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Homological Algebra and Divergent Series*

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Abstract. We study some features of infinite resolutions of Koszul algebras motivated by the developments in the string theory initiated by Berkovits.

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To Friedrich Hirzebruch on his 80-th anniversary, with admiration

1 Introduction

1.1. This article consists of two parts. The first part is a simple exercise on Mellin transform. The second one is a review on some numerical aspects of Koszul duality. An object which lies behind the two parts is a Tate resolution of a commutative ring over a field of characteristic zero.

Let us give some more details on the contents. In the first part we develop the elegant ideas, due to physicists [5], which allow to define the numerical invariants of projective varieties, doing a regularization of some divergent series connected with their homogeneous rings.

Let $R_0 = k[x_0, \dots, x_N]$ be a polynomial algebra over a field k, $f_1, \dots, f_p \in R_0$ homogeneous elements of degrees $d_i = \deg f_i > 0$ which generate the ideal $I = (f_1, \dots, f_p)$. Consider the quotient algebra $A = R_0/I$; it is graded

$$A = \bigoplus_{j=0}^{\infty} A^j.$$

Geometrically the projection $R_0 \longrightarrow A$ corresponds to a closed embedding $i : X := \operatorname{Proj} A \hookrightarrow \mathbb{P}^N := \operatorname{Proj} R_0$; the projective variety X is defined in $\mathbb{P}^N$ by the equations $f_1 = 0, \dots, f_p = 0$. The algebra A is called the homogeneous ring of X (it depends on the embedding into the projective space).

Let us call *a semi-free resolution* of A the following data:

(i) An associative unital bi-graded k-algebra $R = \bigoplus_{i,j=0}^{\infty} R_i^j$ (so $R_i^j \cdot R_l^m \subset R_{i+l}^{j+m}$); the indexes i and j will be called the homological degree and the polynomial degree respectively. We set $R_i := \oplus_j R_i^j$, $R^j := \oplus_i R_i^j$. We introduce a structure of a superalgebra on R by defining the parity to be equal to the parity of the homological degree.

The multiplication has to be super-commutative, i.e. for $x \in R_i$, $y \in R_l$ we have $xy = (-1)^{il} yx$.

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R should be equipped with a differential $d : R \longrightarrow R$, $d^2 = 0$, such that $d(R_i^j) \subset R_{i-1}^j$ and for $x \in R_i$, $y \in R$,

$$d(xy) = dx \cdot y + (-1)^i x \cdot dy.$$

(ii) R_0 should be equipped with a morphism of algebras $\epsilon : R_0 \longrightarrow A$ such that $\epsilon(R_0^j) \subset A^j$.
(iii) For each $j \geq 0$, R^j should be a resolution of A^j, i.e. the complexes

$$\cdots \longrightarrow R_2^j \xrightarrow{d} R_1^j \xrightarrow{d} R_0^j \xrightarrow{\epsilon} A^j \longrightarrow 0$$

should be exact.

(iv) If one forgets the differential d, R should be a polynomial (super)algebra in homogeneous generators. We suppose that in each polynomial degree one has a finite number of generators.

Semi-free resolutions exist for each A but they are not unique (sometimes they are called *Tate resolutions*).

Example 1.1. Let us consider the Koszul complex

$$K(R_0; \mathbf{f}) := k[x_0, \dots, x_N] \otimes \Lambda\langle \xi_1 \cdots \xi_M \rangle,$$

where Λ denotes the exterior algebra; the differential is defined by the formula $d\xi_j = f_j$. The requirements *(i)*, *(ii)*, *(iv)* are fulfilled (the homological degree of x_i is 1). The requirement *(iii)* holds iff the sequence $\mathbf{f} = \{f_1, \dots, f_M\}$ is regular.

If this is the case, the Krull dimension of A is given by a simple formula

$$\dim A = N + 1 - M, \tag{1.1}$$

where $N + 1$ is the number of x_i and M is the number of ξ_j; one can say that $N + 1 - M$ is the "superdimension" of the polynomial ring $k[x_0, \dots, x_N] \otimes \Lambda\langle \xi_1 \cdots \xi_M \rangle$ (the number of even generators minus the number of odd generators).

In general one can consider a semi-free resolution of A as a natural replacement of the Koszul complex. One of the main goals of this paper is *to propose an analog of the equation* (1.1) *valid for not necessarily complete intersections.*

If our algebra A is not a complete intersection then a semi-free resolution will be infinite. However, one can associate with such a resolution a sequence of integer numbers and to show that $\dim A$ can be written as a "sum" of these numbers; this will be the required generalization of (1.1). One puts the sum inside of the quotation marks since we have here a divergent series, the sum of which is calculated be means of a regularization, following a classical procedure of Riemann.

In fact, our formula is nothing else but the Mellin transform of the classical theorem by Hilbert which says that $\dim A$ is equal to the order of pole of the Hilbert series of A at 1. Moreover, we give a similar interpretation for the other numerical invariants of A such as the degree. For details, see Sections 2.1–2.4.

Except for the case of the complete intersection which is rather seldom, there exists a wider class of algebras which admit a remarkable explicit semi-free resolution. Namely, if A is a *Koszul algebra* then one can take for R the Chevalley cochain complex of the graded Lie algebra L Koszul dual to A; one can call it *the Koszul–Chevalley resolution*, cf. Section 3.4 for details.

As has been noted by many the notion of the Koszul duality is of fundamental importance in Physics. For us a motivating reference is the work of M. Movshev and A. Schwarz. It turns out it is closely related to the "gauge fields-strings duality" dear to Polyakov, cf. [28].

In the second part of this article we discuss a (probably) simplest nontrivial case of a Koszul algebra: we take for A the homogeneous ring of the Veronese curve $X = \mathbb{P}^1 \hookrightarrow \mathbb{P}^N$. This part contains no new results; rather it is a review of some classical and modern theorems related to the Koszul–Chevalley resolution of A. This topic turns out remarkably rich; it revolves around the *Gauss cyclotomic identity*. We see here Euler products, Witt theorem, Polya theory and a formula of Polyakov. A particular case of a deep theorem by Kempf and Bezrukavnikov says that A is a Koszul algebra. We will see in Theorem 3.17 that the "numerical" manifestation of this fact is precisely the Gauss cyclotomic identity.

1.2. At the end of this introduction let us say a few words about the algebras interesting to the physicists; we hope to return to these questions later on. In his seminal papers N. Berkovits considers the algebra of functions on the space of *pure spinors* of dimension 10, the quotient of the polynomial ring in 16 variables $R_0 = \mathbb{C}[\lambda^1, \dots, \lambda^{16}]$ by the ideal generated by 10 quadratic elements

$$\sum_{\alpha,\beta} \lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta, \qquad m = 0, \dots, 9.$$

The corresponding projective variety $i : X = \operatorname{Proj} A \hookrightarrow \mathbb{P}^{15}$ is the hermitian symmetric space $X = SO(10, \mathbb{C})/U(5)$. The canonical bundle is $\omega_X = \mathcal{O}_X(-8)$ where $\mathcal{O}_X(1) = i^* O_{\mathbb{P}^{15}}$, cf. [13].

The Hilbert series of A is equal to

$$H(A;t) = \frac{1 - 10t^2 + 16t^3 - 16t^5 + 10t^6 - t^8}{(1-t)^{16}} = \frac{(1+t)(1+4t+t^2)}{(1-t)^{11}}.$$

The algebra A is Koszul[1].

Let $L = \bigoplus_{i=1}^{\infty} L_i$ be the graded Lie algebra Koszul dual to A and $C^*(L)$ its Chevalley cochain complex – the Koszul–Chevalley resolution of A. As a graded algebra it is the tensor product

$$C^*(L) = S(L[1]^*) = SL_1^* \otimes \Lambda L_2^* \otimes SL_3^* \otimes \cdots$$

(more precisely the inductive limit of finite products; here S denotes the symmetric algebra and the star denotes the dual space). Let us consider a graded commutative algebra

$$\mathcal{C}_1(L) = C^*(L) \otimes \Lambda L_1^*. \tag{1.2}$$

One can define a differential on this algebra that will make a dga whose cohomology will be

$$H^*(\mathcal{C}_1(L)) = \operatorname{Tor}^{R_0}_*(A, \mathbb{C}) =: Q. \tag{1.3}$$

The dga $C_1(L)$ is quasi-isomorphic to the *Berkovits algebra* studied by Movshev and Schwarz, cf. [24].

The graded commutative algebra Q is called *the algebra of syzygies*. For $i \geq 1$ denote by $L_{\geq i} \subset L$ the graded Lie subalgebra $\bigoplus_{j \geq i} L_j$. It follows from the above description that Q is isomorphic to the algebra $H^*(L_{\geq 2}; \mathbb{C})$ – this is a result contained in [13].

Generalizing this construction, consider for each $i \geq 1$ a graded commutative algebra

$$\mathcal{C}_i(L) = C^*(L) \otimes \Lambda L_1^* \otimes SL_2^* \otimes \cdots \otimes F_i L_i^*,$$

where $F_i = S$ if i is even and Λ otherwise. One expects that it is possible to introduce a differential on $\mathcal{C}_i(L)$ which provides a dga whose cohomology is $H^*(L_{\geq i})$. For example, an algebra quasi-isomorphic to $\mathcal{C}_3(L)$ has been studied in a very interesting paper [1].

[1]The Kempf–Bezrukavnikov theorem says that the coordinate rings of homogeneous spaces of the form G/P where G is semisimple complex and P is a parabolic, are Koszul. Another manifestation of this is the Kapranov's description (à la Beilinson and Bernstein–Gelfand–Gelfand) of the derived category of coherent sheaves on G/P.

The algebra of syzygies Q is a remarkable object. It is finite-dimensional over $\mathbb{C}$, $Q = \overset{3}{\underset{i=0}{\oplus}} Q_i$ and admits, after [13] a scalar product $Q_i \otimes Q_{3-i} \longrightarrow \mathbb{C}$ compatible with the multiplication. One can imagine Q as the cohomology of a smooth compact oriented variety of dimension 3.

A few words about the aim of the paper and the novelty of the results. The physicists are interested in quantum strings fluctuating on the singular space $\mathrm{Spec}(A)$. To pass to strings one has to study the *chiral* analogs of the above algebras, cf. [1, 3]. The construction of the *chiral* analog of such algebras, the *chiralization*, can be accomplished with the use of free infinite resolutions of the above algebras. In this paper we put together the properties of Koszul algebras and their infinite resolutions which contribute to the properties of the *chiral* algebras [1, 3]. We introduce and study in Part I the regularization technique following the insight from [5] which allows to define a part of the above *chiral* algebra structure. The Part II contains samples of calculations with the equivariant Hilbert series which one needs to define the action of an appropriate Kac–Moody on the *chiralization* of the above algebras. We are planning to return to *chiralization* of these objects in a separate publication.

Otherwise, one can regard this topic from the point of view of the *string topology* of Chas and Sullivan. The string theory is the study of spaces of loops. Let M be a closed oriented manifold of dimension d, ΩM its free loop space. A fundamental theorem of Chas and Sullivan, cf. [8], says that the homology of M shifted by d

$$\mathbb{H}_*(\Omega M) := H_{*+d}(\Omega M)$$

admits a structure of a *Batalin–Vilkovisky* (BV) algebra. If M is simply connected then $\mathbb{H}_*(\Omega M)$ is isomorphic to the Hochschild cohomology

$$\mathbb{H}_*(\Omega M) = HH^*(C^*(M), C^*(M)),$$

where $C^*(M)$ is the complex of singular cochains of M, cf. [7, 21].

An algebraic counterpart of the Chas–Sullivan theory is the following remarkable result ("the cyclic Deligne conjecture"), cf. [18, 32]: let A be an associative algebra with an invariant scalar product; then the complex of Hochschild cochains $CH^*(A)$ admits a structure of a homotopy BV algebra.

Returning to the Berkovits algebras $\mathcal{C}_1(L)$, Q, cf. (1.2), (1.3), it would be very interesting to study their Hochschild (as well as cyclic) cohomology. In view of the previous remarks, it should be closely related to the state space of the Berkovits string[2].

It is worth mentioning certain analogy between the "chiral" and "topological" points of view. For example the Deligne conjecture: the Hochschild cochain complex $CH^*(A)$ of an associative algebra A is a homotopy Gerstenhaber algebra (or more precisely, an algebra over the operad e_2 of chains of little discs), resembles the Lian–Zuckerman conjecture [20]: the space of a topological (i.e. $N = 2$ supersymmetric) vertex algebra is a homotopy Gerstenhaber algebra. If an associative algebra is equipped with an invariant scalar product then $CH^*(A)$ becomes a homotopy BV algebra. Which complementary structure one needs on a topological vertex algebra to become a homotopy BV algebra? In other words, does there exist a vertex counterpart of the cyclic Deligne conjecture?

[2]It seems that the Polyakov's gauge fields-strings correspondence translates in algebra into the assertion: the Hochschild cohomologies of Koszul dual algebras are isomorphic

First part

2 Numerical invariants and regularization

“Les séries divergentes sont en général quelque chose de bien fatal, et c’est une honte qu’on ose y fonder aucune démonstration. On peut démontrer tout ce qu’on veut en les employant. . . ”

N.-H. Abel, a letter to Holmboe, January 16, 1826

2.1 Semi-free resolutions and Hilbert series

2.1.1. Let

$$A = R_0/(f_1, \dots, f_M) = \overset{\infty}{\underset{i=0}{\oplus}} A^i,$$

and $i : X := \operatorname{Proj} A \hookrightarrow \mathbb{P}^N := \operatorname{Proj} R_0$ be as in the Introduction; denote by $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}^N}(1)$ the corresponding very ample line bundle.

Set $h_i = \dim_k A^i$ and let

$$H(A;t) = \sum_{i=0}^{\infty} h_i t^i$$

be the Hilbert series; we suppose X to be connected, so $H(A;0) = h_0 = 1$. Let

$$R: \ \cdots \longrightarrow R_2 \longrightarrow R_1 \longrightarrow R_0 \longrightarrow A \longrightarrow 0 \tag{2.1}$$

be a semi-free resolution of A. For R_1 one can take $R_1 = \overset{M}{\underset{l=1}{\oplus}} R_0\xi_l$, where the variables ξ_l are odd and $d\xi_l = f_l$, so $\xi_l \in R_1^{d_l}$.

Example 2.1. Suppose that $f_1, \dots, f_M$ is a regular sequence, that is, X is a complete intersection. Then one can take for R the Koszul complex, $R_i = R_0 \otimes \Lambda^i W$, where $W = \overset{M}{\underset{l=1}{\oplus}} k\xi_l$.

The exact sequence (2.1) immediately gives an expression of the Hilbert series:

$$H(A;t) = \frac{\prod_{l=1}^{M}(1 - t^{d_l})}{(1-t)^{N+1}} = \frac{\prod_{l=1}^{M}(1 + t + \cdots + t^{d_l - 1})}{(1-t)^{d+1}} = \frac{P(t)}{(1-t)^{d+1}}, \tag{2.2}$$

where

$$d = N - M = \dim X. \tag{2.3}$$

One notes that

$$\deg X = \prod_{l=1}^{M} d_l = P(1). \tag{2.4}$$

2.1.2. Let us return to the general case. Recall that R as an algebra is a polynomial superalgebra in homogeneous variables. Let us define integer numbers a_n = the number of even generators of polynomial degree n minus the number of odd generators of polynomial degree n.

Then the exact sequence (2.1) gives a product expression

$$H(t) = H(A;t) = \prod_{n=1}^{\infty} (1-t^n)^{-a_n} \tag{2.5}$$

similar to (2.2).

This formula shows in particular that the numbers a_n depend only on A but not on the resolution R.

Similarly to (2.3) and (2.4) we want to show that

$$\dim X + 1 = \text{“}a_1 + a_2 + a_3 + \cdots\text{”}, \tag{2.6}$$

$$\log(\deg X) = -\text{“}(\log 1)\cdot a_1 + (\log 2)\cdot a_2 + (\log 3)\cdot a_3 + \cdots\text{”}. \tag{2.7}$$

It is natural also to consider the “higher moments”:

$$\text{“}\sum_{n=1}^{\infty} n^l a_n\text{”}, \qquad l \geq 1. \tag{2.8}$$

However, the series at the right hand side are divergent, so one puts the sums in the quotation marks. Our aim will be to perform the summation of these series[3]. There are several ways of doing the summation.

For example, one can write a Lambert series

$$f(t) = \sum_{n=1}^{\infty} a_n \cdot \frac{nte^{-nt}}{1-e^{-nt}},$$

cf. [14, App. IV, (1.1)].

This series diverges if $|t|$ is small; but one can show that if $|t|$ is sufficiently big then the series absolutely converges and $t^{-1}f(t)$ is a rational function of $y = e^{-t}$. Therefore one extend $f(t)$ to the complex plane; this function will be holomorphic at $t = 0$ (i.e. and $y = 1$). It is natural to define

$$\text{“}\sum a_n\text{”} := f(0) = \operatorname*{res}_{t=0} \frac{f(t)}{t}.$$

The identity (2.6) will hold true. Moreover, the higher moments (2.8) may be expressed in terms of the coefficients of the Taylor series of $f(t)$ and 0; this is noted in [5].

Another classical way of regularization is using the Mellin transform and working with the Dirichlet series. This is what we are going to do.

2.2 Möbius inversion

2.2.1. Take the logarithm of (2.5):

$$\log\ H(t) = -\sum_{n=1}^{\infty} a_n \log(1-t^n) = \sum_{n=1}^{\infty}\sum_{l=1}^{\infty} a_n \frac{t^{ln}}{k} = \sum_{m=1}^{\infty}\left(\sum_{l|m} \frac{a_l}{m/l}\right)\cdot t^m \tag{2.9}$$

[3]The classical sources about divergent series are the books [14, 31].

and then the derivative:

$$\frac{tH'(t)}{H(t)} = \sum_{n=1}^{\infty} \frac{na_n t^n}{1-t^n} = \sum_{m=1}^{\infty} \left(\sum_{l|m} la_l \right) \cdot t^m. \tag{2.10}$$

In other words, if one denotes

$$\frac{tH'(t)}{H(t)} = \sum_{m=1}^{\infty} b_m t^m, \tag{2.11}$$

then

$$b_m = \sum_{l|m} la_l. \tag{2.12}$$

2.2.2. Recall that the Möbius function $\mu : \mathbb{N}_+ = \{1, 2, \ldots\} \longrightarrow \{-1, 0, 1\}$ is defined by: $\mu(1) = 1$, $\mu(n) = (-1)^l$ if $n = p_1 p_2 \cdots p_l$ is a product of l distinct prime numbers, and $\mu(n) = 0$ if n contains squares.

Another definition is by a generating series: if one defines the Riemann ζ function by the Euler product

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} \left(1 - p^{-s}\right)^{-1},$$

then

$$\zeta(s)^{-1} = \prod_{p \text{ prime}} \left(1 - p^{-s}\right) = \sum_{n=1}^{\infty} \mu(n) n^{-s}. \tag{2.13}$$

The Möbius inversion formula says that if $f : \mathbb{N}_+ \longrightarrow \mathbb{C}$ is a function and if a function $g : \mathbb{N}_+ \longrightarrow \mathbb{C}$ is defined by

$$g(m) = \sum_{l|m} f(l),$$

then

$$f(m) = \sum_{l|m} \mu(m/l) g(l). \tag{2.14}$$

Applying this formula to the function $f(m) = ma_m$, we get

$$ma_m = \sum_{l|m} \mu(m/l) b_l. \tag{2.15}$$

2.3 Dirichlet series

2.3.1. Let $P(t)$ be a polynomial with complex coefficients such that $P(0) = 1$, so one can write

$$P(t) = \prod_{l=1}^{p} (1 - \alpha_l t). \tag{2.16}$$

Consider the product (2.5) for $P(t)$:

$$P(t) = \prod_{n=1}^{\infty} (1 - t^n)^{-a_n} .$$

So if

$$\frac{tP'(t)}{P(t)} = \sum_{m=1}^{\infty} b_m t^m,$$

then

$$m a_m = \sum_{l|m} \mu(m/l) b_l.$$

On the other hand

$$\frac{tP'(t)}{P(t)} = - \sum_{r=1}^{p} \frac{\alpha_r t}{1 - \alpha_r t} = - \sum_{r=1}^{p} \sum_{m=1}^{\infty} \alpha_r^m t^m, \tag{2.17}$$

i.e.

$$b_m = - \sum_{r=1}^{p} \alpha_r^m,$$

wherefrom

$$a_m = -m^{-1} \sum_{r=1}^{p} \sum_{l|m} \mu(m/l) \alpha_r^l.$$

One puts

$$\rho(P) = \max_r |\alpha_r|.$$

It follows that if $\rho(P) > 1$, then $|a_m|$ grows as fast as $m^{-1}\rho(P)^m$.

2.3.2. Let us consider the Dirichlet series

$$\sum_{n=1}^{\infty} a_n n^{-s}.$$

We see that if $\rho(P) \le 1$ then this series absolutely converges for $\mathfrak{Re}(s) > 1$, and if $\rho(P) > 1$, it diverges for all s; in this case (which in fact is interesting to us) one has to do something else.

Let us write formally with [5]:

$$- \sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} \sum_{r=1}^{p} \sum_{l|n} \mu(n/l) \alpha_r^l n^{-1-s} = \sum_{r=1}^{p} \sum_{l,m=1}^{\infty} \mu(m) \alpha_r^l (lm)^{-1-s}$$
$$= \sum_{m=1}^{\infty} \mu(m) m^{-1-s} \cdot \sum_{r=1}^{p} \sum_{l=1}^{\infty} \alpha_r^l l^{-1-s} = \zeta(s+1)^{-1} \sum_{r=1}^{p} \sum_{l=1}^{\infty} \alpha_r^l l^{-1-s}.$$

Now if we rewrite, after Riemann,

$$l^{-1-s} = \frac{1}{\Gamma(s+1)} \int_0^{\infty} e^{-lt} t^s dt,$$

wherefrom

$$\sum_{l=1}^{\infty} \alpha_r^l l^{-1-s} = \frac{1}{\Gamma(s+1)} \int_0^{\infty} \sum_{l=1}^{\infty} \alpha_r^l e^{-lt} t^s dt = \frac{1}{\Gamma(s+1)} \int_0^{\infty} \frac{\alpha_r e^{-t} t^s}{1-\alpha_r e^{-t}} dt$$

(justified if $|\alpha_r| < 1$). Because

$$\frac{e^{-t}}{1-\alpha_r e^{-t}} = \frac{1}{e^t - \alpha_r},$$

the integral

$$\int_0^{\infty} \frac{\alpha_r e^{-t} t^s}{1-\alpha_r e^{-t}} dt$$

absolutely converges for $\mathfrak{Re}(s) > -1$ if $\alpha_r \notin [0,1]$ (since $P(1) = 1$, one has $\alpha_r \neq 0$ automatically). One notes that

$$\sum_{r=1}^{p} \frac{\alpha_r e^{-t}}{1-\alpha_r e^{-t}} = -\frac{e^{-t} P'(e^{-t})}{P(e^{-t})}. \tag{2.18}$$

On the other hand, let us write the functional equation for $\zeta(s)$ (cf. [34, 13.151]):

$$2^{1-s}\Gamma(s)\zeta(s)\cos(\pi s/2) = \pi^s \zeta(1-s)$$

and replace s by $s+1$:

$$2^{-s}\Gamma(s+1)\zeta(s+1)\cos(\pi(s+1)/2) = \pi^{s+1}\zeta(-s),$$

i.e.

$$\frac{1}{\Gamma(s+1)\zeta(s+1)} = -2^{-s}\pi^{-s-1}\sin(\pi s/2)\zeta(-s)^{-1}.$$

It follows:

$$\begin{aligned}\sum_{n=1}^{\infty} a_n n^{-s} &= \frac{1}{\Gamma(s+1)\zeta(s+1)} \int_0^{\infty} \frac{e^{-t}P'(e^{-t})}{P(e^{-t})} t^s dt \\ &= -\frac{2^{-s}\pi^{-s-1}\sin(\pi s/2)}{\zeta(-s)} \int_0^{\infty} \frac{e^{-t}P'(e^{-t})}{P(e^{-t})} t^s dt.\end{aligned}$$

2.3.3. Consider the last integral

$$I_P(s) := \int_0^{\infty} \frac{e^{-t}P'(e^{-t})}{P(e^{-t})} t^s dt.$$

It is well defined and represents a holomorphic function of s on the half-plane $\mathfrak{Re}(s) > -1$ as soon as the condition (2.19) below is verified:

The roots of $P(t)$ do not lie in the segment $[0,1]$. (2.19)

As it was kindly pointed out by one of the referees the condition (2.19) holds. Indeed, since the algebra A is affine, it has polynomial growth, so that the radius of convergence of the series $H(A,t)$ is $0 \leq t < 1$. Because the coefficients of the formal power series $H(A,t)$ are nonnegative, we have $H(A,t) > 0$ for all $0 \leq t < 1$. It follows that $P(A,t) > 0$ with $H(A,t)$. Also, the is function $H(A,t)$ is rational (Hilbert–Serre), so that it has a pole at $t = 1$ of the order $d+1$. This means that the value $P(A,1)$ is nonzero as well.

2.3.4. Following Riemann and using the notations of [34], consider the integral

$$J_P(s) = \int_{\infty}^{(0+)} \frac{e^{-t}P'(e^{-t})}{P(e^{-t})}(-t)^s dt, \tag{2.20}$$

where $\int_{\infty}^{(0+)}$ denotes the integral along the contour

$$\begin{aligned} C = \{t = a + \epsilon i,\ \infty > a \geq \epsilon\} \cup \{t = \sqrt{2}\epsilon e^{i\theta},\ \pi/4 \leq \theta \leq 7\pi/4\} \\ \cup \{t = a - \epsilon i,\ \epsilon \leq a < \infty\}, \end{aligned} \tag{2.21}$$

where $\epsilon > 0$ is sufficiently small. Here $(-t)^s = e^{s\log(-t)}$, and we use the branch of the logarithm $\pi \geq \arg t \geq -\pi$ on the small circle. Then

$$J_P(s) = 2i\sin(\pi s) I_P(s),$$

cf. [34, 12.22]. But $J_P(s)$ is an entire function. Therefore, if one sets

$$z_P(s) = -\frac{2^{-s}\pi^{-s-1}\sin(\pi s/2)}{\zeta(-s)} \cdot \frac{1}{2i\sin(\pi s)} J_P(s) = \frac{i \cdot 2^{-s-2}\pi^{-s-1}}{\cos(\pi s/2)\zeta(-s)} J_P(s), \tag{2.22}$$

this defines the analytic continuation of (2.20) to a meromorphic function on the complex plane.

2.3.5. Let us return to the situation of Section 2.1; it is known that the Hilbert series is a rational function of the form

$$H(A;t) = \frac{P(A;t)}{(1-t)^{d+1}},$$

where $d = \dim X = \dim A - 1$ and $P(A;t)$ is a polynomial with integer coefficients (cf. a simple Example 2.2). One has $P(A;0) = H(A;0) = 1$ since we suppose that X connected.

In view of the preceding discussion we define

$$z(X, \mathcal{L}; s) = \frac{1}{\Gamma(s+1)\zeta(s+1)} \int_0^{\infty} \frac{e^{-t}H'(A;e^{-t})}{H(A;e^{-t})} t^s dt = d + 1 + z_P(s) \tag{2.23}$$

(because $\Gamma(s+1)\zeta(s+1) = \int_0^{\infty}(t^s/(e^t-1))dt$). So if

$$H(A;t) = \prod_{n=1}^{\infty}(1-t^n)^{-A_n}, \tag{2.24}$$

then

$$z(X, \mathcal{L}; s) = \text{“}\sum_{n=1}^{\infty} A_n n^{-s}\text{”},$$

the quotation marks mean that the sum is regularized. We define “the Dirichlet summation”:

$$\text{“}\sum_{n=1}^{\infty} A_n\text{”} := z(X, \mathcal{L}; 0),$$

and

$$\text{“}\sum_{n=1}^{\infty} \log n \cdot A_n\text{”} := -z'(X, \mathcal{L}; 0),$$

2.3.6. We have $\zeta(0) = -1/2$ and $\zeta'(0) = -\log\sqrt{2\pi} \neq 0$ (cf. [33, Ch. VII, § 9]). The integral $I_P(0)$ is well defined (below we shall compute its value), so the formula (2.23) gives $z_P(0) = 0$, wherefrom

$$z(X, \mathcal{L}; 0) = d + 1,$$

cf. (2.6). On the other hand, $z'(X, \mathcal{L}; s) = z'_P(s)$, and a unique term giving a nontrivial contribution into $z'(X, \mathcal{L}; 0)$, is

$$-\left(\frac{2^{-s}\pi^{-s-1} \cdot (\pi/2)\cos(\pi s/2)}{\zeta(-s)}\right)\Bigg|_{s=0} \cdot I_P(0).$$

The first factor gives 1, whereas

$$\begin{aligned} I_P(0) &= \int_0^\infty \frac{e^{-t}P'(e^{-t})}{P(e^{-t})}dt = -\int_0^\infty \frac{d\log P(e^{-t})}{dt}dt \\ &= -\log P(e^{-t})\Big|_0^\infty = -\log P(0) + \log P(1) = \log P(1), \end{aligned}$$

wherefrom

$$z'(X, \mathcal{L}; 0) = \log P(1).$$

But it is known that $P(1) = \deg X$, which implies

$$\deg X = e^{z'(X,\mathcal{L};0)},$$

cf. (2.7).

Example 2.2 (Veronese curve and the Weil zeta function). Let $X = \mathbb{P}^1$, $\mathcal{L} = \mathcal{O}(q+1)$, $q \geq 1$. Then $A^n = \Gamma(X, \mathcal{O}((q+1)n)$, which implies

$$H(A; t) = \sum_{n=0}^\infty ((q+1)n+1)t^n = \frac{1+qt}{(1-t)^2},$$

and

$$P(t) = 1 + qt = \prod_{m=1}^\infty (1 - t^m)^{p_m(-q)},$$

where

$$p_m(x) = \frac{1}{m}\sum_{l|m} \mu(m/l)x^l.$$

It is well known that if q is a prime power then $p_m(q)$ is equal to the number of monic irreducible polynomials of degree m in $\mathbb{F}_q[T]$. In fact, we have

$$Z_{\mathbb{F}_q}(\mathbb{A}^1, t) = \frac{1}{P(-t)} = \prod_{m=1}^\infty (1 - t^m)^{-p_m(q)}$$

– this is the Euler product of the zeta function of the affine line over the finite field $\mathbb{F}_q$.

(The idea that the Hilbert series of a variety is equal to the Weil zeta function of another one seems very strange. Cf. [12] however[4].)

[4]We are grateful to Yu.I. Manin who showed us this very interesting paper.

2.4 Values at negative points

2.4.1. Let $s = -m$ be a negative integer. In this case one can close the contour in the integral (2.20):

$$J_P(-m) = (-1)^m \int_{|t|=\epsilon} Q(t)t^{-m}dt = 2\pi i \cdot (-1)^m \operatorname*{res}_{t=0} Q(t)t^{-m},$$

where

$$Q(t) = \frac{e^{-t}P'(e^{-t})}{P(e^{-t})} = -\frac{d\log P(e^{-t})}{dt}.$$

So if $Q(t) = q_0 + q_1 t/1! + q_2 t^2/2! + \cdots$ is the Taylor series at 0 then

$$J_P(-m-1) = 2\pi i \cdot (-1)^{m+1}\frac{q_m}{m!}.$$

Here are some first values:

$$q_0 = \frac{P'(1)}{P(1)}, \qquad q_1 = -q_0 + q_0^2 - \frac{P''(1)}{P(1)},$$
$$q_2 = q_0 - 3q_0^2 + 2q_0^3 + 3(1-q_0)\frac{P''(1)}{P(1)} + \frac{P'''(1)}{P(1)}.$$

2.4.2. One can say this in a different way: if

$$P(t) = 1 + c_1 t + \cdots + c_D t^D,$$

and we imagine the numbers c_i as "Chern classes" then q_m will be the coefficients of the logarithm of the Todd genus ...

2.4.3. Now applying (2.22): for $m = 1$, the function $\zeta(s)$ has a simple pole at $s = 1$ with the residue 1, whence

$$z_P(-1) = \frac{2P'(1)}{P(1)}. \tag{2.25}$$

Remark 2.3. Suppose that $P(t)$ is a "reciprocal" polynomial, i.e.

$$t^{\deg P}P(1/t) = P(t),$$

which happens rather often. In this case it is easy to see that

$$\frac{2P'(1)}{P(1)} = \deg P,$$

so $z_P(-1) = \deg P$.

2.4.4. For $m > 1$ one has to distinguish two cases.

(a) If $m = 2l$ is even then $\cos(\pi l) = (-1)^l$, and after Euler,

$$\zeta(2l) = (-1)^{l-1}\frac{(2\pi)^{2l}}{2(2l)!}b_{2l},$$

where the Bernoulli numbers are defined by the generating series

$$\frac{S}{e^S - 1} = 1 - \frac{S}{2} + \sum_{l=1}^{\infty} \frac{b_{2l}}{(2l)!} S^{2l}.$$

Here are some first values:

$$b_2 = \frac{1}{6}, \qquad b_4 = -\frac{1}{30}, \qquad b_6 = \frac{1}{42}, \qquad b_8 = -\frac{1}{30}.$$

It follows:

$$z_P(-2l) = \frac{2l q_{2l-1}}{b_{2l}}.$$

So if the numbers a_n are defined by (2.24) then

$$\text{“}\sum_{n=1}^{\infty} n^{2l} a_n\text{”} = z(X, \mathcal{L}; -2l) = d + 1 + \frac{2l q_{2l-1}}{b_{2l}} \tag{2.26}$$

– a formula found empirically in [5, (4.21)].

(b) If $m = 2l + 1$, $l > 0$ then $z_P(s)$ has a simple pole at $s = -m$, with the residue

$$\operatorname*{res}_{s=-2l-1} z_P(s) = (-1)^l \frac{(2\pi)^{2l} q_{2l}}{(2l+1)! \zeta(2l+1)}.$$

2.4.5. *Hurwitz zeta function.*

(a) We have $J_P(s) = \sum_{r=1}^{p} J_{\alpha_r}(s)$ where

$$J_\alpha(s) = -\int_\infty^{(0+)} \frac{\alpha}{e^t - \alpha} (-t)^s dt.$$

Let us choose a value $\beta = \log \alpha$. The expression under the integral $f_\alpha(t,s) = (\alpha/e^t - \alpha) \cdot (-t)^s$ has the poles at points $t_n = \beta + 2\pi i n$, $n \in \mathbb{Z}$. By the Cauchy formula,

$$J_\alpha(s) + 2\pi i \sum_{n=-\infty}^{\infty} \operatorname*{res}_{t=t_n} f_\alpha(t,s) = 0$$

if $\Re\mathrm{e}(s) < -1$, then

$$J_\alpha(s) = 2\pi i \sum_{n=-\infty}^{\infty} (-\beta - 2\pi i n)^s, \tag{2.27}$$

which gives $J_P(s)$ in the form of a series absolutely convergent for $\Re\mathrm{e}(s) < -1$, expressible in terms of the Hurwitz *zeta* functions.

(b)

Example 2.4. Take $P(t) = 1 + pt$, where $p \in \mathbb{Z}_{>0}$, cf. Example 2.2. Then $q_0 = p/(p+1)$, $q_1 = -p/(p+1)^2$ (cf. Section 2.4.1), so

$$\frac{J_P(-2)}{2\pi i} = -\frac{p}{(p+1)^2}.$$

On the other hand, take $\beta = \log(-p) = \log p + \pi i$, and the above series will be written as

$$\sum_{n=-\infty}^{\infty} (-\log p - \pi i - 2\pi i n)^{-2} = \sum_{n=0}^{\infty} \left\{ (-\log p + (2n+1)\pi i)^{-2} + (-\log p - (2n+1)\pi i)^{-2} \right\},$$

so one came to an identity

$$\sum_{n=0}^{\infty} \left\{ (-\log p + (2n+1)\pi i)^{-2} + (-\log p - (2n+1)\pi i)^{-2} \right\} = -\frac{p}{(p+1)^2}. \tag{2.28}$$

For example, for $p = 3$, $-p/(p+1)^2 = -0,1875$, whereas an approximate value given by MAPLE is

$$\sum_{n=0}^{1000} \left\{ (-\log p + (2n+1)\pi i)^{-2} + (-\log p - (2n+1)\pi i)^{-2} \right\} = -0,187449\ldots;$$

we see that the convergence is very slow.

(c) If $s = -1$, the series (2.27) in our example is still "Eisenstein summable", and on gets a rational value:

$$\sum_{n=0}^{\infty} \left\{ (-\log p + (2n+1)\pi i)^{-1} + (-\log p - (2n+1)\pi i)^{-1} \right\} = -\frac{2(p-1)}{p+1}. \tag{2.29}$$

This is an easy corollary of the decomposition of $\cot z$ into simple fractions[5]. But this value is different from $-q_0 = -p/(p+1)$: we have "an anomaly".

2.4.6. Suppose that our variety X is smooth, $H^i(X, \mathcal{L}^{\otimes n}) = 0$ for all $i > 0$ and $n \geq 0$. Then $A^n = \Gamma(X, \mathcal{L}^{\otimes n})$, and we can switch on the Riemann–Roch–Hirzebruch [15]:

$$h_n = \dim H^0(X, \mathcal{L}^{\otimes n}) = \int_X e^{nc_1(\mathcal{L})} \mathrm{Td}(\mathcal{T}_X),$$

where $\mathcal{T}_X$ denotes the tangent bundle, Td the Todd genus, given for line bundle E by the formula,

$$\mathrm{Td}\,(E) = \frac{c_1(E)}{1 - e^{-c_1(E)}},$$

so $\mathrm{Td}\,(\mathcal{T}_X) = 1 + \mathrm{Td}_1(\mathcal{T}_X) + \mathrm{Td}_2(\mathcal{T}_X) + \cdots$, where

$$\mathrm{Td}_1(\mathcal{T}_X) = \frac{c_1(\mathcal{T}_X)}{2}, \qquad \mathrm{Td}_2(\mathcal{T}_X) = \frac{c_1^2(\mathcal{T}_X) + c_2(\mathcal{T}_X)}{12}.$$

[5]We thank Oleg Ogievietsky who has noted this.

One writes: $e^{nc_1(\mathcal{L})} = \sum_{i=0}^{\infty} c_1(\mathcal{L})^i n^i/i!$, so $h_n = R(n)$, where

$$R(t) = \left(\int_X c_1(\mathcal{L})^d\right) \cdot \frac{t^d}{d!} + \left(\int_X c_1(\mathcal{L})^{d-1} c_1(\mathcal{T}_X)\right) \cdot \frac{t^{d-1}}{2(d-1)!} +$$
$$+ \left(\int_X c_1(\mathcal{L})^{d-2}(c_1(\mathcal{T}_X)^2 + c_2(\mathcal{T}_X))\right) \cdot \frac{t^{d-2}}{12(d-2)!} + \cdots = r_d \frac{t^d}{d!} + r_{d-1}\frac{t^{d-1}}{(d-1)!} + \cdots$$

is a polynomial of degree $d = \dim X$, the Hilbert polynomial of the ring A; it is a polynomial with rational coefficients which takes integer values at integer argument, whence

$$R(t) = \sum_{i=0}^{d} (-1)^{d-i} e_{d-i} \binom{t+i}{i},$$

where $e_0, e_1, \ldots, e_d \in \mathbb{Z}$. For example,

$$e_0 = r_d = \int_X c_1(\mathcal{L})^d,$$
$$e_1 = \frac{d+1}{2} r_d - r_{d-1} = \frac{1}{2}\int_X \big((d+1)c_1(\mathcal{L})^d - c_1(\mathcal{L})^{d-1}c_1(\mathcal{T}_X)\big),$$
$$e_2 = r_{d-2} - \frac{d}{2} r_{d-1} + \frac{(d+1)(3d-2)}{24} r_d$$

(we use $\sum_{1\le i<j\le d} ij = d(d^2-1)(3d+2)/24$).

On the other hand it is known that

$$e_i = \frac{P^{(i)}(1)}{i!}.$$

It follows:

$$e^{z'(X,\mathcal{L};0)} = P(1) = e_0 = \int_X c_1(\mathcal{L})^d,$$

so one finds the degree of X. Next

$$z_P(-1) = -\frac{2P'(1)}{P(1)} = -\frac{2e_1}{e_0} = -\frac{1}{e_0}\int_X \big((d+1)c_1(\mathcal{L})^d - c_1(\mathcal{L})^{d-1}c_1(\mathcal{T}_X)\big)$$
$$= -d - 1 + \frac{1}{e_0}\int_X c_1(\mathcal{L})^{d-1}c_1(\mathcal{T}_X),$$

i.e.

$$z(X,\mathcal{L};-1) = \frac{1}{e_0}\int_X c_1(\mathcal{L})^{d-1}c_1(\mathcal{T}_X),$$

etc.

One can put these calculations into a formal "generating function" for the moments

$$M_l = \text{“}\sum_{n=1}^{\infty} n^l a_n\text{”}.$$

Note that from the formulas above and using the expansion

$$\log\left(1 - e^t\right) = \log(-t) + \frac{t}{2} + \sum_{l=1}^{\infty} \frac{b_{2l}}{2l(2l)!} t^{2l}$$

one concludes the formal identity:

$$-\log P(e^t) = \text{“}\sum_{l=0}^{\infty} \log(l) a_l + M_1 \frac{t}{2} + \sum_{l=1}^{\infty} (M_{2l}) \frac{b_{2l}}{2l(2l)!} t^{2l}\text{”}. \tag{2.30}$$

The following elegant formula is due to F. Hirzebruch [16]: the polynomial $P(t)$ is a characteristic number defined as

$$P(t) = \int_X \frac{(1-t)e^{(1-t)c_1} \widehat{A}((1-t)^2 p_1, \dots, (1-t)^{2i} p_{2i}, \dots)}{1 - te^{(1-t)g}},$$

where c_1, p_1, p_2, $\dots$ are the first Chern class and the Pontryagin classes of the manifold X.

From this it is immediate to express the left hand side of (2.30) as a characteristic number. Indeed

$$\log P(e^t) = \log \int_X \frac{(1-e^t)e^{(1-e^t)c_1} \widehat{A}((1-e^t)^2 p_1, \dots, (1-e^t)^{2i} p_{2i}, \dots)}{1 - e^{t+(1-e^t)}}.$$

In order to expand the right hand side into a series of t observe that the constant term of the series under the logarithm is

$$\int_X \frac{1}{1-g} = P(1).$$

Therefore

$$\log \frac{1}{\int_X \frac{1}{1-g}} \int_X \frac{(1-e^t)e^{(1-e^t)c_1} \widehat{A}((1-e^t)^2 p_1, \dots, (1-e^t)^{2i} p_{2i}, \dots)}{1 - e^{t+(1-e^t)}}$$

is a series in t equal to $\log P(e^t) - \log P(1)$. This gives the desired generating function.

Example 2.5. Let X_g be the moduli space of semi-stable vector bundles of rank 2 with trivial determinant over a Riemann surface of genus g; it carries the canonical *determinant* line bundle $\mathcal{L}_g$, cf. [2]. Consider a graded algebra

$$A_g = \bigoplus_{i=0}^{\infty} H^0\big(X_g, \mathcal{L}_g^{\otimes i}\big)$$

(we thank Peter Zograf who proposed to consider this example).

The coefficients of the Hilbert series $H_g(t) = H(A_g; t)$ can be calculated using the Verlinde formula [2]. Here are some first examples, cf. [36]:

$$H_2(t) = \frac{1}{(1-t)^4}, \qquad H_3(t) = \frac{1+t+t^2+t^3}{(1-t)^7} = \frac{1-t^4}{(1-t)^8}, \tag{2.31}$$

$$H_4(t) = \frac{1+6t+21t^2+40t^3+21t^4+6t^5+t^6}{(1-t)^{10}}. \tag{2.32}$$

The highest coefficient $r_d/d!$ of the Hilbert polynomial has been calculated in [Wi], who found the value $2\zeta(2g-2)/(2\pi^2)^{g-1}$; here $d = \dim X_g = 3g-3$, whence

$$P_g(1) = (-1)^g \frac{(3g-3)!}{(2g-2)!} b_{2g-2} = (-1)^{g+1} \frac{(3g-3)!}{(2g-1)!} \zeta(-2g+3) \tag{2.33}$$

(one denotes $P_g(t) = H_g(t)(1-t)^{d+1}$). For example, for $g = 4$ one finds $P_4(1) = 96$, which is compatible with (2.32).

The above discussion gives a strange expression of this number as a regularized infinite product (starting from $g = 4$), of the form $\prod_{n=1}^{\infty} n^{a_n}$. For example, the beginning of the product for $P_4(t)$ will be:

$$P_4(t) = (1-t)^{-6}\big(1-t^3\big)^{16}\big(1-t^4\big)^{9}\big(1-t^5\big)^{-144}\big(1-t^6\big)^{360} \\ \times \big(1-t^8\big)^{-2259}\big(1-t^9\big)^{3920}\cdots, \tag{2.34}$$

whence

$$\zeta(-5) = -\frac{9!}{7!}\cdot \text{“}1^6 3^{-16} 4^{-9} 5^{144} 6^{-360} 8^{2259} 9^{-3920}\ldots\text{”}.$$

Second part

3 Koszulness and infinite resolutions

3.1 The Veronese ring

3.1.1. Let us fix an integer $b \geq 0$, and consider the Veronese embedding

$$i_b : \mathbb{P}^1 \longrightarrow \mathbb{P}^{b+1}$$

defined in coordinates by the formula

$$i_b(u_0 : u_1) = (x_0 : \cdots : x_{b+1}) := \big(u_0^{b+1} : u_0^b u_1 : \cdots : u_1^{b+1}\big).$$

If $t = u_1/u_0 \in \mathbb{A}^1 \subset \mathbb{P}^1$, then

$$i_b(t) = \big(t, t^2, \ldots, t^{b+1}\big) \in \mathbb{A}^{b+1} \subset \mathbb{P}^{b+1}.$$

The image $X_b := i_b(\mathbb{P}^1) \subset \mathbb{P}^{b+1}$ of this embedding is called the Veronese curve (or the moment curve, in view of the last formula).

This curve can be defined in $\mathbb{P}^{b+1}$ by the equations $x_i x_j - x_k x_l = 0$ if $i+j = k+l$. A minimal system of equations consists of $b(b+1)/2$ quadratic equations:

$$f_{ij} := x_i x_j - x_{i+1}x_{j-1} = 0, \qquad 0 \leq i \leq b-1, \qquad i+2 \leq j \leq b+1.$$

Example 3.1. $i_0 = \mathrm{Id}_{\mathbb{P}^1}$. If $b = 1$, $X_1 \subset \mathbb{P}^2 = \{(x_0 : x_1 : x_2)\}$ is defined by one equation:

$$x_0 x_2 - x_1^2 = 0.$$

The curve $X_3 \subset \mathbb{P}^4$ is defined by 3 equations

$$x_0x_2 - x_1^2 = 0; \qquad x_0x_3 - x_1x_2 = 0; \qquad x_1x_3 - x_2^2 = 0.$$

3.1.2. We have $i_b^*\mathcal{O}_{\mathbb{P}^{b+1}}(1) = \mathcal{O}_{\mathbb{P}^1}(b+1)$, all the higher cohomology of this sheaf vanish, and $X_b = \mathrm{Spec\,Proj}\, A_b$, where

$$A_b = \bigoplus_{n=0}^{\infty} \Gamma\big(\mathbb{P}^1, \mathcal{O}((b+1)n\big) = \mathbb{C}[x_0, \ldots, x_{b+1}]/(f_{ij}).$$

The Hilbert series of the above algebra is as follows:

$$H(A_b; t) = \sum_{n=0}^{\infty}((b+1)n+1)t^n = \frac{1+bt}{(1-t)^2}. \tag{3.1}$$

It is not difficult to figure out the q-analogue of (3.1): if we set $[b]_q := (q^b - q^{-b})/(q - q^{-1})$, then:

$$H_q(A_b;t) := \sum_{n=0}^{\infty}[(b+1)n+1]_q t^n = \left(q - q^{-1}\right)^{-1} \cdot \sum_n \left\{q q^{(b+1)n} t^n - q^{-1} q^{-(b+1)n} t^n\right\}$$
$$= \left(q - q^{-1}\right)^{-1} \cdot \left(\frac{q}{1 - q^{b+1}t} - \frac{q^{-1}}{1 - q^{-b-1}t}\right) = \frac{1 + [b]_q t}{(1 - q^{b+1}t)(1 - q^{-b-1}t)}. \tag{3.2}$$

We will use these formulas later in Section 3.4.4.

Example 3.2. For $b = 1$ we have,

$$H(A_1;t) = \frac{1+t}{(1-t)^2} = \frac{1-t^2}{(1-t)^3}, \tag{3.3}$$

the ring A_1 is a cone:

$$A_1 = \mathbb{C}[x_0, x_1, x_2]/\left(x_0x_2 - x_1^2\right) = B/(f).$$

It admits a dga resolution of length 1: the Koszul complex:

$$K.(B;f): \; 0 \longrightarrow B \cdot e \longrightarrow B \longrightarrow A \longrightarrow 0, \qquad d(e) = f.$$

Therefore

$$K.(B;f) = B \otimes \Lambda^{\cdot}\langle e\rangle = \mathbb{C}[x_0, x_1, x_2; e]$$

as a graded algebra; the *homological* degree of x_i is 0 and of e is 1. These correspond to the exponents -3 and 1 in (3.3).

There is another interpretation of the Koszul complex. Let V be a vector space $\overset{2}{\underset{i=0}{\oplus}} \mathbb{C} \cdot x_i$, then $B = S^{\cdot}V$. Consider the dual space $V^* = \overset{2}{\underset{i=0}{\oplus}} \mathbb{C} \cdot y_i$. Define a graded Lie algebra L on 3 generators y_i, $i = 0, 1, 2$, of degrees 1, obeying 5 relations:

$$[y_0, y_0] = 0, \qquad [y_0, y_1] + [y_1, y_0] = 0, \qquad [y_0, y_2] + [y_1, y_1] + [y_2, y_0] = 0,$$
$$[y_1, y_2] + [y_2, y_1] = 0, \qquad [y_2, y_2] = 0.$$

It is concentrated in degrees 1 and 2: $L = L_1 \oplus L_2$, where $L_1 = V^*$ and $L_2 = \mathbb{C} \cdot [y_1, y_1]$.

If $L' = \mathbb{C} \cdot y_1 \oplus \mathbb{C} \cdot [y_1, y_1] \subset L$ is a Lie subalgebra generated by y_1, then L' is free (sic!); it is an ideal, and the quotient algebra $\bar{L} = L/L'$ is Abelian, on 2 generators $\bar{y}_0$, $\bar{y}_2$.

Consider the complex of the Chevalley cochains:

$$C^{\cdot}(L) = \Lambda^{\cdot}(L^*) = S^{\cdot}L_1^* \otimes \Lambda^{\cdot}L_2^* = S^{\cdot}V \otimes \Lambda^{\cdot}\langle e\rangle, \qquad e = [y_1, y_1]^*$$

then one can identify $C^{\cdot}(L)$ with $K(B;f)$.

Example 3.3. Starting from $b = 2$ the product decomposition of the Hilbert series becomes infinite: for example, if $b = 2$

$$H(A_2;t) = \frac{1+2t}{(1-t)^2} = \frac{(1-t^2)^3(1-t^4)^3(1-t^6)^{11}(1-t^8)^{30}(1-t^{10})^{105}\cdots}{(1-t)^4(1-t^3)^2(1-t^5)^6(1-t^7)^{18}(1-t^9)^{56}\cdots}.$$

The first two exponents are “koszul”: the number of unknowns and the number of equations. The exponents grow exponentially.

3.2 Gauss cyclotomic identity

3.2.1. Necklaces. The *necklace polynomial* is defined as

$$M_n(x) = \frac{1}{n} \sum_{d|n} \mu(d) x^{n/d}.$$

Example 3.4. If p is a prime, then $M_p(x) = (x^p - x)/p$.

Let a *necklace* c be made of n beads; suppose that each bead can be one of m colors. A necklace is called *primitive* if it is not of the form $c = dc'$ where $d|n$, $d > 1$.

Theorem 3.5 (C. Moreau, 1872). *The number of primitive necklaces made of n beads in b colors is equal to $M_n(b)$.*

The proof is an exercise. C. Moreau was an artillery captain from Constantine, cf. [21].

Corollary 3.6. *The number of all necklaces made of n beads in b colors is equal to $\Phi_n(b)$, where*

$$\Phi_n(x) = \frac{1}{n} \sum_{d|n} \phi(d) x^{n/d},$$

$\phi(d)$ is the Euler function.

Proof. If this number is $C(n;b)$ then

$$C(n;b) = \sum_{l|n} M_l(b) = \sum_{l|n} \sum_{d|l} \frac{\mu(d)}{d} \cdot \frac{b^{l/d}}{l/d} \overset{(\text{we set } p = l/d)}{=} \sum_{p|n} b^p \cdot \left(\frac{1}{p} \sum_{d|(n/p)} \frac{\mu(d)}{d} \right)$$

or,

$$\sum_{d|(n/p)} \frac{\mu(d)}{d} = \frac{\phi(n/p)}{n/p},$$

which proves the corollary. ∎

3.2.2. A theorem of Pólya. Following Polyakov (cf. [28]), one can consider the same numbers from the point of view of Pólya theory [27].

Suppose we are given two finite sets X and Y as well as a weight function $w : Y \to \mathbb{N}$. If $n = |X|$, without loss of generality we can assume that $X = \{1, 2, \ldots, n\}$. Consider the set of all mappings $F = \{f \mid f : X \to Y\}$. We can define the weight of a function $f \in F$ to be

$$w(f) = \sum_{x \in X} w\left(f(x)\right).$$

Every subgroup of the symmetric group on n elements, S_n, acts on X through permutations. If A is one such subgroup, an equivalence relation $\sim_A$ on F is defined as $f \sim_A g \iff f = g \circ a$ for some $a \in A$.

Denote by $[f] = \{g \in F \mid f \sim_A g\}$ the equivalence class of f with respect to this equivalence relation. $[f]$ is also called the orbit of f. Since each $a \in A$ acts bijectively on X, then

$$w(g) = \sum_{x \in X} w\left(g(x)\right) = \sum_{x \in X} w\left(g(a \circ x)\right)) = \sum_{x \in X} w\left(f(x)\right) = w(f).$$

Therefore we can safely define $w([f]) = w(f)$. In other words, permuting the summands of a sum does not change the value of the sum.

Let $c_k = |\{y \in Y \mid w(y) = k\}|$ be the number of elements of Y of weight k. The generating function by weight of the source objects is $c(t) = \sum_k c_k \cdot t^k$. Let $C_k = |\{[f] \mid w([f]) = k\}|$ be the number of orbits of weight k. The generating function of the filled slot configurations is $C(t) = \sum_k C_k \cdot t^k$.

Theorem 3.7. *Given all the above definitions, Pólya's enumeration theorem asserts that*

$$C(t) = Z(A)\left(c(t), c\left(t^2\right), \dots, c(t^n)\right),$$

where $Z(A)$ *is the cycle index (Zyklenzeiger) of* A

$$Z(A)(t_1, t_2, \dots, t_n) = \frac{1}{|A|} \sum_{g \in A} t_1^{j_1(g)} t_2^{j_2(g)} \cdots t_n^{j_n(g)}.$$

Consider the group of cyclic permutations $G \stackrel{\sim}{=} \mathbb{Z}/n$ as a subgroup of the symmetric group S_n. Define, after Pólya, the cycle index polynomial of G, in n variables, as

$$P_G(x_1, \dots, x_n) = \frac{1}{n} \sum_{\sigma \in G} \prod_{i=1}^{n} x_i^{c_i(\sigma)},$$

where $c_i(\sigma)$ is the number of cycles of length i in σ.

In other words, one associates to each $\sigma \in G$ a monomial. For example, for $n = 6$ there are following permutations:

$$\begin{aligned}
&\sigma_0 = (1)(2)(3)(4)(5)(6), \quad \text{the corresponding monomial is } x_1^6,\\
&\sigma_1 = (123456): \quad x_6,\\
&\sigma_2 = \sigma_1^2 = (135)(246): \quad x_3^2,\\
&\sigma_3 = \sigma_1^3 = (14)(25)36): \quad x_2^3,\\
&\sigma_4 = \sigma_1^4 = (153)(264): \quad x_3^2,\\
&\sigma_5 = \sigma_1^5 = (654321): \quad x_6.
\end{aligned}$$

The Zyklenzeiger is equal to $P_{\mathbb{Z}/6}(x) = \frac{1}{6}(x_1^6 + x_2^3 + 2x_3^2 + 2x_6)$.

In general,

$$P_{\mathbb{Z}/n}(x_1, \dots, x_n) = \frac{1}{n} \sum_{d|n} \phi(d) x_d^{n/d}.$$

After a change of variables $x_i = \sum\limits_{j=1}^{b} y_j^i$, one obtains a polynomial

$$W_G(y_1, \dots, y_b) := P_G\left(\sum y_j, \sum y_j^2, \dots, \sum y_j^n\right).$$

As a special case of the theorem of Pólya we obtain that the number of necklaces made of n beads in b colors is equal to $W_{\mathbb{Z}/n}(1, \dots, 1) = P_{\mathbb{Z}/n}(b, \dots, b) = \Phi_n(b)$.

3.2.3. Cyclotomic identity.

Theorem 3.8 (Gauss, cf. [11]). *One has the following formula:*

$$1 - bt = \prod_{n=1}^{\infty} (1 - t^n)^{M_n(b)}.$$

It is proved by the application of the Möbius inversion. There is a useful generalization of this identity, found by Pieter Moree [22]. We will need it later studying the equivariant Hilbert series, see Section 3.4.4.

Theorem 3.9. *Let $f(q) \in \mathbb{C}[q, q^{-1}]$ be an arbitrary Laurent polynomial. Introduce the polynomials*

$$M_n(f; q) = \frac{1}{n} \sum_{d|n} \mu(d) f(q^d)^{n/d}.$$

Then

$$1 - f(q)t = \prod_{n=1}^{\infty} \prod_{i=-\infty}^{\infty} \left(1 - q^i t^n\right)^{a_{in}},$$

where for each n, the number a_{in} are defined by the equations

$$\sum_i a_{in} q^i = M_n(f; q).$$

Proof. Applying $-\log$ to the both sides of the above formula we obtain:

$$-\log(1 - f(q)t) = \sum_{m=1}^{\infty} \frac{f(q)^m t^m}{m}$$

and

$$-\log\left(\prod_{i,n} \left(1 - q^i t^n\right)^{a_{in}}\right) = \sum_i \sum_{n,k=1}^{\infty} a_{in} \frac{q^{ik} t^{nk}}{k} = \sum_{m=1}^{\infty} t^m \cdot \left(\sum_{n|m} \sum_i a_{in} \frac{q^{im/n}}{m/n}\right),$$

therefore

$$f(q)^m = \sum_{n|m} \sum_i n a_{in} q^{im/n}$$

for each $m = 1, 2, \ldots$. Make a change of variables: $p = q^m$,

$$f(p^{1/m})^m = \sum_{n|m} \sum_i n a_{in} p^{i/n}.$$

Using the Möbius inversion we get:

$$\sum_i n a_{in} p^{i/n} = \sum_{d|n} \mu(n/d) f(p^{1/d})^d,$$

and making a substitution: $q = p^{1/n}$, we obtain

$$\sum_i n a_{in} q^i = \sum_{d|n} \mu(n/d) f(q^{n/d})^d,$$

which is the formula required. ∎

Moreover, O. Ogievetsky noticed that the theorem can be generalized to the case of several variables:

Theorem 3.10. *Let $f(q_1,\dots,q_p)$ be a Laurent polynomial in p variables. Introduce the Laurent polynomials*

$$M_n(f;q_1,\dots,q_p)=\frac{1}{n}\sum_{d|n}\mu(d)f\left(q_1^d,\dots,q_p^d\right)^{n/d}.$$

Then

$$1-f(q,\dots,q_p)t=\prod_{n=1}^{\infty}\prod_{i=-\infty}^{\infty}\left(1-q_1^{i_1}\cdots q_p^{i_p}t^n\right)^{a_{i_1\dots i_p;n}},$$

where for each n, the numbers $a_{i_1\dots i_p;n}$ are the coefficients of M_n:

$$\sum_i a_{i_1\dots i_p;n}q_1^{i_1}\cdots q_p^{i_p}=M_n(f;q_1,\dots,q_p).$$

The proof is the same as before.

Example 3.11. Take $f(q)=-q$. Then

$$1+qt=(1-qt)^{-1}\left(1-q^2t^2\right),$$

where

$$\frac{1}{n}\sum_{d|n}(-1)^d\mu(d)$$

is equal to 0 if $n\geq 3$, -1 if $n=1$, and 1 if $n=2$.

3.2.4. Application: the ζ function of the affine line. Let $A:=\mathbb{F}_p[x]$; this ring is similar in many ways to $\mathbb{Z}$.

Nonzero ideals $I\subset A$ are in bijection with unitary polynomials $f(x)$, $I=(f)$, and principal ideals correspond to irreducible ones. Set

$$N(I):=\sharp(A/I)=p^{\deg f},$$

and define

$$\zeta(A;s)=\sum_{I\subset A,\, I\neq 0}N(I)^{-s}=\sum_{f\ \text{unitary}}p^{-s\deg f}.$$

There is p^n unitary polynomials of degree n, therefore

$$\zeta(A;s)=\sum_{n=1}^{\infty}p^n\cdot p^{-sn}=\frac{1}{1-p\cdot p^{-s}}=\frac{1}{1-pT}, \tag{3.4}$$

where $T:=p^{-s}$.

The Euler product formula for $\zeta(A;s)$ can be written in the following form

$$\zeta(A;s)=\prod_{f\ \text{unitary, irreducible}}\frac{1}{1-p^{-\deg f\cdot s}}$$

$$= \prod_{d=1}^{\infty} \prod_{f \text{ un., irr., } \deg f = d} \frac{1}{1-p^{-ds}} = \prod_{d=1}^{\infty} \frac{1}{(1-T^d)^{N_d(p)}},$$

where $N_d(p)$ denotes the number of unitary irreducible polynomials of degree d in A.

On the other hand, applying the cyclotomic identity to (3.4), one gets

$$\zeta(A;s) = \frac{1}{1-pT} = \frac{1}{\prod_{d=1}^{\infty}(1-T^d)^{M_d(p)}},$$

proving therefore

Theorem 3.12 (Gauss [11]). *The number $N_d(p)$ of irreducible unitary polynomials of degree d in $\mathbb{F}_p[x]$ is equal to*

$$M_d(p) = \frac{1}{d}\sum_{l|d} \mu(l) p^{d/l}.$$

Corollary 3.13. *For $d \geq 1$, $N_d(p) > 0$, i.e. for each $d \geq 1$ there is an irreducible polynomial of degree d.*

It is interesting to compare this theorem of Gauss with Riemann's explicit formula:

$$\pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left\{ \mathrm{Li}(x^{1/n}) + \sum_{\Im\mathrm{m}(\rho)>0} (x^{\rho/n} + x^{1-\rho}) + \int_{x^{1/n}}^{\infty} \frac{dt}{t(t^2-1)\log t} - \log 2 \right\}.$$

Here $\pi(x) =$ the number of primes $p \leq x$, the summation is over the non-trivial roots of $\zeta(s)$, and

$$\mathrm{Li}\, x = \int_1^x \frac{dt}{\log t}.$$

3.3 Witt formula

La Nature est un temple où de vivants piliers
Laissent parfois sortir de confuses paroles ...

Ch. Baudelaire

Theorem 3.14 (Witt [35]). *Let L be a free Lie algebra on b generators. Then*

$$\dim L_n = M_n(b) = \frac{1}{n}\sum_{d|n} \mu(d) b^{n/d}.$$

In other words,

$$H(L;t) := \sum_{n=0}^{\infty} \dim L_n \cdot t^n = -\sum_{m=1}^{\infty} \frac{\mu(m)}{m} \log\left(1 - bt^m\right).$$

Proof. If $x_1, \ldots, x_b$ are the generators of L, then the universal enveloping algebra $UL = \mathbb{C}\langle x_1, \ldots, x_b\rangle$ (a free associative algebra), has the following Hilbert series

$$H(UL;t) := \sum_{i=0}^{\infty} \dim UL_i \cdot t^i = \frac{1}{1-bt}.$$

On the other hand, due to Poincaré–Birkhoff–Witt, $UL = S^{\cdot}L$, therefore, if one denotes $a_n := \dim L_n$, then

$$H(UL;t) = \prod_{n=1}^{\infty} \frac{1}{(1-t^n)^{a_n}},$$

and applying the cyclotomic identity finishes the proof. ■

3.3.1. This theorem admits a nice generalization. Let L be a free Lie superalgebra with b even generators $x_1, \dots, x_b$ and c odd generators $y_1, \dots, y_c$.

Equip L with a grading, assigning to x_i and to y_j the degree 1, and setting $\deg([a,b]) = \deg(a) + \deg(b)$. If $L_n \subset L$ is a subspace of the elements of the degree n, then $L = \overset{\infty}{\underset{n=1}{\oplus}} L_n$.

Explicitly, the Lie monomial $\alpha = [a_1, [a_2, \dots a_n] \dots]$, where $a_i = x_j$ or y_k, has the degree n. On the other hand, its *parity* is equal to the number of y's among a_i. For example, $[x_i, y_j]$ is an element odd of degree 2 (sic!).

Each homogenous summand L_n therefore is decomposed into two summands: even and odd elements, $L_n = L_n^p \oplus L_n^i$. One defines the *super-dimension* as $\dim^{\sim} L_n = \dim L_n^p - \dim L_n^i$.

Theorem 3.15 (Petrogradsky [25]). *One has the formula*

$$\dim^{\sim} L_n = M_n(b-c) = \frac{1}{n} \sum_{d|n} \mu(d)(b-c)^{n/d}. \tag{3.5}$$

Example 3.16. Let $b = 0$, $c = 1$; then $L = \mathbb{C} \cdot y \oplus \mathbb{C} \cdot [y,y]$. In the right hand side, one sees that $M_n(-1) = -1$ if $n = 1$, 1 if $n = 2$, and 0 if $n \geq 3$, cf. Examples 3.11 and 3.2.

3.3.2. Hochschild homology. Let A be an associative unitary algebra. Recall that the Hochschild homology $HH_i(A)$ of A is defined as the homology of the complex

$$CH.(A): \ \cdots \longrightarrow A \otimes A \otimes A \xrightarrow{b} A \otimes A \xrightarrow{b} A \longrightarrow 0,$$

where $CH_n(A) = A^{\otimes n+1}$, and the differential b is defined by

$$\begin{aligned} b(a_1 \otimes a_2 \cdots \otimes a_n) &= a_1 a_2 \otimes a_3 \otimes \cdots \otimes a_n - a_1 \otimes a_2 a_3 \otimes a_4 \cdots \otimes a_n + \cdots \\ &\quad + (-1)^{n-1} a_1 \otimes \cdots \otimes a_{n-1} a_n + (-1)^n a_n a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

In particular, $d(a \otimes b) = ab - ba$ and $HH_0(A) = A/[A,A]$, where $[A,A] \subset A$ is the subspace generated by the elements $ab - ba$.

In fact, one can replace A by $\bar{A} := A/\mathbb{C}$: the complex $CH.(A)$ is quasi-isomorphic to

$$\bar{CH}.(A): \ \cdots \longrightarrow A \otimes \bar{A} \otimes \bar{A} \longrightarrow \bar{A} \otimes A \longrightarrow A \longrightarrow 0.$$

One has

$$HH_i(A) = \mathrm{Tor}_i^{A \otimes A^o}(A, A).$$

Let us introduce the operators $B : A^{\otimes n+1} \longrightarrow A^{\otimes n+2}$ by the formula:

$$\begin{aligned} B(a_0, \dots, a_n) = \sum_{i=0}^{n-1} \big\{ &(-1)^{ni}(1, a_i, \dots, a_n, a_0, \dots, a_{i-1}) \\ &- (-1)^{n(i-1)}(a_{i-1}, 1, a_i, \dots, a_n, a_0, \dots, a_{i-2}) \big\}. \end{aligned}$$

For example,

$$B(a_0) = (1, a_0) + (a_0, 1). \tag{3.6}$$

Then $Bb + bB = 0$ (cf. [17, 2.1]); therefore it induces the morphisms

$$B: \ HH_n(A) \longrightarrow HH_{n+1}(A). \tag{3.7}$$

Recall that the cyclic homology can be defined as the homology of the bi-complex

$$HC_i(A) = H_i\big(CH.(A) \xrightarrow{B} CH.(A)[1] \xrightarrow{B} CH.(A)[2] \xrightarrow{B} \cdots\big).$$

3.3.3. Now one can give an algebraic interpretation of the polynomials $\Phi_n(x) = \sum\limits_{d|n} \phi(d)x^{n/d}/n$.

Let $A = \mathbb{C}\langle x_1, \ldots, x_b\rangle$, then $HH_0(A)$ inherits a grading from A, $HH_0(A) = \overset{\infty}{\underset{n=0}{\oplus}} HH_0(A)_n$; if one has $\bar{A} = \underset{n\geq 1}{\oplus} A_n$, then:

$$\bar{HH}_0(A) = \underset{n\geq 1}{\oplus} HH_0(A)_n = \bar{A}/[\bar{A}, \bar{A}].$$

One can think of $HH_0(A)_n$ as a space of *cyclic words* $x_{i_1}x_{i_2}\ldots x_{i_n}$ of length n in letters x_i, where two words are identified if one is a cyclic permutation of another. Therefore cyclic words are identified with necklaces of n beads in b colors.

Hence $HH_0(A)_n$ can be viewed as a linear space with a basis indexed by necklaces of n beads in b colors. It follows that

$$\dim HH_0(A)_n = \Phi_n(b) = \frac{1}{n}\sum_{d|n} \phi(d)b^{n/d}, \qquad n \geq 1,$$

(cf. Corollary 3.6), or

$$H(\bar{HH}_0(A); t) = -\sum_{m=1}^{\infty} \frac{\phi(m)}{m} \log(1 - bt^m)$$

(*the Polyakov formula*).

3.3.4. Let $V = \overset{b}{\underset{i=1}{\oplus}} \mathbb{C}\cdot x_i$, therefore $A = TV$. For each $n \geq 1$ one defines an automorphism $\tau : V^{\otimes n} \longrightarrow V^{\otimes n}$ as

$$\tau(v_1 \otimes \cdots \otimes v_n) = v_n \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_{n-1}.$$

One observes that

$$HH_0(TV)_n = V_\tau^{\otimes n} := \mathrm{Coker}(1 - \tau).$$

Following [17, 3.1], one defines a complex of length 1:

$$CH^{sm}_{.}(TV): \ 0 \longrightarrow TV \otimes V \longrightarrow TV \longrightarrow 0$$

equipped with a differential $d(a \otimes v) = av - va$. Define also a morphism of complexes ϕ : $CH.(TV) \longrightarrow CH^{sm}_{.}(TV)$ as: $\phi_0 = \mathrm{Id}_{TV}$,

$$\phi_1(a \otimes v_1 \ldots v_n) = \sum_{i=1}^{n} v_{i+1}\cdots v_n a v_1 \cdots v_{i-1} \otimes v_i. \tag{3.8}$$

On the other hand, there is an evident inclusion $\iota : CH^{sm}_{\cdot}(TV) \longrightarrow CH_{\cdot}(A)$ such that $\phi \circ \iota = \mathrm{Id}$, and, as one can check, $\iota \circ \phi$ is homotopic to the identity.

It follows that

$$HH_1(TV)_n = \left(V^{\otimes n}\right)^{\tau} := \mathrm{Ker}(1 - \tau)$$

and $HH_i(TV) = 0$ for $i \geq 2$.

Instead of the Hochschild homology one can also consider the (reduced) cyclic homology. Then: $HC_0(TV) = HH_0(TV)$ and $\bar{HC}_i(TV) = 0$ for $i > 0$, cf. [23].

3.3.5. Partial derivatives. Let $m = \dots x_i x_j x_k \dots$ be a cyclic word (i.e. $m \in TV/[TV, TV]$) such that the letter x_i appears once in it; then one can define a usual word (i.e. an element de TV) $\partial m/\partial x_i$, by "cutting" m and removing the letter x_i:

$$\partial m/\partial x_i = x_j x_k \dots .$$

When x_i appears in a word several times, the result will be a sum, by the Leibnitz rule, cf. [19]. In this way we define a map

$$\frac{\partial}{\partial x_i} : \ TV/[TV, TV] \longrightarrow TV.$$

Consider an operator $\sum\limits_{i=1}^{b} x_i \partial/\partial x_i$ (cf. (3.8)). It respects the polynomial degree, and it is not hard to verify that its image is contained in TV^{τ}. For example:

$$e(xyz) = xyz + yzx + zxy.$$

One observes that the map

$$\sum_{i=1}^{b} x_i \frac{\partial}{\partial x_i} : \ HH_0(TV) = TV_{\tau} \longrightarrow TV^{\tau} = HH_1(TV) \tag{3.9}$$

coincides with homomorphism B, cf. (3.6), (3.7). (Compare [23, the line before (14), p. 8].)

It was pointed out to us by V. Ginzburg that there is another interpretation of the above map. Consider the composition

$$d : \ TV_{\tau} = TV/[TV, TV] \stackrel{\sum x_i \partial/\partial x_i}{\longrightarrow} TV^{\tau} \subset TV \otimes V. \tag{3.10}$$

It was already Quillen who studied this map in the 80's, cf. [30]. In the notations of [30, § 3] $TV/[TV, TV] = TV_{\natural}$ and $TV \otimes V = \Omega^1_{TV,\natural}$; one can consider these spaces as the space of cyclic functions (differentiable 1-forms respectively) over the "non-commutative space Spec TV"; the morphism (3.10) is called "the Karoubi–de Rham differential".

3.4 Koszul duality

3.4.1. Let A be an associative *quadratic* algebra, which means that, it is a quotient $A = TV/(R)$ of the free associative algebra over the space V of finite dimension by the two-sided ideal (R) generated by a subspace of relations $R \subset V \otimes V$.

Recall that the quadratic dual algebra is defined as $A^! = TV^*/(R^{\perp})$, where $R^{\perp} \subset V^* \otimes V^* \stackrel{\sim}{=} (V \otimes V)^*$ is the annihilator of R.

We are interested in the case when A is commutative; in this case $R \supset \Lambda^2 V$, and $R^{\perp} \subset \Lambda^2 V^{\perp} = S^2 V^*$. In other words, if $\{x_i\}$ form a basis of V, $\{y_i\}$ form a basis of V^*, then $R^{\perp}$

is contained in $S^2V^* \subset V^{\otimes 2}$; in other words it is generated by odd commutators $[y_i, y_j] = y_iy_j + y_iy_j$. Define a Lie algebra L as a *graded* Lie algebra generated by y_i of degree 1 and relations $g = 0$, $g \in R^\perp$. Then $A^! = UL$ by definition. The Lie algebra L is called the *Koszul, or Quillen dual* of A.

For example, if $A = H^*(X; \mathbb{C})$ is the cohomology ring of a simply connected topological space X, then L is its homotopy Lie algebra, $\oplus\, \pi_i(X)_\mathbb{C}$, under some additional conditions, cf. [29].

Therefore we have, $L = \overset{\infty}{\underset{n=1}{\oplus}} L_n$; moreover one observes that $L_1 = V^*$, $L_2 = S^2V^*/R^\perp$. Set $L_{\le 2} = L_1 \oplus L_2$; it is a quotient Lie algebra of L.

3.4.2. *The Chevalley cochain complex* of L is by definition the space

$$C^\cdot(L) = SL_1^* \otimes \Lambda L_2^* \otimes SL_3^* \otimes \cdots$$

equipped with the Chevalley differential. This complex is double graded:

$$C^\cdot(L) = \overset{\infty}{\underset{a,b=0}{\oplus}} C^\cdot(L)_{ab},$$

where the first (homological) degree of L_i^* is set to be equal $i - 1$, and the second (polynomial) degree of L_i^* is i, both gradings are compatible with the product. The Chevalley differential preserves the second grading and decreases the first by 1:

$$dC^\cdot(L)_{ab} \subset C^\cdot(L)_{a-1,b}.$$

Here are the components of $C^\cdot(L)$ of polynomial degree ≤ 3:

$$\begin{aligned} &L_3^* \longrightarrow L_2^* \otimes L_1^* \longrightarrow S^3L_1^*,\\ &L_2^* \longrightarrow S^2L_1^*,\\ &L_1^*. \end{aligned} \tag{3.11}$$

One has: $C^\cdot(L)_{0\cdot} = SL_1^*$ and the complex starts as:

$$\cdots \longrightarrow (L_3^* \oplus \Lambda^2L_2^*) \otimes SL_1^* \longrightarrow L_2^* \otimes SL_1^* \longrightarrow SL_1^*. \tag{3.12}$$

Since $L_1^* = V$ and $L_2^* = R \subset S^2V$, the first differential in (3.12) is a SV-linear map which sends $f \otimes 1 \in L_2^*$ to $f \in R \subset S^2L_1^*$.

One observes that:

(a) there is a natural augmentation $C^\cdot(L) \longrightarrow A = SV/(R)$, cf. [24].

(A similar picture arises in the construction of the mixed Tate motivic cohomology of a field k, cf. [4]. There, the analogues of complexes $C^\cdot(L)_{n\cdot}$ are the Beilinson motivic complexes $\mathbb{Z}(n)$, the analogue of A is the Milnor K-theory $K_\cdot^{\mathrm{Miln}}(k)$; the analogue of L is (the Lie algebra of) the "mixted Tate motivic fundamental group".)

(b) If one chooses a basis $\{f_1, \ldots, f_d\}$ of R, then one can identify the Chevalley complex of $L_{\le 2}$ with the Koszul complex

$$C^\cdot(L_{\le 2}) \overset{\sim}{=} K(SV; (f)),$$

cf. Example 3.2.

(c) There is a natural inclusion $C^\cdot(L_{\le 2}) \subset C^\cdot(L)$ compatible with the augmentation to A.

Define a generating series

$$H(C^\cdot(L); u, t) := \sum_{a,b=0}^{\infty} \dim C^\cdot(L)_{ab} u^a t^b$$

and its specialization, the Euler–Hilbert series:

$$EH(C^{\cdot}(L);t) := H(C^{\cdot}(L);-1,t) = \sum_{n=0}^{\infty} EP(C^{\cdot}(L)_{?,n})t^n,$$

where EP stands for the Euler–Poincaré characteristic. One observes immediately that

$$EH(SL_n^*) = (1-t^n)^{-\dim L_n} \qquad \text{and} \qquad EH(\Lambda L_n^*) = (1-t^n)^{\dim L_n},$$

where

$$EH(C^{\cdot}(L);t) = \prod_{n=1}^{\infty} (1-t^n)^{(-1)^n \dim L_n}. \tag{3.13}$$

3.4.3. For example, consider the Veronese ring A_b, cf. Section 3.1.1. Recall that A_b is generated by $x_0, \ldots, x_{b+1}$, subject to relations $x_i x_j - x_k x_l = 0$ if $i+j = k+l$. Then $A^!$ is generated by $y_i = x_i^*, 0 \le i \le b+1$ obeying the relations

$$\sum_{i+j=k} [y_i, y_j] = 0, \qquad k = 0, \ldots, 2(b+1), \tag{3.14}$$

where $[y_i, y_j] = y_i y_j + y_j y_i$ (sic!). Therefore if one defines a graded Lie algebra L^b with generators $y_i \in L_1^b = A_{b1}^*$ and relations (3.14), then $A_b^! = UL^b$.

This Lie algebra has a nice structure, cf. [13] (the case $b=1$ was considered in Example 3.2). It admits an involution $i(y_j) = y_{b+1-j}$. Let $\hat{L}^b \subset L^b$ be a Lie subalgebra generated by $y_1, \ldots, y_b$. Then $\hat{L}^b = [L^b, L^b]$ is a Lie ideal (stable under i); as a graded Lie algebra it is *free.*

The quotient algebra $\bar{L} = L^b/\hat{L}^b$ is a graded Abelian Lie algebra on 2 generators $\bar{y}_0$, $\bar{y}_{b+1}$. One sees that $\hat{L}^b_{\ge 2} = L^b_{\ge 2}$.

We have noticed in Example 3.2 that L_1 is finite-dimensional; as opposed to that, if $b \ge 2$, than L^b is infinite-dimensional (with an exponential growth). Namely, one has $L^b = \overset{\infty}{\underset{n=1}{\oplus}} L_{bn}$, where the dimensions of the homogenous components can be calculated easily using an odd analogue of the Witt theorem, see Theorem 3.14. One obtains:

$$\dim L_1 = -M_1(-b) + 2 = b + 2,$$
$$\dim^{\sim} L_n = \dim^{\sim} \hat{L}_n = M_n(-b) = \frac{1}{n} \sum_{d|n} \mu(d)(-b)^{n/d} \qquad (n \ge 2).$$

Here $\hat{L}_{b1}$ is generated by b odd elements, therefore

$$\dim^{\sim} \hat{L}_n = (-1)^n \dim \hat{L}_n.$$

One observes that the signs od $\dim^{\sim} L_n$ alternate. For example, $\dim L_2 = M_2(-b) = (b^2+b)/2$, $\dim L_3 = -M_3(-b) = (b^3-b)/3$, etc.

Now consider the Chevalley complex of L^b: (3.13) implies that its Euler–Hilbert is as follows:

$$\begin{aligned} EH\big(C^{\cdot}\big(L^b\big);t\big) &= (1-t)^{-\dim L_{b1}} \big(1-t^2\big)^{\dim L_{b2}} \big(1-t^3\big)^{-\dim L_{b3}} \cdots \\ &= (1-t)^{-2} \prod_{n=1}^{\infty} (1-t^n)^{M_n(-b)} = \frac{1+bt}{(1-t)^2}. \end{aligned} \tag{3.15}$$

It is clear that it coincides with the Hilbert series of the ring A_b, (3.1). the last equality is due to the cyclotomic identity. It is not surprising.

In fact, a deep theorem of Bezrukavnikov says:

Theorem 3.17 (cf. [6]). *The algebra A_b is Koszul.*

Formula (3.15) can be viewed as a "numerical evidence" that the theorem holds.

Corollary 3.18. *There are natural isomorphisms $A_b^! \cong \mathrm{Ext}^{\cdot}_{A_b}(\mathbb{C},\mathbb{C})$, $A_b \cong \mathrm{Ext}^{\cdot}_{A_b^!}(\mathbb{C},\mathbb{C}) = H^{\cdot}(L;\mathbb{C})$.*

It follows that $A_b^{\sim} := C^{\cdot}(L^b)$ is a dga resolution of A_b, cf. [24].

3.4.4. Characters. The Lie algebra sl(2) acts on A_b in such a way that A_{b1} is an irreducible sl(2)-module; therefore its character is

$$\mathrm{Ch}(A_{b1}) = [b+1]_q = \frac{q^{b+1} - q^{-b-1}}{q - q^{-1}}.$$

The character of A_b is given by the equivariant Hilbert series, cf. (3.2):

$$H_q(A_b;t) = \frac{1 + [b]_q t}{(1 - q^{b+1}t)(1 - q^{-b-1}t)}.$$

The above action induces an action of sl(2) on L^b and therefore on $A_b^{\sim} := C^{\cdot}(L^b)$. Applying now the "$q$-cyclotomic" identity: Theorem 3.9 for $f(q) = -[b]_q$, one obtains the sl(2)-character $\mathrm{Ch}(A_b^{\sim})$.

On the other hand:the subalgebra Lie $\hat{L}_b \subset L^b$ is free on b generators, therefore the Lie algebra gl(b) acts on it. One can use the theorem of Ogievetsky (see Theorem 3.10) for $f(q_1,\dots,q_b) = -\mathrm{Ch}_{\mathrm{gl}(b)}(\hat{L}_{b1})$ for calculation of the character $\mathrm{Ch}_{\mathrm{gl}(b)}(C^{\cdot}(\hat{L}_b))$.

The gl(b)-character of the free Lie algebra on *odd* b generators was calculated by Angeline Brandt in [9].

The free *group* on b generators is isomorphic to the fundamental group of the Riemann sphere with $b+1$ points removed; its nilpotent completion is a fundamental object of the theory of Grothendieck–Drinfeld–Ihara, cf. [10].

3.4.5. Returning to the case of an arbitrary commutative quadratic algebra, one can show that a complete intersection is Koszul, cf. [26]. The other way around, if A is commutative Koszul, and L is its dual Lie algebra, then A is a complete intersection if and only if $L = L_{\leq 2}$.

One can say that commutative (or maybe also noncommutative?) Koszul algebras are natural generalizations of the quadratic complete intersections; and one would expect that all the results which hold for quadratic complete intersections will generalize to Koszul algebras.

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