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Discrete Cocompact Subgroups of the Five-Dimensional Connected and Simply Connected Nilpotent Lie Groups

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Abstract. The discrete cocompact subgroups of the five-dimensional connected, simply connected nilpotent Lie groups are determined up to isomorphism. Moreover, we prove if $G = N \times A$ is a connected, simply connected, nilpotent Lie group with an Abelian factor A, then every uniform subgroup of G is the direct product of a uniform subgroup of N and $\mathbb{Z}^r$ where $r = \dim(A)$.

Key words: nilpotent Lie group; discrete subgroup; nil-manifold; rational structures, Smith normal form; Hermite normal form

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1 Introduction

To solve many problems in various branches of mathematics, for example, in differential geometry (invariant geometric structures on Lie groups), in spectral geometry, in physics (multidimensional models of space-times) etc. (see [4]), we are led to study the explicit description of uniform subgroups (i.e., discrete cocompact) of solvable Lie groups. Towards that purpose, we focus attention to the description of uniform subgroups of a connected, simply connected non Abelian nilpotent Lie groups satisfying the rationality criterion of Malcev (see [9]). At present we have no progress on this difficult problem although the classification for certain groups is already exists. To attack this problem, we will be in a first step interested in giving an explicit description of uniform subgroups of nilpotent Lie groups of dimension less or equal to 6. Another motivation comes from [13, p. 339]. For the case of 3 or 4 dimensions, the classification is given in [1, 15, 11]. For 5-dimensional nilpotent Lie groups, there are eight real connected and simply connected non Abelian nilpotent Lie groups which are $G_3 \times \mathbb{R}^2$, $G_4 \times \mathbb{R}$ and $G_{5,i}$ for $1 \leq i \leq 6$ [3]. The uniform subgroups of $G_3 \times \mathbb{R}^2$, $G_{5,1}$, $G_{5,3}$ and $G_{5,5}$ are determined respectively in [15, 5, 12, 6]. The aim of this paper is to complete the classification of the discrete cocompact subgroups of the Lie groups $G_4 \times \mathbb{R}$, $G_{5,2}$, $G_{5,4}$ and $G_{5,6}$.

This paper is organized as follows. In Section 2, we fix some notation which will be of use later and we record few standard facts about rational structures and uniform subgroups of a connected, simply connected nilpotent Lie groups. Section 3 is devoted to the study of the rationality of certain subalgebras of a given nilpotent Lie algebra with a rational structure. In Section 4, in order to determine the uniform subgroups of $G_4 \times \mathbb{R}$, we study a more general class when G is a connected, simply connected, nilpotent Lie group with an Abelian factor, that is $G = N \times A$ where A is an Abelian normal subgroup of G. Theorem 5 shows that every uniform subgroup Γ of G is a direct product of a uniform subgroup of N and $\mathbb{Z}^r$ where $r = \dim(A)$. As an immediate application of this result, we determine the uniform subgroups of $G_4 \times \mathbb{R}$. We determine afterwards all uniform subgroups of the Lie groups $G_{5,2}$, $G_{5,4}$ and $G_{5,6}$.

2 Notations and basic facts

The aim of this section is to give a brief review of certain results from rational structures and uniform subgroups of connected and simply connected nilpotent Lie groups which will be needed later. The reader who is interested in detailed proof is referred to standard texts [2, 16, 9].

2.1 Rational structures and uniform subgroups

Let G be a nilpotent, connected and simply connected real Lie group and let $\mathfrak{g}$ be its Lie algebra. We say that $\mathfrak{g}$ (or G) has a *rational structure* if there is a Lie algebra $\mathfrak{g}_{\mathbb{Q}}$ over $\mathbb{Q}$ such that $\mathfrak{g} \cong \mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R}$. It is clear that $\mathfrak{g}$ has a rational structure if and only if $\mathfrak{g}$ has an $\mathbb{R}$-basis $\{X_1, \ldots, X_n\}$ with rational structure constants.

Let $\mathfrak{g}$ have a fixed rational structure given by $\mathfrak{g}_{\mathbb{Q}}$ and let $\mathfrak{h}$ be an $\mathbb{R}$-subspace of $\mathfrak{g}$. Define $\mathfrak{h}_{\mathbb{Q}} = \mathfrak{h} \cap \mathfrak{g}_{\mathbb{Q}}$. We say that $\mathfrak{h}$ is *rational* if $\mathfrak{h} = \mathbb{R}\text{-span}\{\mathfrak{h}_{\mathbb{Q}}\}$, and that a connected, closed subgroup H of G is *rational* if its Lie algebra $\mathfrak{h}$ is rational. The elements of $\mathfrak{g}_{\mathbb{Q}}$ (or $G_{\mathbb{Q}} = \exp(\mathfrak{g}_{\mathbb{Q}})$) are called *rational elements* (or *rational points*) of $\mathfrak{g}$ (or G).

A discrete subgroup Γ is called *uniform* in G if the quotient space G/Γ is compact. The homogeneous space G/Γ is called a *compact nilmanifold*. A proof of the next result can be found in Theorem 7 of [9] or in Theorem 2.12 of [16].

Theorem 1 (The Malcev rationality criterion). *Let G be a simply connected nilpotent Lie group, and let $\mathfrak{g}$ be its Lie algebra. Then G admits a uniform subgroup Γ if and only if $\mathfrak{g}$ admits a basis $\{X_1, \ldots, X_n\}$ such that*

$$[X_i, X_j] = \sum_{\alpha=1}^{n} c_{ij\alpha} X_\alpha \qquad \textit{for all} \quad i, \ j,$$

where the constants $c_{ij\alpha}$ are all rational. (The $c_{ij\alpha}$ are called the structure constants of $\mathfrak{g}$ relative to the basis $\{X_1, \ldots, X_n\}$.)

More precisely, we have, if G has a uniform subgroup Γ, then $\mathfrak{g}$ (hence G) has a rational structure such that $\mathfrak{g}_{\mathbb{Q}} = \mathbb{Q}\text{-span}\{\log(\Gamma)\}$. Conversely, if $\mathfrak{g}$ has a rational structure given by some $\mathbb{Q}$-algebra $\mathfrak{g}_{\mathbb{Q}} \subset \mathfrak{g}$, then G has a uniform subgroup Γ such that $\log(\Gamma) \subset \mathfrak{g}_{\mathbb{Q}}$ (see [2, 9]). If we endow G with the rational structure induced by a uniform subgroup Γ and if H is a Lie subgroup of G, then H is rational if and only if $H \cap \Gamma$ is a uniform subgroup of H. Note that the notion of rational depends on Γ.

2.1.1 Weak and strong Malcev basis

Let $\mathfrak{g}$ be a nilpotent Lie algebra and let $\mathscr{B} = \{X_1, \ldots, X_n\}$ be a basis of $\mathfrak{g}$. We say that $\mathscr{B}$ is a weak (resp. strong) Malcev basis for $\mathfrak{g}$ if $\mathfrak{g}_i = \mathbb{R}\text{-span}\{X_1, \ldots, X_i\}$ is a subalgebras (resp. an ideal) of $\mathfrak{g}$ for each $1 \leq i \leq n$ (see [2]).

Let Γ be a uniform subgroup of G. A strong Malcev (or Jordan–Hölder) basis $\{X_1, \ldots, X_n\}$ for $\mathfrak{g}$ is said to be *strongly based on* Γ if

$$\Gamma = \exp(\mathbb{Z}X_1) \cdots \exp(\mathbb{Z}X_n).$$

Such a basis always exists (see [2, 10]).

The *lower central series* (or the *descending central series*) of $\mathfrak{g}$ is the decreasing sequence of characteristic ideals of $\mathfrak{g}$ defined inductively as follows

$$\mathcal{C}^1(\mathfrak{g}) = \mathfrak{g}; \qquad \mathcal{C}^{p+1}(\mathfrak{g}) = [\mathfrak{g}, \mathcal{C}^p(\mathfrak{g})] \qquad (p \geq 1).$$

The characteristic ideal $\mathcal{C}^2(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$ is called the derived ideal of the Lie algebra $\mathfrak{g}$ and denoted by $\mathcal{D}(\mathfrak{g})$. Let $\mathcal{D}(\mathrm{G}) = \exp(\mathcal{D}(\mathfrak{g}))$, observe that we have $\mathcal{D}(\mathrm{G}) = [G, G]$.

The Lie algebra $\mathfrak{g}$ is called *k-step nilpotent Lie algebra* or *nilpotent Lie algebra of class k* if there is an integer k such that

$$\mathcal{C}^{k+1}(\mathfrak{g}) = \{0\} \qquad \text{and} \qquad \mathcal{C}^{k}(\mathfrak{g}) \neq \{0\}\,.$$

We denote the center of G by $Z(G)$ and the center of $\mathfrak{g}$ by $\mathfrak{z}(\mathfrak{g})$.

Proposition 1 ([10, 2]). *If $\mathfrak{g}$ has rational structure, all the algebras in the descending central series are rational.*

Let G be a group. The center $Z(G)$ of G is a normal subgroup. Let $\mathscr{C}_2(G)$ be the inverse image of $Z(G/Z(G))$ under the canonical projection $G \longrightarrow G/Z(G)$. Then $\mathscr{C}_2(G)$ is normal in G and contains $Z(G)$. Continue this process by defining inductively: $\mathscr{C}_1(G) = Z(G)$ and $\mathscr{C}_i(G)$ is the inverse image of $Z(G/\mathscr{C}_{i-1}(G))$ under the canonical projection $G \longrightarrow G/\mathscr{C}_{i-1}(G)$. Thus we obtain a sequence of normal subgroups of G, called the ascending central series of G.

Proposition 2 ([2]). *If G is a nilpotent Lie group with rational structure, all the algebras in the ascending central series are rational. In particular, the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ is rational.*

A proof of the next result can be found in Proposition 5.3.2 of [2].

Proposition 3. *Let Γ be uniform subgroup in a nilpotent Lie group G, and let $H_1 \subsetneqq H_2 \subsetneqq \cdots \subsetneqq H_k = G$ be rational Lie subgroups of G. Let $\mathfrak{h}_1, \ldots, \mathfrak{h}_{k-1}, \mathfrak{h}_k = \mathfrak{g}$ be the corresponding Lie algebras. Then there exists a weak Malcev basis $\{X_1, \ldots, X_n\}$ for $\mathfrak{g}$ strongly based on Γ and passing through $\mathfrak{h}_1, \ldots, \mathfrak{h}_{k-1}$. If the H_j are all normal, the basis can be chosen to be a strong Malcev basis.*

A rational structure on $\mathfrak{g}$ induces a rational structure on the dual space $\mathfrak{g}^*$ (for further details, see [2, Chaper 5]). If $\mathfrak{g}$ has a rational structure given by the uniform subgroup Γ, a real linear functional $f \in \mathfrak{g}^*$ is rational ($f \in \mathfrak{g}^*_{\mathbb{Q}}, \mathfrak{g}_{\mathbb{Q}} = \mathbb{Q}\text{-span}\,\{\log(\Gamma)\}$) if $\langle f, \mathfrak{g}_{\mathbb{Q}}\rangle \subset \mathbb{Q}$, or equivalently $\langle f, \log(\Gamma)\rangle \subset \mathbb{Q}$. Let $\mathrm{Aut}(G)$ (respectively $\mathrm{Aut}(\mathfrak{g})$) denote the group of automorphism of G (respectively $\mathfrak{g}$). If $\varphi \in \mathrm{Aut}(G)$, φ_* will denote the derivative of φ at identity. The mapping $\mathrm{Aut}(G) \longrightarrow \mathrm{Aut}(\mathfrak{g})$, $\varphi \longmapsto \varphi_*$ is a groups isomorphism (since G is simply connected).

Theorem 2 ([9, Theorem 5]). *Let G_1 and G_2 be connected simply connected nilpotent Lie groups and Γ_1, Γ_2 uniform subgroups of G_1 and G_2. Any abstract group isomorphism f between Γ_1 and Γ_2 extends uniquely to an isomorphism $\overline{f}$ of G_1 on G_2; that is, the following diagram*

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\;f\;} & \Gamma_2 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ G_1 & \xrightarrow[\;\overline{f}\;]{} & G_2 \end{array}$$

is commutative, where i is the inclusion mapping.

We conclude this review with two results.

2.2 Smith normal form

A commutative ring $\mathbf{R}$ with identity $1_{\mathbf{R}} \neq 0$ and no zero divisors is called an *integral domain*. A principal ideal ring which is an integral domain is called a *principal ideal domain*.

Theorem 3 (elementary divisors theorem). *If A is an $n \times n$ matrix of rank $r > 0$ over a principal ideal domain $\mathbf{R}$, then A is equivalent to a matrix of the form*

$$\begin{pmatrix} L_r & 0 \\ 0 & 0 \end{pmatrix},$$

where L_r is an $r \times r$ diagonal matrix with nonzero diagonal entries $d_1, \ldots, d_r$ such that $d_1|d_2|\cdots|d_r$.

The notation $d_1|d_2|\cdots|d_r$ means d_1 divides d_2, d_2 divides d_3, etc. The elements $d_1, \ldots, d_r$ are called the elementary divisors of A. See [7] for a more precise result.

2.3 Hermite normal form

Definition 1 (Hermite normal form). A matrix $(a_{ij}) \in \mathrm{Mat}(m, n, \mathbb{R})$ with $m \leq n$ is in Hermite normal form if the following conditions are satisfied:

(1) $a_{ij} = 0$ for $i > j$;

(2) $a_{ii} > 0$ for $i = 1, \ldots, m$;

(3) $0 \leq a_{ij} < a_{ii}$ for $i < j$.

Theorem 4 (Hermite 1850). *For every matrix $A \in \mathrm{Mat}(m, n, \mathbb{R})$ with $\mathrm{rank}(A) = m \leq n$, there is a matrix $T \in \mathrm{GL}(n, \mathbb{Z})$, so that AT is in Hermite normal form. The Hermite normal form AT is unique.*

3 Rationality of certain subalgebras

The following generalizes the Proposition 5.2.4 of [2].

Proposition 4. *Let G be a simply connected nilpotent Lie group and $\Gamma \subset G$ a uniform subgroup. Let $\Delta \subset \Gamma$ be a finite set and let $\mathbf{C}(\Delta)$ denote the centralizer of Δ in G. Then $\mathbf{C}(\Delta)$ is rational.*

Proof. This follows immediately from Lemma 1.14, Theorem 2.1 of [16] and Theorem 4.5 of [14]. ∎

Next, we prove the following proposition which will play an important role below.

Proposition 5. *Let Γ be a uniform subgroup of a nilpotent Lie group $G = \exp(\mathfrak{g})$. Let $H = \exp(\mathfrak{h})$ be a rational subgroup of G. Then the centralizer $\mathfrak{c}(\mathfrak{h})$ of $\mathfrak{h}$ in $\mathfrak{g}$ is rational.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a weak Malcev basis for $\mathfrak{g}$ strongly based on Γ passing through $\mathfrak{h}$ (see [2, Proposition 5.3.2]). We note $\mathfrak{h} = \mathbb{R}\text{-span}\{e_1, \ldots, e_p\}$. Let $\{e_1^*, \ldots, e_n^*\}$ be the dual basis of $\{e_1, \ldots, e_n\}$. For $i = 1, \ldots, n$ and $j = 1, \ldots, p$, we define

$$f_{ij} : \mathfrak{g} \longrightarrow \mathbb{R}; \qquad X \longmapsto \langle e_i^*, [X, e_j] \rangle.$$

The functionals f_{ij} are rational. Then the kernels $\ker(f_{ij})$ are rational subspaces in $\mathfrak{g}$ (see [2, Lemma 5.1.2]). On the other hand, it is easy to see that

$$\mathfrak{c}(\mathfrak{h}) = \bigcap_{\substack{1 \leq i \leq n \\ 1 \leq j \leq p}} \ker(f_{ij}).$$

Therefore, we conclude from Lemma 5.1.2 of [2] that $\mathfrak{c}(\mathfrak{h})$ is rational. ∎

4 Uniform subgroups of nilpotent Lie group with an Abelian factor. Uniform subgroups of $G_4 \times \mathbb{R}$

The aim of this section is to describe the classification of the uniform subgroups of $G_4 \times \mathbb{R}$. Let us begin with a more general situation. First, we introduce the following definition.

Definition 2 ([8, 4]). An *Abelian factor* of a Lie algebra $\mathfrak{g}$ is an Abelian ideal $\mathfrak{a}$ for which there exists an ideal $\mathfrak{n}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a}$ (i.e., $[\mathfrak{n}, \mathfrak{a}] = \{0\}$).

Let $m(\mathfrak{g})$ denote the maximum dimension over all Abelian factors of $\mathfrak{g}$. If $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$ then the maximal Abelian factors are precisely the linear direct complements of $\mathfrak{z}(\mathfrak{g}) \cap \mathcal{D}(\mathfrak{g})$ in $\mathfrak{z}(\mathfrak{g})$, that is, those subspaces $\mathfrak{a} \subset \mathfrak{z}(\mathfrak{g})$ such that $\mathfrak{z}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}) \cap \mathcal{D}(\mathfrak{g}) \oplus \mathfrak{a}$. Therefore

$$m(\mathfrak{g}) = \dim(\mathfrak{z}(\mathfrak{g})) - \dim(\mathfrak{z}(\mathfrak{g}) \cap \mathcal{D}(\mathfrak{g})).$$

Let $\mathfrak{g}$ be a nilpotent Lie algebra and $\mathfrak{a}$ an Abelian factor of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{n} \oplus \mathbb{R}^r$ be any decomposition in ideals of $\mathfrak{g}$. The next simple lemma establishes a relation between the derived algebra of $\mathfrak{g}$ and the derived algebra of $\mathfrak{n}$. Its value will be immediately apparent in the proof of Theorem 5.

Lemma 1. *With the above notation, we have*

$$\mathcal{D}(\mathfrak{g}) = \mathcal{D}(\mathfrak{n}).$$

Proof. Let $X_1, X_2 \in \mathfrak{g}$. Write $X_i = a_i + b_i$, $i = 1, 2$, where $a_i \in \mathfrak{a}$ and $b_i \in \mathfrak{n}$. We calculate

$$[X_1, X_2] = [a_1 + b_1, a_2 + b_2] = [b_1, b_2]$$

since $\mathfrak{a} \subset \mathfrak{z}(\mathfrak{g})$. Therefore $\mathcal{D}(\mathfrak{g}) = \mathcal{D}(\mathfrak{n})$. The proof of the lemma is complete. ■

In the sequel, the symbol $\simeq$ denotes abstract group isomorphism.

Theorem 5. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with maximal Abelian factor of dimension $m(\mathfrak{g}) = r$ and let $\mathfrak{g} = \mathfrak{n} \oplus \mathbb{R}^r$ be any decomposition in ideals, that is $\mathbb{R}^r$ is a maximal Abelian factor of $\mathfrak{g}$. Then we have the following:*

(1) *The group $G = \exp(\mathfrak{g})$ admits a uniform subgroup if and only if $N = \exp(\mathfrak{n})$ admits a uniform subgroup.*

(2) *If Γ is a uniform subgroup of G, then there exists a uniform subgroup H of N such that*

$$\Gamma \simeq H \times \mathbb{Z}^r.$$

Proof. Let $\mathfrak{a} = \mathbb{R}^r$.

(1) If H is a uniform subgroup of N then it is clear that $H \times \mathbb{Z}^r$ is a uniform subgroup of G. Conversely, we suppose that G admits a uniform subgroup Γ. Let $\{e_1, \ldots, e_n\}$ be a strong Malcev basis for $\mathfrak{g}$ strongly based on Γ passing through $\mathcal{D}(\mathfrak{g})$ and $\mathcal{D}(\mathfrak{g}) + \mathfrak{z}(\mathfrak{g})$. Put

$$\mathcal{D}(\mathfrak{g}) = \mathbb{R}\text{-span}\,\{e_1, \ldots, e_q\}$$

and

$$\mathcal{D}(\mathfrak{g}) + \mathfrak{z}(\mathfrak{g}) = \mathbb{R}\text{-span}\,\{e_1, \ldots, e_q, e_{q+1}, \ldots, e_{q+r}\}.$$

For every $i = q + r + 1, \ldots, n$, we note

$$e_i = u_i + v_i,$$

where $u_i \in \mathfrak{n}$ and $v_i \in \mathfrak{a}$. It is clear that $\{e_1, \ldots, e_q, u_{q+r+1}, \ldots, u_n\}$ is a basis for $\mathfrak{n}$ and has the same structure constants as $\{e_1, \ldots, e_n\}$. By the Malcev rationality criterion, N admits a uniform subgroup.

(2) Let Γ be a uniform subgroup of G. Since $\mathfrak{z}(\mathfrak{g})$ and $\mathcal{D}(\mathfrak{g})$ are rational then there exists a strong Malcev basis $\{e_1, \ldots, e_n\}$ for $\mathfrak{g}$ strongly based on Γ passing through $\mathfrak{z}(\mathfrak{g}) \cap \mathcal{D}(\mathfrak{g})$, $\mathfrak{z}(\mathfrak{g})$ and $\mathcal{D}(\mathfrak{g}) + \mathfrak{z}(\mathfrak{g})$. Put

$$\mathfrak{z}(\mathfrak{g}) \cap \mathcal{D}(\mathfrak{g}) = \mathbb{R}\text{-span}\,\{e_1, \ldots, e_p\}\,.$$

Since $\dim(\mathfrak{z}(\mathfrak{g})/\mathfrak{z}(\mathfrak{g}) \cap \mathcal{D}(\mathfrak{g})) = m(\mathfrak{g}) = r$, then

$$\mathfrak{z}(\mathfrak{g}) = \mathbb{R}\text{-span}\,\{e_1, \ldots, e_p, \ldots, e_{p+r}\}\,.$$

Since $\mathfrak{z}(\mathfrak{g})$ is Abelian, we can assume that $(e_1, \ldots, e_p)$ are independent modulo $\mathfrak{a}$, i.e., they span a complement of $\mathfrak{a}$ in $\mathfrak{z}(\mathfrak{g})$. Let for every $i = p+1, \ldots, n$

$$e_i = a_i + b_i,$$

where $a_i \in \mathfrak{a}$ and $b_i \in \mathfrak{n}$. On the other hand, we have $\mathcal{D}(\mathfrak{g}) + \mathfrak{z}(\mathfrak{g}) = \mathcal{D}(\mathfrak{g}) \oplus \mathfrak{a}$, then $\mathfrak{a} = \mathbb{R}\text{-span}\,\{a_{p+1}, \ldots, a_{p+r}\}$. Let $\Phi_* : \mathfrak{g} \longrightarrow \mathfrak{g}$ be the function defined by

$$e_i \longmapsto \begin{cases} e_i, & \text{if } 1 \le i \le p, \\ a_i, & \text{if } p+1 \le i \le p+r, \\ b_i, & \text{if } p+r+1 \le i \le n. \end{cases}$$

We will show that Φ_* is a Lie automorphism of $\mathfrak{g}$. In fact, if $i \le p+r$ or $j \le p+r$, we have $\Phi_*([e_i, e_j]) = [\Phi_*(e_i), \Phi_*(e_j)] = 0$. Next, we suppose that $i, j > p+r$. It is easy to verify that

$$[\Phi_*(e_i), \Phi_*(e_j)] = [b_i, b_j].$$

On the other hand, we have

$$[e_i, e_j] = [a_i + b_i, a_j + b_j] = [b_i, b_j]$$

and hence

$$\Phi_*([e_i, e_j]) = [b_i, b_j].$$

Consequently, we obtain

$$\Phi_*([e_i, e_j]) = [\Phi_*(e_i), \Phi_*(e_j)].$$

Then

$$\begin{aligned} \Gamma \simeq \Phi(\Gamma) &= \prod_{i=1}^{p} \exp(\mathbb{Z}e_i) \prod_{i=p+1}^{p+r} \exp(\mathbb{Z}a_i) \prod_{i=p+r+1}^{n} \exp(\mathbb{Z}b_i) \\ &= \prod_{i=1}^{p} \exp(\mathbb{Z}e_i) \prod_{i=p+r+1}^{n} \exp(\mathbb{Z}b_i) \prod_{i=p+1}^{p+r} \exp(\mathbb{Z}a_i). \end{aligned}$$

Evidently, $H = \prod_{i=1}^{p} \exp(\mathbb{Z}e_i) \prod_{i=p+r+1}^{n} \exp(\mathbb{Z}b_i)$ is a uniform subgroup of N and $K = \prod_{i=p+1}^{p+r} \exp(\mathbb{Z}a_i)$ is a uniform subgroup of $A = \exp(\mathfrak{a})$. It follows that

$$\Gamma \simeq H \times \mathbb{Z}^r.$$

This completes the proof of the theorem. ■

A consequence of the above theorem, we deduce the uniform subgroups of $G_4 \times \mathbb{R}$. We recall that the Lie algebra $\mathfrak{g}_4$ of G_4 is spanned by the vectors $X_1, \dots, X_4$ such that the only non vanishing brackets are

$$[X_4, X_3] = X_2, \qquad [X_4, X_2] = X_1.$$

Corollary 1. *Let $\{e\}$ be the canonical basis of $\mathbb{R}$. Every uniform subgroup Γ of $G_4 \times \mathbb{R}$ has the following form*

$$\Gamma \simeq \exp(\mathbb{Z}e) \exp(\mathbb{Z}X_1) \exp(p_1 \mathbb{Z}X_2) \exp\left(\mathbb{Z}\left(p_1 p_2 X_3 - \frac{p_3}{2} X_2\right)\right) \exp(\mathbb{Z}X_4),$$

where p_1, p_2, p_3 are integers satisfying $p_1 > 0$, $p_2 > 0$, $p_1 p_2 + p_3 \in 2\mathbb{Z}$ and $0 \leq p_3 < 2p_1$. Furthermore, different choices for the p's give non isomorphic subgroups.

Proof. The proof follows from Theorem 5 and [6, Proposition 4.1] (see also [11, Theorem 1]). ■

Remark 1. We note that Proposition B.1 of [15] becomes a direct consequence of Theorem 5 and Theorem 2.4 of [5].

5 Uniform subgroups of $G_{5,2}$

Let $\mathfrak{g}_{5,2}$ be the two-step nilpotent Lie algebra with basis

$$\mathscr{B} = \{X_1, \dots, X_5\}$$

and non-trivial Lie brackets defined by

$$[X_5, X_4] = X_2, \qquad [X_5, X_3] = X_1. \tag{1}$$

Let $G_{5,2}$ be the corresponding connected and simply connected nilpotent Lie group. First, we introduce the following set:

$$\mathscr{D}_2 = \{r = (r_1, r_2) \in (\mathbb{N}^*)^2 : \ r_1 \text{ divides } r_2\}.$$

Theorem 6.

1. *Let $r = (r_1, r_2) \in \mathscr{D}_2$. Then*
$$\Gamma_r = \exp\left(\frac{1}{r_1}\mathbb{Z}X_1\right) \exp\left(\frac{1}{r_2}\mathbb{Z}X_2\right) \exp(\mathbb{Z}X_3) \exp(\mathbb{Z}X_4) \exp(\mathbb{Z}X_5)$$
is a uniform subgroup of $G_{5,2}$.
2. *If Γ is a uniform subgroup of $G_{5,2}$, then there exist $r \in \mathscr{D}_2$ and $\Phi \in \mathrm{Aut}(G_{5,2})$ such that $\Phi(\Gamma) = \Gamma_r$.*
3. *For r and s in $\mathscr{D}_2$, Γ_r and Γ_s are isomorphic groups if and only if $r = s$.*

Proof. The proof of this theorem will be achieved through a sequence of partial results.

Lemma 2. *The group $\mathrm{Aut}(\mathfrak{g}_{5,2})$ is the set of all matrices of the form*

$$\begin{pmatrix} \alpha A & B & u \\ 0 & A & v \\ 0 & 0 & \alpha \end{pmatrix},$$

where $A \in \mathrm{GL}(2, \mathbb{R}), B \in \mathrm{Mat}(2, \mathbb{R}), \alpha \in \mathbb{R}^*$ and $u, v \in \mathbb{R}^2$.

Proof. It is easy to verify that $\mathbb{R}\text{-span}\{X_1,\dots,X_4\}$ is the unique one co-dimensional Abelian ideal of $\mathfrak{g}$. Therefore any automorphism A of $\mathfrak{g}$ leaves invariant the subalgebras $\mathbb{R}\text{-span}\{X_1,X_2\}$ and $\mathbb{R}\text{-span}\{X_1,\dots,X_4\}$. The remainder of the proof follows from (1). ■

Lemma 3. *Every uniform subgroup Γ of $G_{5,2}$ has the following form: there are integers p, q and α satisfying $p,q>0$ and $0\le\alpha<q$ such that*

$$\Gamma\simeq\Gamma(p,q,\alpha)=\exp\left(\frac{1}{p}\mathbb{Z}X_1\right)\exp\left(\mathbb{Z}\left(\frac{1}{q}X_2-\frac{\alpha}{pq}X_1\right)\right)\exp(\mathbb{Z}X_3)\exp(\mathbb{Z}X_4)\exp(\mathbb{Z}X_5).$$

Proof. Let $G=G_{5,2}$, $\mathfrak{g}=\mathfrak{g}_{5,2}$ and let Γ be a uniform subgroup of G. Let

$$\Omega=\left\{l=\sum_{i=1}^{5}l_iX_i^*\in\mathfrak{g}^*:\ l_1\neq0\right\}$$

be the layer of the generic coadjoint orbits (see [2, Chapter 3]). As $\mathfrak{g}^*_{\mathbb{Q}}$ is dense in $\mathfrak{g}^*$ and Ω is a non-empty Zariski open set in $\mathfrak{g}^*$ then $\mathfrak{g}^*_{\mathbb{Q}}\cap\Omega\neq\varnothing$. Let $l\in\mathfrak{g}^*_{\mathbb{Q}}\cap\Omega$. Since l is rational then by Proposition 5.2.6 of [2], the radical $\mathfrak{g}(l)$ of the skew-symmetric bilinear form B_l on $\mathfrak{g}$ defined by $B_l(X,Y)=\langle l,[X,Y]\rangle$ $(X,Y\in\mathfrak{g})$, is also rational. It follows by Proposition 5 that $\mathbf{c}(\mathfrak{g}(l))$ is also rational subalgebra of $\mathfrak{g}$. As

$$\mathfrak{g}(l)=\mathbb{R}\text{-span}\{X_1,X_2,l_2X_3-l_1X_4\}$$

then a simple calculation shows that

$$\mathbf{c}(\mathfrak{g}(l))=\mathbb{R}\text{-span}\{X_1,\dots,X_4\}.$$

On the other hand, the derived ideal $[\mathfrak{g},\mathfrak{g}]=\mathbb{R}\text{-span}\{X_1,X_2\}$ is also rational subalgebra in $\mathfrak{g}$ (see [2, Corollary 5.2.2]). It follows by Proposition 5.3.2 of [2] that there exists a strong Malcev basis $\{Y_1,\dots,Y_5\}$ of $\mathfrak{g}$ strongly based on Γ passing through $[\mathfrak{g},\mathfrak{g}]$ and $\mathbb{R}\text{-span}\{X_1,\dots,X_4\}$. Then, we have $Y_1=a_{11}X_1+a_{12}X_2$, $Y_2=a_{21}X_1+a_{22}X_2$, $Y_i=\sum_{j=1}^{4}a_{ij}X_j$ $(i=3,4)$ and $Y_5=\sum_{j=1}^{5}a_{5j}X_j$, where $a_{ij}\in\mathbb{R}$. The mapping $(\Phi_1)_*:\mathfrak{g}\longrightarrow\mathfrak{g}$ defined by

$$\begin{aligned}
&(\Phi_1)_*(Y_i)=Y_i \qquad (i=1,2),\\
&(\Phi_1)_*(Y_3)=\frac{1}{a_{55}}(a_{33}X_3+a_{34}X_4),\\
&(\Phi_1)_*(Y_4)=\frac{1}{a_{55}}(a_{43}X_3+a_{44}X_4),\\
&(\Phi_1)_*(Y_5)=X_5
\end{aligned}$$

is a Lie algebra automorphism. Then

$$\Gamma\simeq\Gamma_1=\exp(\mathbb{Z}Y_1)\exp(\mathbb{Z}Y_2)\exp((\Phi_1)_*(Y_3))\exp((\Phi_1)_*(Y_4))\exp(\mathbb{Z}X_5).$$

On the other hand, let the Lie algebra automorphism $(\Phi_2)_*:\mathfrak{g}\longrightarrow\mathfrak{g}$ defined by

$$(\Phi_2)_*(X_5)=X_5,\qquad(\Phi_2)_*(X_4)=\frac{a_{43}}{a_{55}}X_3+\frac{a_{44}}{a_{55}}X_4,\qquad(\Phi_2)_*(X_3)=\frac{a_{33}}{a_{55}}X_3+\frac{a_{34}}{a_{55}}X_4.$$

It follows that there exist $a,b,c,d\in\mathbb{R}$ such that

$$\Gamma_1\simeq\Phi_2^{-1}(\Gamma_1)=\Gamma_2=\exp(\mathbb{Z}(aX_1+bX_2))\exp(\mathbb{Z}(cX_1+dX_2))\prod_{i=3}^{5}\exp(\mathbb{Z}X_i).$$

Next, since $\exp(X_5)\exp(X_3)\exp(-X_5) \in \Gamma_2$, then $\exp(X_1) \in \Gamma_2$ and hence the ideal $\mathbb{R}$-span$\{X_1\}$ is rational relative to Γ_2. It follows that there exist $x, y_1, y_2 \in \mathbb{R}$ such that

$$\Gamma_2 = \exp(\mathbb{Z}(xX_1))\exp(\mathbb{Z}(y_1X_1 + y_2X_2))\prod_{i=3}^{5}\exp(\mathbb{Z}X_i).$$

Also, we use that $\exp(X_1)$ belongs to Γ_2, we deduce that there exists $p \in \mathbb{N}^*$ such that $x = \frac{1}{p}$. Similarly, since $\exp(X_5)\exp(X_4)\exp(-X_5) \in \Gamma_2$, then $\exp(X_2) \in \Gamma_2$. Therefore there exists $(q, m) \in \mathbb{N}^* \times \mathbb{Z}$ such that $y_2 = \frac{1}{q}$ and $y_1 = -\frac{m}{pq}$. Then

$$\Gamma_2 = \exp\left(\frac{1}{p}\mathbb{Z}X_1\right)\exp\left(\mathbb{Z}\left(\frac{1}{q}X_2 - \frac{m}{pq}X_1\right)\right)\prod_{i=3}^{5}\exp(\mathbb{Z}X_i).$$

Finally, we observe that, if α is the remainder of the division of m by q, then

$$\Gamma_2 = \exp\left(\frac{1}{p}\mathbb{Z}X_1\right)\exp\left(\mathbb{Z}\left(\frac{1}{q}X_2 - \frac{\alpha}{pq}X_1\right)\right)\prod_{i=3}^{5}\exp(\mathbb{Z}X_i). \qquad \blacksquare$$

Lemma 4. *A necessary and sufficient condition that two subgroups $\Gamma(p, q, \alpha)$ and $\Gamma(p', q', \alpha')$ are isomorphic is that there exist $A, B \in GL(2, \mathbb{Z})$ such that*

$$\begin{pmatrix} p' & \alpha' \\ 0 & q' \end{pmatrix} = A\begin{pmatrix} p & \alpha \\ 0 & q \end{pmatrix}B. \tag{2}$$

Proof. Let $\Gamma(p, q, \alpha) \simeq \Gamma(p', q', \alpha')$. It is well known that any abstract isomorphism of $\Gamma(p, q, \alpha)$ onto $\Gamma(p', q', \alpha')$ is the restriction of an automorphism Φ of G (see [9, Theorem 5, p. 292]). Since $\mathbb{R}$-span $\{X_1, \ldots, X_4\}$ is the unique one co-dimensional Abelian ideal of $\mathfrak{g}$, then we can suppose that $\Phi_*(X_5) = X_5$, $\Phi_*(X_3) = aX_3 + bX_4$ and $\Phi_*(X_4) = cX_3 + dX_4$ $(a, b, c, d \in \mathbb{R})$. We deduce that

$$\mathbb{Z}\text{-span}\,\{aX_3 + bX_4, cX_3 + dX_4\} = \mathbb{Z}\text{-span}\,\{X_3, X_4\}$$

and

$$\begin{aligned}&\mathbb{Z}\text{-span}\left\{\frac{1}{p}(aX_1 + bX_2), \frac{1}{q}(cX_1 + dX_2) - \frac{\alpha}{pq}(aX_1 + bX_2)\right\}\\ &\quad= \mathbb{Z}\text{-span}\left\{\frac{1}{p'}X_1, \frac{1}{q'}X_2 - \frac{\alpha'}{p'q'}X_1\right\}.\end{aligned}$$

Consequently, we obtain

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}) \qquad \text{and} \qquad \begin{pmatrix} p' & \alpha' \\ 0 & q' \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} \frac{1}{p} & \frac{-\alpha}{pq} \\ 0 & \frac{1}{q} \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}).$$

Conversely, suppose that there exist $A, B \in \mathrm{GL}(2, \mathbb{Z})$ satisfy the relation (2). The mapping ϕ_* defined by

$$\mathrm{Mat}(\phi_*, \mathscr{B}) = \begin{pmatrix} B^{-1} & 0 & 0 \\ 0 & B^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

belongs to $\mathrm{Aut}(\mathfrak{g})$ and it is clear that $\phi(\Gamma(p, q, \alpha)) = \Gamma(p', q', \alpha')$. The lemma is completely proved. $\blacksquare$

Finally, an appeal to Lemma 4 and Smith normal form completes the proof of Lemma 3. $\blacksquare$

6 Uniform subgroups of $G_{5,4}$

Let $\mathfrak{g}_{5,4}$ be the three-step nilpotent Lie algebra with basis

$$\mathscr{B} = (X_1, \dots, X_5)$$

with Lie brackets are given by

$$[X_5, X_4] = X_3, \qquad [X_5, X_3] = X_2, \qquad [X_4, X_3] = X_1. \tag{3}$$

and the non-defined brackets being equal to zero or obtained by antisymmetry. Let $G_{5,4}$ be the corresponding connected and simply connected nilpotent Lie group. For $u \in \mathbb{Z}$, let

$$\mathbf{A}(u) = \begin{pmatrix} 1 & 0 & \frac{u}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{u}{2} & \frac{u}{2} \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and let

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We denote by $\mathcal{A}_4$ the subset of $\mathrm{Mat}(4, \mathbb{Z})$ consisting of all matrices of the form

$$[\![D, m]\!] = \begin{pmatrix} D & 0 \\ 0 & m \end{pmatrix}$$

satisfying

$$[\![D, m]\!]^{-1} \mathbf{A}(m) [\![D, m]\!] \in \mathrm{SL}(4, \mathbb{Z}) \tag{4}$$

and

$$D^{-1} \mathbf{B} D \in \mathrm{SL}(3, \mathbb{Z}), \tag{5}$$

where $m \in \mathbb{N}^*$, the block matrix $D = (\alpha_{ij}; 1 \le i, j \le 3)$ is an upper-triangular integer invertible matrix and $\mathrm{SL}(3, \mathbb{Z})$ is the set of all integer matrices with determinant 1.

Proposition 6. *If $[\![D, m]\!] \in \mathcal{A}_4$ and if H is the Hermite normal form of D, then $[\![H, m]\!] \in \mathcal{A}_4$.*

Proof. Let H be the Hermite normal form of D and let $T \in \mathrm{GL}(3, \mathbb{Z})$ such that $H = DT$. It is clear that $[\![H, m]\!]$ is the Hermite normal form of $[\![D, m]\!]$ and

$$[\![H, m]\!] = [\![D, m]\!] \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}.$$

Consequently

$$[\![H, m]\!]^{-1} \mathbf{A}(m) [\![H, m]\!] = \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}^{-1} [\![D, m]\!]^{-1} \mathbf{A}(m) [\![D, m]\!] \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}.$$

As $[\![D, m]\!]^{-1} \mathbf{A}(m) [\![D, m]\!] \in \mathrm{SL}(3, \mathbb{Z})$, then $[\![H, m]\!]^{-1} \mathbf{A}(m) [\![H, m]\!] \in \mathrm{SL}(3, \mathbb{Z})$. The second condition (5) is shown similarly. ■

Next, we define

$$\mathcal{B}_4 = \Big\{ [\![D, m]\!] \in \mathcal{A}_4 : \; D \text{ is in Hermite normal form} \Big\}.$$

We are ready to formulate our result.

Theorem 7. *With the above notations, we have*

1. *If* $[\![D, m]\!] \in \mathcal{B}_4$, *then*

$$\Gamma_{[\![D,m]\!]} = \exp(\mathbb{Z}\epsilon_1)\exp(\mathbb{Z}\epsilon_2)\exp(\mathbb{Z}\epsilon_3)\exp(\mathbb{Z}\epsilon_4)\exp(\mathbb{Z}\epsilon_5),$$

where the vectors ϵ_j $(1 \le j \le 4)$ *are the column vectors of* $[\![D, m]\!]$ *in the basis* $(X_1, \ldots, X_4)$ *and* $\epsilon_5 = X_5$, *is a discrete uniform subgroup of* $G_{5,4}$.

2. *If* Γ *is a uniform subgroup of* $G_{5,4}$, *then there exist* $[\![D, m]\!] \in \mathcal{B}_4$ *and* $\Phi \in \mathrm{Aut}(G_{5,4})$ *such that* $\Phi(\Gamma) = \Gamma_{[\![D,m]\!]}$.

In the proof of the theorem we need the following lemma.

Lemma 5. *The group* $\mathrm{Aut}(\mathfrak{g}_{5,4})$ *is the set of all invertible matrices of the form*

$$\begin{pmatrix} a_{44}\delta & a_{45}\delta & a_{45}a_{34} - a_{35}a_{44} & a_{14} & a_{15} \\ a_{54}\delta & a_{55}\delta & a_{55}a_{34} - a_{35}a_{54} & a_{24} & a_{25} \\ 0 & 0 & \delta & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{54} & a_{55} \end{pmatrix},$$

where $\delta = a_{55}a_{44} - a_{45}a_{54}$.

Proof. Any automorphism A of $\mathfrak{g}_{5,4}$ leaves invariant the subalgebras $\mathbb{R}\text{-span}\,\{X_1, X_2\}$ and $\mathbb{R}\text{-span}\,\{X_1, X_2, X_3\}$. The remainder of the proof follows from (3). ∎

Proof of Theorem 7. Let $G = G_{5,4}$ and $\mathfrak{g} = \mathfrak{g}_{5,4}$. Assertion 1 is obvious. To prove the second, let Γ be a uniform subgroup of G. Since the ideals

$$\mathfrak{z}(\mathfrak{g}) = \mathbb{R}\text{-span}\,\{X_1, X_2\} \qquad \text{and} \qquad \mathcal{D}(\mathfrak{g}) = \mathbb{R}\text{-span}\,\{X_1, X_2, X_3\}$$

are rational, then there exists a strong Malcev basis $\mathscr{B}'$ of $\mathfrak{g}$ strongly based on Γ and passing through $\mathfrak{z}(\mathfrak{g})$ and $\mathcal{D}(\mathfrak{g})$. Let

$$\mathsf{P}_{\mathscr{B}\to\mathscr{B}'} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{54} & a_{55} \end{pmatrix}$$

be the change of basis matrix from the basis $\mathscr{B}$ to the basis $\mathscr{B}'$. Let $\Phi_* \in \mathrm{Aut}(\mathfrak{g}_{5,4})$ defined by

$$\Phi_*(X_5) = a_{55}X_5 + a_{45}X_4 + a_{35}X_3 + a_{25}X_2 + a_{15}X_1,$$
$$\Phi_*(X_4) = a_{54}X_5 + a_{44}X_4 + a_{34}X_3 + a_{24}X_2 + a_{14}X_1.$$

Then it is not hard to verify that there exist $b_{11}, b_{21}, b_{12}, b_{22}, b_{13}, b_{23}, b_{33} \in \mathbb{R}$ such that

$$\Gamma \simeq \Phi^{-1}(\Gamma) = \exp(\mathbb{Z}e_1)\exp(\mathbb{Z}e_2)\exp(\mathbb{Z}e_3)\exp(\mathbb{Z}X_4)\exp(\mathbb{Z}X_5),$$

where $e_1 = b_{11}X_1 + b_{21}X_2$, $e_2 = b_{12}X_1 + b_{22}X_2$, $e_3 = b_{13}X_1 + b_{23}X_2 + b_{33}X_3$. On the other hand, as $\Phi^{-1}(\Gamma)$ is a subgroup of G, then we have

$$\exp(X_5)\exp(X_4)\exp(-X_5) \in \Phi^{-1}(\Gamma).$$

Then there exist integer coefficients t_1, t_2, t_3, such that

$$\exp(X_5)\exp(X_4)\exp(-X_5) = \exp(t_1e_1)\exp(t_2e_2)\exp(t_3e_3)\exp(X_4).$$

The above equation may obviously rewritten as

$$X_3 + \frac{1}{2}X_2 = t_1e_1 + t_2e_2 + t_3e_3 - \frac{1}{2}t_3b_{33}X_1. \tag{6}$$

Using a similar technique, we obtain that there exist integer coefficients x_1, x_2, y_1, y_2 satisfy $x_1^2 + x_2^2 \neq 0$ and $y_1^2 + y_2^2 \neq 0$ such that

$$b_{33}X_2 = x_1e_1 + x_2e_2 \tag{7}$$

and

$$b_{33}X_1 = y_1e_1 + y_2e_2. \tag{8}$$

From the equations (6), (7) and (8), it is clear that the coefficients b_{ij} belong to $\mathbb{Q}$. Let m be the least common multiple of the denominators of the rational numbers b_{ij} and let $\pi_* \in \mathrm{Aut}(\mathfrak{g}_{5,4})$ such that $\pi_*(X_5) = X_5$ and $\pi_*(X_4) = mX_4$. Then, we obtain

$$\begin{aligned}\Gamma \simeq{}& \exp(\mathbb{Z}(m^2b_{11}X_1 + mb_{21}X_2))\exp(\mathbb{Z}(m^2b_{12}X_1 + mb_{22}X_2))\\ &\times \exp(\mathbb{Z}(m^2b_{13}X_1 + mb_{23}X_2 + mb_{33}X_3))\exp(\mathbb{Z}(mX_4))\exp(\mathbb{Z}X_5).\end{aligned}$$

By Theorem 4, let

$$D = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & c_{33} \end{pmatrix}$$

be the Hermite normal form of

$$\begin{pmatrix} m^2b_{11} & m^2b_{12} & m^2b_{13} \\ mb_{21} & mb_{22} & mb_{23} \\ 0 & 0 & mb_{33} \end{pmatrix}.$$

It follows that

$$\begin{aligned}\Gamma \simeq \Gamma_{[\![D,m]\!]} ={}& \exp(\mathbb{Z}(c_{11}X_1))\exp(\mathbb{Z}(c_{12}X_1 + c_{22}X_2))\\ &\times \exp(\mathbb{Z}(c_{13}X_1 + c_{23}X_2 + c_{33}X_3))\exp(\mathbb{Z}(mX_4))\exp(\mathbb{Z}X_5).\end{aligned}$$

It remains to prove that the block matrix $[\![D,m]\!] = \begin{pmatrix} D & 0 \\ 0 & m \end{pmatrix} \in \mathcal{B}_4$. We need only to prove (4) and (5). Since

$$\exp(X_5)\exp(mX_4)\exp(-X_5) \in \Gamma_{[\![D,m]\!]}$$

then there exist integer coefficients a_{14}, a_{24}, a_{34}, such that

$$e^{\mathrm{ad}\,X_5}(mX_4) = a_{14}(c_{11}X_1) + a_{24}(c_{12}X_1 + c_{22}X_2)$$

$$+ a_{34}(c_{13}X_1 + c_{23}X_2 + c_{33}X_3) + mX_4 - \frac{1}{2}a_{34}c_{33}mX_1. \tag{9}$$

Similarly, we establish

$$e^{\operatorname{ad} X_5}(c_{13}X_1 + c_{23}X_2 + c_{33}X_3) = a_{13}(c_{11}X_1) + a_{23}(c_{12}X_1 + c_{22}X_2) + c_{13}X_1 + c_{23}X_2 + c_{33}X_3, \tag{10}$$

$$e^{\operatorname{ad} X_5}(c_{12}X_1 + c_{22}X_2) = a_{12}(c_{11}X_1) + (c_{12}X_1 + c_{22}X_2), \tag{11}$$

$$e^{\operatorname{ad} X_5}(c_{11}X_1) = c_{11}X_1. \tag{12}$$

We see that the conditions (9)–(12) boil down to the matrix equation

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} [\![D, m]\!] = \begin{pmatrix} 1 & 0 & -\frac{m}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} [\![D, m]\!] \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This proves (4). The proof of the second equality (5) is shown similarly (repeat the proof of (4) verbatim, using mX_4 instead of X_5). The proof of theorem is complete. ∎

7 Uniform subgroups of $G_{5,6}$

Let $\mathfrak{g}_{5,6}$ be the three-step nilpotent Lie algebra with basis

$$\mathscr{B} = (X_1, \dots, X_5)$$

with Lie brackets are given by

$$[X_5, X_4] = X_3, \qquad [X_5, X_3] = X_2, \qquad [X_5, X_2] = X_1, \qquad [X_4, X_3] = X_1 \tag{13}$$

and the non-defined brackets being equal to zero or obtained by antisymmetry. Let $G_{5,6}$ be the simply connected Lie group with Lie algebra $\mathfrak{g}_{5,6}$. Let $\mathcal{A}_6$ be the set of all invertible integer matrices of the form

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 \\ 0 & \alpha_{22} & \alpha_{23} & 0 \\ 0 & 0 & \alpha_{33} & 0 \\ 0 & 0 & 0 & \alpha_{44} \end{pmatrix}.$$

We denote by $\mathcal{B}_6$ the subset of $\mathrm{Mat}(5, \mathbb{Z})$ consisting of all invertible integer matrices of the form

$$[\![D, m]\!] = \begin{pmatrix} D & 0 \\ 0 & m \end{pmatrix}$$

satisfying

$$D^{-1} \begin{pmatrix} \frac{\alpha_{44}m^3}{6} + \frac{\alpha_{44}^2}{2} & m\alpha_{23} + \frac{m^2\alpha_{33}}{2} & m\alpha_{22} & \alpha_{44}\alpha_{23} \\ \frac{\alpha_{44}m^2}{2} & m\alpha_{33} & 0 & 0 \\ \alpha_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}(4, \mathbb{Z}),$$

where $m \in \mathbb{N}^*$, the block matrix $D = (\alpha_{ij} : 1 \le i, j \le 4) \in \mathcal{A}_6$. Let $\sim$ be the equivalence relation on $\mathcal{B}_6$ given by

$$[\![D, m]\!] \sim [\![D', m']\!] \iff [\![D, m]\!][\![D', m']\!]^{-1} = \mathrm{diag}[a^5, a^4, a^3, a^2, a], \qquad \text{for some} \quad a \in \mathbb{Q}^*.$$

We now state the final result of this paper.

Theorem 8. *With the above notation, we have*

1. *If* $[\![D, m]\!] \in \mathcal{B}_6$*, then*

$$\Gamma_{[\![D,m]\!]} = \exp(\mathbb{Z}e_1)\exp(\mathbb{Z}e_2)\exp(\mathbb{Z}e_3)\exp(\mathbb{Z}e_4)\exp(\mathbb{Z}e_5),$$

where the vectors e_j $(1 \leq j \leq 5)$ *are the column vectors of* $[\![D, m]\!]$ *in the basis* $(X_1, \ldots, X_5)$*, is a uniform subgroup of* $G_{5,6}$*.*

2. *If* Γ *is a uniform subgroup of* $G_{5,6}$*, then there exist* $[\![D, m]\!] \in \mathcal{B}_6$ *and* $\Phi \in \operatorname{Aut}(G_{5,6})$ *such that* $\Phi(\Gamma) = \Gamma_{[\![D,m]\!]}$*.*
3. *For* $[\![D, m]\!], [\![D', m']\!] \in \mathcal{B}_6$*, the subgroups* $\Gamma_{[\![D,m]\!]}$ *and* $\Gamma_{[\![D',m']\!]}$ *are isomorphic if and only if* $[\![D, m]\!] \sim [\![D', m']\!]$*.*

We first state and prove two lemmas.

Lemma 6. *The following ideals*

$$\mathfrak{a}_i = \mathbb{R}\text{-span}\,\{X_1, \ldots, X_i\} \qquad (1 \leq i \leq 4)$$

are rational with respect to any rational structure on $\mathfrak{g}_{5,6}$*.*

Proof. It is clear that $\mathfrak{a}_1 = \mathfrak{z}(\mathfrak{g}_{5,6})$, $\mathfrak{a}_2 = \mathscr{C}_2(\mathfrak{g}_{5,6})$ and $\mathfrak{a}_3 = \mathcal{D}(\mathfrak{g}_{5,6})$. Then the rationality of $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$ follows from Proposition 2 and Proposition 1. On the other hand, we have

$$\mathbf{c}(\mathfrak{a}_2) = \mathfrak{a}_4.$$

We conclude from Proposition 5 that $\mathfrak{a}_4$ is rational. ∎

Lemma 7. *We have*

$$\operatorname{Aut}(\mathfrak{g}_{5,6}) = \left\{ \Phi \in \operatorname{End}(\mathfrak{g}_{5,6}) :\ \operatorname{Mat}(\Phi, \mathscr{B}) = \begin{pmatrix} a_{55}^5 & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{55}^4 & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{55}^3 & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{55}^2 & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix} \in \operatorname{GL}(5, \mathbb{R}), \right.$$
$$\left. a_{23} = a_{55}a_{34},\ a_{13} = a_{55}a_{24} + a_{45}a_{34} - a_{35}a_{55}^2,\ a_{12} = a_{45}a_{55}^3 + a_{34}a_{55}^2 \right\}.$$

Proof. Let $\Phi \in \operatorname{Aut}(\mathfrak{g}_{5,6})$. Since $\mathfrak{a}_2$ is invariant under Φ, then also $\mathbf{c}(\mathfrak{a}_2) = \mathfrak{a}_4$ is invariant under Φ. It follows that the matrix $\operatorname{Mat}(\Phi, \mathscr{B})$ of Φ has the following form

$$\operatorname{Mat}(\Phi, \mathscr{B}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix} \in \operatorname{GL}(5, \mathbb{R}).$$

The remainder of the proof follows from (13). ∎

Proof of Theorem 8. Assertion 1. is obvious. To prove the second, let Γ be a uniform subgroup of $G_{5,6}$. Since for every $i = 1, \ldots, 4$, the ideal $\mathfrak{a}_i = \mathbb{R}\text{-span}\,\{X_1, \ldots, X_i\}$ is rational, then there exists a strong Malcev basis $\mathscr{B}'$ of $\mathfrak{g}$ strongly based on Γ and passing through $\mathfrak{a}_1, \ldots, \mathfrak{a}_3$ and $\mathfrak{a}_4$. Let

$$\mathsf{P}_{\mathscr{B}\to\mathscr{B}'} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix}$$

be the change of basis matrix from the basis $\mathscr{B}$ to the basis $\mathscr{B}'$. Let $\Phi_* \in \mathrm{Aut}(\mathfrak{g}_{5,6})$ defined by

$$\Phi_*(X_5) = a_{55}X_5 + a_{45}X_4 + \cdots + a_{15}X_1,$$
$$\Phi_*(X_4) = a_{55}^2 X_4 + \frac{a_{55}^2}{a_{44}}(a_{34}X_3 + a_{24}X_2 + a_{14}X_1).$$

Then there exist $b_{11}, b_{12}, b_{22}, b_{13}, b_{23}, b_{33}, b_{44} \in \mathbb{R}$ such that

$$\Gamma \simeq \Phi^{-1}(\Gamma) = \exp(\mathbb{Z}\epsilon_1)\exp(\mathbb{Z}\epsilon_2)\exp(\mathbb{Z}\epsilon_3)\exp(\mathbb{Z}\epsilon_4)\exp(\mathbb{Z}X_5),$$

where $\epsilon_1 = b_{11}X_1$, $\epsilon_2 = b_{12}X_1 + b_{22}X_2$, $\epsilon_3 = b_{13}X_1 + b_{23}X_2 + b_{33}X_3$, $\epsilon_4 = b_{44}X_4$. By similar arguments to those used in (9)–(12) we derive the following system

$$\begin{aligned}
&b_{44} = x_1 b_{33},\\
&\frac{1}{2}b_{44} = x_1 b_{23} + x_2 b_{22},\\
&\frac{1}{6}b_{44} = x_1 b_{13} + x_2 b_{12} + x_3 b_{11} - \frac{1}{2}x_1 b_{44} b_{33},\\
&b_{33} = y_1 b_{22},\\
&b_{23} + \frac{1}{2}b_{33} = y_1 b_{12} + y_2 b_{11},\\
&b_{44}b_{33} = z b_{11},\\
&b_{22} = t b_{11},
\end{aligned}$$

where $x_1, y_1, z, t \in \mathbb{N}^*$, $x_2, x_3, y_2 \in \mathbb{Z}$. By calculation one has

$$(b_{11}, b_{12}, b_{22}, b_{13}, b_{23}, b_{33}, b_{44}, 1) \in \mathscr{G}.$$

where

$$\mathscr{G} = \Bigg\{ (\alpha, m) \in \mathbb{Q}^7 \times \mathbb{N}^* : \ \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) \ \text{ such that}$$
$$\alpha_1 = \frac{bm^4}{zs^2a^2},\ \alpha_2 = \frac{bm^3}{2zs^2a} - \frac{ybm^4}{z^2s^3a} + \frac{bm^4}{2zs^2a} - \frac{tbm^4}{zs^3a^2},\ \alpha_3 = \frac{bm^3}{zs^2a},$$
$$\alpha_4 = \frac{bm^2}{6zsa} - \frac{ybm^3}{2z^2s^2a} + \frac{y^2bm^4}{z^3s^3a} - \frac{ybm^4}{2z^2s^2a} + \frac{ytbm^4}{z^2s^3a^2} - \frac{xbm^4}{z^2s^2a^2} + \frac{b^2m^4}{2zs^2a^2},$$
$$\alpha_5 = \frac{bm^2}{2zsa} - \frac{ybm^3}{z^2s^2a},\ \alpha_6 = \frac{bm^2}{zsa},\ \alpha_7 = \frac{bm^2}{sa},\ a, b, s, z \in \mathbb{Z}^*,\ x, y, t \in \mathbb{Z} \Bigg\}.$$

From this we deduce that the coefficients b_{ij} belong to $\mathbb{Q}$. Let m be the least common multiple of the denominators of the rational numbers b_{ij}. Let $\pi \in \mathrm{Aut}(G_{5,6})$ such that

$$\mathrm{Mat}(\pi_*, \mathscr{B}) = \mathtt{diag}\,[m^5, m^4, m^3, m^2, m].$$

Then

$$\Gamma \simeq \pi(\Phi^{-1}(\Gamma)) = \exp(\mathbb{Z}e_1)\exp(\mathbb{Z}e_2)\exp(\mathbb{Z}e_3)\exp(\mathbb{Z}e_4)\exp(m\mathbb{Z}X_5)$$

such that the matrix $[e_1,\dots,e_4]$ with column vectors $e_1,\dots,e_4$ expressed in the basis $\{X_1,\dots,X_4\}$, belongs to $\mathcal{A}_6$. Write

$$[e_1,\dots,e_4] = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 \\ 0 & \alpha_{22} & \alpha_{23} & 0 \\ 0 & 0 & \alpha_{33} & 0 \\ 0 & 0 & 0 & \alpha_{44} \end{pmatrix}.$$

By similar techniques as above we can prove that $(\alpha_{11},\alpha_{12},\alpha_{13},\alpha_{22},\alpha_{23},\alpha_{33},\alpha_{44},m) \in \mathscr{G}$. This equivalent that

$$[e_1,\dots,e_4]^{-1} \begin{pmatrix} \frac{\alpha_{44}m^3}{6} + \frac{\alpha_{44}^2}{2} & m\alpha_{23} + \frac{m^2\alpha_{33}}{2} & m\alpha_{22} & \alpha_{44}\alpha_{23} \\ \frac{\alpha_{44}m^2}{2} & m\alpha_{33} & 0 & 0 \\ \alpha_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}(4,\mathbb{Z}).$$

Finally, we achieve with the proof of 3. We show both directions. Let $a \in \mathbb{Q}^*$ such that

$$[\![D,m]\!][\![D',m']\!]^{-1} = \mathrm{diag}\,[a^5,a^4,a^3,a^2,a].$$

Consider the linear mapping $\phi_* : \mathfrak{g}_{5,6} \longrightarrow \mathfrak{g}_{5,6}$ defined by

$$\mathrm{Mat}(\phi_*,\mathscr{B}) = \mathrm{diag}\,[a^5,a^4,a^3,a^2,a].$$

It is clear that $\phi \in \mathrm{Aut}(G_{5,6})$ (see Lemma 7) and $\phi(\Gamma_{[\![D,m]\!]}) = \Gamma_{[\![D',m']\!]}$. Conversely, suppose there exists $\phi : \Gamma_{[\![D,m]\!]} \longrightarrow \Gamma_{[\![D',m']\!]}$ an isomorphism between $\Gamma_{[\![D,m]\!]}$ and $\Gamma_{[\![D',m']\!]}$. By Theorem 2, ϕ has an extension $\overline{\phi} \in \mathrm{Aut}(G_{5,6})$. As $\overline{\phi}_*(mX_5) = m'X_5$ and $\overline{\phi}_*(\alpha_{44}X_4) = \alpha'_{44}X_4$ where the α_{ij} (resp. α'_{ij}) are the entries of D (resp. of D'), then by Lemma 7, the matrix of $\overline{\phi}_*$ in the basis $\mathscr{B}$ has the following form

$$\mathrm{Mat}(\overline{\phi}_*,\mathscr{B}) = \mathrm{diag}\,\left[a^5,a^4,a^3,a^2,a\right]$$

for some $a \in \mathbb{Q}^*$. Consequently, we obtain

$$[\![D,m]\!] = \mathrm{diag}\,\left[a^5,a^4,a^3,a^2,a\right][\![D',m']\!].$$

This completes the proof. ■

Remark 2. In the statement 2 of Theorem 8, the element $[\![D,m]\!]$ of $\mathcal{B}_6$ is not unique.

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