} = 0 \tag{B.2}$$

for $i,j\in C$, which associates with the line passing through the point p_9. The factor d_L corresponds to the irreducible component of the anti-canonical devisor $\mathcal{D}_L=\mathrm{e}_0-\mathrm{e}_7-\mathrm{e}_8-\mathrm{e}_9$, and is invariant under the action of $W\big(E_7^{(1)}\big)$. The action of the Dynkin diagram automorphism π on this bilinear equation gives us the second equation in (2.4) with $d_C=\pi(d_L)$.

One can get another representation $\widetilde{W}\big(E_7^{(1)}\big)$ for the τ-variables, by using the bilinear equations (B.1) and (B.2).

Theorem B.1. *Let us introduce the variables σ and $\widetilde{\sigma}$ by*

$$\sigma=d_r\frac{e(\frac14\varepsilon_{23})\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_3}-e(-\frac14\varepsilon_{23})\tau_{\mathrm{e}_2}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}}{[\varepsilon_{23}]},$$
$$\widetilde{\sigma}=d_l\frac{e(\frac14\varepsilon_{23})\tau_{\mathrm{e}_2}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}-e(-\frac14\varepsilon_{23})\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_3}}{[\varepsilon_{23}]},$$

where the factors d_l and d_r are given by $d_l=e\left(\frac{1}{16}\alpha_l-\frac{1}{16}\alpha_r\right)$ and $d_r=d_l^{-1}$ with $\alpha_l=3\varepsilon_{127}+2\varepsilon_{78}+\varepsilon_{89}$ and $\alpha_r=3\varepsilon_{34}+2\varepsilon_{45}+\varepsilon_{56}$. Then, the action of $\widetilde{W}\big(E_7^{(1)}\big)$ on the variables τ_{e_3}, τ_{e_4}, τ_{e_5}, τ_{e_6}, τ_{e_7}, τ_{e_8}, τ_{e_9}, $\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}$, σ and $\widetilde{\sigma}$ is described as follows:

$$s_{89}:\ \tau_{\mathrm{e}_8}\leftrightarrow\tau_{\mathrm{e}_9},\qquad s_{78}:\ \tau_{\mathrm{e}_7}\leftrightarrow\tau_{\mathrm{e}_8},\qquad s_{127}:\ \tau_{\mathrm{e}_7}\leftrightarrow\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2},$$
$$s_{34}:\ \tau_{\mathrm{e}_3}\leftrightarrow\tau_{\mathrm{e}_4},\qquad s_{45}:\ \tau_{\mathrm{e}_4}\leftrightarrow\tau_{\mathrm{e}_5},\qquad s_{56}:\ \tau_{\mathrm{e}_5}\leftrightarrow\tau_{\mathrm{e}_6},$$
$$s_{23}(\tau_{\mathrm{e}_3})=\frac{e(-\frac14\varepsilon_{23})\,d_r^{-1}\sigma+e(\frac14\varepsilon_{23})\,d_l^{-1}\widetilde{\sigma}}{\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}},$$
$$s_{23}(\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2})=\frac{e(\frac14\varepsilon_{23})\,d_r^{-1}\sigma+e(-\frac14\varepsilon_{23})\,d_l^{-1}\widetilde{\sigma}}{\tau_{\mathrm{e}_3}},\tag{B.3}$$
$$s_{12}(\sigma)=\frac{e(-\frac14\varepsilon_{12})\tau_{\mathrm{e}_7}\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_9}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}+e(\frac14\varepsilon_{12})\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_4}\tau_{\mathrm{e}_5}\tau_{\mathrm{e}_6}}{\widetilde{\sigma}},$$
$$s_{12}(\widetilde{\sigma})=\frac{e(\frac14\varepsilon_{12})\tau_{\mathrm{e}_7}\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_9}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}+e(-\frac14\varepsilon_{12})\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_4}\tau_{\mathrm{e}_5}\tau_{\mathrm{e}_6}}{\sigma},\tag{B.4}$$
$$\pi:\ \tau_{\mathrm{e}_3}\leftrightarrow\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2},\qquad\tau_{\mathrm{e}_4}\leftrightarrow\tau_{\mathrm{e}_7},\qquad\tau_{\mathrm{e}_5}\leftrightarrow\tau_{\mathrm{e}_8},\qquad\tau_{\mathrm{e}_6}\leftrightarrow\tau_{\mathrm{e}_9},\qquad\sigma\leftrightarrow\widetilde{\sigma}.$$

These also give rise to another representation of $\widetilde{W}\big(E_7^{(1)}\big)$.

Proof. We immediately get (B.3) from the definition of σ and $\widetilde{\sigma}$. It is easy to see that σ and $\widetilde{\sigma}$ are invariant under the action of s_{89}, s_{78}, s_{127}, s_{23}, s_{34}, s_{45} and s_{56}. The bilinear equations (B.1) and (B.2) yield

$$\begin{aligned}&\left[e\left(\tfrac14\varepsilon_{23}\right)\tau_{\mathrm{e}_2}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}-e\left(-\tfrac14\varepsilon_{23}\right)\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_3}\right]\\&\quad\times\left[e\left(\tfrac14\varepsilon_{13}\right)\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_0-\mathrm{e}_2-\mathrm{e}_3}-e\left(-\tfrac14\varepsilon_{13}\right)\tau_{\mathrm{e}_1}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}\right]\\&\quad=[\varepsilon_{13}][\varepsilon_{23}]\left[e\left(-\tfrac14\varepsilon_{12}\right)\tau_{\mathrm{e}_7}\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_9}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}+e\left(\tfrac14\varepsilon_{12}\right)\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_4}\tau_{\mathrm{e}_5}\tau_{\mathrm{e}_6}\right],\end{aligned}$$

from which we get the first equation of (B.4). Since we see that $\pi:\sigma\leftrightarrow\widetilde{\sigma}$ by the definition, we immediately get the second equation of (B.4). ∎

Note that this representation coincides with that constructed by Tsuda [18]. The above theorem gives us the following proposition.

Proposition B.1. *Define the variables f and g by*

$$f=\frac{\widetilde{\sigma}}{\sigma}\frac{e(\frac14\varepsilon_{12})\tau_{\mathrm{e}_7}\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_9}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}+e(-\frac14\varepsilon_{12})\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_4}\tau_{\mathrm{e}_5}\tau_{\mathrm{e}_6}}{e(-\frac14\varepsilon_{12})\tau_{\mathrm{e}_7}\tau_{\mathrm{e}_8}\tau_{\mathrm{e}_9}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}+e(\frac14\varepsilon_{12})\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_4}\tau_{\mathrm{e}_5}\tau_{\mathrm{e}_6}},\qquad g=\frac{\widetilde{\sigma}}{\sigma}.$$

Then, the action of $\widetilde{W}(E_7^{(1)})$ *on these variables is described by*

$$s_{12}:\ f \leftrightarrow g, \qquad \pi:\ f \mapsto \frac{1}{f}, \qquad g \mapsto \frac{1}{g}, \qquad s_{23}:\ f \mapsto h,$$

where h *is a rational function determined by*

$$\frac{h + d_l^2 e(\frac{1}{2}\varepsilon_{12})}{h + d_l^2 e(-\frac{1}{2}\varepsilon_{12})} = \frac{f + d_l^2 e(\frac{1}{2}\varepsilon_{13})}{f + d_l^2 e(-\frac{1}{2}\varepsilon_{13})} \frac{g + d_l^2 e(-\frac{1}{2}\varepsilon_{23})}{g + d_l^2 e(\frac{1}{2}\varepsilon_{23})}. \tag{B.5}$$

Note that the variable f can be expressed by

$$f = d_l^2 \frac{e(\frac{1}{4}\varepsilon_{13})\tau_{\mathrm{e}_1}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2} - e(-\frac{1}{4}\varepsilon_{13})\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_0-\mathrm{e}_2-\mathrm{e}_3}}{e(\frac{1}{4}\varepsilon_{13})\tau_{\mathrm{e}_3}\tau_{\mathrm{e}_0-\mathrm{e}_2-\mathrm{e}_3} - e(-\frac{1}{4}\varepsilon_{13})\tau_{\mathrm{e}_1}\tau_{\mathrm{e}_0-\mathrm{e}_1-\mathrm{e}_2}}, \tag{B.6}$$

and $h = s_{13}(g)$.

C A derivation of the difference equations

By writing down the action of the translation operator $T_{21} \in W(E_7^{(1)})$ on the variables f and g, we will get the system of q-difference equations (1.1). Hereafter, we denote the time evolution of a variable x by $\overline{x} = T_{21}(x)$ and $\underline{x} = T_{21}^{-1}(x)$. Let us introduce the transformation μ by $\mu = s_{12}s_{23}s_{147}s_{158}s_{169}$. It is easy to see that $T_{21} = \mu^2$ and $\mu(g) = f$. We also introduce the auxiliary variables $k = s_{247}(h)$ and $l = s_{258}(k)$. Note that we have $\overline{g} = s_{269}(l)$.

Lemma C.1. *We have*

$$\frac{f + d_l^2 e(-\frac{1}{2}\varepsilon_{13})}{f + d_l^2 e(\frac{1}{2}\varepsilon_{13})} = \frac{f/g - e(-\frac{1}{2}\varepsilon_{12})}{f/g + e(-\frac{1}{2}\varepsilon_{12})} \frac{f/h - e(-\frac{1}{2}\varepsilon_{23})}{f/h - e(\frac{1}{2}\varepsilon_{23})},$$
$$\frac{f + d_l^2 \kappa_{47} e(-\frac{1}{2}\varepsilon_{247})}{f + d_l^2 \kappa_{47} e(\frac{1}{2}\varepsilon_{247})} = \frac{f/h - e(\frac{1}{2}\varepsilon_{23})}{f/h - e(-\frac{1}{2}\varepsilon_{23})} \frac{f/k - e(-\frac{1}{2}\varepsilon_{347})}{f/k - e(\frac{1}{2}\varepsilon_{347})},$$
$$\frac{f + d_l^2 \kappa_{58} e(-\frac{1}{2}\varepsilon_{258})}{f + d_l^2 \kappa_{58} e(\frac{1}{2}\varepsilon_{258})} = \frac{f/k - e(\frac{1}{2}\varepsilon_{347})}{f/k - e(-\frac{1}{2}\varepsilon_{347})} \frac{f/l - e(-\frac{1}{2}\varepsilon_{347} - \frac{1}{2}\varepsilon_{258})}{f/l - e(\frac{1}{2}\varepsilon_{347} + \frac{1}{2}\varepsilon_{258})},$$
$$\frac{f + d_l^2 \kappa_{69} e(-\frac{1}{2}\varepsilon_{269})}{f + d_l^2 \kappa_{69} e(\frac{1}{2}\varepsilon_{269})} = \frac{f/l - e(\frac{1}{2}\varepsilon_{347} + \frac{1}{2}\varepsilon_{258})}{f/l - e(-\frac{1}{2}\varepsilon_{347} - \frac{1}{2}\varepsilon_{258})} \frac{f/\overline{g} - e(-\frac{1}{2}(\varepsilon_{12} + \delta))}{f/\overline{g} - e(\frac{1}{2}(\varepsilon_{12} + \delta))}.$$

Proof. The first equation is reduced to (B.5). The other expressions can be rewritten as

$$\frac{f + d_l^2 \kappa_{47} e(-\frac{1}{2}\varepsilon_{247})}{f + d_l^2 \kappa_{47} e(\frac{1}{2}\varepsilon_{247})} \frac{h + d_l^2 \kappa_{47} e(\frac{1}{2}\varepsilon_{347})}{h + d_l^2 \kappa_{47} e(-\frac{1}{2}\varepsilon_{347})} \frac{k + d_l^2 \kappa_{47} e(-\frac{1}{2}\varepsilon_{23})}{k + d_l^2 \kappa_{47} e(\frac{1}{2}\varepsilon_{23})} = 1,$$
$$\frac{f + d_l^2 \kappa_{58} e(-\frac{1}{2}\varepsilon_{258})}{f + d_l^2 \kappa_{58} e(\frac{1}{2}\varepsilon_{258})} \frac{k + d_l^2 \kappa_{58} e(\frac{1}{2}\varepsilon_{347} + \frac{1}{2}\varepsilon_{258})}{k + d_l^2 \kappa_{58} e(-\frac{1}{2}\varepsilon_{347} - \frac{1}{2}\varepsilon_{258})} \frac{l + d_l^2 \kappa_{58} e(-\frac{1}{2}\varepsilon_{347})}{l + d_l^2 \kappa_{58} e(\frac{1}{2}\varepsilon_{347})} = 1,$$
$$\frac{f + d_l^2 \kappa_{69} e(-\frac{1}{2}\varepsilon_{269})}{f + d_l^2 \kappa_{69} e(\frac{1}{2}\varepsilon_{269})} \frac{l + d_l^2 \kappa_{69} e(\frac{1}{2}(\varepsilon_{12} + \delta))}{l + d_l^2 \kappa_{69} e(-\frac{1}{2}(\varepsilon_{12} + \delta))} \frac{\overline{g} + d_l^2 \kappa_{69} e(-\frac{1}{2}\varepsilon_{347} - \frac{1}{2}\varepsilon_{258})}{\overline{g} + d_l^2 \kappa_{69} e(\frac{1}{2}\varepsilon_{347} + \frac{1}{2}\varepsilon_{258})} = 1, \tag{C.1}$$

where $\kappa_{47} = e(\frac{1}{2}\varepsilon_{34} - \frac{1}{2}\varepsilon_{127})$, $\kappa_{58} = e(\frac{1}{2}\varepsilon_{35} - \frac{1}{2}\varepsilon_{128})$ and $\kappa_{69} = e(\frac{1}{2}\varepsilon_{36} - \frac{1}{2}\varepsilon_{129})$. From the expressions (B.6), we get

$$\frac{f + d_l^2 \kappa_{47} e(-\frac{1}{2}\varepsilon_{247})}{f + d_l^2 \kappa_{47} e(\frac{1}{2}\varepsilon_{247})} = e\left(-\tfrac{1}{2}\varepsilon_{247}\right) \frac{\tau_{\mathrm{e}_7}\tau_{\mathrm{e}_0-\mathrm{e}_2-\mathrm{e}_7}}{\tau_{\mathrm{e}_4}\tau_{\mathrm{e}_0-\mathrm{e}_2-\mathrm{e}_4}}.$$

By applying $s_{13}s_{12}$ and s_{247} successively, we also get

$$\frac{h+d_l^2\kappa_{47}e(\frac{1}{2}\varepsilon_{347})}{h+d_l^2\kappa_{47}e(-\frac{1}{2}\varepsilon_{347})}=e\left(\tfrac{1}{2}\varepsilon_{347}\right)\frac{\tau_{e_4}\tau_{e_0-e_3-e_4}}{\tau_{e_7}\tau_{e_0-e_3-e_7}},$$

$$\frac{k+d_l^2\kappa_{47}e(-\frac{1}{2}\varepsilon_{23})}{k+d_l^2\kappa_{47}e(\frac{1}{2}\varepsilon_{23})}=e\left(-\tfrac{1}{2}\varepsilon_{23}\right)\frac{\tau_{e_0-e_2-e_4}\tau_{e_0-e_3-e_7}}{\tau_{e_0-e_2-e_7}\tau_{e_0-e_3-e_4}},$$

and then the first equation of (C.1). The second and third equations of (C.1) can be obtained by a similar way. ■

The above lemma immediately gives us

$$\frac{fg-e(\frac{1}{2}\varepsilon_{12})}{fg-e(-\frac{1}{2}\varepsilon_{12})}\frac{f\overline{g}-e(\frac{1}{2}(\varepsilon_{12}+\delta))}{f\overline{g}-e(-\frac{1}{2}(\varepsilon_{12}+\delta))}$$
$$=\frac{f+d_l^2e(\frac{1}{2}\varepsilon_{13})}{f+d_l^2e(-\frac{1}{2}\varepsilon_{13})}\frac{f+d_l^2\kappa_{47}e(\frac{1}{2}\varepsilon_{247})}{f+d_l^2\kappa_{47}e(-\frac{1}{2}\varepsilon_{247})}\frac{f+d_l^2\kappa_{58}e(\frac{1}{2}\varepsilon_{258})}{f+d_l^2\kappa_{58}e(-\frac{1}{2}\varepsilon_{258})}\frac{f+d_l^2\kappa_{69}e(\frac{1}{2}\varepsilon_{269})}{f+d_l^2\kappa_{69}e(-\frac{1}{2}\varepsilon_{269})},$$

where we replace g with $1/g$. Applying μ^{-1} to the above equation, we also get

$$\frac{\underline{f}g-e(-\frac{1}{2}(\varepsilon_{12}-\delta))}{\underline{f}g-e(\frac{1}{2}(\varepsilon_{12}-\delta))}\frac{fg-e(-\frac{1}{2}\varepsilon_{12})}{fg-e(\frac{1}{2}\varepsilon_{12})}$$
$$=\frac{g+d_r^2e(\frac{1}{2}\varepsilon_{23})}{g+d_r^2e(-\frac{1}{2}\varepsilon_{23})}\frac{g+d_r^2\kappa_{47}^{-1}e(\frac{1}{2}\varepsilon_{147})}{g+d_r^2\kappa_{47}^{-1}e(-\frac{1}{2}\varepsilon_{147})}\frac{g+d_r^2\kappa_{58}^{-1}e(\frac{1}{2}\varepsilon_{158})}{g+d_r^2\kappa_{58}^{-1}e(-\frac{1}{2}\varepsilon_{158})}\frac{g+d_r^2\kappa_{69}^{-1}e(\frac{1}{2}\varepsilon_{169})}{g+d_r^2\kappa_{69}^{-1}e(-\frac{1}{2}\varepsilon_{169})}.$$

Let us introduce the parameters b_i $(i=1,2,\dots,8)$ and the independent variable t by

$$b_1=-q^{1/8}d_l^2e\left(\tfrac{1}{4}(\varepsilon_{13}+\varepsilon_{23})\right),\qquad b_2=-q^{1/8}d_l^2e\left(\tfrac{1}{4}(\varepsilon_{14}+\varepsilon_{24})+\tfrac{1}{2}\varepsilon_{34}\right),$$
$$b_3=-q^{1/8}d_l^2e\left(\tfrac{1}{4}(\varepsilon_{15}+\varepsilon_{25})+\tfrac{1}{2}\varepsilon_{35}\right),\qquad b_4=-q^{1/8}d_l^2e\left(\tfrac{1}{4}(\varepsilon_{16}+\varepsilon_{26})+\tfrac{1}{2}\varepsilon_{36}\right),$$
$$b_5=-q^{1/8}d_l^2e\left(-\tfrac{1}{4}(\varepsilon_{13}+\varepsilon_{23})\right),\qquad b_6=-q^{1/8}d_l^2e\left(-\tfrac{1}{4}(\varepsilon_{137}+\varepsilon_{237})-\tfrac{1}{2}\varepsilon_{127}\right),$$
$$b_7=-q^{1/8}d_l^2e\left(-\tfrac{1}{4}(\varepsilon_{138}+\varepsilon_{238})-\tfrac{1}{2}\varepsilon_{128}\right),\qquad b_8=-q^{1/8}d_l^2e\left(-\tfrac{1}{4}(\varepsilon_{139}+\varepsilon_{239})-\tfrac{1}{2}\varepsilon_{129}\right)$$

and $t=e(\frac{1}{2}\varepsilon_{12})$, respectively. Replacing the dependent variables f and g with $q^{-1/8}t^{-1/2}f$ and $q^{1/8}t^{-1/2}g$, respectively, we get the system of difference equations (1.1).

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