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Self-Consistent-Field Method and τ-Functional Method on Group Manifold in Soliton Theory: a Review and New Results[⋆]

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Abstract. The maximally-decoupled method has been considered as a theory to apply an basic idea of an integrability condition to certain multiple parametrized symmetries. The method is regarded as a mathematical tool to describe a symmetry of a collective submanifold in which a canonicity condition makes the collective variables to be an orthogonal coordinate-system. For this aim we adopt a concept of curvature unfamiliar in the conventional time-dependent (TD) self-consistent field (SCF) theory. Our basic idea lies in the introduction of a sort of Lagrange manner familiar to fluid dynamics to describe a collective coordinate-system. This manner enables us to take a one-form which is linearly composed of a TD SCF Hamiltonian and infinitesimal generators induced by collective variable differentials of a canonical transformation on a group. The integrability condition of the system read the curvature $C = 0$. Our method is constructed manifesting itself the structure of the group under consideration. To go beyond the maximaly-decoupled method, we have aimed to construct an SCF theory, i.e., υ (external parameter)-dependent Hartree–Fock (HF) theory. Toward such an ultimate goal, the υ-HF theory has been reconstructed on an affine Kac–Moody algebra along the soliton theory, using infinite-dimensional fermion. An infinite-dimensional fermion operator is introduced through a Laurent expansion of finite-dimensional fermion operators with respect to degrees of freedom of the fermions related to a υ-dependent potential with a Υ-periodicity. A bilinear equation for the υ-HF theory has been transcribed onto the corresponding τ-function using the regular representation for the group and the Schur-polynomials. The υ-HF SCF theory on an infinite-dimensional Fock space F_∞ leads to a dynamics on an infinite-dimensional Grassmannian Gr_∞ and may describe more precisely such a dynamics on the group manifold. A finite-dimensional Grassmannian is identified with a Gr_∞ which is affiliated with the group manifold obtained by reducting $gl(\infty)$ to $sl(N)$ and $su(N)$. As an illustration we will study an infinite-dimensional matrix model extended from the finite-dimensional $su(2)$ Lipkin–Meshkov–Glick model which is a famous exactly-solvable model.

Key words: self-consistent field theory; collective theory; soliton theory; affine KM algebra

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Contents

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1 Introduction

1.1 Historical background on microscopic study of nuclear collective motions

A standard description of fermion many-body systems starts with the most basic approximation that is based on an independent-particle (IP) picture, i.e., self-consistent-field (SCF) approximation for the fermions. The Hartree–Fock theory (HFT) is typical one of such an approximation for ground states of the systems. Excited states are treated with the random phase approximation (RPA). The time dependent Hartree–Fock (TDHF) equation and time dependent Hartree–Bogoliubov (TDHB) equation are nonlinear equations owing to their SCF characters and may have no unique solution. The HFT and HBT are given by variational method to optimize energy expectation value by a Slater determinant (S-det) and an HB wave function, respectively [2]. Particle-hole (p-h) operators of the fermions with N single-particle states form a Lie algebra $u(N)$ [3] and generate a Thouless transformation [4] which induces a representation of the corresponding Lie group $U(N)$. The $U(N)$ canonical transformation transforms an S-det with M particles to another S-det. Any S-det is obtained by such a transformation of a given reference S-det, i.e., Thouless theorem provides an exact wave function of fermion state vector which is the generalized coherent state representation (CS rep) of $U(N)$ Lie group [5]. Following Yamamura and Kuriyama [6], we give a brief history of methods extracting collective motions out of fully parametrized TDHF/TDHB manifolds in SCF. Arvieu and Veneroni, and Baranger and independently Marumori have proposed a theory for spherical even nuclei [7] called quasi-particle RPA (QRPA) and it has been a standard approximation for the excited states of the systems. In nuclei, there exist a short-range correlation and a long-range one [8]. The former is induced by a pairing interaction and generates a superconducting state. The excited state is classified by a seniority-scheme and described in terms of quasi-particles given by the BCS-Bogoliubov theory [9]. The latter is occured by p-h interactions and gives rise to collective motions related to a density fluctuation around equilibrium states. The p-h RPA (RPA) describes such collective motions like vibrational and rotational motions. It, however, stands on a harmonic approximation and should be extended to take some nonlinear effects into account. To solve such a problem, the boson expansion HB theory (BEHBT) has been developed by Belyaev and Zelevinsky [10], and Marumori, Yamamura and Tokunaga [11]. The essence of the BEHBT is to express the fermion-pairs in terms of boson operators keeping a pure boson-character. The boson representation is constructed to reproduce the Lie algebra of the fermion-pairs. The state vector in the fermion Fock space corresponds to the one in the boson Fock space by one-to-one mapping. Such a boson representation makes any transition-matrix-valued quantity for the boson-state vectors coincide with that for the fermion-state ones. The algebra of fermion-pairs and the boson representation have been extensively investigated. The fermion-pairs form an algebra $so(2N)$. As for the boson representation, e.g., da Providência and Weneser and Marshalek [12] have proposed boson operators basing on p-h pairs forming an algebra $su(N)$. By Fukutome, Yamamura and Nishiyama [13, 14], the fermions were found to span the algebras $so(2N+1)$ and $so(2N+2)$ accompanying with $u(N+1)$. The BET expressed by Schwinger-type and Dyson-type bosons has been intensively studied by Fukutome and Nishiyama [15, 16, 17, 18]. However, the above BET's themselves do not contain any scheme under which collective degree of freedom can be selected from the whole degrees of freedom.

On the contrary, we have a traditional approach to the microscopic theory of collective motion, the TDHF theory (TDHFT) and TDHB theory (TDHBT), e.g., [19, 20]. The pioneering idea of the TDHFT was suggested by Marumori [7] for the case of small amplitude vibrational motions. Using this idea, one can determine the time dependence of any physical quantity, e.g., frequency of the small fluctuation around a static HF/HB field. The equation for the frequency has the same form as that given by RPA. A quantum energy given by this method means an excitation energy of the first excited state. Then, the RPA is a possible quantization of the TDHFT/TDHBT in the small amplitude limit. In fact, as was proved by Marshalek and Horzwarth [21], the BEHBT is reduced to the TDHBT under the replacement of boson operators with classical canonical variables. Using a canonical transformation in a classical mechanics, it is expected to obtain a scheme for choosing the collective degree of freedom in the SCF. Historically, there was another stream, i.e., an adiabatic perturbation approach. This approach starts from an assumption that the speed of collective motion is much slower than that of any other non-collective motion. At an early stage of the study of this stream, the adiabatic treatment of the TDHFT (ATDHFT) was presented by Thouless and Valatin [22]. Such a theory has a feature common to the one of the theory for large-amplitude collective motion. Later the ATDHFT was developed mainly by Baranger and Veneroni, Brink, Villars, Goeke and Reinhard, and Mukherjee and Pal [23]. The most important point of the ATDHFT by Villars is in introducing a “collective path” into a phase space. A collective motion corresponds to a trajectory in the phase space which moves along the collective path. Standing on the same spirit, Holtzwarth and Yukawa, Rowe and Bassermann [24], gave the TDHFT and Marumori, Maskawa, Sakata and Kuriyama so-called “maximally decoupled” method in a canonical form [25]. So, various techniques of classical mechanics are useful and then canonical quantization is expected. By solving the equation of collective path, one can obtain some corrections to the TDHF result. The TDHFT has a possibility to illustrate not only collective modes but also intrinsic modes. However, the following three points remain to be solved yet: (i) to determine a microscopic structure of collective motion, which may be a superposition of each particle motion, in relation to dynamics under consideration (ii) to determine IP motion which should be orthogonal to collective motion and (iii) to give a coupling between both the motions. The canonical-formed TDHFT enables us to select the collective motion in relation to the dynamics, though it makes no role to take IP motion into account, because the TD S-det contains only canonical variables to represent the collective motion. Along the same way as the TDHFT, Yamamura and Kuriyama have extended the TDHFT to that on a fermion CS constructed on the TD S-det. The CS rep contains not only the usual canonical variables but also the Grassmann variables. A classical image of fermions can be obtained by regarding the Grassmann variables as canonical ones [26]. The constraints governing the variables to remove the overcounted degrees of freedom were decided under the physical consideration. Owing to the Dirac’s canonical theory for a constrained system, the TDHFT was successfully developed for a unified description of collective and IP motions in the classical mechanics [27].

1.2 Viewpoint of symmetry of evolution equations

The TDHF/TDHB can be summarized to find optimal coordinate-systems on a group manifold basing on Lie algebras of the finite-dimensional fermion-pairs and to describe dynamics on the manifold. The boson operators in BET are generators occurring in the coordinate system of tangent space on the manifold in the fermion Fock space. But the BET’s themselves do not contain any scheme under which collective degrees of freedom can be selected from the whole degrees of freedom. Approaches to collective motions by the TDHFT suggest that the coordinate system on which collective motions is describable deeply relates not only to the global symmetry of the finite-dimensional group manifold itself but also to *hidden local symmetries*, besides the Hamiltonian. Various collective motions may be well understood by taking the local symmetries into account. The local symmetries may be closely connected with infinite-dimensional Lie algebras. However, there has been little attempts to *manifestly* understand collective motions in relation to the local symmetries. From the viewpoint of symmetry of evolution equations, we will study the algebro-geometric structures toward a unified understanding of both the collective and IP motions.

The first issue is to investigate fundamental "curvature equations" to extract collective submanifolds out of the full TDHF/TDHB manifold. We show that the expression in a quasi-particle frame (QPF) of the zero-curvature equations described later becomes the nonlinear RPA which is the natural extension of the usual RPA. We abbreviate RPA and QRPA to only RPA. We had at first started from a question whether soliton equations exist in the TDHF/TDHB manifold or do not, in spite of the difference that the solitons are described in terms of infinite degrees of freedom and the RPA in terms of finite ones. We had met with the inverse-scattering-transform method by AKNS [28] and the differential geometrical approaches on group manifolds [29]. An integrable system is explained by the zero-curvature, i.e., integrability condition of connection on the corresponding Lie group. Approaches to collective motions had been little from the viewpoint of the curvature. If a collective submanifold is a collection of collective paths, an infinitesimal condition to transfer a path to another may be nothing but the integrability condition for the submanifold with respect to a parameter time t describing a trajectory of an SCF Hamiltonian and to other parameters specifying any point on the submanifold. However the trajectory of the SCF Hamiltonian is unable to remain on the manifold. Then the curvature may be able to work as a criterion of effectiveness of the collective submanifold. From a wide viewpoint of symmetry the RPA is extended to any point on the manifold because an equilibrium state which we select as a starting point must be equipotent with any other point on the manifold. The well-known RPA had been introduced as a linear approximation to treat excited states around a ground state (the equilibrium state), which is essentially a harmonic approximation. When an amplitude of oscillation becomes larger and then an anharmonicity appears, then we have to treat the anharmonicity by taking nonlinear effects in the equation of motion into account. It is shown that *equations defining the curvature* of the collective submanifold becomes *fundamental equations* to treat the anharmonicity. We call them "the formal RPA equation". It will be useful to understand algebro-geometric meanings of large-amplitude collective motions.

The second issue is to go beyond the perturbative method with respect to the collective variables [25]. For this aim, we investigate an interrelation between the SCF method (SCFM) extracting *collective motions* and τ-functional method (τ-FM) [30] constructing *integrable equations* in solitons. In a soliton theory on a group manifold, transformation groups governing solutions for soliton equations become infinite-dimensional Lie groups whose generators of the corresponding Lie algebras are expressed as infinite-order differential operators of affine Kac–Moody algebras. An infinite-dimensional fermion Fock space F_∞ is realized in terms of a space of complex polynomial algebra. The infinite-dimensional fermions are given in terms of the infinite-order differential operators and the soliton equation is nothing but the differential equation to determine the group orbit of the highest weight vector in the F_∞ [30]. The generalized CS rep gives a key to elucidate relationship of a HF wave function to a τ-function in the soliton theory. This has been pointed out first by D'Ariano and Rasetti [31] for an infinite-dimensional harmonic electron gas. Standing on their observation, for the SCFM one can give a theoretical frame for an integrable sub-dynamics on an abstract F_∞. The relation between SCFM in finite-dimensional fermions and τ-FM in infinite ones, however, has not been investigated because dynamical descriptions of fermion systems by them have looked very different manners. In the papers [32, 33, 34, 35, 1], we have first tried to clarify it using SCFM on $U(N)$ group and τ-FM on that group. To attain this object we will have to solve the following main problems:

first, how we embed the finite-dimensional fermion system into a certain infinite one and how we rebuilt the TDHFT on it; second, how any algebraic mechanism works behind particle and collective motions and how any relation between collective variables and a spectral parameter in soliton theory is there; last, how the SCF Hamiltonian selects various subgroup-orbits and how a collective submanifold is made from them and further how the submanifold relates to the formal RPA. To understand microscopically cooperative phenomena, the concept of collective motion is introduced in relation to a TD variation of SC mean-field. IP motion is described in terms of particles referring to a stationary mean-field. The variation of a TD mean-field gives rise to couplings between collective and IP motions and couplings among quantum fluctuations of the TD mean-field itself [6], while in τ-FM a soliton equation is derived as follows: Consider an infinite-dimensional Lie algebra and its representation on a functional space. The group-orbit of the highest weight vector becomes an *infinite-dimensional Grassmannian* G_∞. The bilinear equation (Plücker relation) is nothing else than the soliton equation. This means that a solution space of the soliton equation corresponds to a group-orbit of the vacuum state. The SCFM does not use the Plücker relation in the context of a bilinear differential equation defining *finite-dimensional Grassmannian* G_M but seems to use implicitly such a relation. In the SCFM a physical concept of quasi-particle and vacuum and a coset space is used instead. If we develop a perturbative theory for large-amplitude collective motion [25], an infinite-dimensional Lie algebra might been necessarily used. The sub-group orbits consisting of several *loop-group* paths [36] classified by the Plücker relation exist innumerably in G_M so that the SCFM is related to the soliton theory in G_∞. The Plücker relation in a coset space $\frac{U(N)}{U(M)\times U(N-M)}$ [37] becomes analogous with the *Hirota's bilinear form* [38, 39]. Toward an ultimate goal we aim to reconstruct a theoretical frame for a υ (external parameter)-dependent SCFM to describe more precisely the dynamics on the F_∞. In the abstract fermion Fock space, we find common features in both SCFM and τ-FM. (i) *Each solution space* is described as *Grassmannian* that is group orbit of the corresponding vacuum state. (ii) The former may implicitly explain the Plücker relation not in terms of bilinear differential equations defining G_M but in terms of the physical concept of quasi-particle and vacuum and mathematical language of coset space and coset variable. The various BETs are built on the Plücker relation to hold the Grassmannian. The latter asserts that the soliton equations are nothing but the bilinear differential equations giving *a boson representation of the Plücker relation*. The relation, however, has been unsatisfactorily investigated yet within the framework of the usual SCFM. We study it and show that both the methods stand on the common features, Plücker relation or bilinear differential equation defining the Grassmannian. On the contrary, we observe different points: (i) The former is built on a *finite*-dimensional Lie algebra but the latter on an *infinite*-dimensional one. (ii) The former has an SCF Hamiltonian consisting of a fermion one-body operator, which is derived from a functional derivative of an expectation value of a fermion Hamiltonian by a ground-state wave function. The latter introduces artificially *a fermion Hamiltonian* of one-body type operator as *a boson mapping operator* from states on fermion Fock space to corresponding ones on τ-functional space (τ-FS).

The last issue is, despite a difference due to the dimension of fermions, to aim at obtaining a *close connection* between *concept of mean-field potential* and *gauge of fermions* inherent in the SCFM and at making a role of a loop group [36] to be clear. Through the observation, we construct infinite-dimensional fermion operators from the finite-dimensional ones by Laurent expansion with respect to a circle S^1. Then with the use of an affine Kac–Moody (KM) algebra according to the idea of Dirac's positron theory [40], we rebuilt a TDHFT in F_∞. The TDHFT results in a gauge theory of fermions and the collective motion, fluctuation of the mean-field potential, appears as the motion of fermion gauges with a common factor. The physical concept of the quasi-particle and vacuum in the SCFM on the S^1 connects to the "Plücker relations" due to the Dirac theory, in other words, the algebraic mechanism extracting various sub-group orbits consisting of *loop* path out of the full TDHF manifold is just the "Hirota's bilinear form" [39] which is an $su(N)(\in sl(N))$ reduction of $gl(N)$ in the τ-FM. As a result, it is shown that an infinite-dimensional fermion many-body system is also realizable in a finite-dimensional one and that roles of the soliton equation (Plücker relation) and the TDHF equation are made clear. We also understand an SCF dynamics through gauge of interacting infinite-dimensional fermions. A bilinear equation for the υ-HFT has been transcribed onto the corresponding τ-function using the regular representation for the group and the Schur polynomials. The υ-HF SCFM on an infinite-dimensional Fock space F_∞ leads to a dynamics on an infinite-dimensional Grassmannian Gr_∞ and may describe more precisely such a dynamics on the group manifold. A finite-dimensional Grassmannian is identified with a Gr_∞ which is affiliated with the group manifold obtained by reducting $gl(\infty)$ to $sl(N)$ and $su(N)$. We have given explicit expressions for Laurent coefficients of soliton solutions for $\widehat{sl}(N)$ and $\widehat{su}(N)$ on the Gr_∞ using Chevalley bases for $sl(N)$ and $su(N)$ [41]. As an illustration we will attempt to make a υ-HFT approach to an infinite-dimensional matrix model extended from the finite-dimensional $su(2)$ Lipkin–Meshkov–Glick (LMG) model [42]. For this aim, we give an affine KM algebra $\widehat{sl}(2, C)$ (complexification of $\widehat{su}(2)$) to which the LMG generators subject, and their τ representations and the σ_K mappings for them. We can represent an infinite-dimensional matrix of the LMG Hamiltonian and its HF Hamiltonian in terms of the Schur polynomials. Its infinite-dimensional HF operator is also given through the mapping σ_M for $\psi_i \psi_j^*$ of infinite-dimensional fermions ψ_i and ψ_i^*, which is expressed by the Schur polynomials $S_k(x)$ and $S_k(\partial_x)$. Further its τ-function for a simple case is provided by the Plücker coordinates and Schur polynomials.

In Section 2, we propose curvature equations as fundamental equations to extract a collective submanifold out of the full TDHB manifold. Basing on these ideas, we construct the curvature equations and study the relation between the maximal decoupled method and the curvature equations. We further investigate the role of the non-zero curvature arising from the residual Hamiltonian. Making use of the expression of the zero-curvature equations in the QPF, we find the formal RPA equation. In Section 3, we present a simply unified aspect for the SCFM and the τ-FM and show a simple idea connecting both the methods. We study the algebraic relation between coset coordinate and Plücker coordinate. Basing on the above idea, we attempt to rebuilt the TDHFT in τ-FS. We introduce υ-dependent infinite-dimensional fermion operators and a F_∞ through Laurent expansion with respect to the degrees of freedom of the original fermions. The algebraic relation between both the methods is manifestly described. We embed a HF $u(N)$ Lie algebra into a $gl(\infty)$ by means of infinite-dimensional fermions. The υ-SCFM in τ-FS is developed. The role of the shift operators in the τ-FM is studied. As an illustration, explicit expressions for Laurent coefficients of soliton solutions for $\widehat{sl}(N)$ and $\widehat{su}(N)$ are presented. A problem related to a nonlinear Schrödinger equation is also discussed. In Section 4, we construct a formal RPA equation on F_∞ and also argue about the relation between a *loop* collective path and a formal RPAEQ. Consequently, it can be proved that the usual perturbative method with respect to periodic collective variables in the TDHFT is involved in the present method which aims for constructing the TDHFT on the affine KM algebra. In Section 5, we introduce infinite-dimensional "particle" and "hole" operators and operators $\widehat{K}_0$ and $\widehat{K}_\pm$ defined by infinite-dimensional "particle-hole" pair operators. Using these operators, we construct an infinite-dimensional Heisenberg subalgebra of the affine KM algebra $\widehat{sl}(2, C)$. The LMG Hamiltonian and its HF Hamiltonian are expressed in terms of the Heisenberg basic-elements whose representations are isomorphic to those in the corresponding boson space. They are given in terms of infinite numbers of variables x_k and derivatives ∂_{x_k} through the Schur polynomials $S_k(x)$. We give also an infinite-dimensional representation of $SU(2N)_\infty$ transformation of the particle and hole operators. Finally, in Section 6, we summarize and discuss the results and future problems. We give Appendices A–J. Especially in Appendix J, we show an explicit expression for Plücker coordinate for the LMG model and calculate a quantity, $\det(1_N + p^\dagger p)$ (p : coset variable) for the LMG model, in terms of the Schur polynomials.

2 Integrability conditions and collective submanifolds

2.1 Introduction

Let us consider an abstract evolution equation $\partial_t u(t) = K(u(t))$ for u, which is dependent only on a parameter, time t. If there exists a symmetry operation to transfer a solution for u to another one, then introducing another parameter s specifying various solutions, we can derive another form of evolution equation with respect to s, $\partial_s u(t,s) = \overline{K}(u(t,s))$ for which we should want to search. The infinitesimal condition for the existence of such a symmetry appears as the well-known integrability condition $\partial_s K(u(t,s)) = \partial_t \overline{K}(u(t,s))$. The "maximally decoupled" method proposed by Marumori et al. [25], invariance principle of Schrödinger equation and canonicity condition, can be considered as a theory to apply the above basic idea to certain multiple parametrized symmetries. The method is regarded as a mathematical tool to describe the symmetries of the collective submanifold in terms of t and collective variables, in which the canonicity conditions make the collective variables to be orthogonal coordinate-systems.

Therefore we adopt a concept of curvature unfamiliar in the conventional TDHBT. The reason why we take such a thing is the following: let us consider a description of motions of systems on a group manifold. An arbitrary state of the system induced by a transitive group action corresponds to any point of the full group parameter space and therefore its time evolution is represented by an integral curve in this space. In the whole representation space adopted, we assume the existence of $2m$ parameters specifying the proper subspace in which the original motion of the system can be approximated well, the existence of the well-defined symmetries. Suppose we start from a given point on a space, which consists of t and the $2m$ parameters, and end at the same point again along the closed curve. Then we have the value of the group parameter different from the one at an initial point on the proper subspace. We search for some quantities characterizing the difference of the value. For our aim, we introduce a differential geometrical viewpoint. The our basic idea lies in the introduction of a sort of Lagrange manner familiar to fluid dynamics to describe collective coordinate systems. This manner enables us to take a one-form which is linearly composed of TDHB Hamiltonian and infinitesimal generators induced by collective variable differentials of an $SO(2N)$ canonical transformation. The integrability conditions of the system read the curvature $C = 0$. Our methods are constructed manifesting themselves the structure of the group under consideration to make easy to understand physical characters at any point on the group manifold.

2.2 Integrability conditions

We consider many fermion systems with pair correlations. Let c_α and $c_\alpha^\dagger$ ($\alpha = 1, \ldots, N$) be the annihilation-creation operators of the fermion. Owing to the anti-commutation relations among them, some sets of fermion operators with simple construction become the basis of a Lie algebra. The operators in the fermion $so(2N)$ Lie algebra, $\bar{E}^\alpha_{\ \beta} = c_\alpha^\dagger c_\beta - \frac{1}{2}\delta_{\alpha\beta}$, $\bar{E}_{\alpha\beta} = c_\alpha c_\beta$, $\bar{E}^{\alpha\beta} = c_\alpha^\dagger c_\beta^\dagger$ generate a canonical transformation $U(g)$ (the Bogoliubov transformation [9]) which is specified by an $SO(2N)$ matrix g:

$$[d, d^\dagger] = U(g)[c, c^\dagger]U^\dagger(g) = [c, c^\dagger]g, \qquad g = \begin{bmatrix} a & b^\star \\ b & a^\star \end{bmatrix}, \qquad gg^\dagger = g^\dagger g = 1_{2N}, \tag{2.1}$$
$$U^{-1}(g) = U^\dagger(g), \qquad U(g)U(g') = U(gg'),$$

where $(c, c^\dagger) = ((c_\alpha), (c_\alpha^\dagger))$ and $(d, d^\dagger) = ((d_i), (d_i^\dagger))$ are $2N$-dimensional row vectors and $a = (a^\alpha_{\ i})$ and $b = (b^\alpha_{\ i})$ ($i = 1, \ldots, N$) are $N \times N$ matrices. 1_{2N} is a $2N$-dimensional unit matrix. The symbols $\dagger$, $\star$ and T mean the hermitian conjugate, the complex conjugation and the transposition, respectively. The explicit expressions of the canonical transformations are given by Fukutome for the various types of the fermion Lie algebra [16]. The fermion Lie operators of the quasi-particles $(E^i{}_j, E_{ij}, E^{ij})$ are constructed from the operators d and $d^\dagger$ in (2.1) by the same way as the one to define the set $(\bar{E}^\alpha{}_\beta, \bar{E}_{\alpha\beta}, \bar{E}^{\alpha\beta})$. The set E in the quasi-particle frame is transformed into the set $\bar{E}$ in the particle frame as follows:

$$\begin{bmatrix} E^\bullet{}_\bullet & E^{\bullet\bullet} \\ E_{\bullet\bullet} & -E^\bullet{}_\bullet{}^\dagger \end{bmatrix} = g^\dagger \begin{bmatrix} \bar{E}^\bullet{}_\bullet & \bar{E}^{\bullet\bullet} \\ \bar{E}_{\bullet\bullet} & -\bar{E}^\bullet{}_\bullet{}^\dagger \end{bmatrix} g. \tag{2.2}$$

The $E^\bullet{}_\bullet = (E^i{}_j)$ and $\bar{E}^\bullet{}_\bullet = (\bar{E}^\alpha{}_\beta)$ etc. are $N \times N$ matrices. A TDHB Hamiltonian of the system is given by

$$H_{\mathrm{HB}} = \tfrac{1}{2}[c, c^\dagger]\mathcal{F} \begin{bmatrix} c^\dagger \\ c \end{bmatrix}, \qquad \mathcal{F} = \begin{bmatrix} -F^\star & -D^\star \\ D & F \end{bmatrix} = \mathcal{F}^\dagger, \tag{2.3}$$

where the HB matrices $F = (F_{\alpha\beta})$ and $D = (D_{\alpha\beta})$ are related to the quasi-particle vacuum expectation values of the Lie operators $\langle \bar{E} \rangle$ as

$$\begin{aligned} &F_{\alpha\beta} = h_{\alpha\beta} + [\alpha\beta|\gamma\delta](\langle \bar{E}^\gamma{}_\delta \rangle + \tfrac{1}{2}\delta_{\gamma\delta}) \qquad (F^\dagger = F), \\ &D_{\alpha\beta} = \tfrac{1}{2}[\alpha\gamma|\beta\delta]\langle \bar{E}_{\delta\gamma} \rangle \qquad (D^{\mathrm{T}} = -D), \\ &[\alpha\beta|\gamma\delta] = -[\alpha\delta|\gamma\beta] = [\gamma\delta|\alpha\beta] = [\beta\alpha|\delta\gamma]^\star. \end{aligned}$$

The quantities $h_{\alpha\beta}$ and $[\alpha\beta|\gamma\delta]$ are the matrix element of the single-particle Hamiltonian and the antisymmetrized one of the interaction potential, respectively. Here and hereafter we use the dummy index convention to take summation over the repeated index.

Let $|0\rangle$ be the free-particle vacuum satisfying $c_\alpha|0\rangle = 0$. The $SO(2N)$(HB) wave function $|\phi(\check{g})\rangle$ is constructed by a transitive action of the $SO(2N)$ canonical transformation $U(\check{g})$ on $|0\rangle$: $|\phi(\check{g})\rangle = U^{-1}(\check{g})|0\rangle$, $\check{g} \in SO(2N)$. In the conventional TDHBT, the TD wave function $|\phi(\check{g})\rangle$ is given through that of the TD group parameters a $(a^\star)$ and b $(b^\star)$. They characterize the TD self-consistent mean HB fields F and D whose dynamical changes induce the collective motions of the many fermion systems. As was made in the TDHF case [25] and [27], we introduce a TD $SO(2N)$ canonical transformation $U(\check{g}) = U[\check{g}(\check{\Lambda}(t), \check{\Lambda}^\star(t))]$. A set of TD complex variables $(\check{\Lambda}(t), \check{\Lambda}^\star(t)) = (\check{\Lambda}_n(t), \check{\Lambda}^\star_n(t);\ n = 1, \ldots, m)$ associated with the collective motions specifies the group parameters. The number m is assumed to be much smaller than the order of the $SO(2N)$ Lie algebra, which means there exist only a few "collective degrees of freedom". The above is the natural extension of the method in TDHF case to the TDHB case.

However, differing from the above usual manner, we have another way, may be called a *Lagrange-like manner*, to introduce a set of complex variables. This is realized if we regard the above-mentioned variables $(\check{\Lambda}(t), \check{\Lambda}^\star(t))$ as functions of independent variables $(\Lambda, \Lambda^\star) = (\Lambda_n, \Lambda^\star_n)$ and t, where *time-independent variables* $(\Lambda, \Lambda^\star)$ are introduced as local coordinates to specify any point of a $2m$-dimensional collective submanifold. A collective motion in the $2m$-dimensional manifold is possibly determined in the usual manner if we could know the explicit forms of $\check{\Lambda}$ and $\check{\Lambda}^\star$ in terms of $(\Lambda, \Lambda^\star)$ and t. The above manner seems to be very analogous to the Lagrange manner in the fluid dynamics. The pair of variables $(\Lambda, \Lambda^\star)$ specifies variations of the SCF associated with the collective motion described by a pair of collective coordinates α and their conjugate π in the *Lagrange-like manner*, $\widehat{\alpha} = \frac{1}{\sqrt{2}}(\Lambda^\star + \Lambda)$ and $\widehat{\pi} = i\frac{1}{\sqrt{2}}(\Lambda^\star - \Lambda)$[25]. Thus, the $SO(2N)$ canonical transformation is rewritten as $U(\check{g}) = U[g(\Lambda, \Lambda^\star, t)] \in SO(2N)$. Notice that a functional form $\check{g}(\check{\Lambda}(t), \check{\Lambda}^\star(t))$ changes into another form $g(\Lambda, \Lambda^\star, t)$ due to an adoption of the Lagrange-like manner. This manner enables us to take a one-form Ω which is linearly composed of the infinitesimal generators induced by the time differential and the collective variable ones $(\partial_t, \partial_\Lambda, \partial_{\Lambda^\star})$ of the $SO(2N)$ canonical transformation $U[g(\Lambda, \Lambda^\star, t)]$. By introducing the one-form Ω, it is possible to search for the collective path and the collective hamiltonian almost separated from other remaining degrees of freedom of the systems. It may be achieved to study the integrability conditions of our systems which are expressed as the set of the Lie-algebra-valued equations.

We define the Lie-algebra-valued infinitesimal generators of collective submanifolds as follows:

$$H_{\mathrm{c}}/\hbar \overset{d}{=} (i\partial_t U^{-1}(g))U(g),$$
$$O_n^\dagger \overset{d}{=} (i\partial_{\Lambda_n} U^{-1}(g))U(g), \qquad O_n \overset{d}{=} (i\partial_{\Lambda_n^\star} U^{-1}(g))U(g). \tag{2.4}$$

Here and hereafter, for simplicity we abbreviate $g(\Lambda, \Lambda^\star, t)$ as g. The explicit form of the infinitesimal generators for the TDHF was first given by Yamamura and Kuriyama [27]. In our TDHB, the Lie-algebra-valued infinitesimal generators are expressed by the trace form as

$$H_{\mathrm{c}}/\hbar = -\tfrac{1}{2}\mathrm{Tr}\left\{(i\partial_t g \cdot g^\dagger)\begin{bmatrix} \bar{E}^{\bullet}_{\bullet} & \bar{E}^{\bullet\bullet} \\ \bar{E}_{\bullet\bullet} & -\bar{E}^{\bullet}_{\bullet}{}^\dagger \end{bmatrix}\right\} = \tfrac{1}{2}[c, c^\dagger](i\partial_t g \cdot g^\dagger)\begin{bmatrix} c^\dagger \\ c \end{bmatrix}, \tag{2.5}$$

$$O_n^\dagger = -\tfrac{1}{2}\mathrm{Tr}\left\{(i\partial_{\Lambda_n} g \cdot g^\dagger)\begin{bmatrix} \bar{E}^{\bullet}_{\bullet} & \bar{E}^{\bullet\bullet} \\ \bar{E}_{\bullet\bullet} & -\bar{E}^{\bullet}_{\bullet}{}^\dagger \end{bmatrix}\right\} = \tfrac{1}{2}[c, c^\dagger](i\partial_{\Lambda_n} g \cdot g^\dagger)\begin{bmatrix} c^\dagger \\ c \end{bmatrix},$$
$$O_n = -\tfrac{1}{2}\mathrm{Tr}\left\{(i\partial_{\Lambda_n^\star} g \cdot g^\dagger)\begin{bmatrix} \bar{E}^{\bullet}_{\bullet} & \bar{E}^{\bullet\bullet} \\ \bar{E}_{\bullet\bullet} & -\bar{E}^{\bullet}_{\bullet}{}^\dagger \end{bmatrix}\right\} = \tfrac{1}{2}[c, c^\dagger](i\partial_{\Lambda_n^\star} g \cdot g^\dagger)\begin{bmatrix} c^\dagger \\ c \end{bmatrix}. \tag{2.6}$$

Multiplying the $SO(2N)$ wave function $|\phi(g)\rangle$ on the both sides of (2.4), we get a set of equations on the $so(2N)$ Lie algebra:

$$D_t|\phi(g)\rangle \overset{d}{=} (\partial_t + iH_{\mathrm{c}}/\hbar)|\phi(g)\rangle = 0,$$
$$D_{\Lambda_n}|\phi(g)\rangle \overset{d}{=} (\partial_{\Lambda_n} + iO_n^\dagger)|\phi(g)\rangle = 0, \qquad D_{\Lambda_n^\star}|\phi(g)\rangle \overset{d}{=} (\partial_{\Lambda_n^\star} + iO_n)|\phi(g)\rangle = 0. \tag{2.7}$$

We regard these equations (2.7) as partial differential equations for $|\phi(g)\rangle$. In order to discuss the conditions under which the differential equations (2.7) can be solved, the mathematical method well known as integrability conditions is useful. For this aim, we take a one-form Ω linearly composed of the infinitesimal generators (2.4): $\Omega = -i(H_{\mathrm{c}}/\hbar \cdot dt + O_n^\dagger \cdot d\Lambda_n + O_n \cdot d\Lambda_n^\star)$. With the aid of the Ω, the integrability conditions of the system read $C \overset{d}{=} d\Omega - \Omega \wedge \Omega = 0$, where d and $\wedge$ denote the exterior differentiation and the exterior product, respectively. From the differential geometrical viewpoint, the quantity C means the curvature of a connection. Then the integrability conditions may be interpreted as the vanishing of the curvature of the connection $(D_t, D_{\Lambda_n}, D_{\Lambda_n}^\star)$. The detailed structure of the curvature is calculated to be

$$\begin{aligned} C = {} & C_{t,\Lambda_n} d\Lambda_n \wedge dt + C_{t,\Lambda_n^\star} d\Lambda_n^\star \wedge dt + C_{\Lambda_{n'},\Lambda_n^\star} d\Lambda_n^\star \wedge d\Lambda_{n'} \\ & + \tfrac{1}{2} C_{\Lambda_{n'},\Lambda_n} d\Lambda_n \wedge d\Lambda_{n'} + \tfrac{1}{2} C_{\Lambda_{n'}^\star,\Lambda_n^\star} d\Lambda_n^\star \wedge d\Lambda_{n'}^\star, \end{aligned}$$

where

$$\begin{aligned} C_{t,\Lambda_n} &\overset{d}{=} [D_t, D_{\Lambda_n}] = i\partial_t O_n^\dagger - i\partial_{\Lambda_n} H_{\mathrm{c}}/\hbar + [O_n^\dagger, H_{\mathrm{c}}/\hbar], \\ C_{t,\Lambda_n^\star} &\overset{d}{=} [D_t, D_{\Lambda_n^\star}] = i\partial_t O_n - i\partial_{\Lambda_n^\star} H_{\mathrm{c}}/\hbar + [O_n, H_{\mathrm{c}}/\hbar], \\ C_{\Lambda_{n'},\Lambda_n^\star} &\overset{d}{=} [D_{\Lambda_{n'}}, D_{\Lambda_n^\star}] = i\partial_{\Lambda_{n'}} O_n - i\partial_{\Lambda_n^\star} O_{n'}^\dagger + [O_n, O_{n'}^\dagger], \\ C_{\Lambda_{n'},\Lambda_n} &\overset{d}{=} [D_{\Lambda_{n'}}, D_{\Lambda_n}] = i\partial_{\Lambda_{n'}} O_n^\dagger - i\partial_{\Lambda_n} O_{n'}^\dagger + [O_n^\dagger, O_{n'}^\dagger], \\ C_{\Lambda_{n'}^\star,\Lambda_n^\star} &\overset{d}{=} [D_{\Lambda_{n'}^\star}, D_{\Lambda_n^\star}] = i\partial_{\Lambda_{n'}^\star} O_n - i\partial_{\Lambda_n^\star} O_{n'} + [O_n, O_{n'}]. \end{aligned} \tag{2.8}$$

The vanishing of the curvature C means $C_{\bullet,\bullet} = 0$.

Finally with the use of the explicit forms of (2.5) and (2.6), we can get the set of Lie-algebra-valued equations as the integrability conditions of partial differential equations (2.7)

$$\begin{aligned}
&C_{t,\Lambda_n} = \tfrac{1}{2}[c, c^\dagger]\mathcal{C}_{t,\Lambda_n}\begin{bmatrix} c^\dagger \\ c \end{bmatrix}, \qquad && \mathcal{C}_{t,\Lambda_n} = i\partial_t\theta_n^\dagger - i\partial_{\Lambda_n}\mathcal{F}_c/\hbar + [\theta_n^\dagger, \mathcal{F}_c/\hbar],\\
&C_{t,\Lambda_n^\star} = \tfrac{1}{2}[c, c^\dagger]\mathcal{C}_{t,\Lambda_n^\star}\begin{bmatrix} c^\dagger \\ c \end{bmatrix}, && \mathcal{C}_{t,\Lambda_n^\star} = i\partial_t\theta_n - i\partial_{\Lambda_n^\star}\mathcal{F}_c/\hbar + [\theta_n, \mathcal{F}_c/\hbar],\\
&C_{\Lambda_{n'},\Lambda_n^\star} = \tfrac{1}{2}[c, c^\dagger]\mathcal{C}_{\Lambda_{n'},\Lambda_n^\star}\begin{bmatrix} c^\dagger \\ c \end{bmatrix}, && \mathcal{C}_{\Lambda_{n'},\Lambda_n^\star} = i\partial_{\Lambda_{n'}}\theta_n - i\partial_{\Lambda_n^\star}\theta_{n'}^\dagger + [\theta_n, \theta_{n'}^\dagger],\\
&C_{\Lambda_{n'},\Lambda_n} = \tfrac{1}{2}[c, c^\dagger]\mathcal{C}_{\Lambda_{n'},\Lambda_n}\begin{bmatrix} c^\dagger \\ c \end{bmatrix}, && \mathcal{C}_{\Lambda_{n'},\Lambda_n} = i\partial_{\Lambda_{n'}}\theta_n^\dagger - i\partial_{\Lambda_n}\theta_{n'}^\dagger + [\theta_n^\dagger, \theta_{n'}^\dagger],\\
&C_{\Lambda_{n'}^\star,\Lambda_n^\star} = \tfrac{1}{2}[c, c^\dagger]\mathcal{C}_{\Lambda_{n'}^\star,\Lambda_n^\star}\begin{bmatrix} c^\dagger \\ c \end{bmatrix}, && \mathcal{C}_{\Lambda_{n'}^\star,\Lambda_n^\star} = i\partial_{\Lambda_{n'}^\star}\theta_n - i\partial_{\Lambda_n^\star}\theta_{n'} + [\theta_n, \theta_{n'}].
\end{aligned} \tag{2.9}$$

Here the quantities $\mathcal{F}_c$, $\theta_n^\dagger$, θ_n are defined through partial differential equations,

$$i\hbar\partial_t g = \mathcal{F}_c g \qquad \text{and} \qquad i\partial_{\Lambda_n} g = \theta_n^\dagger g, \qquad i\partial_{\Lambda_n^\star} g = \theta_n g. \tag{2.10}$$

The quantity $\mathcal{C}_{\bullet,\bullet}$ may be naturally regarded as the curvature of the connection on the group manifold. The reason becomes clear if we take the following procedure quite parallel with the above: Starting from (2.10), we are led to a set of partial differential equations on the $SO(2N)$ Lie group,

$$\begin{aligned}
&\mathcal{D}_t g \stackrel{d}{=} (\partial_t + i\mathcal{F}_c/\hbar)g = 0,\\
&\mathcal{D}_{\Lambda_n} g \stackrel{d}{=} (\partial_{\Lambda_n} + i\theta_n^\dagger)g = 0, \qquad \mathcal{D}_{\Lambda_n^\star} g \stackrel{d}{=} (\partial_{\Lambda_n^\star} + i\theta_n)g = 0.
\end{aligned} \tag{2.11}$$

The curvature $\mathcal{C}_{\bullet,\bullet}$ ($\stackrel{d}{=} [\mathcal{D}_\bullet, \mathcal{D}_\bullet]$) of the connection $(\mathcal{D}_t, \mathcal{D}_{\Lambda_n}, \mathcal{D}_{\Lambda_n^\star})$ is easily shown to be equivalent to the quantity $\mathcal{C}_{\bullet,\bullet}$ in (2.9). The above set of the Lie-algebra-valued equations (2.9) evidently leads us to putting all the curvatures $\mathcal{C}_{\bullet,\bullet}$ in (2.9) equal to zero. On the other hand, the TDHB Hamiltonian (2.3), being the full Hamiltonian on the full $SO(2N)$ wave function space, can be represented in the same form as (2.4), $H_{\mathrm{HB}}/\hbar = (i\partial_t U^{-1}(g'))U(g')$, where g' is any point on the $SO(2N)$ group manifold. This Hamiltonian is also transformed into the same form as (2.5). It is self-evident that the above fact leads us to the well-known TDHBEQ, $i\hbar\partial_t g' = \mathcal{F}g'$. The full TDHB Hamiltonian can be decomposed into two components at the reference point $g' = g$:

$$H_{\mathrm{HB}}|_{U^{-1}(g')=U^{-1}(g)} = H_c + H_{\mathrm{res}}, \qquad \mathcal{F}|_{g'=g} = \mathcal{F}_c + \mathcal{F}_{\mathrm{res}},$$

where the second part $H_{\mathrm{res}}(\mathcal{F}_{\mathrm{res}})$ means a residual component out of a *well-defined* collective submanifold for which we should search now.

For our purpose, let us introduce another curvature $\mathcal{C}'_{t,\Lambda_n}$ and $\mathcal{C}'_{t,\Lambda_n^\star}$ with the same forms as those in (2.9), except that the Hamiltonian $\mathcal{F}_c$ is replaced by $\mathcal{F}|_{g'=g}$ ($= \mathcal{F}_c + \mathcal{F}_{\mathrm{res}}$). The quasi-particle vacuum expectation values of the Lie-algebra-valued curvatures are easily calculated as

$$\langle C'_{t,\Lambda_n}\rangle_g = -i\partial_{\Lambda_n}\langle H_{\mathrm{res}}/\hbar\rangle_g, \qquad \langle C'_{t,\Lambda_n^\star}\rangle_g = -i\partial_{\Lambda_n^\star}\langle H_{\mathrm{res}}/\hbar\rangle_g, \tag{2.12}$$

$$\langle H_{\mathrm{res}}\rangle_g = -\tfrac{1}{4}\mathrm{Tr}\left\{ g\begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix} g^\dagger(\mathcal{F}_g - i\hbar\partial_t g\cdot g^\dagger)\right\}, \tag{2.13}$$

where we have used $\langle C_{t,\Lambda_n}\rangle = 0$ and $\langle C_{t,\Lambda_n^\star}\rangle = 0$. The above equations (2.12) and (2.13) are interpreted that the values of $\langle C'{}_{t,\Lambda_n}\rangle_g$ and $\langle C'{}_{t,\Lambda_n^\star}\rangle_g$ represent the *gradient* of energy of the residual Hamiltonian in the $2m$-dimensional manifold. Suppose there exists the *well-defined* collective submanifold. Then it will be not so wrong to deduce the following remarks: the energy value of the residual Hamiltonian becomes almost constant on the collective submanifold, i.e.,

$$\delta_g \langle H_{\rm res} \rangle_g \cong 0 \qquad \text{and} \qquad \partial_{\Lambda_n} \langle H_{\rm res} \rangle_g \cong 0, \qquad \partial_{\Lambda^\star_n} \langle H_{\rm res} \rangle_g \cong 0,$$

where δ_g means g-variation, regarding g as function of $(\Lambda, \Lambda^\star)$ and t. It may be achieved if we should determine g (collective path) and $\mathcal{F}_{\rm c}$ (collective Hamiltonian) through auxiliary quantity $(\theta, \theta^\dagger)$ so as to satisfy $H_{\rm c} + \text{const} = H_{\rm HB}$ as far as possible. Putting $\mathcal{F}_{\rm c} = \mathcal{F}$ in (2.9), we seek for g and $\mathcal{F}_{\rm c}$ satisfying

$$\mathcal{C}_{t,\Lambda_n} \cong 0, \qquad \mathcal{C}_{t,\Lambda^\star_n} \cong 0, \qquad \mathcal{C}_{\Lambda_{n'},\Lambda^\star_n} = 0, \qquad \mathcal{C}_{\Lambda_{n'},\Lambda_n} = 0, \qquad \mathcal{C}_{\Lambda^\star_{n'},\Lambda^\star_n} = 0. \tag{2.14}$$

The set of the equations $\mathcal{C}_{\bullet,\bullet} = 0$ makes an essential role to determine the collective submanifold in the TDHBT. The set of the equations (2.14) and (2.11) becomes our fundamental equation for describing the collective motions, under the restrictions (2.21).

If we want to describe the collective motions through the TD complex variables $(\check{\Lambda}(t), \check{\Lambda}^\star(t))$ in the usual manner, we must inevitably know $\check{\Lambda}$ and $\check{\Lambda}^\star$ as functions of $(\Lambda, \Lambda^\star)$ and t. For this aim, it is necessary to discuss the correspondence of the Lagrange-like manner to the usual one.

First let us define the Lie-algebra-valued infinitesimal generator of collective submanifolds as

$$\check{O}^\dagger_n \stackrel{d}{=} (i\partial_{\check{\Lambda}_n} U^{-1}(\check{g}))U(\check{g}), \qquad \check{O}_n \stackrel{d}{=} (i\partial_{\check{\Lambda}^\star_n} U^{-1}(\check{g}))U(\check{g}), \qquad (\check{g} \in g),$$

whose form is the same as the one in (2.4). To guarantee $\check{\Lambda}_n(t)$ and $\check{\Lambda}^\star_n(t)$ to be canonical, according to [25, 27], we set up the following expectation values with use of the $SO(2N)$ (HB) wave function $|\phi(\check{g})\rangle$:

$$\begin{aligned} &\langle \phi(\check{g})| i\partial_{\check{\Lambda}_n} |\phi(\check{g})\rangle = \langle \phi(\check{g})| \check{O}^\dagger_n |\phi(\check{g})\rangle = i\tfrac{1}{2}\check{\Lambda}^\star_n, \\ &\langle \phi(\check{g})| i\partial_{\check{\Lambda}^\star_n} |\phi(\check{g})\rangle = \langle \phi(\check{g})| \check{O}_n |\phi(\check{g})\rangle = -i\tfrac{1}{2}\check{\Lambda}_n. \end{aligned} \tag{2.15}$$

The above relation leads us to the *weak* canonical commutation relation

$$\begin{aligned} &\langle \phi(\check{g})| [\check{O}_n, \check{O}^\dagger_{n'}] |\phi(\check{g})\rangle = \delta_{nn'}, \\ &\langle \phi(\check{g})| [\check{O}^\dagger_n, \check{O}^\dagger_{n'}] |\phi(\check{g})\rangle = 0, \qquad \langle \phi(\check{g})| [\check{O}_n, \check{O}_{n'}] |\phi(\check{g})\rangle = 0 \qquad (n, n' = 1, \dots, m) \end{aligned} \tag{2.16}$$

the proof of which was shown in [25] and [27].

Using (2.7), the collective Hamiltonian $H_{\rm c}/\hbar$ and the infinitesimal generators $O^\dagger_n$ and O_n in the Lagrange-like manner are expressed in terms of infinitesimal ones $\check{O}^\dagger_n$ and $\check{O}_n$ in the usual way as follows:

$$\begin{aligned} &H_{\rm c}/\hbar = \partial_t \check{\Lambda}_n \check{O}^\dagger_n + \partial_t \check{\Lambda}^\star_n \check{O}_n, \\ &O^\dagger_n = \partial_{\Lambda_n} \check{\Lambda}_{n'} \check{O}^\dagger_{n'} + \partial_{\Lambda_n} \check{\Lambda}^\star_{n'} \check{O}_{n'}, \qquad O_n = \partial_{\Lambda^\star_n} \check{\Lambda}_{n'} \check{O}^\dagger_{n'} + \partial_{\Lambda^\star_n} \check{\Lambda}^\star_{n'} \check{O}_{n'}. \end{aligned} \tag{2.17}$$

Substituting (2.17) into (2.8), it is easy to evaluate the expectation values of the Lie-algebra-valued curvatures $C_{\bullet,\bullet}$ by the $SO(2N)$ (HB) wave function $|\phi[\check{g}(\check{\Lambda}(t), \check{\Lambda}^\star(t))]\rangle$ $(= |\phi[g(\Lambda, \Lambda^\star, t)]\rangle)$. A *weak* integrability condition requiring the expectation values $\langle \phi(\check{g})| C_{\bullet,\bullet} |\phi(\check{g})\rangle = 0$ yields the following set of partial differential equations with aid of the quasi-particle vacuum property, $d|\phi(g)\rangle = 0$:

$$\partial_{\Lambda_n} \check{\Lambda}_{n'} \partial_t \check{\Lambda}^\star_{n'} - \partial_{\Lambda_n} \check{\Lambda}^\star_{n'} \partial_t \check{\Lambda}_{n'} = \partial_{\Lambda^\star_n} \check{\Lambda}_{n'} \partial_t \check{\Lambda}^\star_{n'} - \partial_{\Lambda^\star_n} \check{\Lambda}^\star_{n'} \partial_t \check{\Lambda}_{n'} = \tfrac{1}{4}\mathrm{Tr}\{\mathcal{R}(g)[\theta^\dagger_n, \mathcal{F}_{\rm c}/\hbar]\}, \tag{2.18}$$

$$\partial_{\Lambda^\star_n} \check{\Lambda}_{n''} \partial_{\Lambda_{n'}} \check{\Lambda}^\star_{n''} - \partial_{\Lambda^\star_n} \check{\Lambda}^\star_{n''} \partial_{\Lambda_{n'}} \check{\Lambda}_{n''} = \tfrac{1}{4}\mathrm{Tr}\{\mathcal{R}(g)[\theta_n, \theta^\dagger_{n'}]\},$$

$$\begin{aligned}
&\partial_{\Lambda_n}\check{\Lambda}_{n''}\partial_{\Lambda_{n'}}\check{\Lambda}^\star_{n''} - \partial_{\Lambda_n}\check{\Lambda}^\star_{n''}\partial_{\Lambda_{n'}}\check{\Lambda}_{n''} = \tfrac{1}{4}\mathrm{Tr}\{\mathcal{R}(g)[\theta^\dagger_n, \theta^\dagger_{n'}]\},\\
&\partial_{\Lambda^\star_n}\check{\Lambda}_{n''}\partial_{\Lambda^\star_{n'}}\check{\Lambda}^\star_{n''} - \partial_{\Lambda^\star_n}\check{\Lambda}^\star_{n''}\partial_{\Lambda^\star_{n'}}\check{\Lambda}_{n''} = \tfrac{1}{4}\mathrm{Tr}\{\mathcal{R}(g)[\theta_n, \theta_{n'}]\},
\end{aligned} \tag{2.19}$$

where an $SO(2N)$ (HB) density matrix $\mathcal{R}(g)$ is defined as

$$\mathcal{R}(g) \stackrel{d}{=} g \begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix} g^\dagger, \qquad \mathcal{R}^\dagger(g) = \mathcal{R}(g), \qquad \mathcal{R}^2(g) = 1_{2N}, \tag{2.20}$$

in which g becomes function of the complex variables $(\Lambda, \Lambda^\star)$ and t. We here have used the transformation property (2.2), the trace formulae equations (2.5) and (2.6) and the differential formulae, i.e.,

$$\begin{aligned}
&\langle\phi(\check{g})|i\partial_{\check{\Lambda}_{n'}}\check{O}_n|\phi(\check{g})\rangle = -\tfrac{1}{2}\delta_{nn'}, \qquad \langle\phi(\check{g})|i\partial_{\check{\Lambda}^\star_{n'}}\check{O}^\dagger_n|\phi(\check{g})\rangle = \tfrac{1}{2}\delta_{nn'},\\
&\langle\phi(\check{g})|i\partial_{\check{\Lambda}_{n'}}\check{O}^\dagger_n|\phi(\check{g})\rangle = 0, \qquad \langle\phi(\check{g})|i\partial_{\check{\Lambda}^\star_{n'}}\check{O}_n|\phi(\check{g})\rangle = 0,
\end{aligned}$$

which owe to the canonicity condition (2.15) and *weak* canonical commutation relation (2.16).

Through the above procedure, as a final goal, we get the correspondence of the Lagrange-like manner to the usual one. We have no unknown quantities in the r.h.s. of equations (2.18) and (2.19), if we could completely solve our fundamental equations to describe the collective motion. Then we come up to be able to know in principle the explicit forms of $(\check{\Lambda}, \check{\Lambda}^\star)$ in terms of $(\Lambda, \Lambda^\star)$ and t by solving the partial differential equations (2.18) and (2.19). However we should take enough notice of roles different from each other made by equations (2.18) and (2.19), respectively, to construct the solutions. Especially, it turns out that the l.h.s. in (2.19) has a close connection with Lagrange bracket. From the outset we have set up the canonicity condition to guarantee the complex variables $(\check{\Lambda}, \check{\Lambda}^\star)$ in the usual manner to be canonical. Thus the variables $(\check{\Lambda}, \check{\Lambda}^\star)$ are interpreted as functions giving a canonical transformation from $(\check{\Lambda}, \check{\Lambda}^\star)$ to another complex variables $(\Lambda, \Lambda^\star)$ in the Lagrange-like manner. From this interpretation, we see that the canonical invariance requirements impose the following restrictions on the r.h.s. of (2.19):

$$-\tfrac{1}{4}\mathrm{Tr}\{\mathcal{R}(g)[\theta_n, \theta^\dagger_{n'}]\} = \delta_{nn'}, \qquad \tfrac{1}{4}\mathrm{Tr}\{\mathcal{R}(g)[\theta^\dagger_n, \theta^\dagger_{n'}]\} = 0, \qquad \tfrac{1}{4}\mathrm{Tr}\{\mathcal{R}(g)[\theta_n, \theta_{n'}]\} = 0. \tag{2.21}$$

Using (2.19) and (2.21), we get Lagrange brackets for canonical transformation of $(\check{\Lambda}, \check{\Lambda}^\star)$ to $(\Lambda, \Lambda^\star)$.

2.3 Validity of maximally-decoupled theory

First we transform the set of the fundamental equations in the particle frame into the one in the quasi-particle frame. The $SO(2N)$ (TDHB) Hamiltonian of the system is expressed as

$$H_{\mathrm{HB}} = \tfrac{1}{2}[d, d^\dagger]\mathcal{F}_o \begin{bmatrix} d^\dagger \\ d \end{bmatrix}, \qquad \mathcal{F}_o = \begin{bmatrix} -F^\star_o & -D^\star_o \\ D_o & F_o \end{bmatrix}, \qquad \mathcal{F}^\dagger_o = \mathcal{F}_o, \tag{2.22}$$

the relation of which to the original TDHB Hamiltonian $\mathcal{F}$ is given by $\mathcal{F}_o = g^\dagger\mathcal{F}g$, $g \in SO(2N)$. The infinitesimal generators of collective submanifolds and their integrability conditions expressed as the Lie-algebra-valued equations are also rewritten into the ones in the quasi-particle frame as follows:

$$\begin{aligned}
&H_{\mathrm{c}} = \tfrac{1}{2}[d, d^\dagger]\mathcal{F}_{o-\mathrm{c}} \begin{bmatrix} d^\dagger \\ d \end{bmatrix},\\
&O^\dagger_n = \tfrac{1}{2}[d, d^\dagger]\theta^\dagger_{o-n} \begin{bmatrix} d^\dagger \\ d \end{bmatrix}, \qquad O_n = \tfrac{1}{2}[d, d^\dagger]\theta_{o-n} \begin{bmatrix} d^\dagger \\ d \end{bmatrix},
\end{aligned}$$

$$
\begin{aligned}
&C_{t,\Lambda_n} = \tfrac{1}{2}[d,d^\dagger]\mathcal{C}_{o-t,\Lambda_n}\begin{bmatrix} d^\dagger \\ d \end{bmatrix} = 0,\\
&\mathcal{C}_{o-t,\Lambda_n} = i\partial_t\theta^\dagger_{o-n} - i\partial_{\Lambda_n}\mathcal{F}_{o-\mathrm{c}}/\hbar - [\theta^\dagger_{o-n},\mathcal{F}_{o-\mathrm{c}}/\hbar],\\
&C_{t,\Lambda^\star_n} = \tfrac{1}{2}[d,d^\dagger]\mathcal{C}_{o-t,\Lambda^\star_n}\begin{bmatrix} d^\dagger \\ d \end{bmatrix} = 0,\\
&\mathcal{C}_{o-t,\Lambda^\star_n} = i\partial_t\theta_{o-n} - i\partial_{\Lambda^\star_n}\mathcal{F}_{o-\mathrm{c}}/\hbar - [\theta_{o-n},\mathcal{F}_{o-\mathrm{c}}/\hbar],\\
&C_{\Lambda_{n'},\Lambda_n} = \tfrac{1}{2}[d,d^\dagger]\mathcal{C}_{o-\Lambda_{n'},\Lambda_n}\begin{bmatrix} d^\dagger \\ d \end{bmatrix} = 0,\\
&\mathcal{C}_{o-\Lambda_{n'},\Lambda_n} = i\partial_{\Lambda_{n'}}\theta^\dagger_{o-n} - i\partial_{\Lambda_n}\theta^\dagger_{o-n'} - [\theta^\dagger_{o-n},\theta^\dagger_{o-n'}],\\
&C_{\Lambda_{n'},\Lambda^\star_n} = \tfrac{1}{2}[d,d^\dagger]\mathcal{C}_{o-\Lambda_{n'},\Lambda^\star_n}\begin{bmatrix} d^\dagger \\ d \end{bmatrix} = 0,\\
&\mathcal{C}_{o-\Lambda_{n'},\Lambda^\star_n} = i\partial_{\Lambda_{n'}}\theta_{o-n} - i\partial_{\Lambda^\star_n}\theta^\dagger_{o-n'} - [\theta_{o-n},\theta^\dagger_{o-n'}],\\
&C_{\Lambda^\star_{n'},\Lambda^\star_n} = \tfrac{1}{2}[d,d^\dagger]\mathcal{C}_{o-\Lambda^\star_{n'},\Lambda^\star_n}\begin{bmatrix} d^\dagger \\ d \end{bmatrix} = 0,\\
&\mathcal{C}_{o-\Lambda^\star_{n'},\Lambda^\star_n} = i\partial_{\Lambda^\star_{n'}}\theta_{o-n} - i\partial_{\Lambda^\star_n}\theta_{o-n'} - [\theta_{o-n},\theta_{o-n'}].
\end{aligned}
\tag{2.23}
$$

The quantities $\mathcal{F}_{o-\mathrm{c}}$, $\theta^\dagger_{o-n}$ $(= g^\dagger\theta^\dagger_n g)$ and θ_{o-n} $(= g^\dagger\theta_n g)$ are defined through partial differential equations on the $SO(2N)$ Lie group manifold,

$$
-i\hbar\partial_t g^\dagger = \mathcal{F}_{o-\mathrm{c}}g^\dagger \qquad \text{and} \qquad -i\partial_{\Lambda_n}g^\dagger = \theta^\dagger_{o-n}g^\dagger, \qquad -i\partial_{\Lambda^\star_n}g^\dagger = \theta_{o-n}g^\dagger. \tag{2.24}
$$

In the above set of (2.23), all the curvatures $\mathcal{C}_{o-\bullet,\bullet}$ should be made equal to zero.

The full TDHB Hamiltonian is decomposed into the collective one and the residual one as

$$
H_{\mathrm{HB}} = H_{\mathrm{c}} + H_{\mathrm{res}}, \qquad \mathcal{F}_o = \mathcal{F}_{o-\mathrm{c}} + \mathcal{F}_{o-\mathrm{res}}, \tag{2.25}
$$

at the reference point g on the $SO(2N)$ group manifold. Following the preceding section, let us introduce other curvatures C'_{t,Λ_n} and $C'_{t,\Lambda^\star_n}$ with the same forms as those in (2.23) except that $\mathcal{F}_{o-\mathrm{c}}$ is replaced by $\mathcal{F}_o$. Then the corresponding curvatures $\mathcal{C}'_{o-t,\Lambda_n}$ and $\mathcal{C}'_{o-t,\Lambda^\star_n}$ are also divided into two terms,

$$
\mathcal{C}'_{o-t,\Lambda_n} = \mathcal{C}^{\mathrm{c}}_{o-t,\Lambda_n} + \mathcal{C}^{\mathrm{res}}_{o-t,\Lambda_n}, \qquad \mathcal{C}'_{o-t,\Lambda^\star_n} = \mathcal{C}^{\mathrm{c}}_{o-t,\Lambda^\star_n} + \mathcal{C}^{\mathrm{res}}_{o-t,\Lambda^\star_n}.
$$

Here the collective curvatures $\mathcal{C}^{\mathrm{c}}_{o-t,\Lambda_n}$ and $\mathcal{C}^{\mathrm{c}}_{o-t,\Lambda^\star_n}$ arising from $\mathcal{F}_{o-\mathrm{c}}$ are defined as the same forms as the ones in (2.23). The residual curvatures $\mathcal{C}^{\mathrm{res}}_{o-t,\Lambda_n}$ and $\mathcal{C}^{\mathrm{res}}_{o-t,\Lambda^\star_n}$ arising from $\mathcal{F}_{o-\mathrm{res}}$ are defined as

$$
\begin{aligned}
&\mathcal{C}^{\mathrm{res}}_{o-t,\Lambda_n} = -i\partial_{\Lambda_n}\mathcal{F}_{o-\mathrm{res}}/\hbar - [\theta^\dagger_{o-n},\mathcal{F}_{o-\mathrm{res}}/\hbar],\\
&\mathcal{C}^{\mathrm{res}}_{o-t,\Lambda^\star_n} = -i\partial_{\Lambda^\star_n}\mathcal{F}_{o-\mathrm{res}}/\hbar - [\theta_{o-n},\mathcal{F}_{o-\mathrm{res}}/\hbar].
\end{aligned}
\tag{2.26}
$$

Using (2.24) and (2.25), the Lie-algebra-valued forms of the curvatures are calculated as

$$
C^{\mathrm{res}}_{t,\Lambda_n} = -i\partial_{\Lambda_n}H_{\mathrm{res}}/\hbar, \qquad C^{\mathrm{res}}_{t,\Lambda^\star_n} = -i\partial_{\Lambda^\star_n}H_{\mathrm{res}}/\hbar.
$$

Supposing there exist the well-defined collective submanifolds satisfying (2.24), we should demand that the following curvatures are made equal to zero:

$$
\mathcal{C}^{\mathrm{c}}_{o-t,\Lambda_n} = 0, \qquad \mathcal{C}^{\mathrm{c}}_{o-t,\Lambda^\star_n} = 0, \tag{2.27}
$$

$$
\mathcal{C}_{o-\Lambda_{n'},\Lambda_n} = 0, \qquad \mathcal{C}_{o-\Lambda_{n'},\Lambda^\star_n} = 0, \qquad \mathcal{C}_{o-\Lambda^\star_{n'},\Lambda^\star_n} = 0, \tag{2.28}
$$

the first equation (2.27) of which lead us to the Lie-algebra-valued relations,

$$C'_{t,\Lambda_n} = C^{\mathrm{res}}_{t,\Lambda_n} = -i\partial_{\Lambda_n} H_{\mathrm{res}}/\hbar, \qquad C'_{t,\Lambda^\star_n} = C^{\mathrm{res}}_{t,\Lambda^\star_n} = -i\partial_{\Lambda^\star_n} H_{\mathrm{res}}/\hbar. \tag{2.29}$$

Then the curvature C'_{t,Λ_n} and $C'_{t,\Lambda^\star_n}$ can be regarded as the *gradients* of quantum-mechanical potentials due to the existence of the residual Hamiltonian H_{res} on the collective submanifolds. The potentials become almost flat on the collective submanifolds, i.e., $H_{\mathrm{HB}} = H_{\mathrm{c}} + \mathrm{const}$, if the proper subspace determined is an almost invariant subspace of the full TDHB Hamiltonian. This collective subspace is an almost degenerate eigenspace of the residual Hamiltonian. Therefore it is naturally deduced that, provided there exists the well-defined collective subspace, the residual curvatures at a point on the subspace are extremely small. Thus, the way of extracting the collective submanifolds out of the full TDHB manifold is made possible by the minimization of the residual curvature, for which a deep insight into (2.29) becomes necessary.

Finally, the restrictions to assure the Lagrange bracket for the usual collective variables and Lagrange-like ones are transformed into the following forms represented in the QPF:

$$-\tfrac{1}{4}\mathrm{Tr}\left\{\begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix}[\theta_{o-n}, \theta^\dagger_{o-n'}]\right\} = \delta_{nn'},$$
$$\tfrac{1}{4}\mathrm{Tr}\left\{\begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix}[\theta^\dagger_{o-n}, \theta^\dagger_{o-n'}]\right\} = 0, \qquad \tfrac{1}{4}\mathrm{Tr}\left\{\begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix}[\theta_{o-n}, \theta_{o-n'}]\right\} = 0.$$

We discuss here how the Lagrange-like manner picture is transformed into the usual one. First let us regard any point on the collective submanifold as a set of initial points (initial value) in the usual manner. Suppose we observe the time evolution of the system with various initial values. Then we have the following relations which make a connection between the Lagrange-like manner and the usual one

$$\mathcal{F}_{o-\mathrm{c}}/\hbar = \partial_t \check{\Lambda}_n \check{\theta}^\dagger_{o-n} + \partial_t \check{\Lambda}^\star_n \check{\theta}_{o-n}, \tag{2.30}$$
$$\theta^\dagger_{o-n} = \partial_{\Lambda_n}\check{\Lambda}_{n'}\check{\theta}^\dagger_{o-n'} + \partial_{\Lambda_n}\check{\Lambda}^\star_{n'}\check{\theta}_{o-n'}, \qquad \theta_{o-n} = \partial_{\Lambda^\star_n}\check{\Lambda}_{n'}\check{\theta}^\dagger_{o-n'} + \partial_{\Lambda^\star_n}\check{\Lambda}^\star_{n'}\check{\theta}_{o-n'}, \tag{2.31}$$

in which the transformation functions are set up by the initial conditions,

$$\check{\Lambda}_n(t)|_{t=0} = \check{\Lambda}_n(\Lambda, \Lambda^\star, t)|_{t=0} = \Lambda_n, \qquad \check{\Lambda}^\star_n(t)|_{t=0} = \check{\Lambda}^\star_n(\Lambda, \Lambda^\star, t)|_{t=0} = \Lambda^\star_n,$$
$$\partial_{\Lambda_n}\check{\Lambda}_{n'}|_{t=0} = \delta_{nn'}, \qquad \partial_{\Lambda^\star_n}\check{\Lambda}^\star_{n'}|_{t=0} = \delta_{nn'}, \qquad \partial_{\Lambda_n}\check{\Lambda}^\star_{n'}|_{t=0} = 0, \qquad \partial_{\Lambda^\star_n}\check{\Lambda}_{n'}|_{t=0} = 0,$$

in order to guarantee both pictures to coincide at time $t = 0$. On the other hand, our collective Hamiltonian $\mathcal{F}_{o-\mathrm{c}}$ can also be expressed in the form

$$\mathcal{F}_{o-\mathrm{c}}/\hbar = v_n(\Lambda, \Lambda^\star, t)\theta^\dagger_{o-n} + v^\star_n(\Lambda, \Lambda^\star, t)\theta_{o-n}, \tag{2.32}$$

where the expansion coefficients v_n and $v^\star_n$ are interpreted as velocity fields in the Lagrange-like manner. Substituting (2.31) into (2.32) and comparing with (2.30), we can get the relations

$$\dot{\check{\Lambda}}_n = \partial_t \check{\Lambda}_n = v_{n'}\partial_{\Lambda_{n'}}\check{\Lambda}_n + v^\star_{n'}\partial_{\Lambda^\star_{n'}}\check{\Lambda}_n, \qquad \dot{\check{\Lambda}}^\star_n = \partial_t \check{\Lambda}^\star_n = v_{n'}\partial_{\Lambda_{n'}}\check{\Lambda}^\star_n + v^\star_{n'}\partial_{\Lambda^\star_{n'}}\check{\Lambda}^\star_n,$$

from which the initial conditions of the velocity fields are given as

$$\dot{\check{\Lambda}}_n(t)|_{t=0} = \partial_t \check{\Lambda}_n|_{t=0} = v_n(\Lambda, \Lambda^\star, t)|_{t=0}, \qquad \dot{\check{\Lambda}}^\star_n(t)|_{t=0} = \partial_t \check{\Lambda}^\star_n|_{t=0} = v^\star_n(\Lambda, \Lambda^\star, t)|_{t=0}.$$

Then we obtain the correspondence of the time derivatives of the collective co-ordinates in the usual manner to the velocity fields in the Lagrange-like one.

Finally we impose the canonicity conditions in the usual manner,

$$\langle\phi(\breve{g})|\breve{O}_n^\dagger|\phi(\breve{g})\rangle = i\tfrac{1}{2}\breve{\Lambda}_n^\star, \qquad \langle\phi(\breve{g})|\breve{O}_n|\phi(\breve{g})\rangle = -i\tfrac{1}{2}\breve{\Lambda}_n, \tag{2.33}$$

which leads us to the *weak* canonical commutation relation with the aid of (2.28) and (2.21).

The TDHBT for *maximally-decoupled* collective motions can be formulated parallel with TDHFT [25]. The basic concept of the theory lies in an introduction of the *invariance principle of the Schrödinger equation*, and the TDHBEQ is solved under the canonicity condition and the vanishing of non-collective dangerous terms. However, as we have no justification on the validity of the *maximally-decoupled* method, we must give a criterion how it extracts the collective submanifold effectively out of the full TDHB manifold. We are now in a position to derive some quantities by which the criterion is established. For this aim, we express the collective Hamiltonian $\mathcal{F}_{o-\mathrm{c}}$ and the residual one $\mathcal{F}_{o-\mathrm{res}}$ in the same form as the one of the TDHB Hamiltonian $\mathcal{F}_o$ given in (2.22). We also represent quantities $\theta_{o-n}^\dagger$, $\mathcal{C}_{o-t,\Lambda_n}^{\mathrm{res}}$ and $\mathcal{C}_{o-t,\Lambda_n^\star}^{\mathrm{res}}$ which consist of $N\times N$ block matrices as follows:

$$\theta_{o-n}^\dagger = \begin{bmatrix} \xi_o & \varphi_o \\ \psi_o & -\xi_o^{\mathrm{T}} \end{bmatrix}_n, \qquad \psi_o^{\mathrm{T}} = -\psi_o, \qquad \varphi_o^{\mathrm{T}} = -\varphi_o, \tag{2.34}$$

$$\begin{aligned} &\mathcal{C}_{o-t,\Lambda_n}^{\mathrm{res}} = \begin{bmatrix} \mathcal{C}_\xi^{\mathrm{res}} & \mathcal{C}_\varphi^{\mathrm{res}} \\ \mathcal{C}_\psi^{\mathrm{res}} & -\mathcal{C}_\xi^{\mathrm{resT}} \end{bmatrix}_n, \qquad \mathcal{C}_\psi^{\mathrm{resT}} = -\mathcal{C}_\psi^{\mathrm{res}}, \qquad \mathcal{C}_\varphi^{\mathrm{resT}} = -\mathcal{C}_\varphi^{\mathrm{res}}, \\ &\mathcal{C}_{o-t,\Lambda_n^\star}^{\mathrm{res}} = \begin{bmatrix} \mathcal{C}_{\xi^\star}^{\mathrm{res}} & \mathcal{C}_{\varphi^\star}^{\mathrm{res}} \\ \mathcal{C}_{\psi^\star}^{\mathrm{res}} & -\mathcal{C}_{\xi^\star}^{\mathrm{resT}} \end{bmatrix}_n, \qquad \mathcal{C}_{\xi^\star}^{\mathrm{res}} = -\mathcal{C}_\xi^{\mathrm{res}\dagger}, \qquad \mathcal{C}_{\psi^\star}^{\mathrm{res}} = \mathcal{C}_\psi^{\mathrm{res}\star}, \qquad \mathcal{C}_{\varphi^\star}^{\mathrm{res}} = \mathcal{C}_\varphi^{\mathrm{res}\star}. \end{aligned} \tag{2.35}$$

Substitution of the explicit form of $\mathcal{F}_{o-\mathrm{res}}$ and equations (2.34) and (2.35) into (2.26) yields

$$\begin{aligned} \mathcal{C}_{\xi,n}^{\mathrm{res}} &= i\partial_{\Lambda_n}F_{o-\mathrm{res}}^\star/\hbar + [\xi_{o-n}, F_{o-\mathrm{res}}^\star/\hbar] - \varphi_{o,n}\mathcal{D}_{o-\mathrm{res}}/\hbar - D_{o-\mathrm{res}}^\star/\hbar\psi_{o,n}, \\ \mathcal{C}_{\psi,n}^{\mathrm{res}} &= -i\partial_{\Lambda_n}D_{o-\mathrm{res}}/\hbar + \xi_{o,n}^{\mathrm{T}}D_{o-\mathrm{res}}/\hbar + D_{o-\mathrm{res}}/\hbar\xi_{o,n} + \psi_{o,n}F_{o-\mathrm{res}}^\star/\hbar + F_{o-\mathrm{res}}/\hbar\psi_{o,n}, \\ \mathcal{C}_{\varphi,n}^{\mathrm{res}} &= i\partial_{\Lambda_n}D_{o-\mathrm{res}}^\star/\hbar + \xi_{o,n}D_{o-\mathrm{res}}^\star/\hbar + D_{o-\mathrm{res}}^\star/\hbar\xi_{o,n}^{\mathrm{T}} - \varphi_{o,n}F_{o-\mathrm{res}}/\hbar - F_{o-\mathrm{res}}^\star/\hbar\varphi_{o,n}. \end{aligned} \tag{2.36}$$

The quantity $\breve{\theta}_{0-n}^\dagger$ can also be expressed in the same form as the one defined in (2.34). Substituting this expression into (2.30) and (2.33), we obtain the relations

$$\begin{aligned} F_{o-\mathrm{res}}/\hbar &= F_o/\hbar + \partial_t\breve{\Lambda}_n\breve{\xi}_{o,n}^{\mathrm{T}} + \partial_t\breve{\Lambda}_n^\star\breve{\xi}_{o,n}^\star, \\ D_{o-\mathrm{res}}/\hbar &= D_o/\hbar + \partial_t\breve{\Lambda}_n\breve{\psi}_{o,n}^{\mathrm{T}} + \partial_t\breve{\Lambda}_n^\star\breve{\varphi}_{o,n}^\star, \end{aligned} \tag{2.37}$$

together with their complex conjugate and

$$\mathrm{Tr}\,\breve{\xi}_{o,n} = i\breve{\Lambda}_n^\star, \qquad \mathrm{Tr}\,\breve{\xi}_{o,n}^\dagger = -i\breve{\Lambda}_n, \tag{2.38}$$

where we have used (2.25) and the explicit forms of the Hamiltonian.

As was mentioned, the way of extracting collective submanifolds out of the full TDHB manifold is made possible by minimization of the residual curvature. This is achieved if we require at least expectation values of the residual curvatures to be minimized as much as possible, i.e.,

$$\langle\phi(g)|C_{t,\Lambda_n}^{\mathrm{res}}|\phi(g)\rangle = \tfrac{1}{2}\mathrm{Tr}\,\mathcal{C}_{\xi,n}^{\mathrm{res}} \cong 0, \qquad \langle\phi(g)|C_{t,\Lambda_n^\star}^{\mathrm{res}}|\phi(g)\rangle = \tfrac{1}{2}\mathrm{Tr}\,\mathcal{C}_{\xi^\star,n}^{\mathrm{res}} \cong 0. \tag{2.39}$$

We here adopt a condition similar to one of the stationary HB method as was done in the TDHF [27]: The so-called dangerous terms in the residual Hamiltonian $\mathcal{F}_{o-\mathrm{res}}$ are made to vanish,

$$D_{o-\mathrm{res}} = 0, \qquad D_{o-\mathrm{res}}^\star = 0. \tag{2.40}$$

With aid of equations (2.36), (2.37) and (2.38), equations (2.40) and (2.39) are rewritten as

$$D_o/\hbar = -\partial_t \check{\Lambda}_n \psi^{\mathrm{T}}_{o,n} - \partial_t \check{\Lambda}^\star_n \varphi^\star_{o,n}, \qquad D^\star_o/\hbar = -\partial_t \check{\Lambda}_n \varphi_{o,n} - \partial_t \check{\Lambda}^\star_n \psi^\dagger_{o,n}, \tag{2.41}$$

$$\begin{aligned} &i\partial_{\Lambda_n} \mathrm{Tr}\, F_{o-\mathrm{res}}/\hbar = i\partial_{\Lambda_n} \mathrm{Tr}\, F_o/\hbar - \partial_{\Lambda_n}\left(\partial_t \check{\Lambda}_{n'} \check{\Lambda}^\star_{n'} - \partial_t \check{\Lambda}^\star_{n'} \check{\Lambda}_{n'}\right) \cong 0, \\ &i\partial_{\Lambda^\star_n} \mathrm{Tr}\, F_{o-\mathrm{res}}/\hbar = i\partial_{\Lambda^\star_n} \mathrm{Tr}\, F_o/\hbar - \partial_{\Lambda^\star_n}\left(\partial_t \check{\Lambda}_{n'} \check{\Lambda}^\star_{n'} - \partial_t \check{\Lambda}^\star_{n'} \check{\Lambda}_{n'}\right) \cong 0. \end{aligned} \tag{2.42}$$

First, we will discuss how equation (2.41) leads us to the equation of path for the collective motion. Notice that the quantities $\check{\theta}^\dagger_{o-n}$ and $\check{\theta}_{o-n}$ are subjected to satisfy the same type of partial differential equation as that of (2.24). Remember the explicit representation of an $SO(2N)$ matrix g given in the previous section. Then we have partial differential equations

$$\check{\psi}_{0,n} = -i(\partial_{\check{\Lambda}_n} \check{b}^{\mathrm{T}} \check{a} + \partial_{\check{\Lambda}_n} \check{a}^{\mathrm{T}} \check{b}), \qquad \check{\varphi}_{0,n} = -i(\partial_{\check{\Lambda}_n} \check{a}^\dagger \check{b}^\star + \partial_{\check{\Lambda}_n} \check{b}^\dagger \check{a}^\star). \tag{2.43}$$

together with its complex conjugate. Putting the relation $\mathcal{F}_o = g^\dagger \mathcal{F} g$ and (2.43) into (2.41), we get

$$\begin{aligned} &\check{a}^{\mathrm{T}}\big\{(\check{D}\check{a} + \check{F}\check{b})/\hbar - (i\dot{\check{\Lambda}}_n \partial_{\check{\Lambda}_n} \check{b} + i\dot{\check{\Lambda}}^\star_n \partial_{\check{\Lambda}^\star_n} \check{b})\big\} \\ &\quad + \check{b}^{\mathrm{T}}\big\{-(\check{F}^\star \check{a} + \check{D}^\star \check{b})/\hbar - (i\dot{\check{\Lambda}}_n \partial_{\check{\Lambda}_n} \check{a} + i\dot{\check{\Lambda}}^\star_n \partial_{\check{\Lambda}^\star_n} \check{a})\big\} = 0, \end{aligned} \tag{2.44}$$

Let H be an exact Hamiltonian of the system with certain two-body interaction and let us denote the expectation value of H by $|\phi(g)\rangle$ as $\langle H\rangle_g$. It can be easily proved that the relations

$$\partial_{a^\star}\langle H\rangle_g = -\tfrac{1}{2}(F^\star a + D^\star b), \qquad \partial_{b^\star}\langle H\rangle_g = \tfrac{1}{2}(Fb + Da), \tag{2.45}$$

and their complex conjugate relations do hold, through which the well-known TDHBEQ is converted into a matrix form as

$$i\dot{g}/\sqrt{2} = \begin{bmatrix} \partial_{a^\star/\sqrt{2}}, & -\partial_{b/\sqrt{2}} \\ \partial_{b^\star/\sqrt{2}}, & -\partial_{a/\sqrt{2}} \end{bmatrix} \langle H\rangle_{g/\sqrt{2}}/\hbar. \tag{2.46}$$

The quantity $\langle H\rangle_{g/\sqrt{2}}$ means now in turn an expectation value of H, being a function of $a/\sqrt{2}$, $b/\sqrt{2}$ and their complex conjugate. With the aid of a relation similar to (2.45), equation (2.44) is reduced to

$$\begin{aligned} &\check{a}^{\mathrm{T}}\left\{\partial_{\check{b}^\star/\sqrt{2}}\langle H\rangle_{\check{g}/\sqrt{2}}/\hbar - \left(i\dot{\check{\Lambda}}_n \partial_{\check{\Lambda}_n} \frac{\check{b}}{\sqrt{2}} + i\dot{\check{\Lambda}}^\star_n \partial_{\check{\Lambda}^\star_n} \frac{\check{b}}{\sqrt{2}}\right)\right\} \\ &\quad + \check{b}^{\mathrm{T}}\left\{\partial_{a^\star/\sqrt{2}}\langle H\rangle_{\check{g}/\sqrt{2}}/\hbar - \left(i\dot{\check{\Lambda}}_n \partial_{\check{\Lambda}_n} \frac{\check{a}}{\sqrt{2}} + i\dot{\check{\Lambda}}^\star_n \partial_{\check{\Lambda}^\star_n} \frac{\check{a}}{\sqrt{2}}\right)\right\} = 0, \end{aligned} \tag{2.47}$$

As one way of satisfying (2.47), we may adopt the following type of partial differential equations:

$$\begin{aligned} &\partial_{\check{a}^\star/\sqrt{2}}\langle H\rangle_{\check{g}/\sqrt{2}}/\hbar - \left(i\dot{\check{\Lambda}}_n \partial_{\check{\Lambda}_n} \check{a}/\sqrt{2} + i\dot{\check{\Lambda}}^\star_n \partial_{\check{\Lambda}^\star_n} \check{a}/\sqrt{2}\right) = 0, \\ &\partial_{\check{b}^\star/\sqrt{2}}\langle H\rangle_{\check{g}/\sqrt{2}}/\hbar - \left(i\dot{\check{\Lambda}}_n \partial_{\check{\Lambda}_n} \check{b}/\sqrt{2} + i\dot{\check{\Lambda}}^\star_n \partial_{\check{\Lambda}^\star_n} \check{b}/\sqrt{2}\right) = 0. \end{aligned} \tag{2.48}$$

Here we notice the *invariance principle of the Schrödinger equation* and the canonicity condition which leads us necessarily to the equation of collective motion expressed in the canonical forms

$$i\dot{\check{\Lambda}}^\star_n = -\partial_{\check{\Lambda}_n}\langle H\rangle_{\check{g}/\sqrt{2}}/\hbar, \qquad i\dot{\check{\Lambda}}_n = \partial_{\check{\Lambda}^\star_n}\langle H\rangle_{\check{g}/\sqrt{2}}/\hbar, \tag{2.49}$$

which can be easily derived with the use of (2.46) and (2.38). Instead of solving approximately our nonlinear time evolution equation (2.27), we adopt the above canonical equation. Then, as is clear from the structure of (2.48), it is self-evident that equation (2.48) becomes the equation of path for the collective motion under substitution of (2.49). In this sense, equation (2.48) is the natural extension of the equation of path in the TDHF case [25, 27] to the one in the TDHB case. The set of (2.48) and (2.49) is expected to determine the behaviour of the *maximally decoupled* collective motions in the TDHB case. However, it means nothing else than the rewriting of the TDHBEQ with the use of canonicity condition, if we are able to assume only the existence of invariant subspace in the full TDHB solution space. The above interpretation is due to the natural consequence of the *maximally decoupled* theory because there exists, as a matter of case, the invariant subspace, if the *invariance principle of the Schrödinger equation* does hold true. The *maximally decoupled* equation can be solved with the additional RPA boundary condition, though its solution is, strictly speaking, different from the true motion of the system on the full $SO(2N)$ group manifold. But how can we convince that the solution describes the well-defined *maximally decoupled* collective motions from the other remaining degrees of freedom of motion? Therefore, in order to answer such a question, we must establish a criterion how we extract the collective submanifolds effectively out of the full TDHB manifold.

Up to the present stage, equation (2.42) remains unused yet and makes no role for approaching to our aim. Finally with the aid of (2.42), we will derive some quantity by which the range of the validity of the *maximally decoupled* theory can be evaluated. As was mentioned previously, we demanded that the expectation values of the residual curvatures are minimized as far as possible and adopted the canonical equation in place of our fundamental equation (2.27). Then, by combining both the above propositions, it may be expected that we can reach our final goal of the present task. Further substitution of the equation of motion (2.49) (rewrite $\langle H\rangle_{\check{g}/\sqrt{2}}$ in the original form $\langle H\rangle_{\check{g}}$ again) into (2.42) yields

$$
\begin{aligned}
&\partial_{\Lambda_n}\check{\Lambda}_{n'}\mathrm{Tr}\Bigg[\big(\partial_{\check{\Lambda}_{n'}}\check{\mathcal{R}}-\check{\Lambda}^\star_{n''}\partial^2_{\check{\Lambda}^\star_{n''},\check{\Lambda}_{n'}}\check{\mathcal{R}}-\check{\Lambda}_{n''}\partial^2_{\check{\Lambda}_{n''},\check{\Lambda}_{n'}}\check{\mathcal{R}}\big)\frac{\check{\mathcal{F}}}{\hbar}\\
&\qquad+\big(2\check{\mathcal{R}}-\check{\Lambda}^\star_{n''}\partial_{\check{\Lambda}^\star_{n''}}\check{\mathcal{R}}-\check{\Lambda}_{n''}\partial_{\check{\Lambda}_{n''}}\check{\mathcal{R}}\big)\partial_{\check{\Lambda}_{n'}}\frac{\check{\mathcal{F}}}{\hbar}\Bigg]\\
&\qquad+\partial_{\Lambda_n}\check{\Lambda}^\star_{n'}\mathrm{Tr}\Bigg[\big(\partial_{\check{\Lambda}^\star_{n'}}\check{\mathcal{R}}-\check{\Lambda}^\star_{n''}\partial^2_{\check{\Lambda}^\star_{n''},\check{\Lambda}^\star_{n'}}\check{\mathcal{R}}-\check{\Lambda}_{n''}\partial^2_{\check{\Lambda}_{n''},\check{\Lambda}^\star_{n'}}\check{\mathcal{R}}\big)\frac{\check{\mathcal{F}}}{\hbar}\\
&\qquad+\big(2\check{\mathcal{R}}-\check{\Lambda}^\star_{n''}\partial_{\check{\Lambda}^\star_{n''}}\check{\mathcal{R}}-\check{\Lambda}_{n''}\partial_{\check{\Lambda}_{n''}}\check{\mathcal{R}}\big)\partial_{\check{\Lambda}^\star_{n'}}\frac{\check{\mathcal{F}}}{\hbar}\Bigg]\cong 0,
\end{aligned}
\tag{2.50}
$$

Here we have used the transformation property of the differential $\partial_{\Lambda_n}=\partial_{\Lambda_n}\check{\Lambda}_{n'}\partial_{\check{\Lambda}_{n'}}+\partial_{\Lambda_n}\check{\Lambda}^\star_{n'}\partial_{\check{\Lambda}^\star_{n'}}$ and the differential formulae for the expectation values of the Hamiltonians H and H_{HB}

$$
\begin{aligned}
&\partial_{\check{\Lambda}_n}\langle H\rangle_{\check{g}}=-\tfrac{1}{4}\mathrm{Tr}\left[\partial_{\check{\Lambda}_n}\check{\mathcal{R}}(\check{g})\check{\mathcal{F}}\right],\\
&\partial_{\check{\Lambda}_n}\langle H_{\mathrm{HB}}\rangle_{\check{g}}=-\tfrac{1}{2}\partial_{\check{\Lambda}_n}\mathrm{Tr}\check{\mathcal{F}}_o=\partial_{\check{\Lambda}_n}\langle H\rangle_{\check{g}}-\tfrac{1}{4}\mathrm{Tr}\left[\check{\mathcal{R}}(\check{g})\partial_{\check{\Lambda}_n}\check{\mathcal{F}}\right].
\end{aligned}
$$

2.4 Nonlinear RPA theory arising from zero-curvature equation

Our fundamental equation may work well *in the large scale* beyond the RPA as the small-amplitude limit. A linearly approximate solution of the TDHBEQ becomes the RPAEQ. Suppose we solve the fundamental equation by expanding it in the form of a power series of the collective variables Λ and $\Lambda^\star(n=1,\ldots,m;\ m\ll N(2N-1)/2)$ defined in the Lagrange-like manner. Then we must show that the fundamental equation has necessarily the RPA solution at the lowest power of the collective variables which approach in the small amplitude limit. A paired mode amplitude $g(\Lambda,\Lambda^\star,t)$ is separated into stationary and fluctuating components as $g=g^{(o)}\tilde{g}$. This means that the $SO(2N)$ matrix g is decomposed into a product of stationary matrix $g^{(o)}$

and $\tilde{g}(\Lambda,\Lambda^\star,t)$ ($\simeq \tilde{g}$). The stationary $g^{(o)}$ satisfies the usual static $SO(2N)$(HB) eigenvalue equation.

Using the above decomposition of g, an original $SO(2N)$(HB) density matrix $\mathcal{R}(\Lambda,\Lambda^\star,t)$ and a HB matrix $\mathcal{F}(\Lambda,\Lambda^\star,t)$ are decomposed as $\mathcal{R}=g^{(o)}\widetilde{\mathcal{R}}g^{(o)\dagger}$ and $\mathcal{F}=g^{(o)}\widetilde{\mathcal{F}}g^{(o)\dagger}$, respectively. The fluctuating $\widetilde{\mathcal{R}}$ and the HB matrix $\widetilde{\mathcal{F}}$ in fluctuating QPF are given in the following forms:

$$\widetilde{\mathcal{R}}(\tilde{g})=g^{(o)\dagger}\mathcal{R}(g)g^{(o)},\qquad \widetilde{\mathcal{R}}(\tilde{g})=\begin{bmatrix} 2\widetilde{R}(\tilde{g})-1_N & -2\widetilde{K}^\star(\tilde{g}) \\ 2\widetilde{K}(\tilde{g}) & -2\widetilde{R}^\star(\tilde{g})+1_N \end{bmatrix}, \tag{2.51}$$

$$\widetilde{\mathcal{F}}=g^{(o)\dagger}\mathcal{F}g^{(o)},\qquad \widetilde{\mathcal{F}}=\begin{bmatrix} -\epsilon^{(o)}-f^\star & -d^\star \\ d & \epsilon^{(o)}+f \end{bmatrix}, \tag{2.52}$$

in which all the quantities are redefined in [14]. Quasi-particle energies $\varepsilon_i^{(o)}$ include a chemical potential.

Introducing fluctuating auxiliary quantities $\tilde{\theta}_n=g^{(o)\dagger}\theta_n g^{(o)}$ and $\tilde{\theta}_n^\dagger=g^{(o)\dagger}\theta_n^\dagger g^{(o)}$, then under the decomposition $g=g^{(o)}\tilde{g}$, the zero-curvature equation $C_{\bullet,\bullet}=0$ in (2.9) is transformed to

$$i\partial_t\tilde{\theta}_n^\dagger - i\partial_{\Lambda_n}\widetilde{\mathcal{F}}_{\mathrm{c}}/\hbar + [\tilde{\theta}_n^\dagger,\widetilde{\mathcal{F}}_{\mathrm{c}}/\hbar]=0,\qquad i\partial_t\tilde{\theta}_n - i\partial_{\Lambda_n^\star}\widetilde{\mathcal{F}}_{\mathrm{c}}/\hbar + [\tilde{\theta}_n,\widetilde{\mathcal{F}}_{\mathrm{c}}/\hbar]=0, \tag{2.53}$$

$$\begin{aligned} &i\partial_{\Lambda_{n'}}\tilde{\theta}_n - i\partial_{\Lambda_n^\star}\tilde{\theta}_{n'}^\dagger + [\tilde{\theta}_n,\tilde{\theta}_{n'}^\dagger]=0,\\ &i\partial_{\Lambda_{n'}}\tilde{\theta}_n^\dagger - i\partial_{\Lambda_n}\tilde{\theta}_{n'}^\dagger + [\tilde{\theta}_n^\dagger,\tilde{\theta}_{n'}^\dagger]=0,\qquad i\partial_{\Lambda_{n'}^\star}\tilde{\theta}_n - i\partial_{\Lambda_n^\star}\tilde{\theta}_{n'} + [\tilde{\theta}_n,\tilde{\theta}_{n'}]=0, \end{aligned} \tag{2.54}$$

$$-\tfrac{1}{4}\mathrm{Tr}\{\widetilde{\mathcal{R}}(\tilde{g})[\tilde{\theta}_n,\tilde{\theta}_{n'}^\dagger]\}=\delta_{nn'},\qquad \tfrac{1}{4}\mathrm{Tr}\{\widetilde{\mathcal{R}}(\tilde{g})[\tilde{\theta}_n^\dagger,\tilde{\theta}_{n'}^\dagger]\}=0,\qquad \tfrac{1}{4}\mathrm{Tr}\{\widetilde{\mathcal{R}}(\tilde{g})[\tilde{\theta}_n,\tilde{\theta}_{n'}]\}=0, \tag{2.55}$$

where the quantities $\widetilde{\mathcal{F}}_{\mathrm{c}}$, $\tilde{\theta}_n^\dagger$ and $\tilde{\theta}_n$ satisfy partial differential equations,

$$i\hbar\partial_t\tilde{g}=\widetilde{\mathcal{F}}_{\mathrm{c}}\tilde{g},\qquad i\partial_{\Lambda_n}\tilde{g}=\tilde{\theta}_n^\dagger\tilde{g}\qquad\text{and}\qquad i\partial_{\Lambda_n^\star}\tilde{g}=\tilde{\theta}_n\tilde{g}. \tag{2.56}$$

Putting $\widetilde{\mathcal{F}}_{\mathrm{c}}=\widetilde{\mathcal{F}}$ (2.52) in (2.53), we are able to look for a collective path ($\tilde{g}$) and a collective Hamiltonian ($\widetilde{\mathcal{F}}_{\mathrm{c}}$) under the minimization of the residual curvature arising from a residual Hamiltonian ($\widetilde{\mathcal{F}}_{\mathrm{res}}$). Next, for convenience of further discussion, we introduce modified fluctuating auxiliary quantities $\tilde{\theta}_{o-n}^\dagger=\tilde{g}^\dagger\tilde{\theta}_n^\dagger\tilde{g}$ and $\tilde{\theta}_{o-n}=\tilde{g}^\dagger\tilde{\theta}_n\tilde{g}$. Then we can rewrite our fundamental equations (2.53), (2.54) and (2.55) in terms of the above quantities as follows:

$$i\partial_t\tilde{\theta}_{o-n}^\dagger - i\tilde{g}^\dagger\left(\partial_{\Lambda_n}\widetilde{\mathcal{F}}_{\mathrm{c}}/\hbar\right)\tilde{g}=0,\qquad i\partial_t\tilde{\theta}_{o-n} - i\tilde{g}^\dagger\left(\partial_{\Lambda_n^\star}\widetilde{\mathcal{F}}_{\mathrm{c}}/\hbar\right)\tilde{g}=0, \tag{2.57}$$

$$\begin{aligned} &i\partial_{\Lambda_{n'}}\tilde{\theta}_{o-n} - i\partial_{\Lambda_n^\star}\tilde{\theta}_{o-n'}^\dagger - [\tilde{\theta}_{o-n},\tilde{\theta}_{o-n'}^\dagger]=0,\qquad i\partial_{\Lambda_{n'}}\tilde{\theta}_{o-n}^\dagger - i\partial_{\Lambda_n}\tilde{\theta}_{o-n'}^\dagger - [\tilde{\theta}_{o-n}^\dagger,\tilde{\theta}_{o-n'}^\dagger]=0,\\ &i\partial_{\Lambda_{n'}^\star}\tilde{\theta}_{o-n} - i\partial_{\Lambda_n^\star}\tilde{\theta}_{o-n'} - [\tilde{\theta}_{o-n},\tilde{\theta}_{o-n'}]=0, \end{aligned} \tag{2.58}$$

$$\begin{aligned} &-\tfrac{1}{4}\mathrm{Tr}\left\{\begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix}[\tilde{\theta}_{o-n},\tilde{\theta}_{o-n'}^\dagger]\right\}=\delta_{nn'},\\ &\tfrac{1}{4}\mathrm{Tr}\left\{\begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix}[\tilde{\theta}_{o-n}^\dagger,\tilde{\theta}_{o-n'}^\dagger]\right\}=0,\qquad \tfrac{1}{4}\mathrm{Tr}\left\{\begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix}[\tilde{\theta}_{o-n},\tilde{\theta}_{o-n'}]\right\}=0. \end{aligned} \tag{2.59}$$

In the derivation of equations (2.57) and (2.58), we have used (2.56). The equation (2.59) is easily obtained with the aid of another expression for the fluctuating density matrix $\widetilde{\mathcal{R}}$ (2.51),

$$\widetilde{\mathcal{R}}(\tilde{g})=\tilde{g}\begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix}\tilde{g}^\dagger. \tag{2.60}$$

In order to investigate the set of the matrix-valued nonlinear time evolution equation (2.57) arising from the zero curvature equation, we give here the ∂_{Λ_n} and $\partial_{\Lambda_n^\star}$ differential forms of the TDHB density matrix and collective hamiltonian. First, using (2.60), we have

$$\partial_{\Lambda_n}\widetilde{\mathcal{R}}(\tilde{g})=\partial_{\Lambda_n}\tilde{g}\begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix}\tilde{g}^\dagger+\tilde{g}\begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix}\partial_{\Lambda_n}\tilde{g}^\dagger$$

$$= \partial_{\Lambda_n}\tilde{g}(\tilde{g}^\dagger\tilde{g})\begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix}\tilde{g}^\dagger + \tilde{g}\begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix}\partial_{\Lambda_n}\tilde{g}^\dagger(\tilde{g}\tilde{g}^\dagger), \tag{2.61}$$

where we have used the relation $\tilde{g}^\dagger\tilde{g} = \tilde{g}\tilde{g}^\dagger = 1$. Using (2.56), the above equation is written as

$$\begin{aligned} \partial_{\Lambda_n}\widetilde{\mathcal{R}}(\tilde{g}) &= -i\tilde{\theta}_n^\dagger\partial_{\Lambda_n}\tilde{g}\begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix}\tilde{g}^\dagger + i\tilde{g}\begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix}\tilde{g}^\dagger\tilde{\theta}_n^\dagger \\ &= -i(\tilde{g}\tilde{g}^\dagger)\tilde{\theta}_n^\dagger\tilde{g}\begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix}\tilde{g}^\dagger + i\tilde{g}\begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix}\tilde{g}^\dagger\tilde{\theta}_n^\dagger(\tilde{g}\tilde{g}^\dagger). \end{aligned}$$

Next, by using $\tilde{\theta}_{o-n}^\dagger = \tilde{g}^\dagger\tilde{\theta}_n^\dagger\tilde{g}$ and $\tilde{\theta}_{o-n} = \tilde{g}^\dagger\tilde{\theta}_n\tilde{g}$, equation (2.61) is transformed into

$$\partial_{\Lambda_n}\widetilde{\mathcal{R}}(\tilde{g}) = -i\tilde{g}\left[\tilde{\theta}_{o-n}^\dagger, \begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix}\right]\tilde{g}^\dagger. \tag{2.62}$$

On the other hand, from (2.51), we easily obtain

$$\partial_{\Lambda_n}\widetilde{\mathcal{R}}(\tilde{g}) = \begin{bmatrix} 2\partial_{\Lambda_n}\widetilde{R}(\tilde{g}) & -2\partial_{\Lambda_n}\widetilde{K}^\star(\tilde{g}) \\ 2\partial_{\Lambda_n}\widetilde{K}(\tilde{g}) & -2\partial_{\Lambda_n}\widetilde{R}^\star(\tilde{g}) \end{bmatrix}. \tag{2.63}$$

Let us substitute explicit representations for $\tilde{g}$ $\left(= \begin{bmatrix} \tilde{a} & \tilde{b}^\star \\ \tilde{b} & \tilde{a}^\star \end{bmatrix}\right)$ and $\tilde{\theta}_{o-n}^\dagger$ $\left(= \begin{bmatrix} \xi_{o-n} & \varphi_{o-n} \\ \psi_{o-n} & -\xi_{o-n}^{\mathrm{T}} \end{bmatrix}\right)$, into the r.h.s. of (2.62) and combine it with (2.63). Then, we obtain the final ∂_{Λ_n} differential form of the TDHB ($SO(2N)$) density matrix as follows:

$$\begin{aligned} &\partial_{\Lambda_n}\widetilde{R}(\tilde{g}) = i\big(\tilde{b}^\star\psi_{o-n}\tilde{a}^\dagger - \tilde{a}\varphi_{o-n}\tilde{b}^{\mathrm{T}}\big), \qquad &&\partial_{\Lambda_n}\widetilde{R}^\star(\tilde{g}) = -i\big(\tilde{a}^\star\psi_{o-n}\tilde{b}^\dagger - \tilde{b}\varphi_{o-n}\tilde{a}^{\mathrm{T}}\big), \\ &\partial_{\Lambda_n}\widetilde{K}(\tilde{g}) = i\big(\tilde{a}^\star\psi_{o-n}\tilde{a}^\dagger - \tilde{b}\varphi_{o-n}\tilde{b}^{\mathrm{T}}\big), \qquad &&\partial_{\Lambda_n}\widetilde{K}^\star(\tilde{g}) = -i\big(\tilde{b}^\star\psi_{o-n}\tilde{b}^\dagger - \tilde{a}\varphi_{o-n}\tilde{a}^{\mathrm{T}}\big). \end{aligned}$$

The $\partial_{\Lambda_n^\star}$ differentiation of the $SO(2N)$ density matrix is also made analogously to the above.

As shown in [14], the fluctuating components of the HB matrix $\widetilde{\mathcal{F}}$ (2.52) are linear functionals of $\widetilde{R}(\tilde{g})$ and $\widetilde{K}(\tilde{g})$. We can easily calculate the ∂_{Λ_n} differential as follows:

$$\begin{aligned} \partial_{\Lambda_n}d &= (\mathbf{D})\partial_{\Lambda_n}\widetilde{K}(\tilde{g}) + (\overline{\mathbf{D}})\partial_{\Lambda_n}\widetilde{K}^\star(\tilde{g}) + (\mathbf{d})\partial_{\Lambda_n}\widetilde{R}(\tilde{g}) = i(\mathbf{D}\}\psi_{o-n} + i(\overline{\mathbf{D}}\}\varphi_{o-n}, \\ \partial_{\Lambda_n}f^\star &= (\mathbf{F})\partial_{\Lambda_n}\widetilde{K}(\tilde{g}) + (\overline{\mathbf{F}}^\star)\partial_{\Lambda_n}\widetilde{K}^\star(\tilde{g}) + (\mathbf{f}^\star)\partial_{\Lambda_n}\widetilde{R}(\tilde{g}) = i(\mathbf{F}^\star\}\psi_{o-n} + i(\overline{\mathbf{F}}^\star\}\varphi_{o-n}. \end{aligned}$$

Here the matrices $(\mathbf{D})$ etc. are given in [14]. Similarly, new matrices $(\mathbf{D}\}$ etc. are defined by

$$\begin{aligned} &(\mathbf{D}\} = \|(ij|D|kl\}\|, \qquad (\overline{\mathbf{D}}\} = \|(ij|\overline{D}|kl\}\| \qquad \text{and} \\ &(\mathbf{F}^\star\} = \|(ij|F^\star|kl\}\|, \qquad (\overline{\mathbf{F}}^\star\} = \|(ij|\overline{F}^\star|kl\}\|, \\ &(ij|D|kl\} = (ij|D|k'l')\tilde{a}_{k'k}^\star\tilde{a}_{l'l}^\star - (ij|\overline{D}|k'l')\tilde{b}_{k'k}^\star\tilde{b}_{l'l}^\star + (ij|d|k'l')\tilde{b}_{k'k}^\star\tilde{a}_{l'l}^\star, \\ &-(ij|\overline{D}|kl\} = (ij|D|k'l')\tilde{b}_{k'k}\tilde{b}_{l'l} - (ij|\overline{D}|k'l')\tilde{a}_{k'k}\tilde{a}_{l'l} + (ij|d|k'l')\tilde{a}_{k'k}\tilde{b}_{l'l}, \\ &(ij|F^\star|kl\} = (ij|F^\star|k'l')\tilde{a}_{k'k}^\star\tilde{a}_{l'l}^\star - (ij|\overline{F}^\star|k'l')\tilde{b}_{k'k}^\star\tilde{b}_{l'l}^\star + (ij|f^\star|k'l')\tilde{b}_{k'k}^\star\tilde{a}_{l'l}^\star, \\ &-(ij|\overline{F}^\star|kl\} = (ij|F^\star|k'l')\tilde{b}_{k'k}\tilde{b}_{l'l} - (ij|\overline{F}^\star|k'l')\tilde{a}_{k'k}\tilde{a}_{l'l} + (ij|f^\star|k'l')\tilde{a}_{k'k}\tilde{b}_{l'l}. \end{aligned} \tag{2.64}$$

The above summation is made with indices k' and l' ($1 \sim N$). Putting $\widetilde{\mathcal{F}}_{\mathrm{c}} = \widetilde{\mathcal{F}}$, we have

$$\partial_{\Lambda_n}\widetilde{\mathcal{F}}_{\mathrm{c}} = i\begin{bmatrix} -(\mathbf{F}^\star\}\psi_{o-n} - (\overline{\mathbf{F}}^\star\}\varphi_{o-n}, & -(\overline{\mathbf{D}}\}^\star\psi_{o-n} - (\mathbf{D}\}^\star\varphi_{o-n} \\ (\mathbf{D}\}\psi_{o-n} + (\overline{\mathbf{D}}\}\varphi_{o-n}, & {}^{\mathrm{T}}(\mathbf{F}^\star\}\psi_{o-n} + {}^{\mathrm{T}}(\overline{\mathbf{F}}^\star\}\varphi_{o-n} \end{bmatrix}, \tag{2.65}$$

where ${}^{\mathrm{T}}(\mathbf{F}^\star\}$ etc. stand for the matrices in which the indices i and j in (2.64) are exchanged.

We here derive a new equation formally analogous to the $SO(2N)$ RPA equation. To achieve this, we first further decompose the fluctuating pair mode amplitude $\tilde{g}$ into a product of a fluctuating $SO(2N)$ matrix and a $2N$-dimensional diagonal matrix with an exponential time dependence as follows:

$$\tilde{g} \rightarrow \tilde{g}\tilde{g}(\varepsilon, -\varepsilon), \qquad \tilde{g}(\varepsilon, -\varepsilon) = \begin{bmatrix} \exp[i\varepsilon t/\hbar], & 0 \\ 0, & \exp[-i\varepsilon t/\hbar] \end{bmatrix},$$
$$\varepsilon = (\delta_{ij}\varepsilon_i), \qquad \varepsilon_i = \varepsilon_i(\Lambda, \Lambda^\star), \tag{2.66}$$

where we redenote a new fluctuating pair mode as $\tilde{g}$ and ε_i is the Λ and $\Lambda^\star$ dependent quasi-particle energy including the chemical potential. Next, using (2.56), $\tilde{\theta}^\dagger_{o-n} = \tilde{g}^\dagger \tilde{\theta}^\dagger_n \tilde{g}$ and $\tilde{\theta}_{o-n} = \tilde{g}^\dagger \tilde{\theta}_n \tilde{g}$, the modified fluctuating auxiliary quantities $\tilde{\theta}^\dagger_{o-n}$ can be written as

$$\tilde{\theta}^\dagger_{o-n} \rightarrow \tilde{g}^\dagger(\varepsilon, -\varepsilon)\tilde{\theta}^\dagger_{o-n}\tilde{g}(\varepsilon, -\varepsilon) + \begin{bmatrix} -\partial_{\Lambda_n}\varepsilon t/\hbar, & 0 \\ 0, & \partial_{\Lambda_n}\varepsilon t/\hbar \end{bmatrix},$$

where we again redenote the new fluctuating auxiliary quantities as $\tilde{\theta}^\dagger_{o-n}$. Accompanying the above change, $\partial_t\tilde{\theta}^\dagger_{o-n}$ are modified to the following forms by using the explicit expression for $\tilde{\theta}^\dagger_{o-n}$:

$$\begin{aligned} \partial_t\tilde{\theta}^\dagger_{o-n} \rightarrow\ & \tilde{g}^\dagger(\varepsilon, -\varepsilon) \\ & \times \begin{bmatrix} \partial_t\xi_{o-n} - \partial_{\Lambda_n}\varepsilon/\hbar - i[\varepsilon/\hbar, \xi_{o-n}], & \partial_t\varphi_{o-n} - i[\varepsilon/\hbar, \varphi_{o-n}]_+ \\ \partial_t\psi_{o-n} + i[\varepsilon/\hbar, \psi_{o-n}]_+, & -\partial_t\xi^{\mathrm{T}}_{o-n} + \partial_{\Lambda_n}\varepsilon/\hbar - i[\varepsilon/\hbar, \xi^{\mathrm{T}}_{o-n}] \end{bmatrix} \tilde{g}(\varepsilon, -\varepsilon). \end{aligned} \tag{2.67}$$

In the above, hereafter we adopt $(\Lambda,\ \Lambda^\star)$-independent $\varepsilon^{(o)}$ given in (2.52) as the quasi-particle energy ε. If we substitute equations (2.65), (2.66) and (2.67) into the set of the matrix-valued nonlinear time evolution equation, i.e., the equation of (2.57), we finally obtain the following set of matrix-valued equations:

$$\begin{aligned} \tilde{g}^\dagger\big(\varepsilon^{(o)}, -\varepsilon^{(o)}\big) & \begin{bmatrix} \begin{array}{l} i\hbar\partial_t\xi_{o-n} + [\varepsilon^{(o)}, \xi_{o-n}] \\ \quad - \{\mathbf{F}^\star\}\psi_{o-n} - \{\overline{\mathbf{F}}^\star\}\varphi_{o-n} \end{array} & \begin{array}{l} i\hbar\partial_t\varphi_{o-n} + [\varepsilon^{(o)}, \varphi_{o-n}]_+ \\ \quad - \{\overline{\mathbf{D}}\}^\star\psi_{o-n} - \{\mathbf{D}\}^\star\varphi_{o-n} \end{array} \\ \begin{array}{l} i\hbar\partial_t\psi_{o-n} - [\varepsilon^{(o)}, \psi_{o-n}]_+ \\ \quad + \{\mathbf{D}\}\psi_{o-n} + \{\overline{\mathbf{D}}\}\varphi_{o-n} \end{array} & \begin{array}{l} -i\partial_t\xi^{\mathrm{T}}_{o-n} + [\varepsilon^{(o)}, \xi^{\mathrm{T}}_{o-n}] \\ \quad + {}^{\mathrm{T}}\{\mathbf{F}^\star\}\psi_{o-n} + {}^{\mathrm{T}}\{\overline{\mathbf{F}}^\star\}\varphi_{o-n} \end{array} \end{bmatrix} \\ & \times \tilde{g}\big(\varepsilon^{(0)}, -\varepsilon^{(0)}\big) = 0, \end{aligned} \tag{2.68}$$

with the modified new matrices $\{\mathbf{D}\}$ etc. defined through

$$\begin{aligned} &\{\mathbf{D}\} = \|\{ij|D|kl\}\|, \qquad \{\overline{\mathbf{D}}\} = \|\{ij|\overline{D}|kl\}\| \qquad \text{and} \\ &\{\mathbf{F}^\star\} = \|\{ij|F^\star|kl\}\|, \qquad \{\overline{\mathbf{F}}^\star\} = \|\{ij|\overline{F}^\star|kl\}\|, \end{aligned}$$

whose matrix elements are given by

$$\begin{aligned} \{ij|D|kl\} &= \tilde{a}_{i'i}\tilde{a}_{j'j}(i'j'|D|kl\} - \tilde{b}_{i'i}\tilde{b}_{j'j}(i'j'|\overline{D}|kl\}^\star \\ &\quad - \tilde{b}_{i'i}\tilde{a}_{j'j}(i'j'|F^\star|kl\} + \tilde{a}_{i'i}\tilde{b}_{j'j}(i'j'|\overline{F}^\star|kl\}^\star, \\ -\{ij|\overline{D}|kl\} &= \tilde{b}_{i'i}\tilde{b}_{j'j}(i'j'|D|kl\}^\star - \tilde{a}_{i'i}\tilde{a}_{j'j}(i'j'|\overline{D}|kl\} \\ &\quad - \tilde{a}_{i'i}\tilde{b}_{j'j}(i'j'|F^\star|kl\}^\star + \tilde{b}_{i'i}\tilde{a}_{j'j}(i'j'|\overline{F}^\star|kl\}, \\ -\{ij|F^\star|kl\} &= \tilde{b}^\star_{i'i}\tilde{a}_{j'j}(i'j'|D|kl\} - \tilde{a}^\star_{i'i}\tilde{b}_{j'j}(i'j'|\overline{D}|kl\}^\star \\ &\quad - \tilde{a}^\star_{i'i}\tilde{a}_{j'j}(i'j'|F^\star|kl\} + \tilde{b}^\star_{i'i}\tilde{b}_{j'j}(i'j'|\overline{F}^\star|kl\}^\star, \\ \{ij|\overline{F}^\star|kl\} &= \tilde{a}^\star_{i'i}\tilde{b}_{j'j}(i'j'|D|kl\}^\star - \tilde{b}^\star_{i'i}\tilde{a}_{j'j}(i'j'|\overline{D}|kl\} \end{aligned}$$

$$- \tilde{b}^{\star}_{i'i}\tilde{b}_{j'j}(i'j'|F^{\star}|kl\}^{\star} + \tilde{a}^{\star}_{i'i}\tilde{a}_{j'j}(i'j'|\overline{F}^{\star}|kl\},$$

in which summation is made over indices i' and j' running from 1 to N. In (2.68) by making block off-diagonal matrices vanish, we get a TD equation with respect to ψ_{o-n} and φ_{o-n} which is formally analogous to that of the $SO(2N)$ RPA, though our TD amplitude and matrices $\{\mathbf{D}\}$ etc. have $(\Lambda, \Lambda^{\star}, t)$-dependence.

2.5 Summary and discussions

We have studied integrability conditions of the TDHBEQ to determine collective submanifolds from the group theoretical viewpoint. As we have seen above, the basic idea lies in the introduction of the Lagrange-like manner to describe the collective coordinates. It should be noted that the variables are nothing but the parameters to describe the symmetry of TDHBEQ. By introducing the one-form, we gave the integrability conditions, the vanishing of the curvatures of the connection, expressed as the Lie-algebra-valued equations. The full TDHB Hamiltonian H_{HB} is decomposed into the collective Hamiltonian H_{c} and the residual one H_{res}. To search for the *well-defined* collective submanifold, we have demanded that the expectation value of the curvature is minimized so as to satisfy $H_{\mathrm{res}} \cong \mathrm{const}$ or $H_{\mathrm{c}} + \mathrm{const} = H_{\mathrm{HB}}$ as far as possible. Further we have imposed the restriction to assure the Lagrange bracket for the usual variables and Lagrange-like ones. Our fundamental equation together with the restricted condition describes the collective motion of the system.

We have proposed the minimization of the residual curvature arising from the residual part of the full TDHB Hamiltonian to determine the collective submanifold. With our theory it is also possible to investigate the range of the validity of the *maximally decoupled* theory of the TDHBT with use of the condition to satisfy (2.50). This condition makes an essential role to give the criterion how we extract well the collective submanifold out of the full TDHB manifold. The reason why the condition occurs in our theory which did not appear in the *maximally decoupled* theory lies in the consideration of the $d^{\dagger}d$-type in the residual Hamiltonian to calculate the residual curvature and in the adoption of the canonical equation. Since the *maximally decoupled* theory has no consideration of such type from the outset, the condition is trivially fulfilled. This is the essential difference between the *maximally decoupled* theory and ours.

We have investigated the nonlinear time-evolution equation arising from zero-curvature equation on TDHB ($SO(2N)$ Lie group) manifold. It is self-evident that the new equation has an $SO(2N)$ RPA solution as a small-amplitude limit. The new equation depends on the collective variables $(\Lambda, \Lambda^{\star})$ defined in a Lagrange-like manner. It works well *in the large scale* beyond the $SO(2N)$ RPA under appropriate boundary and initial conditions. The integrability condition is just the infinitesimal condition to transfer a solution to another solution for the evolution equation under consideration. The usual treatment of the RPA for small amplitude around ground state is nothing but a method of determining an infinitesimal transformation of symmetry under the assumption that fluctuating fields are composed of only normal-modes. We conclude that the set of equations defining the symmetry of the SCF equation and the weak boson commutation relations on the QPF becomes the nonlinear RPA theory.

Finally, following Rajeev [43], we also show the existence of the homogeneous symplectic 2-form ω. From (2.20), using the $\frac{SO(2N)}{U(N)}$ coset variable $q\ (= ba^{-1}) = -q^{\mathrm{T}}$, the $SO(2N)$ (HB) density matrix $\mathcal{R}(g)$ is expressed as

$$\begin{gathered}
\mathcal{R}(g) = g \begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix} g^{\dagger} = \begin{bmatrix} 2R(g) - 1_N & -2K^{\star}(g) \\ 2K(g) & -2R^{\star}(g) + 1_N \end{bmatrix}, \\
R(g) = q^{\dagger}q\big(1_N + q^{\dagger}q\big)^{-1}, \qquad K(g) = -q\big(1_N + q^{\dagger}q\big)^{-1}.
\end{gathered} \tag{2.69}$$

Introducing a new $N\times 2N$ matrix $\mathcal{Z}(g)$ as $\mathcal{Z}(g)=\left(1_N+q^\dagger q\right)^{-\frac{1}{2}}\left[1_N,q^\dagger\right]$ with $\mathcal{Z}(g)^\dagger\mathcal{Z}(g)=1_{2N}$, we have a very simple expression for $\mathcal{R}(g)$ as

$$\begin{aligned}&\mathcal{R}(g)=1_{2N}-2\mathcal{Z}(g)\mathcal{Z}(g)^\dagger=\left[\begin{array}{cc}1_N&0\\0&1_N\end{array}\right]-2\left[\begin{array}{cc}\left(1_N+q^\dagger q\right)^{-1}&\left(1_N+q^\dagger q\right)^{-1}q^\dagger\\q\left(1_N+q^\dagger q\right)^{-1}&q\left(1_N+q^\dagger q\right)^{-1}q^\dagger\end{array}\right],\\&\mathrm{Tr}\,\mathcal{R}(g)=0,\end{aligned}$$

which has quite the same form as the one given by Rajeev [43]. The two-form ω is given as

$$\begin{aligned}&\omega=-\tfrac{i}{8}\mathrm{Tr}\{\,(d\mathcal{R}(g))^3\,\}=-\tfrac{i}{8}\mathrm{Tr}\{\,(d\mathcal{R}(g))^3\,\mathcal{R}(g)^2\},\\&d\omega=-d\omega=0\quad\text{(closed form)}.\end{aligned}$$

If we introduce hermitian matrices $U=\left[\begin{array}{cc}0&u\\u^\dagger&0\end{array}\right]$ and $V=\left[\begin{array}{cc}0&v\\v^\dagger&0\end{array}\right]$, then we have

$$\omega(U,V)=-\tfrac{i}{8}\mathrm{Tr}\left\{\left[\begin{array}{cc}1_N&0\\0&-1_N\end{array}\right][U,V]\right\}=\tfrac{i}{4}\mathrm{Tr}\left\{u^\dagger v-v^\dagger u\right\},$$

which is a symplectic form and makes it possible to discuss geometric quantization on a finite/infinite-dimensional Grassmannian [44, 45].

3 SCF method and τ-functional method on group manifolds

3.1 Introduction

Despite the difference due to the dimension of fermions mentioned in Section 1, we ask the following: How is a *collective submanifold*, truncated through the SCF equation, related to a *subgroup orbit* in the infinite-dimensional Grassmannian by the τ-FM? To get a microscopic understanding of cooperative phenomena, the concept of collective motion is introduced in relation to TD variation of a SCF. Independent-particle (IP) motion is described in terms of particles referring to a stationary MF. The TD variation of the TD SCF is attributed to couplings between the collective and the IP motions and couplings among quantal fluctuations of the TD SCF [6]. There is a one-to-one correspondence between *MF potentials* and vacuum states of the system. Decoupling of collective motion out of full-parameterized TDHF dynamics corresponds to truncation of the *integrable sub-dynamics* from a full-parameterized TDHF manifold. The collective submanifold is a collection of collective paths developed by the SCF equation. The collectivity of each path reflects the *geometrical attribute of the Grassmannian*, which is independent of the characteristic of the SCF Hamiltonian. Then the collective submanifold should be understood in relation to the collectivity of various subgroup orbits in the Grassmannian. The collectivity arises through interference among interacting fermions and links with the concept of the MF potential. The perturbative method has been considered to be useful to describe the *periodic* collective motion with large amplitude [25, 6]. If we do not break the group structure of the Grassmannian in the perturbative method, the *loop group* may work under that treatment.

Thus we notice the following point in both methods: Various subgroup orbits consisting of *loop* path may *infinitely exist* in the full-parameterized TDHF manifold. They must satisfy an infinite set of Plücker relations to hold the Grassmannian. As a result, the finite-dimensional Grassmannian on the circle S^1 is identified with an infinite-dimensional one. Namely the τ-FM works as an algebraic tool to classify the subgroup orbits. The SCF Hamiltonian is able to exist in the infinite-dimensional Grassmannian. Then the SCFT can be rebuilt on the infinite-dimensional fermion Fock space and also on the τ-functional space. The infinite-dimensional fermions are introduced through Laurent expansion of the finite-dimensional fermions with respect to the degrees of freedom of the fermions related to the MF potential. Inversely, the collectivity of the MF potential is attributed to gauges of interacting infinite-dimensional fermions and interference among fermions is elucidated via the Laurent parameter. These are described with the use of affine KM algebra according to the Dirac theory [40]. Algebro-geometric structure of *infinite*-dimensional fermion many-body systems is realized in the *finite*-dimensional ones.

3.2 Bilinear differential equation in SCF method

Owing to the anti-commutation relations $\{c_\alpha, c_\beta^\dagger\} = \delta_{\alpha\beta}$, $\{c_\alpha, c_\beta\} = \{c_\alpha^\dagger, c_\beta^\dagger\} = 0$, fermion pair operators $e_{\alpha\beta} \equiv c_\alpha^\dagger c_\beta$ satisfy a Lie commutation relation $[e_{\alpha\beta}, e_{\gamma\delta}] = \delta_{\beta\gamma} e_{\alpha\delta} - \delta_{\alpha\delta} e_{\gamma\beta}$ and span the $u(N)$ Lie algebra. A canonical transformation $U(g) = e^{\gamma_{\alpha\beta} c_\alpha^\dagger c_\beta}$ ($\gamma^\dagger = -\gamma$, $g = e^\gamma \ni U(N)$) generates a transformation such that

$$
\begin{aligned}
&U(g) c_\alpha^\dagger U^{-1}(g) = c_\beta^\dagger g_{\beta\alpha}, \qquad U(g) c_\alpha U^{-1}(g) = c_\beta g_{\beta\alpha}^*, \\
&U^{-1}(g) = U(g^{-1}) = U(g^\dagger), \qquad U(gg') = U(g)U(g'), \qquad g^\dagger g = g g^\dagger = 1_N.
\end{aligned} \tag{3.1}
$$

Let $|0\rangle$ be a free vacuum and $|\phi_M\rangle$ be an M particle S-det

$$
\begin{aligned}
&c_\alpha |0\rangle = 0, \qquad \alpha = 1, \ldots, N, \qquad |\phi_M\rangle = c_M^\dagger \cdots c_1^\dagger |0\rangle, \\
&U(g)|\phi_M\rangle = (c^\dagger g)_M \cdots (c^\dagger g)_1 |0\rangle \stackrel{d}{=} |g\rangle, \qquad U(g)|0\rangle = |0\rangle,
\end{aligned} \tag{3.2}
$$

where the $c^\dagger$ is an N-dimensional row vector $c^\dagger = (c_1^\dagger, \ldots, c_N^\dagger)$. Equation (3.2) shows that the M particle S-det is an exterior product of M single-particle states and that $U(g)$ transforms $|\phi_M\rangle$ to another S-det (Thouless transformation) [4] under (3.1). Such states are called “simple” states. The set of all the “simple” states of unit modulus together with the equivalence relation, identifying distinct states only in phases with the same state, constitutes a manifold known as Grassmannian Gr_M. The Gr_M is an orbit of the group given through (3.2). Any simple state $|\phi_M\rangle \in \mathrm{Gr}_M$ defines a decomposition of single-particle Hilbert spaces into sub-Hilbert spaces of occupied and unoccupied states [46]. Thus, the Gr_M corresponds to a coset space $\mathrm{Gr}_M \sim \frac{U(N)}{(U(M)\times U(N-M))}$. Using a variable p of the coset space, following [15, 16] and [37], we express the third equation of (3.2) as

$$
U(g)|\phi_M\rangle = \langle \phi_M | U(g_\zeta g_w) | \phi_M \rangle e^{p_{ia} c_i^\dagger c_a} |\phi_M\rangle, \qquad g = g_\zeta g_w, \tag{3.3}
$$

where we have used the relations

$$
\begin{aligned}
&1 + \sum_{\rho=1}^{M_{\max}} \sum_{\substack{1 \le a_1 < \cdots < a_\rho \le M, \\ M+1 \le i_1 < \cdots < i_\rho \le N}} \mathcal{A}(p_{i_1 a_1} \cdots p_{i_\rho a_\rho}) c_{i_1}^\dagger c_{a_1} \cdots c_{i_\rho}^\dagger c_{a_\rho} = e^{p_{ia} c_i^\dagger c_a}, \\
&\langle \phi_M | U(g_\zeta g_w) | \phi_M \rangle = [\det(1 + p^\dagger p)]^{-\frac{1}{2}} \cdot \det w
\end{aligned} \tag{3.4}
$$

and the definition

$$
\mathcal{A}(p_{i_1 a_1} \cdots p_{i_\rho a_\rho}) \stackrel{d}{=} \det \begin{bmatrix} p_{i_1 a_1} & \cdots & p_{i_1 a_\rho} \\ \vdots & & \vdots \\ p_{i_\rho a_1} & \cdots & p_{i_\rho a_\rho} \end{bmatrix}.
$$

In (3.4) a maximum value $M_{\max}$ is given by $M_{\max} = \min(N - M, M)$ and $\mathcal{A}(\cdots)$ is an anti-symmetrizer. $\det w$ is a determinant of matrix w and is a phase appearing in the decomposition of any $U(N)$ matrix as $g = g_\zeta g_w$. The indices i and a denote unoccupied states $(M+1, \ldots, N)$ and occupied states $(1, \ldots, M)$, respectively. The matrices p and w are defined in Appendix A.

In the Gr_M we can introduce an expression called the Plücker coordinate which has played important roles for an algebraic construction of a soliton theory in its early stage [38],

$$
\begin{aligned}
&U(g)|\phi_M\rangle = \sum_{1\le\alpha_1,\ldots,\alpha_M\le N} v^{1,\ldots,M}_{\alpha_1,\ldots,\alpha_M}(g) c^\dagger_{\alpha_M}\cdots c^\dagger_{\alpha_1}|0\rangle,\\
&v^{1,\ldots,M}_{\alpha_1,\ldots,\alpha_M}(g) = \det\begin{bmatrix} g_{\alpha_1,1} & \cdots & g_{\alpha_1,M}\\ \vdots & & \vdots\\ g_{\alpha_M,1} & \cdots & g_{\alpha_M,M}\end{bmatrix} \qquad \text{(Plücker coordinate)}.
\end{aligned}
\tag{3.5}
$$

From elementary determinantal calculus, we prove easily the Plücker coordinate has a relation

$$
\sum_{i=1}^{M+1}(-1)^{i-1} v^{1,\ldots,M}_{\alpha_1,\ldots,\alpha_{M-1},\beta_i}\cdot v^{1,\ldots,M}_{\beta_1,\ldots,\beta_{i-1},\beta_{i+1},\ldots,\beta_{M+1}} = 0 \qquad \text{(Plücker relation)},
$$

where the indices denote the distinct sets $1\le\alpha_1,\ldots,\alpha_{M-1}\le N$ and $1\le\beta_1,\ldots,\beta_{M+1}\le N$. The Plücker relation is equivalent to a bilinear identity equation

$$
\sum_{\alpha=1}^{N} c^\dagger_\alpha U(g)|\phi\rangle\otimes c_\alpha U(g)|\phi\rangle = \sum_{\alpha=1}^{N} U(g)c^\dagger_\alpha|\phi\rangle\otimes U(g)c_\alpha|\phi\rangle = 0.
$$

The bilinear equation has a more general form

$$
\sum_{\alpha=1}^{N} c^\dagger_\alpha U(g)|\phi_k\rangle\otimes c_\alpha U(g)|\phi_l\rangle = \sum_{\alpha=1}^{N} U(g)c^\dagger_\alpha|\phi_k\rangle\otimes U(g)c_\alpha|\phi_l\rangle = 0, \qquad N\ge k\ge l\ge 0,
$$

where $|\phi_k\rangle$ and $|\phi_l\rangle$ denote k-particle simple state and l-one, respectively. It is noted that the Gr_M is essentially an $SU(N)$ group manifold since the phase equivalence theorem does hold.

Now we study the relation between the coset coordinate appeared in (3.3) and the Plücker coordinates in (3.5). Both the well-known coordinates make a crucial role to clarify the algebraic relation between the SCFT, i.e. TDHFT, and the soliton theory.

Using the expressions for unoccupied and occupied states in (3.3), we can rewrite (3.5) as

$$
\begin{aligned}
U(g)|\phi_M\rangle &= |\phi_M\rangle + \sum_{\rho=1}^{M_{\max}} \sum_{\substack{1\le a_1<\cdots<a_\rho\le M,\\ M+1\le i_1<\cdots<i_\rho\le N}} v^{1,\ldots,M}_{1,\ldots,a_1-1,a_1+1,\ldots,a_\rho-1,a_\rho+1,\ldots,M,i_1,\ldots,i_\rho}(g_\zeta g_w)\\
&\quad\times c^\dagger_{i_\rho}\cdots c^\dagger_{i_1} c^\dagger_M\cdots c^\dagger_{a_\rho+1}c^\dagger_{a_\rho-1}\cdots c^\dagger_{a_1+1}c^\dagger_{a_1-1}\cdots c^\dagger_1|0\rangle\\
&= |\phi_M\rangle + v^{1,\ldots,M}_{1,\ldots,M}(g_\zeta g_w)\sum_{\rho=1}^{M_{\max}} \sum_{\substack{1\le a_1<\cdots<a_\rho\le M,\\ M+1\le i_1<\cdots<i_\rho\le N}} \frac{v^{1,\ldots,a_1,\ldots,a_\rho,\ldots,M}_{1,\ldots,i_1,\ldots,i_\rho,\ldots,M}(g_\zeta g_w)}{v^{1,\ldots,M}_{1,\ldots,M}(g_\zeta g_w)}\\
&\quad\times c^\dagger_{i_1}c_{a_1}\cdots c^\dagger_{i_\rho}c_{a_\rho}|\phi_M\rangle.
\end{aligned}
\tag{3.6}
$$

The last line of the above is recast again into the form of (3.5) after many time exchanges between $c_{a_1}\cdots c_{a_\rho}$ and all creation operators so that all the annihilation operators are ordered in such a way that they are to the right of all the creation operators including the ones in $|\phi_M\rangle$. Then we have the relation

$$
v^{1,\ldots,M}_{1,\ldots,a_1-1,a_1+1,\ldots,a_\rho-1,a_\rho+1,\ldots,M,i_1,\ldots,i_\rho}(g_\zeta g_w) = (-1)^{\sum_{j=0}^{\rho-1}(M-j-a_{\rho-j})} v^{1,\ldots,a_1,\ldots,a_\rho,\ldots,M}_{1,\ldots,i_1,\ldots,i_\rho,\ldots,M}(g_\zeta g_w),
$$

and the following decompositions:

$$
\begin{aligned}
&v^{1,\dots,a_1,\dots,a_\rho,\dots,M}_{1,\dots,i_1,\dots,i_\rho,\dots,M}(g_\zeta g_w) = v^{1,\dots,a_1,\dots,a_\rho,\dots,M}_{1,\dots,i_1,\dots,i_\rho,\dots,M}(g_\zeta) v^{1,\dots,M}_{1,\dots,M}(g_w),\\
&v^{1,\dots,M}_{1,\dots,M}(g_\zeta g_w) = v^{1,\dots,M}_{1,\dots,M}(g_\zeta) v^{1,\dots,M}_{1,\dots,M}(g_w), \qquad v^{1,\dots,M}_{1,\dots,M}(g_\zeta) = \det C(\zeta) = [\det(1+p^\dagger p)]^{-\frac{1}{2}},
\end{aligned}
$$

where

$$
v^{1,\dots,a_1,\dots,a_\rho,\dots,M}_{1,\dots,i_1,\dots,i_\rho,\dots,M}(g_\zeta) = \det \begin{bmatrix} C(\zeta)_{1,1} & \cdots & C(\zeta)_{1,M} \\ \vdots & & \vdots \\ C(\zeta)_{a_1-1,1} & \cdots & C(\zeta)_{a_1-1,M} \\ S(\zeta)_{i_1,1} & \cdots & S(\zeta)_{i_1,M} \\ C(\zeta)_{a_1+1,1} & \cdots & C(\zeta)_{a_1+1,M} \\ \vdots & & \vdots \\ C(\zeta)_{a_\rho-1,1} & \cdots & C(\zeta)_{a_\rho-1,M} \\ S(\zeta)_{i_\rho,1} & \cdots & S(\zeta)_{i_\rho,M} \\ C(\zeta)_{a_\rho+1,1} & \cdots & C(\zeta)_{a_\rho+1,M} \\ \vdots & & \vdots \\ C(\zeta)_{M,1} & \cdots & C(\zeta)_{M,M} \end{bmatrix}, \qquad v^{1,\dots,M}_{1,\dots,M}(g_w) = \det w. \tag{3.7}
$$

Here matrix elements in the a_1-th, $\dots$ and a_ρ-th rows, $C(\zeta)_{a_1,1\sim M}, \dots$ and $C(\zeta)_{a_\rho,1\sim M}$ are replaced with $S(\zeta)_{i_1,1\sim M}, \dots$ and $S(\zeta)_{i_\rho,1\sim M}$ to describe ρ $(1 < \rho < M)$ times particle-hole excitations from hole state a_1 to particle state $i_1, \dots$ and those of hole state a_ρ to particle state i_ρ, respectively.

Equating equations (3.3) and (3.5) with equations (3.6) and (3.7), respectively, we obtain the anti-symmetrized $\mathcal{A}(\cdots)$ and the coset variable expressed in terms of Plücker coordinates as

$$
\mathcal{A}(p_{i_1a_1}\cdots p_{i_\rho a_\rho}) = \frac{v^{1,\dots,a_1,\dots,a_\rho,\dots,M}_{1,\dots,i_1,\dots,i_\rho,\dots,M}(g_\zeta)}{v^{1,\dots,M}_{1,\dots,M}(g_\zeta)}, \qquad p_{ia} = [S(\zeta)C^{-1}(\zeta)]_{ia} = \frac{v^{1,\dots,a,\dots,M}_{1,\dots,i,\dots,M}(g_\zeta)}{v^{1,\dots,M}_{1,\dots,M}(g_\zeta)}, \tag{3.8}
$$

in the second Plücker coordinate of which, only one row matrix elements of its determinantal form (3.7) $C(\zeta)_{a,1\sim M}$ are replaced with $S(\zeta)_{i,1\sim M}$. Expanding the anti-symmetrized $\mathcal{A}(\cdots)$ in the left-hand side of the first equation of (3.8) with respect to, for example, the first column, we have a decomposition rule

$$
\begin{aligned}
\frac{v^{1,\dots,a_1,\dots,a_\rho,\dots,M}_{1,\dots,i_1,\dots,i_\rho,\dots,M}(g_\zeta)}{v^{1,\dots,M}_{1,\dots,M}(g_\zeta)} &= \sum_{j=1}^{\rho}(-1)^{j+1} p_{i_j a_1}\mathcal{A}(p_{i_1a_2}\cdots p_{i_{j-1},a_j}p_{i_{j+1}a_{j+1}}\cdots p_{i_\rho a_\rho})\\
&= \sum_{j=1}^{\rho}(-1)^{j+1}\frac{v^{1,\dots a_1,\dots,M}_{1,\dots,i_j,\dots,M}(g_\zeta)}{v^{1,\dots,M}_{1,\dots,M}(g_\zeta)}\frac{v^{1,\dots,a_1,\dots,a_2,\dots,a_j,\dots,a_{j+1},\dots,a_\rho,\dots,M}_{1,\dots,a_1,\dots,i_1,\dots,i_{j-1},\dots,i_{j+1},\dots,i_\rho,\dots,M}(g_\zeta)}{v^{1,\dots,M}_{1,\dots,M}(g_\zeta)},
\end{aligned}
$$

which is rewritten to another form (the second Plücker relation)

$$
\begin{aligned}
&v^{1,\dots,M}_{1,\dots,M}(g_\zeta)v^{1,\dots,a_1,\dots,a_\rho,\dots,M}_{1,\dots,i_1,\dots,i_\rho,\dots,M}(g_\zeta)\\
&\quad + \sum_{j=1}^{\rho}(-1)^j v^{1,\dots,a_1,\dots,M}_{1,\dots,i_j,\dots,M}(g_\zeta) v^{1,\dots,a_1,\dots,a_2,\dots,a_j,\dots,a_{j+1},\dots,a_\rho,\dots,M}_{1,\dots,a_1,\dots,i_1,\dots,i_{j-1},\dots,i_{j+1},\dots,i_\rho,\dots,M}(g_\zeta) = 0,
\end{aligned} \tag{3.9}
$$

in which hole state a_1 in the last Plücker coordinate make no changes $(a_1 \to a_1)$ since in the second one particle-hole excitation already occurred from hole state a_1 to particle state i_j [1].

It is well-known that the Plücker relation is equivalent to a bilinear identity equation

$$\sum_{\alpha=1}^{N} c_\alpha^\dagger U(g)|\phi_M\rangle \otimes c_\alpha U(g)|\phi_M\rangle = \sum_{\alpha=1}^{N} U(g)c_\alpha^\dagger|\phi_M\rangle \otimes U(g)c_\alpha|\phi_M\rangle = 0,$$

which have made an important role to construct many kinds of solitons on various group manifolds [30].

Parallel to the regular representation method by Fukutome [15, 16], we can prove that the Lie commutation relation is also satisfied by the differential operators for particle-hole pairs in Appendix B:

$$\begin{aligned}
&e^{ia} \stackrel{d}{=} -\left(p_{ja}^* p_{ib}^* \frac{\partial}{\partial p_{jb}^*} + \frac{\partial}{\partial p_{ia}} - \frac{i}{2} p_{ia}^* \frac{\partial}{\partial \tau}\right), \qquad e_{ai} \stackrel{d}{=} -\left(p_{ja} p_{ib} \frac{\partial}{\partial p_{jb}} + \frac{\partial}{\partial p_{ia}^*} + \frac{i}{2} p_{ia} \frac{\partial}{\partial \tau}\right), \\
&e_{ab} \stackrel{d}{=} p_{ia} \frac{\partial}{\partial p_{ib}} - p_{ib}^* \frac{\partial}{\partial p_{ia}^*} + i\delta_{ab} \frac{\partial}{\partial \tau}, \qquad e_{ij} \stackrel{d}{=} p_{ia}^* \frac{\partial}{\partial p_{ja}^*} - p_{ja} \frac{\partial}{\partial p_{ia}}.
\end{aligned} \tag{3.10}$$

From the calculations in Appendix B, these differential operators are also proved to satisfy relations

$$\begin{aligned}
&e^{ia}\Phi_{M,M}(p,\ p^*,\tau) = p_{ia}^*\Phi_{M,M}(p,\ p^*,\tau), \qquad e_{ai}\Phi_{M,M}(p,p^*,\tau) = 0, \\
&e_{ab}\Phi_{M,M}(p,p^*,\tau) = \delta_{ab}\Phi_{M,M}(p,p^*,\tau), \qquad e_{ij}\Phi_{M,M}(p,p^*,\tau) = 0,
\end{aligned} \tag{3.11}$$

and a commutator $[e^{ia}, p_{jb}^*] = -p_{ib}^* p_{ja}^*$. A free particle-hole vacuum function $\Phi_{M,M}(p,p^*,\tau)$ is given as

$$\Phi_{M,M}(p,p^*,\tau) = [\det(1 + p^\dagger p)]^{-\frac{1}{2}} e^{-i\tau}. \tag{3.12}$$

Further we can introduce higher order differential operators obeying the relation

$$\begin{aligned}
&D_{1,\ldots,i_1,\ldots,i_\mu,\ldots,M}^{1,\ldots,a_1,\ldots,a_\mu,\ldots,M}(p,\partial_p,\partial_{p^*},\partial_\tau) \stackrel{d}{=} e^{i_1 a_1}\cdots e^{i_\mu a_\mu}, \\
&D_{1,\ldots,i_1,\ldots,i_\mu,\ldots,M}^{1,\ldots,a_1,\ldots,a_\mu,\ldots,M}(p,\partial_p,\partial_{p^*},\partial_\tau)\Phi_{M,M}(p,p^*,\tau) = \mathcal{A}(p_{i_1 a_1}^* \cdots p_{i_\mu a_\mu}^*)\Phi_{M,M}(p,p^*,\tau),
\end{aligned}$$

which show that by operating the differential operator D on the vacuum function Φ, we obtain the Plücker coordinate $\mathcal{A}$. The Plücker relation (3.9) becomes a finite set of partial differential equations satisfying

$$\begin{aligned}
&\Phi_{M,M}(p,p^*,\tau) D_{1,\ldots,i_1,\ldots,i_\rho,\ldots,M}^{1,\ldots,a_1,\ldots,a_\rho,\ldots,M}\Phi_{M,M}(p,p^*,\tau) \\
&+\sum_{j=1}^{\rho}(-1)^j D_{1,\ldots,i_1,\ldots,M}^{1,\ldots,a_1,\ldots,M}\Phi_{M,M}(p,p^*,\tau) D_{1,\ldots,a_1,\ldots,i_1,\ldots,i_{j-1},\ldots,i_{j+1},\ldots,i_\rho,\ldots,M}^{1,\ldots,a_1,\ldots,a_2,\ldots,a_j,\ldots,a_{j+1},\ldots,a_\rho,\ldots,M}\Phi_{M,M}(p,p^*,\tau) = 0, \\
&\left(v_{1,\ldots,i_1,\ldots,i_\mu,\ldots,M}^{1,\ldots,a_1,\ldots,a_\mu,\ldots,M}(g_\zeta g_w)\right)^* = \left(v_{1,\ldots,i_1,\ldots,i_\mu,\ldots,M}^{1,\ldots,a_1,\ldots,a_\mu,\ldots,M}(g_\zeta)\det w\right)^* \\
&\qquad = D_{1,\ldots,i_1,\ldots,i_\mu,\ldots,M}^{1,\cdots,a_1,\ldots,a_\mu,\ldots,M}\Phi_{M,M}(p,p^*,\tau).
\end{aligned}$$

Thus, in both the SCFT and the soliton theory on a group, we can find the common feature that the Grassmannian is just identical with the solution space of the bilinear differential equation. The solution space of each differential equation becomes an integral surface [32, 34, 1].

3.3 SCF method in F_∞

We will give here a brief sketch of the SCF equation, i.e., the TDHFM. According to Rowe et al. [46], we start with a geometrical aspect of the method in the following way:

Let us consider the time dependent Schrödinger equation $i\hbar\partial_t\Psi = H\Psi$ with a Hamiltonian

$$H = h_{\beta\alpha}c^\dagger_\beta c_\alpha + \tfrac{1}{2}\langle\gamma\alpha|\delta\beta\rangle c^\dagger_\gamma c^\dagger_\delta c_\beta c_\alpha, \tag{3.13}$$

where $\langle\gamma\alpha|\delta\beta\rangle$ denotes a matrix element of an interaction potential. The starting point for the TDHFT lies in an extremal condition of an action integral

$$\delta\int_{t_1}^{t_2} dt\mathcal{L}(g(t)) = 0, \qquad \mathcal{L}(g(t)) \stackrel{d}{=} \langle\phi_M|U(g^\dagger(t))(i\hbar\partial_t - H)U(g(t))|\phi_M\rangle. \tag{3.14}$$

To get an explicit expression for the TDHFEQ, we calculate an expectation value of one- and two-body operators for the S-det (3.2). Using the canonical transformation (3.1), we have

$$W_{\alpha\beta} \stackrel{d}{=} \langle\phi_M|U(g^\dagger)c^\dagger_\beta c_\alpha U(g)|\phi_M\rangle = (g^\dagger)_{\beta'\beta}(g^{\mathrm{T}})_{\alpha'\alpha}\langle\phi_M|c^\dagger_{\beta'}c_{\alpha'}|\phi_M\rangle = \sum_{\alpha'=1}^{M} g_{\alpha\alpha'}g^\dagger_{\alpha'\beta}, \tag{3.15}$$

$$\begin{aligned}\langle\phi_M|U(g^\dagger)c^\dagger_\gamma c^\dagger_\delta c_\beta c_\alpha U(g)|\phi_M\rangle &= (g^\dagger)_{\gamma'\gamma}(g^\dagger)_{\delta'\delta}(g^{\mathrm{T}})_{\beta'\beta}(g^{\mathrm{T}})_{\alpha'\alpha}\langle\phi_M|c^\dagger_{\gamma'}c^\dagger_{\delta'}c_{\beta'}c_{\alpha'}|\phi_M\rangle\\ &= W_{\alpha\gamma}W_{\beta\delta} - W_{\alpha\delta}W_{\beta\gamma}.\end{aligned} \tag{3.16}$$

Introducing triangular matrices $C(\zeta)$ and $S(\zeta)$ in Gr_M [37, 16] and using an isometric matrix u^{T}

$$u^{\mathrm{T}} = \left[C^{\mathrm{T}}(\zeta), S^{\mathrm{T}}(\zeta)\right], \qquad u^\dagger u = 1_M,$$

W in (3.16) is expressed as $W = uu^\dagger$ and satisfies $W^2 = W$ (idempotency relation). Then, it turns out that the above matrix W is just the density matrix. From (3.16), we get an energy functional, i.e., an expectation value of the Hamiltonian (3.13)

$$\begin{aligned}&H[W] \stackrel{d}{=} \langle\phi_M|U(g^\dagger)HU(g)|\phi_M\rangle = h_{\beta\alpha}W_{\alpha\beta} + \tfrac{1}{2}[\gamma\alpha|\delta\beta]W_{\alpha\gamma}W_{\beta\delta},\\ &[\gamma\alpha|\delta\beta] = \langle\gamma\alpha|\delta\beta\rangle - \langle\gamma\beta|\delta\alpha\rangle.\end{aligned} \tag{3.17}$$

By projecting the original hamiltonian onto the Gr_M, we obtain also a HF Hamiltonian $H_{\mathrm{HF}}[W]$

$$H_{\mathrm{HF}}[W] = \mathcal{F}_{\alpha\beta}[W]c^\dagger_\alpha c_\beta, \qquad \mathcal{F}_{\alpha\beta} = \frac{\delta H[W]}{\delta W_{\beta\alpha}} = h_{\alpha\beta} + [\alpha\beta|\gamma\delta]W_{\delta\gamma}. \tag{3.18}$$

The Lagrange function $\mathcal{L}(g(t))$ in (3.14) is computed as

$$\mathcal{L}(g(t)) = \frac{i\hbar}{2}(g^\dagger_{ab}\dot{g}_{ba} + g^\dagger_{ai}\dot{g}_{ia} - \dot{g}^\dagger_{ab}g_{ba} - \dot{g}^\dagger_{ai}g_{ia}) - H[W], \tag{3.19}$$

using $\partial_t U(g^\dagger(t))U(g(t)) + U(g^\dagger(t))\cdot\partial_t U(g(t)) = 0$. The condition (3.14) gives the TDHFEQ

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{g}^\dagger}\right) - \frac{\partial\mathcal{L}}{\partial g^\dagger} = 0, \qquad \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{g}}\right) - \frac{\partial\mathcal{L}}{\partial g} = 0,$$

and then we obtain a compact form of the TDHFEQ $i\hbar\partial_t g(t) = \mathcal{F}[W\{g(t)\}]g(t)$. The time evolution of the S-det (3.2) is given by

$$i\hbar\partial_t U(g(t))|\phi_M\rangle = H_{\mathrm{HF}}[W(g(t))]U(g(t))|\phi_M\rangle. \tag{3.20}$$

On the other hand, using a υ-dependent fermion operator given soon later, from (3.14) we also obtain a compact form of υ-dependent HF equation, instead of time t, as

$$i\hbar\partial_\upsilon g(\upsilon) = \mathcal{F}[W\{g(\upsilon)\}]g(\upsilon), \qquad (i\hbar\partial_\upsilon u(\upsilon) = \mathcal{F}[W\{g(\upsilon)\}]u(\upsilon))\,. \tag{3.21}$$

Following the observation by D'Ariano and Rasetti's [31] for the relation between "soliton equations and coherent states", we may assert that the SCFM presents the theoretical scheme for an integrable sub-dynamics on a certain infinite-dimensional fermion Fock space, by identifying $|\phi_M\rangle$ with the highest weight vector and by regarding the TDHF-manifold Gr_M as the projection onto a subspace of the τ-function.

We reconstruct a υ-dependent SCFM in a F_∞ and study a relation between soliton equation and υ-dependent HF equation [41]. We start from a single-particle Schrödinger equation with a υ-dependent and a Υ-periodic potential $V(r,\upsilon) = V(r,\upsilon+\Upsilon)$

$$h_{\mathrm{sp}}(r,\upsilon) = -\frac{\hbar^2}{2m}\Delta + V(r,\upsilon), \qquad h_{\mathrm{sp}}(r,\upsilon)\psi_\alpha(r,\upsilon) = \epsilon_\alpha\psi_\alpha(r,\upsilon).$$

Here we have supposed that an eigen-spectrum ϵ_α is υ independent, though the potential is dependent on υ. It holds an iso-spectrum under a υ-evolution of the potential. An eigen-function $\psi_\alpha(r,\upsilon)$ constitutes an orthonormal complete set and satisfies the same periodicity, $\psi_\alpha(r,\upsilon+\Upsilon) = \psi_\alpha(r,\upsilon)$ (Floquet's theorem). *This picture is very different from that in* [1] and [32]. According to Goddard and Olive [47], we can make Laurent expansion of a fermion-field creation-operator $\psi^\dagger(r,\upsilon)$ with a parameter υ as

$$\psi^\dagger(r,\upsilon) = \sum_\alpha\sum_{r\in\mathbb{Z}}\left(\frac{1}{\Upsilon}\right)^{\frac{1}{2}}\psi_{Nr+\alpha}z^{-r}\psi^*_\alpha(r,\upsilon),$$

where $z = \exp\left(i2\pi\frac{\upsilon}{\Upsilon}\right)$ given on a unit circle. Thus, the $\psi_{Nr+\alpha}$ can be regarded as a new fermion creation-operator. We obtain also a new fermion annihilation-operator in the same way. The anti-commutation relations can be rewritten as

$$\{c_\alpha(\upsilon), c^\dagger_\beta(\upsilon')\} = \delta_{\alpha\beta}\delta(\upsilon-\upsilon'), \qquad \{c_\alpha(\upsilon), c_\beta(\upsilon')\} = \{c^\dagger_\alpha(\upsilon), c^\dagger_\beta(\upsilon')\} = 0. \tag{3.22}$$

Through Laurent expansion of the fermion-field operators, infinite-dimensional fermion operators with particle spectra and Laurent spectra can be obtained as

$$\begin{aligned} &c_\alpha(\upsilon) = \sum_{r\in\mathbb{Z}}\left(\frac{1}{\Upsilon}\right)^{\frac{1}{2}}\psi^*_{Nr+\alpha}z^r, \qquad c^\dagger_\alpha(\upsilon) = \sum_{r\in\mathbb{Z}}\left(\frac{1}{\Upsilon}\right)^{\frac{1}{2}}\psi_{Nr+\alpha}z^{-r}, \\ &\delta(\upsilon-\upsilon') = \frac{1}{\Upsilon}\sum_{r\in\mathbb{Z}}\exp\left\{i2\pi\frac{(\upsilon-\upsilon')}{\Upsilon}r\right\}, \end{aligned} \tag{3.23}$$

where $\mathbb{Z}$ means the set of the integers. The indices α and r are called the label on particle spectra and that on Laurent spectra, respectively. Substitution of (3.23) into (3.22) leads to the anti-commutation relations

$$\{\psi^*_{Nr+\alpha}, \psi_{Ns+\beta}\} = \delta_{\alpha\beta}\delta_{rs}, \qquad \{\psi^*_{Nr+\alpha}, \psi^*_{Ns+\beta}\} = \{\psi_{Nr+\alpha}, \psi_{Ns+\beta}\} = 0. \tag{3.24}$$

If the canonical transformation (3.1) has the υ-dependence and generates the υ-evolution of the potential, it is possible to embed a $U(N)$ group induced from (3.1) into a group which can be induced from a canonical transformation of the infinite-dimensional fermion operators (3.24).

According to Kac and Raina [48, 49] in Appendix C, we introduce a F_∞ and an associative affine Kac–Moody algebra. Here we restrict ourselves to the case of the Lie algebra $u(N)$. The corresponding perfect vacuum $|\mathrm{Vac}\rangle$ and "simple" state $|M\rangle$ are defined, respectively as

$$\begin{aligned}
&\psi_{Nr+\alpha}|\mathrm{Vac}\rangle = 0, \qquad \langle\mathrm{Vac}|\psi^*_{Nr+\alpha} = 0 \quad (r \le -1),\\
&\psi^*_{Nr+\alpha}|\mathrm{Vac}\rangle = 0, \qquad \langle\mathrm{Vac}|\psi_{Nr+\alpha} = 0 \qquad (r \ge 0),\\
&|M\rangle = \psi_M \cdots \psi_1|\mathrm{Vac}\rangle, \qquad \langle M|M\rangle = 1, \qquad \langle\mathrm{Vac}|\mathrm{Vac}\rangle = 1.
\end{aligned} \tag{3.25}$$

We embed the free vacuum $|0\rangle$ and simple state $|\phi_M\rangle$ into F_∞ as $|0\rangle \mapsto |\mathrm{Vac}\rangle$, $|\phi_M\rangle \mapsto |M\rangle$ $(M = 1, \dots, N)$. Assume that a state with Laurent spectrum corresponding to $|0\rangle$ is a stable state with minimal energy. This means a choice of gauge under which $|0\rangle$ corresponds to $|\mathrm{Vac}\rangle$. The matrix γ $(\in u(N))$ in $U(g)$ has also the periodicity Υ. Mapping from a unit circle S^1 to $u(N)$ [47], we make Laurent expansion of γ as $\gamma(z) = \sum_{r\in\mathbb{Z}} \gamma_r z^r$ and impose $\gamma^\dagger(z) = -\gamma(z) \mapsto \gamma_r^\dagger = -\gamma_{-r}$ and $z^{-1} = z^*$ $(|z| = 1)$. Using the *correspondence* between basic elements: $c^\dagger_\alpha c_\beta z^r \mapsto \tau\{e_{\alpha\beta}(r)\} \stackrel{d}{=} \sum_{s\in\mathbb{Z}} \psi_{N(s-r)+\alpha}\psi^*_{Ns+\beta}$ and normal-ordered product $:\psi_{Nr+\alpha}\psi^*_{Ns+\beta}: \stackrel{d}{=} \psi_{Nr+\alpha}\psi^*_{Ns+\beta} - \delta_{\alpha\beta}\delta_{rs}$ $(s < 0)$, let us define the following $\widehat{su}(N)(\subset \widehat{sl}(N))$ Lie algebra [1, 32]:

$$\begin{aligned}
&X_\gamma = \widehat{X}_\gamma + \mathbb{C}\cdot c, \ \mathbb{C}^* = -\mathbb{C} \quad \text{(pure imaginary)},\\
&\widehat{X}_\gamma = \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} (\gamma_r)_{\alpha,\beta} :\psi_{N(s-r)+\alpha}\psi^*_{Ns+\beta}:, \qquad \operatorname{Tr}\gamma_r = 0,\\
&[X_\gamma, c]_{\mathrm{KM}} = 0, \qquad [X_\gamma, X_{\gamma'}]_{\mathrm{KM}} = \widehat{X}_{[\gamma,\gamma']} + \alpha(\gamma,\gamma')\cdot c, \qquad c|M\rangle = 1\cdot|M\rangle,\\
&\alpha(\gamma,\gamma') = -\alpha^*(\gamma,\gamma') = \sum_{r\in\mathbb{Z}} r \operatorname{Tr}(\gamma_r\gamma'_{-r}),
\end{aligned} \tag{3.26}$$

where c denotes a center. As for the τ representation (rep) and KM bracket, see Appendix C.

Using equations (3.24) and (3.26), adjoint actions of X_γ for ψ and ψ^* are computed as

$$[X_\gamma, \psi_{Nr+\alpha}] = \sum_{s\in\mathbb{Z}} \psi_{N(r-s)+\beta}(\gamma_s)_{\beta\alpha}, \qquad [X_\gamma, \psi^*_{Nr+\alpha}] = \sum_{s\in\mathbb{Z}} \psi^*_{N(r-s)+\beta}(\gamma^*_s)_{\beta\alpha}. \tag{3.27}$$

Let us introduce a canonical transformation $U(\hat{g}) = e^{X_\gamma}$ satisfying $U^{-1}(\hat{g}) = U(\hat{g}^{-1}) = U(\hat{g}^\dagger)$ and $U(\hat{g}\hat{g}') = U(\hat{g})U(\hat{g}')$. The $\hat{g}$ $(= e^\gamma)$ has a form analogous to g but with infinite dimension and satisfies $\hat{g}^\dagger\hat{g} = \hat{g}\hat{g}^\dagger = 1_\infty$. Further, using (3.27) and the operator identity $e^{X_\gamma}Ae^{-X_\gamma} = A + [X_\gamma, A] + \frac{1}{2!}[X_\gamma, [X_\gamma, A]] + \cdots$, the infinite-dimensional fermion operator is transformed into

$$\psi_{Nr+\alpha}(\hat{g}) \stackrel{d}{=} U(\hat{g})\psi_{Nr+\alpha}U^{-1}(\hat{g}) = \sum_{s\in\mathbb{Z}} \psi_{N(r-s)+\beta}(g_s)_{\beta\alpha}, \qquad \hat{g}_{Nr+\alpha,Ns+\beta} \equiv (g_{s-r})_{\alpha\beta}, \tag{3.28}$$

where g_s means the s-th block matrix of $\hat{g}$ on each diagonal parallel to the principal diagonal and satisfies the ortho-normalization relation. Using the correspondence $|\phi_M\rangle \mapsto |M\rangle$, $U(g) \mapsto U(\hat{g})$ $(= e^{X_\gamma})$ and

$$\sum_{\alpha=1}^N c^\dagger_\alpha \otimes c_\alpha \mapsto \sum_{\alpha=1}^N \sum_{r\in\mathbb{Z}} c^\dagger_\alpha z^{-r} \otimes c_\alpha z^r \simeq \sum_{\alpha=1}^N \sum_{r\in\mathbb{Z}} \psi_{Nr+\alpha} \otimes \psi^*_{Nr+\alpha},$$

the bilinear equation on the finite-dimensional Fock space is embedded into the one on F_∞ as

$$\sum_{\alpha=1}^N \sum_{r\in\mathbb{Z}} \psi_{Nr+\alpha}U(\hat{g})|M\rangle \otimes \psi^*_{Nr+\alpha}U(\hat{g})|M\rangle$$

$$= \sum_{\alpha=1}^{N} \sum_{r\in\mathbb{Z}} U(\hat{g})\psi_{Nr+\alpha}|M\rangle \otimes U(\hat{g})\psi^*_{Nr+\alpha}|M\rangle = 0, \tag{3.29}$$

$$\sum_{\alpha=1}^{N} \sum_{r\in\mathbb{Z}} \psi_{Nr+\alpha}U(\hat{g})|k\rangle \otimes \psi^*_{Nr+\alpha}U(\hat{g})|l\rangle$$
$$= \sum_{\alpha=1}^{N} \sum_{r\in\mathbb{Z}} U(\hat{g})\psi_{Nr+\alpha}|k\rangle \otimes U(\hat{g})\psi^*_{Nr+\alpha}|l\rangle = 0, \qquad k \geq l, \quad k,l = 1,\ldots,N. \tag{3.30}$$

where $M = 1 \sim N$ and $k \geq l$ $(k,l = 1 \sim N)$. Thus we arrive at the following picture: An algebra of extracting sub-group orbits made of *loop* path from Gr_M belongs to the $sl(N)$-reduction of $gl(\infty)$ in the soliton theory. Relieving from restrictions of $su(N)$ and (3.29) and taking $\gamma \in sl(N)$ with M and $k \geq l$ $(\in \mathbb{Z})$, (3.29) and (3.30) can be regarded as the bilinear equations of the reduced KP (Kadomtsev–Petviashvili) hierarchy and the modified KP in the soliton theory [30]. The algebra of extracting various sub-group manifolds made of several *loop-group* pathes [36] from Gr_M belongs to an $sl(N)$-reduction of $gl(\infty)$. This picture suggests us the possibility to construct a υ-dependent HF soliton equation by using the υ-dependent SCFM governed by conditions (3.29) and (3.30) on the space $sl(N)$ bigger than $su(N)$. However we must note that in the SCFM bilinear equations (3.29) and (3.30) are considered to play roles of conditions for assuring of existence of sub-group orbits on Gr_M different from the soliton theory in which boson expressions for them become an infinite set of dynamical equations. Notice also that the concept of quasi-particle and vacuum in the SCFM on S^1 is connected to the Plücker relation.

Following [1] and [32], we embed the original Hamiltonian (3.13) into F_∞. By replacing the annihilation-creation operators of fermions as $c^\dagger_\beta \mapsto \sum_{s\in\mathbb{Z}} \psi_{Ns+\beta}$ and $c_\alpha \mapsto \sum_{r\in\mathbb{Z}} \psi^*_{Nr+\alpha}$, we get

$$H_{F_\infty} = h_{\beta\alpha} \sum_{r,s\in\mathbb{Z}} \psi_{Nr+\beta}\psi^*_{Ns+\alpha} + \tfrac{1}{2}\langle\gamma\alpha|\delta\beta\rangle \sum_{k,l\in\mathbb{Z},\ r,s\in\mathbb{Z}} \psi_{Nk+\gamma}\psi_{Nl+\delta}\psi^*_{Ns+\beta}\psi^*_{Nr+\alpha}. \tag{3.31}$$

Previously we had the υ-dependent SCF Hamiltonian $\mathcal{F}[W\{g(\upsilon)\}]$ through equations (3.18) and (3.21) where υ-dependence is directly brought from $g(\upsilon)$. This is contrast that in the usual TDHFEQ (3.20), the time t-dependence of course arises from $g(t)$. To embed this SCF Hamiltonian, we introduce a general Hamiltonian on F_∞ as

$$H_{F_\infty} = \sum_{r,s\in\mathbb{Z}} h_{Ns+\beta,Nr+\alpha}\psi_{Ns+\beta}\psi^*_{Nr+\alpha}$$
$$+ \tfrac{1}{2} \sum_{r,s\in\mathbb{Z},\ k,l\in\mathbb{Z}} \langle Nk+\gamma, Nr+\alpha|Nl+\delta, Ns+\beta\rangle \psi_{Nk+\gamma}\psi_{Nl+\delta}\psi^*_{Ns+\beta}\psi^*_{Nr+\alpha}, \tag{3.32}$$

which is equivalent to (3.31) if $h_{Ns+\beta,Nr+\alpha} = h_{\beta\alpha}$ and $\langle Nk+\gamma, Nr+\alpha|Nl+\delta, Ns+\beta\rangle = \langle\gamma\alpha|\delta\beta\rangle$ (equivalence conditions for H_{F_∞}) hold. To calculate the formal expectation value of (3.32) for the vector $U(\hat{g})|M\rangle$, first we do it for one-body and two-body operators. Using (3.28) we obtain

$$\langle M|\psi_{Ns+\beta}\psi^*_{Nr+\alpha}|M\rangle = \delta_{sr}\delta_{\beta\alpha} \quad (\text{for } r=0,\ \alpha=1,\ldots,M \text{ and for } r<0,\ \alpha=1,\ldots,N),$$
$$\langle M|\psi_{Nk+\gamma}\psi_{Nl+\delta}\psi^*_{Ns+\beta}\psi^*_{Nr+\alpha}|M\rangle = \delta_{kr}\delta_{\gamma\alpha}\cdot\delta_{ls}\delta_{\delta\beta} - \delta_{ks}\delta_{\gamma\beta}\cdot\delta_{lr}\delta_{\delta\alpha},$$
$$(\text{for } r(s)=0,\ \alpha(\beta)=1,\ldots,M \text{ and for } r(s)<0,\ \alpha(\beta)=1,\ldots,N).$$

Then, for one-body and two-body type operators we obtain

$$\widehat{W}_{Nr+\alpha,Ns+\beta} = \langle M|U(\hat{g}^\dagger)\psi_{Ns+\beta}\psi^*_{Nr+\alpha}U(\hat{g})|M\rangle$$

$$= \sum_{\gamma=1}^{M} (g_{-r})_{\alpha\gamma} (g^{\dagger}_{-s})_{\gamma\beta} + \sum_{t<0} \sum_{\gamma=1}^{N} (g_{t-r})_{\alpha\gamma} (g^{\dagger}_{t-s})_{\gamma\beta}, \tag{3.33}$$

$$\langle M | U(\hat{g}^{\dagger}) \psi_{Nk+\gamma} \psi_{Nl+\delta} \psi^{*}_{Ns+\beta} \psi^{*}_{Nr+\alpha} U(\hat{g}) | M \rangle$$
$$= \widehat{W}_{Nr+\alpha,Nk+\gamma} \widehat{W}_{Ns+\beta,Nl+\delta} \widehat{W}_{Nr+\alpha,Nl+\delta} \widehat{W}_{Ns+\beta,Nk+\gamma}.$$

The $\widehat{W}$ is just the so-called density matrix since it is easily proved to satisfy the idempotency relation $\widehat{W}^2 = \widehat{W}$ which has not been given explicitly in [1] and [32]. It provides a strong tool to develop our SCF scenario on the F_∞. Taking summation over infinite integers, inevitably we have an *anomaly* in the above expectation value. To avoid this *anomaly*, the one-body operator (3.33) must be changed to a normal-ordered product as

$$(\mathcal{W}_k)_{\alpha\beta} \stackrel{d}{=} \langle M | U(\hat{g}^{\dagger}) : \tau\{e_{\beta\alpha}(-k)\} : U(\hat{g}) | M \rangle = \sum_{r\in\mathbb{Z}} \langle M | U(\hat{g}^{\dagger}) : \psi_{N(r+k)+\beta} \psi^{*}_{Nr+\alpha} : U(\hat{g}) | M \rangle$$
$$= \sum_{r\in\mathbb{Z}} \widehat{W}_{Nr+\alpha,N(r+k)+\beta} - \sum_{r<0} \delta_{k,0}\delta_{\beta\alpha} = \sum_{r\in\mathbb{Z}} \sum_{\gamma=1}^{M} (g_r)_{\alpha\gamma} (g^{\dagger}_{r-k})_{\gamma\beta}, \tag{3.34}$$

where we have used the correspondence relation between basic elements. The $\mathcal{W}_k$ is identical with a coefficient of the Laurent expansion of the density matrix W (3.15)

$$\mathcal{W}_{\alpha\beta}(z) = \sum_{k\in\mathbb{Z}} (\mathcal{W}_k)_{\alpha\beta} z^k = \sum_{k\in\mathbb{Z}} \sum_{s\in\mathbb{Z}} \sum_{\gamma=1}^{M} (g_s)_{\alpha\gamma} (g^{\dagger}_{s-k})_{\gamma\beta} z^k.$$

We compute formal expectation value of (3.32) for the state $U(\hat{g})|M\rangle$. Introducing a new integer K, due to the equivalence conditions for H_{F_∞}, from (3.31) we get

$$\langle H_{F_\infty} \rangle [\widehat{W}] = \sum_{K\in\mathbb{Z}} (h_K)_{\beta\alpha} \sum_{s\in\mathbb{Z}} \widehat{W}_{Ns+\alpha,N(s-K)+\beta}$$
$$+ \tfrac{1}{2} \sum_{K,L\in\mathbb{Z}} [(K,\gamma),\alpha|(L,\delta),\beta] \sum_{r,s\in\mathbb{Z}} \widehat{W}_{Nr+\alpha,N(r-K)+\gamma} \sum_{s\in\mathbb{Z}} \widehat{W}_{Ns+\beta,N(s-L)+\delta},$$
$$(h_k)_{\beta\alpha} \equiv h_{\beta\alpha}, \qquad [(k,\gamma),\alpha|(l,\delta),\beta] \equiv [\gamma\alpha|\delta\beta] \qquad \forall\ k,\ l. \tag{3.35}$$

Changing $\widehat{W}$ in (3.35) into its normal-ordered product and using (3.34), we obtain

$$\langle H_{F_\infty} \rangle [\mathcal{W}] = \sum_{k\in\mathbb{Z}} \left\{ h_{\beta\alpha} (\mathcal{W}_{-k})_{\alpha\beta} + \tfrac{1}{2} [\gamma\alpha|\delta\beta] \sum_{l\in\mathbb{Z}} (\mathcal{W}_{-k+l})_{\alpha\gamma} (\mathcal{W}_{-l})_{\beta\delta} \right\}.$$

This result coincides with formal Laurent polynomials of $H[W]$ (3.17) in the sense of the expansion

$$H[\mathcal{W}(z)] = \sum_{l\in\mathbb{Z}} \left\{ h_{\beta\alpha} (\mathcal{W}_l)_{\alpha\beta} + \tfrac{1}{2} [\gamma\alpha|\delta\beta] \sum_{k\in\mathbb{Z}} (\mathcal{W}_{l-k})_{\alpha\gamma} (\mathcal{W}_k)_{\beta\delta} \right\} z^l. \tag{3.36}$$

To get a υ-independent Hamiltonian, in (3.36) it is enough for us to pick up only the term with Laurent spectrum $l = 0$. Then, we may select a density-functional Hamiltonian

$$\langle H_{F_\infty} \rangle [\mathcal{W}] = h_{\beta\alpha} (\mathcal{W}_0)_{\alpha\beta} + \tfrac{1}{2} [\gamma\alpha|\delta\beta] \sum_{k\in\mathbb{Z}} (\mathcal{W}_k)_{\alpha\gamma} (\mathcal{W}_{-k})_{\beta\delta}. \tag{3.37}$$

This is the extraction of the sub-Hamiltonian $H^{\mathrm{sub}}_{F_\infty}$ out of the original Hamiltonian (3.31) as shown below

$$H^{\mathrm{sub}}_{F_\infty} = h_{\beta\alpha}\sum_{s\in\mathbb{Z}}\psi_{Ns+\beta}\psi^*_{Ns+\alpha} + \tfrac{1}{2}[\gamma\alpha|\delta\beta]\sum_{K\in\mathbb{Z}}\sum_{r,s\in\mathbb{Z}}\psi_{N(r-K)+\gamma}\psi_{N(s+K)+\delta}\psi^*_{Ns+\beta}\psi^*_{Nr+\alpha}. \tag{3.38}$$

We here adopt (3.37) as an energy functional for the $u(N)$ HF system on the F_∞. Through the variation

$$\delta\langle H_{F_\infty}\rangle[\mathcal{W}] = \sum_{k\in\mathbb{Z}}(\mathcal{F}_{-k})_{\alpha\beta}\delta(\mathcal{W}_k)_{\beta\alpha}, \qquad (\mathcal{F}_k)_{\alpha\beta} \stackrel{d}{=} h_{\alpha\beta}\delta_{k,o} + [\alpha\beta|\gamma\delta](\mathcal{W}_k)_{\delta\gamma}, \tag{3.39}$$

we get a SCF Hamiltonian on the F_∞ similar to formal Laurent expansion of H_{HF} (3.18) on the Gr_M as

$$H_{F_\infty;\mathrm{HF}} = \sum_{K\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}(\mathcal{F}_K)_{\alpha\beta} : \psi_{N(s-K)+\alpha}\psi^*_{Ns+\beta} : . \tag{3.40}$$

For the υ-dependent HF equation on the F_∞, the state vector $U(\hat{g})|M\rangle$ is required to satisfy the variational principle

$$\delta S = \delta_{\hat{g}}\int_{\upsilon_1}^{\upsilon_2} d\upsilon L(\hat{g}) = 0, \qquad L(\hat{g}) = \langle M|U(\hat{g}^\dagger)(i\partial_\upsilon - H_{F_\infty})U(\hat{g})|M\rangle, \tag{3.41}$$

where we use $\hbar = 1$ here and hereafter. First by using $U(\hat{g}) = e^{X_\gamma}$ we get the following relations:

$$\begin{aligned}\delta_{\hat{g}}\int d\upsilon\langle M|U(\hat{g}^\dagger)i\partial_\upsilon U(\hat{g})|M\rangle &= \delta_{\hat{g}}\int d\upsilon\langle M|i\partial_\upsilon|M\rangle + \delta_{\hat{g}}\int d\upsilon\langle M|i\partial_\upsilon X_\gamma \\ &\quad - \tfrac{1}{2!}[X_\gamma, i\partial_\upsilon X_\gamma] + \cdots |M\rangle,\end{aligned} \tag{3.42}$$

$$\begin{aligned}i\partial_\upsilon X_\gamma = \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}&\left\{(i\partial_\upsilon\gamma_r)_{\alpha\beta} : \psi_{N(s-r)+\alpha}\psi^*_{Ns+\beta} : + (\gamma_r)_{\alpha\beta}i\partial_\upsilon : \psi_{N(s-r)+\alpha}\psi^*_{Ns+\beta} :\right\} \\ &+ i\partial_\upsilon(\mathbb{C}\cdot 1),\end{aligned} \tag{3.43}$$

where we have used (3.28). From the definition of $\tau\{e_{\alpha\beta}(r)\}$ and the normal-ordered product, we can calculate the υ-differentiation of the second term in the curly bracket of (3.43)

$$i\partial_\upsilon\sum_{s\in\mathbb{Z}} : \psi_{N(s-r)+\alpha}\psi^*_{Ns+\beta} := i\partial_\upsilon : \tau\{e_{\alpha\beta}(r)\} := ir\partial_\upsilon \ln z : \tau\{e_{\alpha\beta}(r)\} : .$$

Assume the parameter of Laurent expansion to be $z = e^{-i\omega_c\upsilon}$. Then (3.43) is rewritten as

$$\begin{aligned}i\partial_\upsilon X_\gamma &= \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}(D_{r;\upsilon}(\gamma_r)_{\alpha\beta}) : \psi_{N(s-r)+\alpha}\psi^*_{Ns+\beta} : + i\partial_\upsilon(\mathbb{C}\cdot 1), \\ D_{r;\upsilon} &\stackrel{d}{=} i\partial_\upsilon + r\omega_c.\end{aligned} \tag{3.44}$$

It is seen that in the first term of the r.h.s. of (3.42) a υ-evolution of the reference vacuum through $z(\upsilon)$ has no influence on variation with respect to $\hat{g}$. Concerning the υ-evolution of X_γ (3.44) a υ-differential ∂_υ acting on $\gamma_r(\upsilon)$ and on ψ and ψ^* through $z(\upsilon)$ is transformed to a covariant differential $D_{r;\upsilon}$ with a connection $r\omega_c$ which acts only on the $\gamma_r(\upsilon)$ from the gauge theoretical viewpoint. We denote simply the covariant differential as D. Therefore we can put $\langle M|i\partial_\upsilon|M\rangle = 0$ and $\mathbb{C} = 0$ since it has no influence on the energy functional (3.37). Then the υ-differential term in equation (3.41) is calculated as

$$U(\hat{g}^\dagger)i\partial_\upsilon U(\hat{g}) = i\partial_\upsilon X_\gamma + \tfrac{1}{2!}[i\partial_\upsilon X_\gamma, X_\gamma] + \tfrac{1}{3!}[[i\partial_\upsilon X_\gamma, X_\gamma], X_\gamma] + \cdots$$

$$= \widehat{X}_{D\gamma} + \sum_{k\geq 2} \tfrac{1}{k!}[\cdots[i\partial_\upsilon X_\gamma, X_\gamma], \cdots], X_\gamma] + \cdots . \tag{3.45}$$

Using (3.26) and the symbol D for the covariant differential, each commutator is calculated as

$$\begin{aligned}
&[i\partial_\upsilon X_\gamma, X_\gamma] = \widehat{X}_{[D\gamma,\gamma]} + \sum_{r\in\mathbb{Z}} r\,\mathrm{Tr}\left\{(D\gamma)_r\gamma_{-r}\right\},\\
&[[i\partial_\upsilon X_\gamma, X_\gamma], X_\gamma] = \widehat{X}_{[[D\gamma,\gamma],\gamma]} + \sum_{r\in\mathbb{Z}} r\,\mathrm{Tr}\left\{([D\gamma,\gamma])_r\gamma_{-r}\right\},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&[\cdots[i\partial_\upsilon X_\gamma, X_\gamma], \cdots], X_\gamma] = \widehat{X}_{[\cdots[D\gamma,\gamma],\cdots],\ \gamma]} + \sum_{r\in\mathbb{Z}} r\,\mathrm{Tr}\left\{([\cdots[D\gamma,\gamma],\cdots])_r\gamma_{-r}\right\}.
\end{aligned} \tag{3.46}$$

Substituting (3.46) into (3.45) and using $D_r \stackrel{d}{=} D_{r;\upsilon}$ and $\hat{g} = e^\gamma$, we get

$$\begin{aligned}
&U(\hat{g}^\dagger) i\partial_\upsilon U(\hat{g}) = \widehat{X}_{\hat{g}^\dagger D\hat{g}} + \mathbb{C}(\hat{g}^\dagger D\hat{g}),\\
&\mathbb{C}(\hat{g}^\dagger D\hat{g}) = \sum_{r\in\mathbb{Z}} r\,\mathrm{Tr}\left\{\left(\sum_{k\geq 2}\tfrac{1}{k!}\overline{[\cdots[D\gamma,\gamma],\cdots]},\gamma]\right)_r \gamma_{-r}\right\}.
\end{aligned} \tag{3.47}$$

The expectation value for the reference vacuum is expressed as

$$\langle M|U(\hat{g}^\dagger) i\partial_\upsilon U(\hat{g})|M\rangle = \sum_{s\in\mathbb{Z}}\sum_{\alpha=1}^{M}\sum_{\gamma=1}^{N}(g_s^\dagger)_{\alpha\gamma}(D_{s;\upsilon}g_s)_{\gamma\alpha} + \mathbb{C}(\hat{g}^\dagger D\hat{g}).$$

Using $\mathbb{C}(\hat{g}^\dagger D\hat{g}) - \mathbb{C}(D\hat{g}^\dagger \cdot \hat{g}) = 0$ which is proved later, we obtain an explicit expression for the $L(\hat{g})$ as

$$L(\hat{g}) = \tfrac{1}{2}\sum_{s\in\mathbb{Z}}\sum_{\alpha=1}^{M}\sum_{\gamma=1}^{N}\left\{(g_s^\dagger)_{\alpha\gamma}(D_{s;\upsilon}g_s)_{\gamma\alpha} - (D_{-s;\upsilon}g_s^\dagger)_{\alpha\gamma}(g_s)_{\gamma\alpha}\right\} - \langle H_{F_\infty}\rangle[W].$$

Thus $L(\hat{g})$ is nothing less than the coefficient of z^0 in the Laurent expansion of $L(g(z))$ (3.19).

We give another υ-dependent HF equation for $\hat{g}$: Laurent expansion of

$$i\partial_\upsilon g(\upsilon) = \mathcal{F}[W\{g(\upsilon)\}]g(\upsilon) \qquad \text{and} \qquad i\partial_\upsilon U(g(\upsilon))|\phi\rangle = H_{\mathrm{HF}}[W(g(\upsilon))]U(g(\upsilon))|\phi\rangle.$$

Demand the extremal condition of (3.40) leads to $D_\upsilon\hat{g} = \mathcal{F}(\hat{g})\hat{g}$ where $\mathcal{F}(\hat{g})$ has an infinite N-periodic sequence of block form $\{\dots,\mathcal{F}_{-1},\mathcal{F}_0,\mathcal{F}_1,\dots\}$ like (C.5) in Appendix C. Defining $(\mathcal{F}_r^c)_{\alpha\beta}(\hat{g},\omega_c) \stackrel{d}{=} \omega_c\sum_{s\in\mathbb{Z}} s(g_s g_{s-r}^\dagger)_{\alpha\beta}$, the $D_\upsilon\hat{g} = \mathcal{F}(\hat{g})\hat{g}$ is transformed to

$$\begin{aligned}
&i\partial_\upsilon\hat{g} = \mathcal{F}^p(\hat{g})\hat{g}, \qquad \mathcal{F}^p(\hat{g}) \stackrel{d}{=} \mathcal{F}(\hat{g}) - \mathcal{F}^c(\hat{g}),\\
&(\mathcal{F}_r^p)_{\alpha\beta} \stackrel{d}{=} (\mathcal{F}_r - \mathcal{F}_r^c)_{\alpha\beta} = h_{\alpha\beta}\delta_{r,0} + [\alpha\beta|\gamma\delta](W_r)_{\delta\gamma} - \omega_c\sum_{s\in\mathbb{Z}} s(g_s g_{s-r}^\dagger)_{\alpha\beta},
\end{aligned}$$

introducing $\widehat{D}_\upsilon \stackrel{d}{=} i\partial_\upsilon + H^c_{F_\infty;\mathrm{HF}}$, this time which is cast into that on the state vector $U(\hat{g})|M\rangle$ as

$$\widehat{D}_\upsilon U(\hat{g})|M\rangle = H_{F_\infty;\mathrm{HF}}U(\hat{g})|M\rangle, \qquad H^c_{F_\infty;\mathrm{HF}} \stackrel{d}{=} \sum_{r,s\in\mathbb{Z}}(\mathcal{F}_r^c)_{\alpha\beta} : \psi_{N(s-r)+\alpha}\psi^*_{Ns+\beta} :,$$

$$i\partial_\upsilon U(\hat{g})|M\rangle = H^p_{F_\infty;\mathrm{HF}} U(\hat{g})|M\rangle, \qquad H^p_{F_\infty;\mathrm{HF}} \stackrel{d}{=} \sum_{r,s\in\mathbb{Z}} (\mathcal{F}^p_r)_{\alpha\beta} : \psi_{N(s-r)+\alpha}\psi^*_{Ns+\beta} :, \tag{3.48}$$

which suggest symmetry breaking and arising of collective motion due to recovery of symmetry. Suppose that $\hat{g}$ to diagonalize $\mathcal{F}^p$ in $H^p_{F_\infty;\mathrm{HF}}$ and $U(\hat{g})|M\rangle$ to do $\mathcal{F}^c$ in $H^c_{F_\infty;\mathrm{HF}}$ are determined spontaneously when $\hat{g} \simeq \hat{g}^0 e^{-i\hat{\epsilon}\upsilon}$ and $\partial_\upsilon \hat{g}^0 = 0$. Using the definition of $\mathcal{F}^c$ we have $\omega_c \Gamma(\hat{g}^0) = \mathcal{F}(\hat{g}^0)\hat{g}^0 - \hat{g}^0\hat{\epsilon}$ where $\Gamma(\hat{g}^0)$ has an infinite N-periodic sequence of block form $\{\dots, -g^0_{-1}, 0, g^0_1, \dots\}$ like (C.5) and $\hat{\epsilon} = \mathrm{diag}\{\dots, \epsilon, \dots\}$. We also obtain $g_r z^r \propto e^{-i(\epsilon+\omega_c I_N)\upsilon}$. Thus the quasi-particle energy $\epsilon\,(\epsilon_{\alpha\beta} = \epsilon_\alpha \delta_{\alpha\beta})$ and the boson energy ω_c are unified into gauge phase. The static υ-HFT on Gr_M has obviously no collective term and leads inevitably to $\omega_c \Gamma(\hat{g}^0) = 0$. $\hat{g}^0$ should compose of only a block-diagonal $g^0_0 = e^{\gamma_0}$, γ_0 being a block-diagonal $su(N)$ matrix.

Equation (3.48) brings a υ-evolution of particle degrees of freedom and a common language, *infinite-dimensional* Gr_∞ *and affine KM algebra*, to discuss the relation between SCFT and soliton theory. The SCFT on F_∞ is nothing else than the zero-th order of the Laurent expansion on Gr_M. Through the construction of the SCFT an explicit algebraic structure of the SCFT on F_∞ is made clear since it is just the gauge theory inherent in the SCFT. The mean-field potential degrees of freedom occur from the gauge degrees of freedom of fermions and the fermions make pairs among them absorbing a change of gauges. The sub-Hamiltonian (3.38) exhibits such a phenomenon in $u(N)$ algebra, which allows us to interpret absorption of gauge as a coherent property of fermion pairs. Thus the SCFTM is regarded as a method to determine self-consistently both quasi-particle energy $\epsilon_\alpha(\hat{g})$ and boson energy ω_c, to the υ-evolution of the "fermion gauge". Then we can say that *both the energies have been unified into the gauge phase.*

Let ϵ and ϵ^* be parameters specifying a continuous deformation of *loop* path on the Gr_M and independent on z. Using the notation in (3.26) and calculating in a similar way to (3.47), $e^{-X_\gamma}\partial_\epsilon e^{X_\gamma}$ is obtained as

$$\begin{aligned} &e^{-X_\gamma}\partial_\epsilon e^{X_\gamma} = \widehat{X}_{\hat{g}^{-1}\partial_\epsilon \hat{g}} + \partial_\epsilon(\mathbb{C}\cdot 1) + \mathbb{C}(\hat{g}^{-1}\partial_\epsilon \hat{g}), \\ &\mathbb{C}(\hat{g}^{-1}\partial_\epsilon \hat{g}) = \partial_\epsilon + \partial_\epsilon\gamma + \sum_{k\geq 2} \tfrac{1}{k!}[\cdots[\partial_\epsilon\gamma, \gamma], \cdots], \gamma]. \end{aligned} \tag{3.49}$$

To avoid the *anomaly*, $\widehat{X}_\gamma$ reads

$$\sum_{r\in\mathbb{Z}} (\gamma_r)_{\alpha\beta} \left\{ \sum_{s\in\mathbb{Z}} \psi_{N(s-r)+\alpha}\psi^*_{Ns+\beta} + \delta_{r0} \sum_{s<0} \delta_{\alpha\beta} \right\} (\mathrm{Tr}\,\gamma_r = 0).$$

Then equation (3.49) is computed to be

$$e^{-X_\gamma}\partial_\epsilon e^{X_\gamma} = \sum_{r,s\in\mathbb{Z}} (\hat{g}_r^{-1}\partial_\epsilon \hat{g}_r)_{\alpha\beta} : \psi_{N(s-r)+\alpha}\psi^*_{Ns+\beta} : + \sum_{s<0} \mathrm{Tr}(\hat{g}_0^{-1}\partial_\epsilon \hat{g}_0). \tag{3.50}$$

From (3.50), we get $\mathbb{C}(\hat{g}^{-1}\partial_\epsilon \hat{g}) = \sum_{s<0} \mathrm{Tr}(\hat{g}_0^{-1}\partial_\epsilon \hat{g}_0) = 0$, $\hat{g}_0^{-1}\partial_\epsilon \hat{g}_0 \in sl(N, C)$ and $\mathbb{C}(\hat{g}^\dagger \partial_{\epsilon^*} \hat{g}) = \mathbb{C}(\partial_{\epsilon^*}\hat{g}^\dagger \cdot \hat{g}) = 0$. We obtain also $\mathbb{C}(\hat{g}^\dagger D\hat{g}) = \mathbb{C}(D\hat{g}^\dagger \cdot \hat{g}) = 0$ in the same way as the above. For subsequent discussion it is convenient to define infinitesimal generators of the collective submanifold as follows [50, 17]:

$$\begin{aligned} &X_{\theta^\dagger} \stackrel{d}{=} i\partial_\epsilon U(\hat{g}) \cdot U(\hat{g})^\dagger = \widehat{X}_{\theta^\dagger} + \mathbb{C}(i\partial_\epsilon \hat{g} \cdot \hat{g}^\dagger) = \widehat{X}_{\theta^\dagger}, \qquad \theta^\dagger \stackrel{d}{=} i\partial_\epsilon \hat{g} \cdot \hat{g}^\dagger, \\ &X_\theta \stackrel{d}{=} i\partial_{\epsilon^*} U(\hat{g}) \cdot U(\hat{g})^\dagger = \widehat{X}_\theta + \mathbb{C}(i\partial_{\epsilon^*} \hat{g} \cdot \hat{g}^\dagger) = \widehat{X}_\theta, \qquad \theta \stackrel{d}{=} i\partial_{\epsilon^*} \hat{g} \cdot \hat{g}^\dagger. \end{aligned} \tag{3.51}$$

3.4 SCF method in τ-functional space

Along the soliton theory in the infinite-dimensional fermion Fock space [30, 48, 49, 54], we transcribe the υ-dependent HFT in F_∞ to the one in τ-functional space. We restrict ourselves mainly to the cases of $sl(N)$ and $su(N)$ and the group orbit of the fundamental highest weight vector $|M\rangle$. Let us consider infinite-dimensional charged fermions ψ_i and ψ_i^* $(i \in \mathbb{Z})$ satisfying the canonical anti-commutation relation $\{\psi_i^*, \psi_j\} = \delta_{ij}$ and $\{\psi_i^*, \psi_j^*\} = \{\psi_i, \psi_j\} = 0$. The perfect vacuum $|\mathrm{Vac}\rangle$ and the simple state $|M\rangle$ given by (3.25) are represented in terms of another basis ν_i $(i \in \mathbb{Z})$ and the present fermions ψ_i and ψ_i^* as

$$
\begin{aligned}
&|\mathrm{Vac}\rangle \simeq \nu_0 \wedge \nu_{-1} \wedge \nu_{-2} \wedge \cdots \quad (\wedge : \text{exterior product}), \qquad \langle \mathrm{Vac}|\mathrm{Vac}\rangle = 1,\\
&\psi_i|\mathrm{Vac}\rangle = 0 \quad (i \le 0), \qquad \psi_i^*|\mathrm{Vac}\rangle = 0 \quad (i > 0),\\
&\langle \mathrm{Vac}|\psi_i^* = 0 \quad (i \le 0), \qquad \langle \mathrm{Vac}|\psi_i = 0 \quad (i > 0),\\
&|M\rangle \simeq \nu_M \wedge \nu_{M-1} \wedge \cdots, \qquad \langle M|M\rangle = 1,\\
&|M\rangle = \psi_M \cdots \psi_1 |\mathrm{Vac}\rangle \quad (M > 0), \qquad |M\rangle = \psi_{M+1}^* \cdots \psi_0^* |\mathrm{Vac}\rangle \quad (M < 0).
\end{aligned}
$$

The basis $\{\nu_i | i \in \mathbb{Z}\}$ is given by the column vector with 1 as the i-th row and 0 elsewhere. The number M is called the *charge number*. The fermions ψ_i and ψ_i^* $(i \in \mathbb{Z})$ generate an algebra $gl(\infty) = \{a_{ij}\psi_i\psi_j^*;$ all but a finite number of $a_{ij}'s$ are $0\}$ satisfying $[\psi_i\psi_j^*, \psi_k\psi_l^*] = \delta_{jk}\psi_i\psi_l^* - \delta_{il}\psi_k\psi_j^*$. We consider further a bigger Lie algebra than $gl(\infty)$ so as to include a Heisenberg subalgebra (bosons). Following Appendix C, it is defined as the vector space $a_\infty = \{\sum_{i,j\in\mathbb{Z}} a_{ij} : \psi_i\psi_j^* : + \mathbb{C}\cdot c,\ a_{ij} = 0,$ for $|i-j| > \mathbb{N}\}$ and $: \psi_i\psi_j^* := \psi_i\psi_j^* - \delta_{ij}\cdot c$ $(j \le 0)$. Define a KM bracket among such elements $X_a = \widehat{X}_a + \mathbb{C}\cdot c$ as

$$
[X_a, X_b]_{\mathrm{KM}} = \widehat{X}_{[a,b]} + \alpha(a,b)\cdot c, \qquad [X_a, c]_{\mathrm{KM}} = 0, \qquad \widehat{X}_a = \sum_{i.j\in\mathbb{Z}} a_{ij} : \psi_i\psi_j^* : .
$$

For detail see Appendix C. A Heisenberg subalgebra S [49] is defined as

$$
\mathcal{S} = \oplus_{k\neq 0}\Lambda_k + \mathbb{C}\cdot c, \qquad \Lambda_k \stackrel{d}{=} \sum_{i\in\mathbb{Z}} : \psi_i\psi_{i+k}^* : \qquad (k \in \mathbb{Z}).
$$

From the definition of the normal-ordered product, the boson algebra is obtained as $[\Lambda_k, \Lambda_l] = k\delta_{k+l,0}\cdot c$. Λ_k is called the shift operator and Λ_0 belongs to the center. Suppose level-one $c = 1$. Let $F^{(M)}$ be a linear span of semi-infinite monomials with charge number M. For the representation of Λ_k on $F^{(M)}$, $\Lambda_k|M\rangle = 0$ holds for $k > 0$. All the elements $\Lambda_{-k_s}\cdots\Lambda_{-k_1}|M\rangle$ $(0 < k_1 \le k_2 \le \cdots \le k_s)$ are linear independent with each other. Thus, we have obtained an irreducible representation of the algebra $\mathcal{S}$ in the fermion space $F^{(M)}$. This is isomorphic to the representation of $\mathcal{S}$ in the corresponding boson space $B^{(M)}$ below.

Let σ_M denote this isomorphism

$$
\begin{aligned}
\sigma_M :\quad & F^{(M)} \mapsto B^{(M)} = \mathbb{C}(x_1, x_2, \dots) \qquad (\deg(x_j) = j), \qquad |M\rangle \mapsto 1,\\
& \Lambda_k \mapsto \frac{\partial}{\partial x_k}, \qquad \Lambda_{-k} \mapsto kx_k \qquad (k > 0), \qquad \Lambda_0 \mapsto M.
\end{aligned}
$$

A mapping operator is introduced as $\sigma_M \stackrel{d}{=} \langle M|e^{H(x)}$, $H(x) \equiv \sum_{j\ge 1} x_j\Lambda_j$ (*Hamiltonian* in τ-FM) [30]. Then we see the correspondence of $F^{(M)} = \oplus_{k\ge 0}F_k^{(M)}$ with $B^{(M)} = \oplus_{k\in\mathbb{Z}}B_k^{(M)}$ and $B^{(M)} = \tilde{z}^M\mathbb{C}(x_1, x_2, \dots)$, where we have defined the direct sum of maps $\sigma = \oplus_{M\in\mathbb{Z}}\sigma_M$ and have introduced a new variable $\tilde{z}$ to keep a track of the index M and then $\sigma(\psi^{(M)}) = \tilde{z}^M$ for $\psi^{(M)} \mapsto F^{(M)}$.

The contravariant hermitian form on the $B^{(M)}$ is given as

$$\langle 1|1\rangle = 1, \qquad \left(\frac{\partial}{\partial x_k}\right)^\dagger = kx_k, \qquad \langle P|Q\rangle = P^*\left(\frac{\partial}{\partial x_1}, \frac{1}{2}\frac{\partial}{\partial x_2}, \dots\right) Q(x)|_{x=0}, \tag{3.52}$$

where the P^* means the complex conjugation of all the coefficients of the polynomial P and $x = (x_1, x_2, \dots)$.

We construct a representation in $B^{(M)}$ in reduction to $\widehat{sl}(N)$. Let the generating series be

$$\Psi(p) = \sum_{j\in\mathbb{Z}} p^j \psi_j, \qquad \Psi^*(p) = \sum_{j\in\mathbb{Z}} p^{-j}\psi_j^* \qquad (p \in \mathbb{C}\backslash 0).$$

and introduce Schur polynomials $S_k(x)$ given in Appendix D. It should be emphasized that in [1] and [32], we already have obtained explicit expressions for the basic elements

$$\begin{aligned}
&\sigma_M: \ \ \psi_i\psi_j^* \mapsto z_{ij}(x, \widetilde{\partial}_x), \qquad \sigma_M: \ :\psi_i\psi_j^* :\mapsto \widetilde{z}_{ij}(x,\widetilde{\partial}_x) \qquad (= z_{ij} - \delta_{ij}, \ j\leq 0)\,,\\
&\widetilde{\partial}_x \stackrel{d}{=} \left(\frac{\partial}{\partial x_1}, \frac{1}{2}\frac{\partial}{\partial x_2}, \cdots\right),\\
&z_{ij}(x, \widetilde{\partial}_x) = \sum_{\mu,\nu\geq 0, \ k\geq 0} S_{i+k+\mu-M}(x) S_{-j-k+\nu+M}(-x) S_\mu(-\widetilde{\partial}_x) S_\nu(\widetilde{\partial}_x),
\end{aligned} \tag{3.53}$$

which makes a crucial role to construct a υ-dependent HFT on $U(\hat{g})|M\rangle$ as shown later. For any element of the $gl(\infty)$ and the a_∞ in the $B^{(M)}$, we have got

$$\begin{aligned}
&\sigma_M: \ \ X_a = \sum_{i,j\in\mathbb{Z}} a_{ij}\psi_i\psi_j^* \mapsto \sum_{i,j\in\mathbb{Z}} a_{ij} z_{ij}(x, \widetilde{\partial}_x),\\
&\sigma_M: \ \ X_a = \sum_{i,j\in\mathbb{Z}} a_{ij} :\psi_i\psi_j^*: + \mathbb{C}\cdot 1 \mapsto \sum_{i,j} a_{ij}\widetilde{z}_{ij}(x,\widetilde{\partial}_x) + \mathbb{C}\cdot 1.
\end{aligned} \tag{3.54}$$

Using $\exp\{H(x)\} = \exp\big\{\sum_{j\geq 1} x_j\Lambda_1^j\big\} = \sum_{j\geq\ 0}\Lambda_j S_j(x)$ and $H(x)|M\rangle = 0$ due to $\Lambda_j|M\rangle = 0$, x-evolution of the infinite-dimensional fermion operator is given in terms of the Schur polynomials as

$$\begin{aligned}
&e^{H(x)}\psi_i e^{-H(x)} = \sum_{j=0}^\infty \psi_j S_{i-j}(x), \qquad e^{H(x)}\psi_i^* e^{-H(x)} = \sum_{j=0}^\infty \psi_j^* S_{j-i}(-x),\\
&e^{H(x)}\psi(p)e^{-H(x)} = \psi(p)e^{\sum_{j\geq 1} p^j x_j}, \qquad e^{H(x)}\psi^*(p)e^{-H(x)} = \psi^*(p)e^{\sum_{j\geq 1} p^j x_j}.
\end{aligned} \tag{3.55}$$

Following Appendix D, under the action $U(g)$, the group orbit of the highest weight vector $|M\rangle$ is mapped to a space of τ-function $\tau_M(x,g) = \langle M|e^{H(x)}U(g)|M\rangle$. Let the Plücker coordinates $v^{M,\ M-1,\dots,1}_{i_M,i_{M-1},\dots,i_1}(g)$ be $v^{M,M-1,\dots,1}_{i_M,i_{M-1},\dots,i_1}(g) = \det|g^{M,\ M-1,\dots,1}_{i_M,i_{M-1},\dots,i_1}|$, $g^{M,M-1,\dots,1}_{i_M,i_{M-1},\dots,i_1}$: Matrix located on intersection of rows $i_M, i_{M-1}, \dots, i_1$ and columns $M, M-1, \dots, 1$ of g and $i_M = Nr + \alpha$ etc. The Schur-polynomial expression for the τ-function is given in a compact form as

$$\tau_M(x,g) = \sum_{i_M > i_{M-1} > \cdots > i_1} v^{M,M-1,\dots,1}_{i_M,i_{M-1},\dots,i_1}(g) S_{i_M-M, i_{M-1}-(M-1),\dots,i_1-1}(x). \tag{3.56}$$

By using (3.5) and (3.55), the derivation of the above is made as follows:

$$\tau_M(x,g) = \langle\mathrm{Vac}|\psi_1^*\cdots\psi_M^* e^{H(x)} \sum_{i_M>i_{M-1}>\cdots>i_1} v^{M,M-1,\dots,1}_{i_M,i_{M-1},\dots,i_1}(g)\psi_{i_M}\cdots\psi_{i_1}|\mathrm{Vac}\rangle$$

$$
\begin{aligned}
&= \sum_{i_M > i_{M-1} > \cdots > i_1} v^{M,M-1,\dots,1}_{i_M,i_{M-1},\dots,i_1}(g) \\
&\times \sum_{I_M=M}^{1} \sum_{I_{M-1}=M}^{1} \cdots \sum_{I_1=M}^{1} S_{i_M - I_M}(x) S_{i_{M-1}-I_{M-1}}(x) \cdots S_{i_1 - I_1}(x) \langle \mathrm{Vac} | \psi^*_1 \cdots \psi^*_M \psi_{I_M} \cdots \psi_{I_1} | \mathrm{Vac} \rangle \\
&= \sum_{i_M > i_{M-1} > \cdots > i_1} v^{M,M-1,\dots,1}_{i_M,i_{M-1},\dots,i_1}(g) \qquad (3.57) \\
&\times \det \begin{vmatrix} S_{i_M-M}(x) & S_{i_M-M+1}(x) & S_{i_M-M+2}(x) & \cdots & S_{i_M-M+(M-1)}(x) \\ S_{i_{M-1}-(M-1)-1}(x) & S_{i_{M-1}-(M-1)}(x) & S_{i_{M-1}-(M-1)+1}(x) & \cdots & S_{i_{M-1}-(M-1)+(M-2)}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ S_{i_1-1-(M-1)}(x) & S_{i_1-1-(M-1)+1}(x) & S_{i_1-1-(M-1)+2}(x) & \cdots & S_{i_1-1-(M-1)+(M-1)}x) \end{vmatrix}.
\end{aligned}
$$

This equation reads (3.56), the generalization of which to infinite-dimension is given in [49].

To see that the affine Kac–Moody algebra associated with the Lie algebra $\widehat{gl}_\infty$ is contained as a subalgebra, we give a reduction of $\widehat{gl}_\infty$ to $\widehat{sl}_n$. A subalgebra X_a $\big(= \sum_{i,j\in\mathbb{Z}} a_{ij} : \psi_i \psi^*_j : + \mathbb{C} \cdot 1 \big)$ of a_∞ is called n-reduced if and only if the following two conditions are satisfied:

$$
\text{(i)} \quad a_{i+N,j+N} = a_{ij} \quad (i,j \in \mathbb{Z}) \qquad \text{and} \qquad \text{(ii)} \quad \sum_{i=1}^{N} a_{i,i+Nj} = 0 \quad (j \in \mathbb{Z}).
$$

From (i) and $\Lambda_{Nj} = \sum_{i\in\mathbb{Z}} : \psi_i \psi^*_{i+Nj} :$ $(j \in \mathbb{Z})$, $[X_a, \Lambda_{Nj}] = 0$ is proved. This means $\tau_M(x, \hat{g})$ $(\hat{g} \in \widehat{sl}(N))$ is independent on x_{Nj}, though Λ_{Nj} does not satisfy (ii). As a result, the Hirota's equation includes no x_{Nj}. Adopting the prescription for τ-FM, we transcribe the fundamental equation (3.29) for υ-dependent HFT on $U(\hat{g})|M\rangle \subset F^{(M)}$ into the corresponding function $\subset B^{(M)}$ in the following forms:

$$
\begin{aligned}
&(1) \;\; U(\hat{g})|M\rangle \big(U(\hat{g}) = e^{X_\gamma};\ X_\gamma \in \widehat{sl}(N) \big) \mapsto N\text{-reduced KP } \tau\text{-function}, \\
&\quad \tau_M(x, \hat{g})_{\mathrm{NKP}} = \langle M | e^{H(x)} U(\hat{g}) | M \rangle, \qquad \frac{\partial}{\partial x_{Nj}} \tau_M(x, \hat{g})_{\mathrm{NKP}} = 0.
\end{aligned}
$$

The $\widehat{gl}(\infty)$ group symmetry transformation acting on the τ-function of the KP hierarchy was studied in [51, 30]. The KP hierarchy was also studied extensively together with the KdV hierarchy by Gelfand and Dickey [52]. The generalized KP hierarchy was obtained out by making use of the Gelfand–Dickey approach via the algebra of pseudo-differential operators [53]. The τ-function $\tau_M(x, \hat{g})_{\mathrm{NKP}}$ has a υ-dependence through $\hat{g}$ $(= e^\gamma)$ in which the anti-hermitian matrix γ is given as $\gamma(z) = \sum_{r\in\mathbb{Z}} \gamma_r z^r$ with $z = \exp\big(i2\pi \frac{\upsilon}{\Upsilon}\big)$.

(2) Quasi-particle and vacuum state $\mapsto$ Hirota's bilinear equation (see Appendix E [30, 39]):

$$
\begin{aligned}
&\sum_{\alpha=1}^{N} \sum_{r\in\mathbb{Z}} \psi_{Nr+\alpha} U(\hat{g})|M\rangle \otimes \psi^*_{Nr+\alpha} U(\hat{g})|M\rangle = 0 \\
&\qquad \mapsto \sum_{j\geq 0} S_j(2y) S_{j+N+1}(-\tilde{D}) \exp\left(\sum_{s\geq 1} y_s D_s \right) \tau_M(x, \hat{g})_{\mathrm{NKP}} \cdot \tau_M(x, \hat{g})_{\mathrm{NKP}} = 0. \qquad (3.58)
\end{aligned}
$$

$D = (D_1, D_2, \dots)$ denotes the Hirota's bilinear differential operator and $\widetilde{D} = \big(D_1, \frac{1}{2} D_2, \dots\big)$.

(3) υ-dependent HF equation on $U(\hat{g})|M\rangle \mapsto \upsilon$-dependent HF equation on $\tau_M(x, \hat{g})_{\mathrm{NKP}}$:

$$
i\partial_\upsilon U\{\hat{g}(\upsilon)\}|M\rangle = H_{F_\infty}\{\hat{g}(\upsilon)\} U\{\hat{g}(\upsilon)\}|M\rangle
$$

$$\mapsto i\partial_\upsilon \tau_M\{x, \hat{g}(\upsilon)\}_{N\mathrm{KP}} = H_{F_\infty;\mathrm{HF}}\{x, \widetilde{\partial}_x, \hat{g}(\upsilon)\}\tau_M\{x, \hat{g}(\upsilon)\}_{N\mathrm{KP}}, \tag{3.59}$$

in which it is seen that an explicit and important role of the υ-dependence of τ-function appears. Using (3.53) and (3.54), $H_{F_\infty;\mathrm{HF}}(x, \widetilde{\partial}_x, \hat{g})$ is given as

$$H_{F_\infty;\mathrm{HF}}(x, \widetilde{\partial}_x, \hat{g}) = \sum_{r,s\in\mathbb{Z}} \{\mathcal{F}_r(\hat{g})\}_{\alpha\beta} \widetilde{z}_{N(s-r)+\alpha, Ns+\beta}(x, \widetilde{\partial}_x),$$

$$\{\mathcal{F}_r(\hat{g})\}_{\alpha\beta} = h_{\alpha\beta}\delta_{r,0} + [\alpha\beta|\gamma\delta](\mathcal{W}_r)_{\delta\gamma}, \qquad (\mathcal{W}_r)_{\alpha\beta} = \sum_{\gamma=1}^{M}\sum_{s\in\mathbb{Z}} (g_s)_{\alpha\gamma}(g^\dagger_{s-r})_{\gamma\beta}. \tag{3.60}$$

3.5 Laurent coefficients of soliton solutions for $\widehat{sl}(N)$ and for $\widehat{su}(N)$

We here show typical τ-functions called n-soliton solutions.

On $\widehat{gl}(\infty)$ [51]: We get a τ-function for $\widehat{gl}(\infty)$ as

$$\tau_{M;n;a,p,q}(x) = \langle M|e^{H(x)} e^{\sum_{\mu\geq 1}^{n} a_\mu \psi(p_\mu)\psi^*(q_\mu)}|M\rangle, \tag{3.61}$$

which is a famous solution of the KP hierarchy obtained from (3.58) [54, 48, 49]. As was shown in [51], from the second line of (3.55) and the Wick's theorem, we get a determinantal formula for τ-function as

$$\tau_{M;n;a,p,q}(x) = \det\left\{\delta_{\mu\nu} + a_\mu \frac{p_\mu}{p_\mu - q_\nu}\left(\frac{p_\mu}{q_\nu}\right)^M e^{\xi(x,p_\mu)-\xi(x,q_\nu)}\right\}.$$

If we use $\Gamma(p,q)^2\tau = 0$ for a *good* formal power series of τ, we have an explicit form of $\tau_{M;n;a,p,q}(x)$

$$\begin{aligned}\tau_{M;n;a,p,q}(x) &= e^{\sum_{\mu=1}^{n}\left\{a_\mu \frac{p_\mu}{p_\mu - q_\nu}\left(\frac{p_\mu}{q_\nu}\right)^M \Gamma(p_\mu, q_\mu)\right\}} \cdot 1 \\ &= 1 + \sum_{\mu=1}^{n}\left(\frac{p_\mu}{q_\mu}\right)^M e^{\eta_\mu} + \sum_{1\leq\mu<\nu\leq n}\left(\frac{p_\mu}{q_\mu}\right)^M\left(\frac{p_\nu}{q_\nu}\right)^M \frac{(p_\mu - p_\nu)(q_\mu - q_\nu)}{(p_\mu - q_\nu)(q_\mu - p_\nu)} e^{\eta_\mu + \eta_\nu} + \cdots,\end{aligned}$$

where $\xi(x,p) = \sum_{j\geq 1} x_j p^j$ and $\eta_\mu = \xi(x,p_\mu) - \xi(x,q_\mu) + \ln\left(a_\mu \frac{p_\mu}{p_\mu - q_\mu}\right)$.

On $\widehat{u}_\infty$: We get a τ-function for $\widehat{u}_\infty$ as

$$\tau_{M;n;a,p,q}(x) = \langle M|e^{H(x)} e^{\sum_{\mu=1}^{n}\left\{a_\mu\psi(p_\mu)\psi^*(q_\mu) - a^*_\mu \psi\left(\frac{1}{q^*_\mu}\right)\psi^*\left(\frac{1}{p^*_\mu}\right)\right\}}|M\rangle.$$

On $\widehat{sl}(N)$ [32, 48]: Making a special choice of parameters p_μ and q_μ in (3.61) as $q_\mu = \epsilon^{s_\mu}p_\mu$, with $\epsilon = e^{2\pi i/N}$ and $s_\mu = 1, \ldots, N-1$, and using $[X_a, \Lambda_{Nj}] = 0$, we get a τ-function for $\widehat{sl}(N)$ as

$$\tau_{M;n,a,p,\epsilon^s p}(x) = \langle M|e^{H(x)} e^{\sum_{\mu\geq 1}^{n} a_\mu\psi(p_\mu)\psi^*(\epsilon^{s_\mu}p_\mu)}|M\rangle. \tag{3.62}$$

On $\widehat{su}(N)$: Taking the exponent in (3.62) anti-Hermitian, a τ-function for $\widehat{su}(N)$ is given as

$$\tau_{M;n;a,p,\epsilon^s p}(x) = \langle M|e^{H(x)} e^{\sum_{\mu=1}^{n}\left\{a_\mu\psi(p_\mu)\psi^*(\epsilon^{s_\mu}p_\mu) - a^*_\mu\psi\left(\frac{1}{(\epsilon^{s_\mu}p_\mu)^*}\right)\psi^*\left(\frac{1}{p^*}\right)\right\}}|M\rangle.$$

We have a one-soliton solution on $\widehat{su}(N)$ and on the simplest $\widehat{su}(2)$, respectively as

$$\widehat{su}(N):\ \tau_{M;1;a,p,\epsilon^s p}(x) = \langle M|e^{H(x)} e^{\left\{a\psi(p)\psi^*(\epsilon^s p) - a^*\psi\left(\frac{\epsilon^s}{p^*}\right)\psi^*\left(\frac{1}{p^*}\right)\right\}}|M\rangle$$

$$
\begin{aligned}
&= 1 + \epsilon^{-Ms} e^{\eta(x,p,\epsilon^s p)} + \epsilon^{Ms} e^{\eta(x,\epsilon^s/p^*,1/p^*)} \\
&+ \frac{(pp^* - \epsilon^s)(pp^* - \epsilon^{-s})}{(pp^* - 1)(pp^* - 1)} e^{\sum_j x_j (p^j - p^{*-j})(1-\epsilon^{sj}) + \mathrm{Log}\frac{-|a|^2}{(1-\epsilon^s)(1-\epsilon^{-s})}},
\end{aligned}
$$

$$
\eta(x, p, \epsilon^s p) = \sum_{j\geq 1} x_j p^j (1 - \epsilon^{sj}) + \mathrm{Log}\left\{a/(1-\epsilon^s)\right\},
$$

$$
\eta\left(x, \epsilon^s/p^*, 1/p^*\right) = \sum_{j\geq 1} -x_j p^{*-j}(1-\epsilon^{sj}) + \mathrm{Log}\left\{-a^*/(1-\epsilon^{-s})\right\},
$$

$$
\begin{aligned}
\widehat{su}(2): \quad \tau_{M;1;a,p,-p}(x) &= \langle M| e^{H(x)} e^{\left\{a\psi(p)\psi^*(-p) - a^*\psi\left(-\frac{1}{p^*}\right)\psi^*\left(\frac{1}{p^*}\right)\right\}} |M\rangle \\
&= 1 + e^{2\sum_{j\in\mathrm{odd}} x_j p^j + \mathrm{Log}\frac{a}{2}} + e^{-2\sum_{j\in\mathrm{odd}} x_j \frac{1}{p^{*j}} + \mathrm{Log}\frac{-a^*}{2}} + \frac{(|p|^2+1)^2}{(|p|^2-1)^2} e^{2\sum_{j\in\mathrm{odd}} x_j \frac{|p|^j - 1}{p^{*j}} + \mathrm{Log}\frac{-|a|^2}{4}}.
\end{aligned}
$$

Finally, we give a reduction of soliton solution in a_∞ to the simplest case of $\widehat{sl}(2)$ (KdV). A subalgebra X_a $(= \sum_{r,s\in\mathbb{Z}} a_{rs} : \psi_r i \psi_s^* : + \mathbb{C}\cdot 1)$ of a_∞ and the Chevalley bases for $sl(2)$ [48, 51] are expressed as

$$
X_a = \sum_{r,s\in\mathbb{Z}} (a_r)_{\alpha\beta} : \psi_{2(s-r)+\alpha}\psi^*_{2s+\beta} : + \mathbb{C}\cdot 1 \qquad (\alpha,\beta = 1,2; \mathrm{Tr}\, a_r = 0,\ a_r \in \widehat{sl}(2)),
$$

$$
\widehat{e} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \widehat{f} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \widehat{h} = \widehat{h}_+ + \widehat{h}_- = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{(Chevalley bases for } sl(2)).
$$

Using $\psi(p) = \sum_{r\in\mathbb{Z}} \psi_r p^r$ and $\psi(p)^* = \sum_{r\in\mathbb{Z}} \psi_r^* p^{-r}$, the algebra X_a^n for n-soliton in (3.62) is computed as

$$
\begin{aligned}
X_a^N &= \sum_{\mu=1}^{n} a_\mu \psi(p_\mu)\psi^*(-p_\mu) = \sum_{r,s\in\mathbb{Z}} \sum_{\mu=1}^{n} a_\mu \\
&\times \Big\{ \widehat{h}_+ \cdot \psi_{2(s-r)+1} p_\mu^{2(s-r)+1} \psi^*_{2s+1}(-p_\mu)^{-(2s+1)} + \widehat{e}\cdot \psi_{2(s-r)+1} p_\mu^{2(s-r)+1} \psi^*_{2s+2}(-p_\mu)^{-(2s+2)} \\
&+ \widehat{f}\cdot \psi_{2(s-r)+2} p_\mu^{2(s-r)+2} \psi^*_{2s+1}(-p_\mu)^{-(2s+1)} + \widehat{h}_- \cdot \psi_{2(s-r)+2} p_\mu^{2(s-r)+2} \psi^*_{2s+2}(-p_\mu)^{-(2s+2)} \Big\} \\
&= \sum_{r,s\in\mathbb{Z}} \sum_{\mu=1}^{N} a_\mu \begin{bmatrix} -p_\mu^{-2r}, & p_\mu^{-(2r+1)} \\ -p_\mu^{-(2r-1)}, & p_\mu^{-2r} \end{bmatrix}_{\alpha\beta} \psi_{2(s-r)+\alpha}\psi^*_{2s+\beta}.
\end{aligned} \tag{3.63}
$$

Thus, we obtain an n-soliton solution $a^n(z) = \sum_{r\in\mathbb{Z}} a_r^n z^r + \mathbb{C}\cdot 1$ $\left(\mathbb{C} = \sum_{\mu=1}^{n} a_\mu/2\right)$ for $\widehat{sl}(2)$ in a matrix as

$$
a_r^n = \sum_{\mu=1}^{n} a_r(p_\mu, a_\mu), \qquad a_r(p,a) = a\cdot \begin{bmatrix} -p^{-2r}, & p^{-(2r+1)} \\ -p^{-(2r-1)}, & p^{-2r} \end{bmatrix} \qquad (a_\mu = \mathrm{const}).
$$

Restricting a solution to the case of $\widehat{su}(2)$, along the similar way as the above, from (3.63) we also get

$$
\begin{aligned}
X_\gamma^n = X_a^n - X_a^{n\dagger} &= \sum_{\mu=1}^{n} \left\{ a_\mu \psi(p_\mu)\psi^*(-p_\mu) - a_\mu^* \psi\left(-\frac{1}{p_\mu^*}\right)\psi^*\left(\frac{1}{p_\mu^*}\right) \right\} \\
&= \sum_{r,s\in\mathbb{Z}} (\gamma_r^n)_{\alpha\beta} : \psi_{2(s-r)+\alpha}\psi_{2s+\beta} : + \sum_{\mu=1}^{n} \left(a_\mu - a_\mu^*\right)/2,
\end{aligned}
$$

which reads

$$\gamma_r^n = \sum_{\mu=1}^{n} \gamma_r(p_\mu, a_\mu), \qquad \gamma_r(p,a) = \begin{bmatrix} -(ap^{-2r} - a^* p^{*2r}), & ap^{-(2r+1)} + a^* p^{*(2r+1)} \\ -(ap^{-(2r-1)} + a^* p^{*(2r-1)}), & ap^{-2r} - a^* p^{*2r} \end{bmatrix}.$$

We can generalize the above n-soliton solutions to the cases of $\widehat{sl}(N)$ and $\widehat{su}(N)$. Using the Chevalley bases for $sl(N)$ and for $su(N)$ [48, 51], the Laurent coefficients can be derived for each case as

$$\widehat{sl}(N): \quad a_r^n\big|_{\text{soliton}} = \sum_{\mu=1}^{n} a_r\left(p_\mu, \epsilon^{s_\mu} p_\mu, b_\mu\right) + \sum_{\mu=1}^{n} \frac{b_\mu}{1-\epsilon},$$

$$a_r \{= a_r\left(p, \epsilon^s p, b\right)_{\alpha\beta}(\alpha, \beta = 1, \ldots, N)\} = (bp^{-Nr+\alpha-\beta}\epsilon^{-s\beta})$$

$$\simeq bp^{-Nr} \begin{bmatrix} 1 & p^{-1} & \cdots & \cdots & p^{-(N-1)} \\ p & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & p^{-1} \\ p^{N-1} & \cdots & \cdots & p & 1 \end{bmatrix} \begin{bmatrix} \epsilon^{-s} & & & & \\ & \epsilon^{-2s} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \epsilon^{-Ns} \end{bmatrix},$$

$$\widehat{su}(N): \quad \gamma_r^n\big|_{\text{soliton}} = \sum_{\mu=1}^{n} \gamma_r\left(p_\mu, \epsilon^{s_\mu} p_\mu, b_\mu\right) + \sum_{\mu=1}^{n} \left(\frac{b_\mu}{1-\epsilon} - \frac{b_\mu^*}{1-\epsilon^*}\right),$$

$$\gamma_r \{= \gamma_r\left(p, \epsilon^s p, b\right)_{\alpha\beta}(\alpha, \beta = 1, \ldots, N)\} = a_r - a_{-r}^{*\mathrm{T}} = (bp^{-Nr+\alpha-\beta}\epsilon^{-s\beta} - b^*\epsilon^{-s\alpha}p^{*Nr+\beta-\alpha})$$

$$\simeq bp^{-Nr} \begin{bmatrix} 1 & p^{-1} & \cdots & \cdots & p^{-(N-1)} \\ p & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & p^{-1} \\ p^{N-1} & \cdots & \cdots & p & 1 \end{bmatrix} \begin{bmatrix} \epsilon^{-s} & & & & \\ & \epsilon^{-2s} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \epsilon^{-Ns} \end{bmatrix}$$

$$-b^* p^{*Nr} \begin{bmatrix} \epsilon^{-s} & & & & \\ & \epsilon^{-2s} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \epsilon^{-Ns} \end{bmatrix} \begin{bmatrix} 1 & p^* & \cdots & \cdots & p^{*(N-1)} \\ p^{*-1} & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & p^* \\ p^{*-(N-1)} & \cdots & \cdots & p^{*-1} & 1 \end{bmatrix}.$$

Suppose the external parameter υ to be a time t and solve the TDHFEQ by restricting a solution space to the above spaces. We get a soliton solution, i.e., a solitary wave propagating on a surface rather than a colliding soliton [55]. This is in contrast with a two-dimensional soliton [56], i.e., *dromion* [57], of the Davey–Stewartson equation (DSE) [58] which provides a two-dimensional generalization of the celebrated nonlinear Schrödinger equation (NLSE) [57]. The *dromion* for $gl(2\infty)$ was derived from the standpoint of a Clifford algebra with generators of infinite free fermions, in terms of a reduction of the two-component KP hierarchy [59, 60, 61]. As Kac and van der Leur pointed out [61], the *dromion* solution of the DSE was first studied from the point of view of the spinor formalism by Heredero et al. [62].

3.6 Summary and discussions

We have transcribed a bilinear equation for υ-HFT into the corresponding τ-function using regular representations for groups [3] and Schur polynomials. The concept of quasi-particle and vacuum in SCFT is connected with bilinear differential equations. So far SCFM has focused mainly on construction of various types of boson expansions for quantum fluctuations of mean-field(MF) rather than taking the bilinear differential equations(Plücker relations) into account. These methods turn out to be essentially equivalent with each other. Various subgroup-manifolds consisting of several *loop-group* paths [36] exist innumerably in Gr_M relating to collective motions. To go beyond the perturbative method in terms of the collective variables, we have aimed to construct υ-HFT on affine Kac–Moody algebras along soliton theory, using infinite-dimensional fermions. These fermions have been introduced through Laurent expansion of finite-dimensional fermions with respect to the degrees of freedom of fermions related to MF. Consequently υ-SCFT on F_∞ leads to dynamics on infinite-dimensional Grassmannian Gr_∞. Gr_M is identified with Gr_∞ affiliated with a manifold obtained by reduction of gl_∞ to $sl(N)$ and $su(N)$(reduction of KP hierarchy to DS, NLS and KdV hierarchies). We have given explicit expressions for Laurent coefficients of soliton solutions for $\widehat{sl}(N)$ and $\widehat{su}(N)$ using Chevalley bases for $sl(N)$ and $su(N)$. In this sence the algebraic treatment of extracting subgroup-orbits with $z(|z| = 1)$ from Gr_M exactly forms the differential equation(Hirota's bilinear equation). The υ-SCFT on F_∞ results in *gauge theory of fermions* and *collective motion* due to quantal fluctuations of υ-dependent SCMF potential is attributed to *motion of the gauge of fermions* in which *common gauge factor* causes interference among fermions. *The concept of particle and collective motions* is regarded as the compatible condition for particle and collective modes. The collective variables may have close relation with a spectral parameter in soliton theory. These show that υ-SCFT on F_∞ presents us *new algebraic method on* S^1 for microscopic understanding of fermion many-body systems.

We have studied the relation between υ-SCFT and soliton theory on group manifold and shown that both the theories describe dynamics on each Grassmannian Gr which is the group orbit of highest weight vector. The former stands on the finite-dimensional fermion operators but the latter does on the infinite-dimensional ones. Each Gr is just identical with the solution space for respective finite and infinite set of bilinear differential equations on the boson space mapped from those on the fermion space. We have investigated the dynamics on υ-dependent HF manifold using regular representations for groups [16]. A picture of quasi particle and vacuum in υ-SCFT is connected with the bilinear differential equation. υ-HFT on finite-dimensional Fock space is embedded into υ-HFT on infinite one. The wave function in an SC Υ-periodic MF potential becomes dependent on the Laurent parameter z on a unit circle S^1. This owes to the introduction of affine Kac–Moody algebra by infinite-dimensional fermion operators, Laurent expansion of finite-dimensional fermions with respect to z. The Plücker relation on *coset variables* becomes analogous to Hirota's bilinear form. The υ-SCFM has been mainly devoted to the construction of boson-coordinate systems rather than that of soliton solution by τ-FM. It turns out that both the methods are equivalent with each other due to the Plücker relation defining Gr. From *loop* group viewpoint and with clearer physical picture we have proposed description of particle and collective motions in υ-SCFT on F_∞ in relation to iso-spectral equation in soliton theory. Then the υ-SCFT on F_∞ may be regarded as soliton theory in the sense that it bases on Gr_∞ and may describe dynamics on infinite set of real fermion-harmonic oscillators though the soliton theory describes dynamics on complex ones. The soliton equation is nothing but the bilinear equation and the boson coordinate x_k with highest degree plays a role of an evolutional variable on τ-functional space (FS) on which in υ-HFT, the bilinear equation provides algebraic means to extract subgroup orbits parametrized with z from Gr_M. The infinite set of x_k becomes coordinates on τ-FS and their υ-evolution yield trajectories of the SCF Hamiltonian $H^p_{F_\infty}$.

Though we have started with a periodic potential to introduce infinite-dimensional fermions, it is easy to see that υ-dependence with periodicity Υ is by no means a necessary condition. The fact that Schrödinger function is dependent on an unit circle S^1, however, makes a crucial role for construction of infinite-dimensional fermions. As pointed out in [32], it turns out that the fully parametrized υ-dependent SCF Hamiltonian is made up of only the υ-dependent Hamiltonian $H_{F_\infty;\mathrm{HF}}$. Then, we have a very important question why infinite-dimensional Lie algebras work well in fermion systems. As concerns this problem, Pan and Draayer (PD) [63] have developed an infinite-dimensional algebraic approach using affine Lie algebras $\widehat{su}(2)$ and $\widehat{su}(1,1)$. They have introduced fermion pair operators with two parameters for the general pairing Hamiltonian and boson operators through Jordan–Schwinger fermion-boson mapping for an exactly solvable $su(2)$ Lipkin–Meshkov–Glick (LMG) model [42]. They have obtained analytical expressions for exact eigenvalues and eigenfunctions of this Hamiltonian based on the Bethe anzatz (BA), from which BA equation [64] or Richardson equation [65] is derived.

It is interesting to study a relationship between various subgroup-manifolds of Gr_∞ and collective sub-manifolds of υ-SCF Hamiltonian by using a simple and exactly solvable LMG model. Notwithstanding, it is possible to provide a theoretical frame of formal RPA [34, 35] as a tool of truncating a collective motion with only one normal mode, i.e., a collective submanifold out of Gr_∞. As mentioned in [34, 35], the collective submanifold may be interpreted as a rotator on curved surface in Gr_∞. It is stressed that the υ-HFT on F_∞ describes a dynamics on real fermion-harmonic oscillators while soliton theory does the same but on complex oscillators. This remark gives us an attractive task to extend the υ-HFT on real space $\widehat{su}(N)$ to the theory on complex space $\widehat{sl}(N,C)$ removing the restriction $|z|=1$. We have discussed a close connection between υ-SCFM and τ-FM on an abstract fermion Fock space and denoted them independently on S^1. It means that algebro-geometric structures of *infinite*-dimensional fermion many-body systems is also realisable in *finite*-dimensional ones. The υ-dependent HF equation on $\tau_M(x,\hat{g})$, however, should lead to multi-circles, relating closely to a problem of construction of multi-dimensional soliton theory [66, 67, 61]. It is also a very exciting problem to investigate such new motions on the multi-circles ($\mathbf{d}$: Number of circles) in finite fermion many-body systems. As suggested by the referee, the motions, on the other hand, may be related to the coupled $(\mathbf{d}+1)$D systems [61] or the linear flows on the Birkhoff strata of the universal Sato Grassmannian [68].

4 RPA equation embedded into infinite-dimensional Fock space

4.1 Introduction

The purpose of this section is to give a geometrical aspect of RPA equation (RPAEQ) [50, 69] and an explicit expression for the RPAEQ with a normal mode on F_∞. We also argue about the relation between a *loop* collective path and a formal RPAEQ (FRPAEQ). Consequently, it can be proved that the usual perturbative method with respect to periodic collective variables η and η^* in TDHFT [25], is involved in the present method which aims for constructing TDHFT on the affine KM algebra. It turns out that the collective submanifold is exactly a rotator on a curved surface in the Gr_∞. If we could arrive successfully at our final goal of clarifying relation between the SCFT and the soliton theory on a group, the present work may give us important clues for description of large-amplitude collective motions in nuclei and molecules and for construction of multi-dimensional soliton equations [61, 66] since the collective motions usually occur in multi-dimensional *loop* space.

4.2 Construction of formal RPA equation on F_∞

We construct the FRPAEQ on F_∞. We put the following canonicity conditions which guarantee the variables (ϵ,ϵ^*) to be an orthogonally canonical coordinate system [6, 25, 32]:

$$\langle\hat{g}|\partial_\epsilon|\hat{g}\rangle \stackrel{d}{=} \langle M|U(\hat{g}^\dagger)\partial_\epsilon U(\hat{g})|M\rangle = \tfrac{1}{2}\epsilon^*, \qquad \langle\hat{g}|\partial_{\epsilon^*}|\hat{g}\rangle \stackrel{d}{=} \langle M|U(\hat{g}^\dagger)\partial_{\epsilon^*} U(\hat{g})|M\rangle = -\tfrac{1}{2}\epsilon. \quad (4.1)$$

Previously we define the infinitesimal generators of the collective submanifold X_θ and $X_{\theta^\dagger}$ (3.51) in which the term $\mathbb{C}(\hat{g}^{-1}\partial_\epsilon\hat{g})$ is proved to vanish. From these infinitesimal generators and $\partial_{\epsilon^*}\langle\hat{g}|\partial_\epsilon|\hat{g}\rangle - \partial_\epsilon\langle\hat{g}|\partial_{\epsilon^*}|\hat{g}\rangle$, we obtain the *weak* orthogonality condition

$$1 = \langle\hat{g}|[X_\theta, X_{\theta^\dagger}]|\hat{g}\rangle = \sum_{\alpha=1}^{M}\sum_{\gamma=1}^{N}\sum_{r\in\mathbb{Z}}([\theta,\theta^\dagger]_r)_{\alpha\gamma}(\mathcal{W}_{-r})_{\gamma\alpha} + \sum_{r\in\mathbb{Z}} r\,\mathrm{Tr}(\theta_r\theta^\dagger_{-r}), \tag{4.2}$$

where we have used (3.26) and (3.34).

As shown in [32], using Lax's ideas [70] we recast (3.47) and $D_t\hat{g} = \mathcal{F}(\hat{g})\hat{g}$, and (3.51) into

$$\begin{aligned}
&D_t\hat{g} = \mathcal{F}(\hat{g})\hat{g}, \qquad \partial_t\hat{g}^0 = 0, \qquad \mathcal{F}(\hat{g}) = \mathcal{F}(\hat{g}^0),\\
&i\partial_\epsilon\hat{g} = \theta^\dagger(\hat{g})\hat{g}, \qquad \theta^\dagger(\hat{g}) = \theta^\dagger(\hat{g}^0) + \hat{g}^0(\partial_\epsilon\hat{\mathfrak{e}})\hat{g}^{0\dagger}\cdot t,\\
&i\partial_{\epsilon^*}\hat{g} = \theta(\hat{g})\hat{g}, \qquad \theta(\hat{g}) = \theta(\hat{g}^0) + \hat{g}^0(\partial_{\epsilon^*}\hat{\mathfrak{e}})\hat{g}^{0\dagger}\cdot t.
\end{aligned} \tag{4.3}$$

Upon introduction of $E = \sum_{\alpha=1}^{M}\epsilon_\alpha(\epsilon,\epsilon^*)$, the canonicity condition (4.1) transforms into

$$\begin{aligned}
&\langle\hat{g}|\partial_\epsilon|\hat{g}\rangle = \langle\hat{g}^0|\partial_\epsilon|\hat{g}^0\rangle - i\partial_\epsilon E\cdot t = \tfrac{1}{2}\epsilon^* - i\partial_\epsilon E\cdot t,\\
&\langle\hat{g}|\partial_{\epsilon^*}|\hat{g}\rangle = \langle\hat{g}^0|\partial_{\epsilon^*}|\hat{g}^0\rangle - i\partial_{\epsilon^*} E\cdot t = -\tfrac{1}{2}\epsilon - i\partial_{\epsilon^*} E\cdot t.
\end{aligned} \tag{4.4}$$

From (4.4), the *weak* orthogonality condition (4.2) is expressed as

$$1 = \partial_{\epsilon^*}\langle\hat{g}|\partial_\epsilon|\hat{g}\rangle - \partial_\epsilon\langle\hat{g}|\partial_{\epsilon^*}|\hat{g}\rangle = \partial_{\epsilon^*}\langle\hat{g}^0|\partial_\epsilon|\hat{g}^0\rangle - \partial_\epsilon\langle\hat{g}^0|\partial_{\epsilon^*}|\hat{g}^0\rangle = \langle\hat{g}^0|[X_{\theta(\hat{g}^0)}, X_{\theta^\dagger(\hat{g}^0)}]|\hat{g}^0\rangle.$$

To satisfy integrability conditions for ϵ, ϵ^* and t, curvatures obtained from (4.3) should vanish;

$$\begin{aligned}
&\mathcal{C}_{t,\epsilon} \overset{d}{=} D_t\theta^\dagger(\hat{g}) - i\partial_\epsilon\mathcal{F}(\hat{g}) + [\theta^\dagger(\hat{g}), \mathcal{F}(\hat{g})] = 0,\\
&\mathcal{C}_{t,\epsilon^*} \overset{d}{=} D_t\theta(\hat{g}) - i\partial_{\epsilon^*}\mathcal{F}(\hat{g}) + [\theta(\hat{g}), \mathcal{F}(\hat{g})] = 0,\\
&\mathcal{C}_{\epsilon,\epsilon^*} \overset{d}{=} i\partial_\epsilon\theta(\hat{g}) - i\partial_{\epsilon^*}\theta^\dagger(\hat{g}) + [\theta(\hat{g}), \theta^\dagger(\hat{g})] = 0,
\end{aligned} \tag{4.5}$$

and $\partial_t\hat{g}^0 = 0$. Here $D_t\theta$ and $D_t\theta^\dagger$ are defined as

$$(D_t\theta)_r = D_{r;t}\theta_r = (i\partial_t + r\omega_c)\theta_r, \qquad (D_t\theta^\dagger)_r = D_{r;t}\theta^\dagger_{-r} = (i\partial_t + r\omega_c)\theta^\dagger_{-r}.$$

The expressions for the curvatures on the quasi-particle frame (QPF) have the same form as those of RPAEQs in the finite Fock space [50]. As mentioned before, the TDHFEQ on the F_∞ leads to the RPAEQ if we take into account only a small fluctuation around a stationary ground-state solution. The form of RPAEQ on the QPF has a following simple geometrical interpretation: Relative vector fields made of the SCF Hamiltonian around each point on *loop* pathes also take the form of RPAEQ around the same point which is in turn a fixed point in the QPF. Thus, the curvature equation in the QPF is regarded as the FRPAEQ on the Gr_∞. Using (3.28), the canonical transformation for $\hat{g}$ is given by

$$\psi_{Nr+\alpha}(\hat{g}) = \sum_{s\in\mathbb{Z}}\sum_{\beta=1}^{N}\psi_{N(r-s)+\beta}(g^0_s)_{\beta\alpha}e^{-i\epsilon_\alpha t},$$

together with its hermitian conjugate. Owing to [50], (4.3) is rewritten on the above QPF as

$$-D_t\hat{g}^\dagger = \mathcal{F}(\hat{g}^\dagger)|_{\mathrm{qpf}}\hat{g}^\dagger, \qquad \mathcal{F}(\hat{g}^\dagger)|_{\mathrm{qpf}} \overset{d}{=} \hat{g}^\dagger\mathcal{F}(\hat{g})\hat{g},$$

$$
\begin{aligned}
&-i\partial_\epsilon \hat{g}^\dagger = \theta^\dagger(\hat{g}^\dagger)|_{\mathrm{qpf}}\hat{g}^\dagger, \qquad \theta^\dagger(\hat{g}^\dagger)|_{\mathrm{qpf}} \overset{d}{=} \hat{g}^\dagger\theta^\dagger(\hat{g})\hat{g},\\
&-i\partial_{\epsilon^*} \hat{g}^\dagger = \theta(\hat{g}^\dagger)|_{\mathrm{qpf}}\hat{g}^\dagger, \qquad \theta(\hat{g}^\dagger)|_{\mathrm{qpf}} \overset{d}{=} \hat{g}^\dagger\theta(\hat{g})\hat{g},
\end{aligned}
\tag{4.6}
$$

The subscript “qpf” means the quasi-particle frame (QPF). For (4.5) we obtain also another expression on this QPF as

$$
\begin{aligned}
&(D_t\theta^\dagger - i\partial_\epsilon\mathcal{F} - [\theta^\dagger,\mathcal{F}])|_{\mathrm{qpf}} = 0, \qquad (D_t\theta - i\partial_{\epsilon^*}\mathcal{F} - [\theta,\mathcal{F}])|_{\mathrm{qpf}} = 0,\\
&(i\partial_\epsilon\theta - i\partial_{\epsilon^*}\theta^\dagger - [\theta,\theta^\dagger])|_{\mathrm{qpf}} = 0.
\end{aligned}
\tag{4.7}
$$

Further, using (4.6) and the relation $i\partial_\epsilon\mathcal{F}|_{\mathrm{qpf}} = i\partial_\epsilon(\hat{g}^\dagger\mathcal{F}(\hat{g})\hat{g}) = -[\theta^\dagger,\mathcal{F}]|_{\mathrm{qpf}} + \hat{g}^\dagger i\partial_\epsilon\mathcal{F}\hat{g}$, one can rewrite equations in the first line of (4.7) as

$$
D_t\theta^\dagger|_{\mathrm{qpf}} - \hat{g}^\dagger i\partial_\epsilon\mathcal{F}(\hat{g})\hat{g} = 0, \qquad D_t\theta|_{\mathrm{qpf}} - \hat{g}^\dagger i\partial_{\epsilon^*}\mathcal{F}(\hat{g})\hat{g} = 0.
\tag{4.8}
$$

From (4.6) and (4.3), the infinitesimal operators are expressed as

$$
\theta^\dagger(\hat{g}^\dagger)|_{\mathrm{qpf}} = -i\partial_\epsilon\hat{g}^\dagger\cdot\hat{g} = e^{i\hat{\mathfrak{e}}t}\{\partial_\epsilon\hat{\mathfrak{e}}\cdot t + \theta^\dagger(\hat{g}^{0\dagger})|_{\mathrm{qpf}}\}e^{-i\hat{\mathfrak{e}}t},
\tag{4.9}
$$

together with the same relation for $\theta(\hat{g}^\dagger)|_{\mathrm{qpf}}$. We have also $\theta^\dagger(\hat{g}^{0\dagger})|_{\mathrm{qpf}} = -i\partial_\epsilon\hat{g}^{0\dagger}\cdot\hat{g}^0$ and $\theta(\hat{g}^{0\dagger})|_{\mathrm{qpf}} = -i\partial_{\epsilon^*}\hat{g}^{0\dagger}\cdot\hat{g}^0$. Then, from (4.8) we can derive the FRPAEQ on the Gr_∞ in the form

$$
\omega_c\Gamma\left\{\theta^\dagger(\hat{g}^{0\dagger})|_{\mathrm{qpf}}\right\} + i\partial_\epsilon\hat{\mathfrak{e}} - \left[\hat{\mathfrak{e}},\theta^\dagger(\hat{g}^{0\dagger})|_{\mathrm{qpf}}\right] - i\hat{g}^{0\dagger}\partial_\epsilon\mathcal{F}(\hat{g}^0)\hat{g}^0 = 0,
\tag{4.10}
$$

To obtain an explicit expression for the last term of the l.h.s. of (4.10), we introduce an auxiliary density matrix $\widehat{R} = \hat{g}^0\mathrm{diag}\left[\cdots I_{M\otimes(N-M)}\cdots\right]\hat{g}^{0\dagger}$, where $I_{M\otimes(N-M)} \overset{d}{=} \begin{bmatrix} -I_M & \\ & I_{N-M}\end{bmatrix}$. The $\widehat{R}$ is related to density matrix $\widehat{W}$ as $\widehat{R} = \widehat{I} - 2\widehat{W}$ ($\widehat{I}$: infinite-dimensional unit matrix). Then, we obtain

$$
\begin{aligned}
i\partial_\epsilon\widehat{W} &= -\tfrac{1}{2}\hat{g}^0\{-i\partial_\epsilon\hat{g}^{0\dagger}\cdot\hat{g}^0\widehat{I}_{M\otimes(N-M)} - \widehat{I}_{M\otimes(N-M)}(-i\partial_\epsilon\hat{g}^{0\dagger}\cdot\hat{g}^0)\}\hat{g}^{0\dagger}\\
&= -\tfrac{1}{2}\hat{g}^0\left[\theta^\dagger(\hat{g}^{0\dagger})|_{\mathrm{qpf}},\widehat{I}_{M\otimes(N-M)}\right]\hat{g}^{0\dagger},
\end{aligned}
\tag{4.11}
$$

and we have used (4.9). Further we introduce the following quantities:

$$
\theta_r^{0\dagger}|_{\mathrm{qpf}} \overset{d}{=} \begin{bmatrix}\xi_r^0 & \phi_r^0\\ \psi_r^0 & \bar{\xi}_r^0\end{bmatrix}, \qquad B_r^\dagger|_{\mathrm{qpf}} \overset{d}{=} -\tfrac{1}{2}\left[\theta_r^{0\dagger}|_{\mathrm{qpf}},\ I_{M\otimes(N-M)}\right] = \begin{bmatrix}0 & -\phi_r^0\\ \psi_r^0 & 0\end{bmatrix},
$$

Using these, we rewrite (4.11) as

$$
\begin{aligned}
&i\partial_\epsilon\widehat{W} = \hat{g}^0\widehat{B}^\dagger|_{\mathrm{qpf}}\hat{g}^{0\dagger} = \sum_{r\in\mathbb{Z}}(i\partial_\epsilon W_r)z^r,\\
&i\partial_\epsilon W_r = \sum_{k,l\in\mathbb{Z}} g_k^0 B_{k-l-r}^\dagger|_{\mathrm{qpf}} g_l^{0\dagger} = \sum_{k,l\in\mathbb{Z}} g_k^0\begin{bmatrix}0 & -\phi_{k-l-r}^0\\ \psi_{k-l-r}^0 & 0\end{bmatrix} g_l^{0\dagger}.
\end{aligned}
\tag{4.12}
$$

Let a ($\bar{a}$) and i ($\bar{i}$) be $1,\ldots,m$ hole-states and $m+1,\ldots,N$ particle-states of the QPF, respectively. Substituting the second equation of (4.12) into (3.39), for $r\neq 0$ we get

$$
i\partial_\epsilon(\mathcal{F}_r)_{\alpha\beta} = [\alpha\beta|\gamma\delta]\sum_{k,l\in\mathbb{Z}}\left\{\left(g_k^0\right)_{\delta i}\left(g_l^{0\dagger}\right)_{a\gamma}\left(\psi_{k-l-r}^0\right)_{ia} - \left(g_k^0\right)_{\delta a}\left(g_l^{0\dagger}\right)_{i\gamma}\left(\phi_{k-l-r}^0\right)_{ai}\right\}.
$$

Thus, we can reach the desired form of the equation, part of the FRPAEQ on the Gr_∞ (4.10),

$$i\left(\hat{g}^{0\dagger}\cdot\partial_\epsilon\mathcal{F}\cdot\hat{g}^0\right)_r=\sum_{k,l\in\mathbb{Z}}g_k^{0\dagger}\cdot i\partial_\epsilon\mathcal{F}_{k-l+r}\cdot g_l^0$$

$$=\sum_{k,l\in\mathbb{Z},\ \bar{k},\bar{l}\in\mathbb{Z}}\begin{bmatrix}\begin{bmatrix}kl & & \bar{k}\bar{l}\\ ab & |F| & \bar{i}\bar{a}\end{bmatrix}\left(\psi^0_{(\bar{k}-\bar{l})-(k-l)-r}\right)_{\bar{i}\bar{a}}-\begin{bmatrix}kl & & \bar{k}\bar{l}\\ ab & |\overline{F}| & \bar{a}\bar{i}\end{bmatrix}\left(\phi^0_{(\bar{k}-\bar{l})-(k-l)-r}\right)_{\bar{a}\bar{i}}, & \begin{bmatrix}kl & & \bar{k}\bar{l}\\ ia & |D| & \bar{i}\bar{a}\end{bmatrix}\left(\psi^0_{(\bar{k}-\bar{l})-(k-l)-r}\right)_{\bar{i}\bar{a}}-\begin{bmatrix}kl & & \bar{k}\bar{l}\\ ia & |\overline{D}| & \bar{a}\bar{i}\end{bmatrix}\left(\phi^0_{(\bar{k}-\bar{l})-(k-l)-r}\right)_{\bar{a}\bar{i}},\\ \begin{bmatrix}kl & & \bar{k}\bar{l}\\ ai & |D| & \bar{i}\bar{a}\end{bmatrix}\left(\psi^0_{(\bar{k}-\bar{l})-(k-l)-r}\right)_{\bar{i}\bar{a}}-\begin{bmatrix}kl & & \bar{k}\bar{l}\\ ai & |\overline{D}| & \bar{a}\bar{i}\end{bmatrix}\left(\phi^0_{(\bar{k}-\bar{l})-(k-l)-r}\right)_{\bar{a}\bar{i}} & \begin{bmatrix}kl & & \bar{k}\bar{l}\\ ij & |F| & \bar{i}\bar{a}\end{bmatrix}\left(\psi^0_{(\bar{k}-\bar{l})-(k-l)-r}\right)_{\bar{i}\bar{a}}-\begin{bmatrix}kl & & \bar{k}\bar{l}\\ ij & |\overline{F}| & \bar{a}\bar{i}\end{bmatrix}\left(\phi^0_{(\bar{k}-\bar{l})-(k-l)-r}\right)_{\bar{a}\bar{i}}\end{bmatrix}.$$

Substituting the above result into (4.10), we can derive the FRPAEQ on F_∞.

Finally we show the following equations to determine the collective submanifold and motion: The canonicity condition (4.1):

$$\langle\hat{g}^0|\partial_{\begin{bmatrix}\epsilon\\ \epsilon^*\end{bmatrix}}|\hat{g}^0\rangle=\sum_{\alpha=1}^{M}\sum_{s\in\mathbb{Z}}\left(g_s^{0\dagger}\partial_{\begin{bmatrix}\epsilon\\ \epsilon^*\end{bmatrix}}g_s^0\right)_{\alpha\alpha}=\tfrac{1}{2}\begin{bmatrix}\epsilon^*\\ -\epsilon\end{bmatrix}.$$

The FRPAEQ (4.10):

$$\omega_c\Gamma\{\theta^\dagger(\hat{g}^{0\dagger})|_{\mathrm{qpf}}\}+i\partial_\epsilon\hat{\mathfrak{E}}-\left[\hat{\mathfrak{E}},\theta^\dagger(\hat{g}^{0\dagger})|_{\mathrm{qpf}}\right]-i\hat{g}^{0\dagger}\partial_\epsilon\mathcal{F}(\hat{g}^0)\hat{g}^0=0,\qquad \hat{g}=\hat{g}^0(\epsilon,\epsilon^*)e^{-i\hat{\mathfrak{E}}(\epsilon,\epsilon^*)t}.$$

Through constructions of the TDHFT and the FRPAEQ on F_∞, the following become apparent: The ordinary perturbative method for collective variables η and η^* [25] is involved in the way of construction of the TDHFT on the affine KM algebra if we restrict ourselves to $\widehat{su}(N)$. When the η and η^* are represented as $\eta=\sqrt{\Omega}e^{i\varphi}$, we can always express

$$\gamma(\eta,\eta^*)=\sum_{r,s\in\mathbb{Z}}\bar{\gamma}_{r,s}\eta^{*r}\eta^s=\sum_r\gamma_r z^r$$

on the Lie algebra if we put $z=e^{i\varphi}$. This means that the infinite-dimensional Lie algebra in the SCFT is introduced in a natural way and is useful to study various motions of fermion many-body systems.

4.3 Summary and discussions

FRPAEQ has been provided as a tool for truncating a collective submanifold with only one normal mode out of an Gr_∞. We have given a simple geometrical interpretation for FRPAEQ. The collective submanifold is interpreted as a rotator on a curved surface in the Gr_∞. In F_∞, to study motions of finite fermion systems, it is manifestly natural and useful to introduce an infinite-dimensional Lie algebra arising from anti-commutation relations among fermions. In order to discuss the relation between TDHFT and soliton theory, we have given expressions for TDHFT on τ-FS along soliton theory. From the *loop* group viewpoint and with a clearer physical picture, we have proposed a way of describing particle and collective motions in SCFT on F_∞ in relation to an iso-spectral equation in soliton theory. Then, SCFT on F_∞ may be regarded as soliton theory in the sense that it is based on the Gr_∞ and may describe dynamics on an infinite set of *real fermion-harmonic oscillators*. On the other hand, soliton theory describes dynamics on *complex fermion-harmonic oscillators*. It is one of the most challenging problem to extend real space $\widehat{su}(N)$ to complex space $\widehat{sl}(N)$ in TDHFT on F_∞ together with removal of the restriction $|z| = 1$. Concerning the construction of soliton theory on multi-dimensional space [61, 66], we have an interesting future problem that is to extend the Plücker relation (Hirota's form) with only one circle to the case of multi-circles such that SCFM on F_∞ can describe dynamics of fermion systems in terms of multi-RPA bosons.

5 Infinite-dimensional KM algebraic approach to LMG model

5.1 Introduction

To go beyond the maximaly-decoupled method, we have aimed to construct an SCF theory, i.e., υ-HFT. In constructing the υ-HF theory we must observe, however, the following two different points between the maximaly-decoupled method and the υ-HF SCFM (i) The former is built on the finite-dimensional Lie algebra but the latter on the infinite-dimensional one. (ii) The former has an SCF Hamiltonian consisting of a fermion one-body operator, which is derived from a functional derivative of an expectaion value of a fermion Hamiltonian by a ground-state wave function. The latter has a fermion Hamiltonian with a one-body type operator brought artificially as an operator which maps states on a fermion Fock space into corresponding ones on a τ-FS. Toward such an ultimate goal, the υ-HFT has been reconstructed on an affine KM algebra along the soliton theory, using infinite-dimensional fermion. An infinite-dimensional fermion operator is introduced through a Laurent expansion of finite-dimensional fermion operators with respect to degrees of freedom of the fermions related to a υ-dependent potential with a Υ-periodicity. A bilinear equation for the υ-HFT has been transcribed onto the corresponding τ-function using the regular representation for the group and the Schur-polynomials. The υ-HF SCFM on an infinite-dimensional Fock space F_∞ leads to a dynamics on an infinite-dimensional Grassmannian Gr_∞ and may describe more precisely such a dynamics on the group manifold. A finite-dimensional Grassmannian is identified with a Gr_∞ which is affiliated with the group manifold obtained by reducting $gl(\infty)$ to $sl(N)$ and $su(N)$. We have given explicit expressions for Laurent coefficients of soliton solutions for $\widehat{sl}(N)$ and $\widehat{su}(N)$ on the Gr_∞ using Chevalley bases for $sl(N)$ and $su(N)$. As an illustration we make the υ-HFT approach to an infinite-dimensional matrix model extended from the finite-dimensional $su(2)$ LMG model and represent an infinite-dimensional matrix LMG model in terms of the Schur polynomials.

5.2 Application to Lipkin–Meshkov–Glick model

To show the usefulness of the infinite KM algebra and to avoid an unnecessary complication, we apply it to a simple model, the LMG model consisting of N $(= M)$ particles. Let us introduce the LMG Hamiltonian which has two N-fold degenerate levels with energies $\frac{1}{2}\varepsilon$ and $-\frac{1}{2}\varepsilon$, respectively

$$H = \varepsilon\widehat{K}_0 - \tfrac{1}{2}V(\widehat{K}_+^2 + \widehat{K}_-^2).$$

The operators $\widehat{K}_0$, $\widehat{K}_+$ and $\widehat{K}_-$ are defined by

$$\widehat{K}_0 \equiv \tfrac{1}{2}\left(\sum_{i=1}^{N} c_i^\dagger c_i - \sum_{a=1}^{N} c_a^\dagger c_a\right), \qquad \widehat{K}_+ \equiv \sum_{i=a=1}^{N} c_i^\dagger c_a = \widehat{K}_-^\dagger, \tag{5.1}$$

where the indices i and a stand for particle-state and hole-state, respectively and satisfy the $SU(2)$ quasi-spin algebra

$$[\widehat{K}_0, \widehat{K}_\pm] = \pm\widehat{K}_\pm, \qquad [\widehat{K}_+, \widehat{K}_-] = 2\widehat{K}_0.$$

Then the S-det, $|S^N\rangle$, in which the N particles fill the lower level, satisfies

$$c_i|S^N\rangle = 0, \qquad c_a^\dagger|S^N\rangle = 0 \qquad (i,a = 1,2,\dots,N), \qquad \hat{K}_-|S^N\rangle = 0.$$

We here use a notation $[c^\dagger]$ denoting a $2N$-dimensional row vector $[c_a^\dagger, c_i^\dagger]$ ($i = 1,2,\dots,N$; $a = 1,2,\dots,N$). We introduce the following $SU(2N)$ Thouless transformation:

$$\begin{aligned}
&U(g)[c^\dagger]U^{-1}(g) = [c^\dagger]g, \qquad U(g) = e^{i\psi\widehat{K}_0}e^{\frac{\theta}{2}(\widehat{K}_+-\widehat{K}_-)}e^{i\varphi\widehat{K}_0},\\
&g = \begin{bmatrix} \cos\frac{\theta}{2}\cdot e^{-i\frac{1}{2}(\psi+\varphi)}\cdot 1_N & -\sin\frac{\theta}{2}\cdot e^{-i\frac{1}{2}(\psi-\varphi)}\cdot 1_N \\ \sin\frac{\theta}{2}\cdot e^{i\frac{1}{2}(\psi-\varphi)}\cdot 1_N & \cos\frac{\theta}{2}\cdot e^{i\frac{1}{2}(\psi+\varphi)}\cdot 1_N \end{bmatrix},\\
&g^\dagger g = gg^\dagger = 1_{2N}, \qquad \det g = 1, \qquad\qquad (5.2)\\
&U(g)U(g') = U(gg'), \qquad U(g^{-1}) = U^{-1}(g) = U^\dagger(g), \qquad U(1) = 1.
\end{aligned}$$

The above $SU(2N)$ matrix is essentially the direct sum of the $SU(2)$ matrix. Any N-particle S-det is constructed by the Thouless transformation of a reference S-det, $|S^N\rangle$ (the Thouless theorem) as

$$\begin{aligned}
&|g\rangle = U(g)|S^N\rangle = \langle S^N|U(g)|S^N\rangle \exp[p\exp(i\psi)\widehat{K}_+]|S^N\rangle,\\
&\langle S^N|U(g)|S^N\rangle = \left(\cos\tfrac{\theta}{2}\right)^N, \qquad p = (p_{ia}) = \tan\left(\tfrac{\theta}{2}\right)e^{i\psi}\cdot 1_N, \qquad\qquad (5.3)
\end{aligned}$$

which is the CS rep of fermion state vector on the $SU(2N)$ group [5].

The HF density matrix is given as

$$\begin{aligned}
W &= \begin{bmatrix} \cos^2\frac{\theta}{2}\cdot 1_N & \frac{1}{2}\sin\theta e^{-i\psi}\cdot 1_N \\ \frac{1}{2}\sin\theta e^{i\psi}\cdot 1_N & \sin^2\frac{\theta}{2}\cdot 1_N \end{bmatrix}\\
&= \tfrac{1}{2}\hat{I}_{2N} + \cos\theta\tfrac{1}{2}\hat{h}_{2N} + \tfrac{1}{2}\sin\theta e^{-i\psi}\hat{e}_{2N} + \tfrac{1}{2}\sin\theta e^{i\psi}\hat{f}_{2N},
\end{aligned}$$

where $\hat{I}_{2N}$ is a $2N$-dimensional unit matrix and $\hat{h}_{2N}$, $\hat{e}_{2N}$ and $\hat{f}_{2N}$ are defined in the next subsection. The usual HF energy $\langle g|H|g\rangle$ $(= H[W])$ is obtained as

$$H[W] = \tfrac{\varepsilon N}{2}\left[\sin^2\tfrac{\theta}{2} - \cos^2\tfrac{\theta}{2} - \chi\left\{\tfrac{1}{2}\sin\theta e^{-i\psi}\right\}^2 - \chi\left\{\tfrac{1}{2}\sin\theta e^{i\psi}\right\}^2\right], \qquad \chi \equiv \tfrac{(N-1)V}{\varepsilon}.$$

The Fock operator $F[W]$ $(= \delta H[W]/\delta W^{\mathrm{T}})$, in the HF approximation is represented as

$$\begin{aligned}
F[W] &= \begin{bmatrix} -\varepsilon\frac{1}{2}\cdot 1_N & -\varepsilon\chi\frac{1}{2}\sin\theta e^{i\psi}\cdot 1_N \\ -\varepsilon\chi\frac{1}{2}\sin\theta e^{-i\psi}\cdot 1_N & \varepsilon\frac{1}{2}\cdot 1_N \end{bmatrix}\\
&= -\tfrac{\varepsilon}{2}\hat{h}_{2N} - \tfrac{\varepsilon\chi}{2}\sin\theta e^{i\psi}\hat{e}_{2N} - \tfrac{\varepsilon\chi}{2}\sin\theta e^{-i\psi}\hat{f}_{2N}. \qquad\qquad (5.4)
\end{aligned}$$

5.3 Infinite-dimensional Lipkin–Meshkov–Glick model

Using (3.25), we introduce the following infinite-dimensional "particle"- and "hole"-annihilation operators ψ^*_{Nr+i} for particle-state and ψ_{Nr+a} for hole-state, respectively, and a vacuum state $|M\rangle$:

$$\begin{aligned}
&\psi^*_{Nr+i}|M\rangle = 0 \qquad (i = 1,\dots,N,\ r \geq 0), \qquad \psi_{Nr+a}|M\rangle = 0 \qquad (a = 1,\dots,N,\ r \leq 0),\\
&|M\rangle = \psi_M\cdots\psi_1|\mathrm{Vac}\rangle \qquad (M = N).
\end{aligned}$$

Then the $\widehat{K}_0$ and $\widehat{K}_\pm$ (5.1) are expressed in terms of the above infinite-dimensional operators as follows:

$$\widehat{K}_0 = \sum_{r,s\in\mathbb{Z}} \left(\tfrac{1}{2}\hat{h}_{2N}\right)_{\alpha\beta} : \psi_{N(s-r)+\alpha}\psi^*_{Ns+\beta} :$$

$$= \tfrac{1}{2}\sum_{r,s\in\mathbb{Z}}\left(\sum_{i=1}^{N} : \psi_{N(s-r)+i}\psi^*_{Ns+i} : - \sum_{a=1}^{N} : \psi_{N(s-r)+a}\psi^*_{Ns+a} :\right),$$

$$\widehat{K}_+ = \sum_{r,s\in\mathbb{Z}} (\hat{e}_{2N})_{\alpha\beta} : \psi_{N(s-r)+\alpha}\psi^*_{Ns+\beta} := \sum_{r,s\in\mathbb{Z}}\sum_{i=a=1}^{N} : \psi_{N(s-r)+i}\psi^*_{Ns+a} :,$$

$$\widehat{K}_- = \sum_{r,s\in\mathbb{Z}} (\hat{f}_{2N})_{\alpha\beta} : \psi_{N(s-r)+\alpha}\psi^*_{Ns+\beta} := \sum_{r,s\in\mathbb{Z}}\sum_{a=i=1}^{N} : \psi_{N(s-r)+a}\psi^*_{Ns+i} :,$$

and

$$\widehat{K}_-|M\rangle = \sum_{r,s\in\mathbb{Z}}\sum_{a=i=1}^{N} : \psi_{N(s-r)+a}\psi^*_{Ns+i} : |M\rangle = 0 \qquad (s \geq 0).$$

Let us introduce the following $2N$-dimensional dual elements of the direct sum of the algebra $sl(2,C)$ multiplied by z^r:

$$\widehat{K}_0(r) = \tfrac{1}{2}\hat{h}_{2N}z^r = \tfrac{1}{2}\begin{bmatrix} 1_N & 0 \\ 0 & -1_N \end{bmatrix} z^r = \tfrac{1}{2}\hat{h}(r),$$

$$\widehat{K}_+(r) = \hat{e}_{2N}z^r = \begin{bmatrix} 0 & 1_N \\ 0 & 0 \end{bmatrix} z^r = \hat{e}(r),$$

$$\widehat{K}_-(r) = \hat{f}_{2N}z^r = \begin{bmatrix} 0 & 0 \\ 1_N & 0 \end{bmatrix} z^r = \hat{f}(r);$$

$$K_0(r) = \{\widehat{K}_0(r)\}_{\alpha\beta}e_{\alpha\beta} = \left(\tfrac{1}{2}\hat{h}_{2N}z^r\right)_{\alpha\beta} e_{\alpha\beta} = \left(\tfrac{1}{2}\hat{h}_{2N}\right)_{\alpha\beta} e_{\alpha\beta}(r),$$

$$K_\pm(r) = \{\widehat{K}_\pm(r)\}_{\alpha\beta}e_{\alpha\beta} = \begin{pmatrix}\hat{e}_{2N}z^r \\ \hat{f}_{2N}z^r\end{pmatrix}_{\alpha\beta} e_{\alpha\beta} = \begin{pmatrix}\hat{e}_{2N} \\ \hat{f}_{2N}\end{pmatrix}_{\alpha\beta} e_{\alpha\beta}(r).$$

Using the formulas in Appendix C, the τ reps of the operators $\widehat{K}_0$ and $\widehat{K}_\pm$ are given, respectively, in the following forms:

$$\tau(K_0) = \tau\left\{\sum_{r\in\mathbb{Z}} K_0(r)\right\} = \sum_{r\in\mathbb{Z}}\left(\tfrac{1}{2}\hat{h}_{2N}\right)_{\alpha\beta}\tau\{e_{\alpha\beta}(r)\},$$

$$= \sum_{r,s\in\mathbb{Z}}\left(\tfrac{1}{2}\hat{h}_{2N}\right)_{\alpha\beta} E_{N(s-r)+\alpha,Ns+\beta} \simeq \sum_{r,s\in\mathbb{Z}}\left(\tfrac{1}{2}\hat{h}_{2N}\right)_{\alpha\beta}\psi_{N(s-r)+\alpha}\psi^*_{Ns+\beta},$$

$$\tau(K_\pm) = \tau\left\{\sum_{r\in\mathbb{Z}} K_\pm(r)\right\} = \sum_{r\in\mathbb{Z}}\begin{pmatrix}\hat{e}_{2N} \\ \hat{f}_{2N}\end{pmatrix}_{\alpha\beta}\tau\{e_{\alpha\beta}(r)\}$$

$$= \sum_{r,s\in\mathbb{Z}}\begin{pmatrix}\hat{e}_{2N} \\ \hat{f}_{2N}\end{pmatrix}_{\alpha\beta} E_{N(s-r)+\alpha,Ns+\beta} \simeq \sum_{r,s\in\mathbb{Z}}\begin{pmatrix}\hat{e}_{2N} \\ \hat{f}_{2N}\end{pmatrix}_{\alpha\beta}\psi_{N(s-r)+\alpha}\psi^*_{Ns+\beta},$$

from which we have the KM brackets among the operators $K_0(r)$ and $K_\pm(r)$. For detailed calculations see Appendix F.

$$[K_0(r), K_0(s)]_{\mathrm{KM}} = \tfrac{1}{2}Nr\delta_{r+s,0}\cdot c, \qquad [K_\pm(r), K_\pm(s)]_{\mathrm{KM}} = 0,$$
$$[K_0(r), K_\pm(s)]_{\mathrm{KM}} = \pm K_\pm(r+s), \qquad [K_+(r), K_-(s)]_{\mathrm{KM}} = 2K_0(r+s) + Nr\delta_{r+s,0}\cdot c.$$

Following Lepowsky and Wilson [71], we introduce the elements of $\widehat{sl}(2,C)$ in the linear-combination forms of $\hat{e}_{2N}$ and $\hat{f}_{2N}$:

$$\begin{pmatrix}X_k \\ Y_k\end{pmatrix} \equiv (\hat{e}_{2N})_{\alpha\beta}e_{\alpha\beta}(r) \pm (\hat{f}_{2N})_{\alpha\beta}e_{\alpha\beta}(r+1) = \begin{pmatrix}\widehat{X}_k \\ \widehat{Y}_k\end{pmatrix}_{\alpha\beta} e_{\alpha\beta},$$

$$\widehat{X}_k \equiv \begin{bmatrix} 0 & 1_N \\ z1_N & 0 \end{bmatrix} z^r, \qquad \widehat{Y}_k \equiv \begin{bmatrix} 0 & 1_N \\ -z1_N & 0 \end{bmatrix} z^r \qquad (k = 2r+1),$$

the 2-*cocycle* α's on a pair of the above elements read

$$\alpha\left(\frac{1}{\sqrt{N}}X_k, \frac{1}{\sqrt{N}}X_l\right) = k\delta_{k+l,0}, \qquad \alpha\left(\frac{1}{\sqrt{N}}Y_k, \frac{1}{\sqrt{N}}Y_l\right) = -k\delta_{k+l,0},$$
$$\alpha\left(\frac{1}{\sqrt{N}}X_k, \frac{1}{\sqrt{N}}Y_l\right) = \delta_{k+l,0}, \qquad \alpha\left(\frac{1}{\sqrt{N}}Y_k, \frac{1}{\sqrt{N}}X_l\right) = -\delta_{k+l,0}$$
$$(k = 2r+1,\ l = 2s+1).$$

For details see Appendix F. Then we have the following KM brackets and the map σ_K:

$$\left[\frac{1}{\sqrt{N}}X_k, \frac{1}{\sqrt{N}}X_l\right]_{\mathrm{KM}} = k\delta_{k+l,0}\cdot c, \qquad \left[\frac{1}{\sqrt{N}}Y_k, \frac{1}{\sqrt{N}}Y_l\right]_{\mathrm{KM}} = -k\delta_{k+l,0}\cdot c,$$
$$\left[\frac{1}{\sqrt{N}}X_k, \frac{1}{\sqrt{N}}Y_l\right]_{\mathrm{KM}} = \frac{2}{\sqrt{N}}\frac{1}{\sqrt{N}}Y_{k+l} + \delta_{k+l,0}\cdot c,$$
$$\sigma_K: \ \frac{1}{\sqrt{N}}X_k \to \frac{\partial}{\partial x_k}, \qquad \sigma_K: \ \frac{1}{\sqrt{N}}Y_k \to k y_k,$$
$$\sigma_K: \ \frac{1}{\sqrt{N}}X_{-k} \to k x_k, \qquad \sigma_K: \ \frac{1}{\sqrt{N}}Y_{-k} \to \frac{\partial}{\partial y_k}.$$

Then $\frac{\sqrt{2}}{\sqrt{N}}K_0(r)$, $\frac{1}{\sqrt{N}}X_k$ $(k = 2r+1)$ and $\frac{1}{\sqrt{N}}Y_k$ $(k = 2r+1)$ are clearly an infinite-dimensional Heisenberg subalgebra of the KM algebra $\widehat{sl}(2, C)$. We also introduce the element in the form of $\hat{h}_{2N}$ as

$$Y_{2r} \equiv \big((-\hat{h}_{2N})_{\alpha\beta}e_{\alpha\beta}(r)\big) = (\widehat{Y}_{2r})_{\alpha\beta}e_{\alpha\beta}, \qquad \widehat{Y}_{2r} \equiv \begin{bmatrix} -1_N & 0 \\ 0 & 1_N \end{bmatrix} z^r.$$

5.4 Representation of infinite-dimensional LMG model in terms of Schur polynomials

The expressions for the operators $\widehat{K}_0$ and $\widehat{K}_\pm$ (5.1) in terms of the operators $\frac{1}{\sqrt{N}}X_{2r+1}$, $\frac{1}{\sqrt{N}}Y_{2r+1}$ and $\frac{1}{\sqrt{N}}Y_{2r}$ and the expressions for the map σ_K for the operators $\frac{1}{\sqrt{N}}X_{2r+1}$ and $\frac{1}{\sqrt{N}}X_{-(2r+1)}$ are given as follows:

$$\widehat{K}_0 = \sum_{r\in\mathbb{Z}} K_0(r) = -\frac{\sqrt{N}}{2}\sum_{r\in\mathbb{Z}}\frac{1}{\sqrt{N}}Y_{2r},$$
$$\begin{pmatrix}\widehat{K}_+ \\ \widehat{K}_-\end{pmatrix} = \sum_{r\in\mathbb{Z}}\begin{pmatrix}K_+(r) \\ K_-(r+1)\end{pmatrix} = \frac{\sqrt{N}}{2}\sum_{r\in\mathbb{Z}}\left(\frac{1}{\sqrt{N}}X_{2r+1} \pm \frac{1}{\sqrt{N}}Y_{2r+1}\right),$$
$$\sigma_K: \ \frac{1}{\sqrt{N}}X_{2r+1} \to \frac{\partial}{\partial x_{2r+1}}, \qquad \sigma_K: \ \frac{1}{\sqrt{N}}X_{-(2r+1)} \to (2r+1)x_{2r+1} \qquad (r > 0). \tag{5.5}$$

Consequently we can obtain important sum-rules for the operators Y_{2r+1} and Y_{2r} as

$$2\sum_{r\in\mathbb{Z}}\begin{pmatrix}Y_{2r} \\ Y_{2r+1}\end{pmatrix} = \sum_{r\geq 0}\begin{pmatrix}S_{2r}(2x) \\ S_{2r+1}(2x)\end{pmatrix}\sum_{s\geq 0}S_{2s}(-2\widetilde{\partial}) + \sum_{r\geq 0}\begin{pmatrix}S_{2r+1}(2x) \\ S_{2r}(2x)\end{pmatrix}\sum_{s\geq 0}S_{2s+1}(-2\widetilde{\partial})$$
$$= \frac{1}{2}\left\{e^{2\sum_{m=0}^{\infty}x_{2m+1}}e^{-2\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{\partial}{\partial x_{2n+1}}} \pm e^{-2\sum_{m=0}^{\infty}x_{2m+1}}e^{2\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{\partial}{\partial x_{2n+1}}}\right\}, \tag{5.6}$$

$$\left(2\sum_{r\in\mathbb{Z}} Y_{2r+1}\right)^2 = \frac{1}{4}\left[e^{4\sum_{m=0}^{\infty} x_{2m+1}} e^{-4\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{\partial}{\partial x_{2n+1}}} - e^{-4\sum_{m=0}^{\infty} x_{2m+1}} e^{4\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{\partial}{\partial x_{2n+1}}} - 2\right], \quad (5.7)$$

where we have used the relation

$$\sum_{n\geq 0}\begin{pmatrix} S_{2n}(x) \\ S_{2n+1}(x)\end{pmatrix} = \frac{1}{2}\left\{\exp\left(2\sum_{n=0}^{\infty} x_{2n+1}\right) \pm \exp\left(-2\sum_{n=0}^{\infty} x_{2n+1}\right)\right\} \qquad (x_{2n}=0,\ n\geq 1),$$

which is derived from the definition of the Schur polynomial $S_k(x)$ in Appendix D. Further we have an expression for a quadratic operator $\widehat{K}_+^2 + \widehat{K}_-^2$ as

$$\begin{aligned}\widehat{K}_+^2 + \widehat{K}_-^2 &= \frac{N}{4}\left\{\sum_{r\in\mathbb{Z}}\left(\frac{1}{\sqrt{N}}X_{2r+1} + \frac{1}{\sqrt{N}}Y_{2r+1}\right)\right\}^2 + \frac{N}{4}\left\{\sum_{r\in\mathbb{Z}}\left(\frac{1}{\sqrt{N}}X_{2r+1} - \frac{1}{\sqrt{N}}Y_{2r+1}\right)\right\}^2 \\ &= \frac{N}{2}\left\{\left(\sum_{r\in\mathbb{Z}}\frac{1}{\sqrt{N}}X_{2r+1}\right)^2 + \left(\sum_{r\in\mathbb{Z}}\frac{1}{\sqrt{N}}Y_{2r+1}\right)^2\right\}. \end{aligned} \quad (5.8)$$

Finally from (5.5), (5.6), (5.7) and (5.8) we get an expression for the LMG Hamiltonian as

$$\begin{aligned} H &= \epsilon\widehat{K}_0 - \frac{V}{2}\left(\widehat{K}_+^2 + \widehat{K}_-^2\right) \\ &= -\epsilon\frac{\sqrt{N}}{2}\sum_{r\in\mathbb{Z}}\frac{1}{\sqrt{N}}Y_{2r} - \frac{V}{2}\frac{N}{2}\left\{\left(\sum_{r\in\mathbb{Z}}\frac{1}{\sqrt{N}}X_{2r+1}\right)^2 + \left(\sum_{r\in\mathbb{Z}}\frac{1}{\sqrt{N}}Y_{2r+1}\right)^2\right\} \\ &= -\frac{1}{4}\epsilon\exp\left(2\sum_{m=0}^{\infty}x_{2m+1}\right)\exp\left(-2\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{\partial}{\partial x_{2n+1}}\right) - \frac{V}{2}\frac{N}{2}\left[\sum_{m=0}^{\infty}\frac{\partial}{\partial x_{2m+1}}\sum_{n=0}^{\infty}\frac{\partial}{\partial x_{2n+1}}\right. \\ &+ \sum_{m=0}^{\infty}\left[\frac{\partial}{\partial x_{2m+1}}, \sum_{n=0}^{\infty}(2n+1)x_{2n+1}\right] + \sum_{m=0}^{\infty}(2n+1)x_{2m+1}\sum_{n=0}^{\infty}\frac{\partial}{\partial x_{2n+1}} \\ &+ \left.\sum_{m=0}^{\infty}(2m+1)x_{2m+1}\sum_{n=0}^{\infty}\frac{\partial}{\partial x_{2n+1}} + \sum_{m=0}^{\infty}(2m+1)x_{2m+1}\sum_{n=0}^{\infty}(2n+1)x_{2n+1}\right] + \frac{V}{32} \\ &- \frac{V}{64}\left\{\exp(4\sum_{m=0}^{\infty}x_{2m+1})\exp\left(-4\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{\partial}{\partial x_{2n+1}}\right)\right. \\ &- \left.\exp\left(-4\sum_{m=0}^{\infty}x_{2m+1}\right)\exp\left(4\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{\partial}{\partial x_{2n+1}}\right)\right\}. \end{aligned} \quad (5.9)$$

5.5 Infinite-dimensional representation of $SU(2N)_\infty$ transformation

We prepare operators $\widehat{K}_0(\varphi)$, $\widehat{K}_0(\psi)$ and $\widehat{K}_\pm(\theta)$ to generate an infinite-dimensional representation of an $SU(2N)_\infty$ transformation-matrix. First we give $\hat{K}_0(\varphi)$ and $\hat{K}_0(\psi)$ as

$$\begin{aligned} \widehat{K}_0\begin{pmatrix}\varphi \\ \psi\end{pmatrix} &= \sum_{r,s\in\mathbb{Z}}\left\{\begin{pmatrix}-\varphi_r \\ -\psi_r\end{pmatrix}\frac{1}{2}\hat{h}_{2N}\right\}_{\alpha\beta} : \psi_{N(s-r)+\alpha}\psi^*_{Ns+\beta} : \\ &= \sum_{r,s\in\mathbb{Z}}\begin{pmatrix}-\varphi_r \\ -\psi_r\end{pmatrix}\mathrm{Tr}\begin{bmatrix}\frac{1}{2}1_N & 0 \\ 0 & -\frac{1}{2}1_N\end{bmatrix}\begin{bmatrix} :\psi_{N(s-r)+i}\psi^*_{Ns+j}: & :\psi_{N(s-r)+i}\psi^*_{Ns+b}: \\ :\psi_{N(s-r)+a}\psi^*_{Ns+j}: & :\psi_{N(s-r)+a}\psi^*_{Ns+b}:\end{bmatrix} \\ &= \frac{1}{2}\sum_{r,s\in\mathbb{Z}}\begin{pmatrix}-\varphi_r \\ -\psi_r\end{pmatrix}\left\{\sum_{i=1}^{N} :\psi_{N(s-r)+i}\psi^*_{Ns+i}: - \sum_{a=1}^{N} :\psi_{N(s-r)+a}\psi^*_{Ns+a}:\right\}, \end{aligned}$$

adjoint actions of which for $\psi_{Nr+\alpha}$ and $\psi^*_{Nr+\alpha}$ are computed as

$$\left[\widehat{K}_0(\varphi(\psi)), \begin{pmatrix}\psi_{Nr+\alpha}\\ \psi^*_{Nr+\alpha}\end{pmatrix}\right] = \sum_{s\in\mathbb{Z}} \psi_{N(r-s)+\beta} \left\{ \begin{pmatrix}-\varphi_s(\psi_s)\\ -\varphi_s(\psi_s)\end{pmatrix} \tfrac{1}{2}\hat{h}_{2N} \right\}_{\beta\alpha}.$$

Then the infinite-dimensional fermion operator $\psi_{Nr+\alpha}(\hat{g}_{\binom{\varphi}{\psi}})$ is transformed into

$$\begin{pmatrix}\psi_{Nr+\alpha}(\hat{g}_{\binom{\varphi}{\psi}})\\ \psi^*_{Nr+\alpha}(\hat{g}_{\binom{\varphi}{\psi}})\end{pmatrix} \stackrel{d}{=} U(\hat{g}_{\binom{\varphi}{\psi}}) \begin{pmatrix}\psi_{Nr+\alpha}(\hat{g}_{\binom{\varphi}{\psi}})\\ \psi^*_{Nr+\alpha}(\hat{g}_{\binom{\varphi}{\psi}})\end{pmatrix} U^{-1}(\hat{g}_{\binom{\varphi}{\psi}}) = e^{i\widehat{K}_0\binom{\varphi}{\psi}} \begin{pmatrix}\psi_{Nr+\alpha}(\hat{g}_{\binom{\varphi}{\psi}})\\ \psi^*_{Nr+\alpha}(\hat{g}_{\binom{\varphi}{\psi}})\end{pmatrix} e^{-i\widehat{K}_0\binom{\varphi}{\psi}}$$

$$= \sum_{s\in\mathbb{Z}} \begin{pmatrix}\psi_{N(r-s)+\beta}(g_{\binom{\varphi_s}{\psi_s}})\\ \psi^*_{N(r-s)+\beta}(g_{\binom{\varphi_s}{\psi_s}})\end{pmatrix}_{\beta\alpha}, \qquad \begin{pmatrix} g_{\binom{\varphi_s}{\psi_s}} = e^{-i\binom{\varphi_s}{\psi_s}\frac{1}{2}\hat{h}_{2N}} \\ g^*_{\binom{\varphi_s}{\psi_s}} = e^{i\binom{\varphi_s}{\psi_s}\frac{1}{2}\hat{h}_{2N}} \end{pmatrix}. \tag{5.10}$$

In a similar way as the above we also give $\widehat{K}_\pm(\theta)$ as

$$\begin{aligned}
\widehat{K}_+(\theta) = \widehat{K}^\dagger_-(\theta) &= \sum_{r,s\in\mathbb{Z}} (-\theta_r \hat{e}_{2N})_{\alpha\beta} : \psi_{N(s-r)+\alpha}\psi^*_{Ns+\beta} : \\
&= \sum_{r,s\in\mathbb{Z}} (-\theta_r)\, \mathrm{Tr} \begin{bmatrix} 0 & 0 \\ 1_N & 0 \end{bmatrix} \begin{bmatrix} :\psi_{N(s-r)+i}\psi^*_{Ns+j}: & :\psi_{N(s-r)+i}\psi^*_{Ns+b}: \\ :\psi_{N(s-r)+a}\psi^*_{Ns+j}: & :\psi_{N(s-r)+a}\psi^*_{Ns+b}: \end{bmatrix} \\
&= \sum_{r,s\in\mathbb{Z}} (-\theta_r) \sum_{i=a=1}^{N} :\psi_{N(s-r)+i}\psi^*_{Ns+a}:;
\end{aligned}$$

and from (I.2) adjoint actions of which for $\psi_{Nr+\alpha}$ are given as

$$\left[\begin{pmatrix}\widehat{K}_+(\theta)\\ \widehat{K}_-(\theta)\end{pmatrix}, \psi_{Nr+\alpha}\right] = \sum_{s\in\mathbb{Z}} \psi_{N(s-r)+\beta} \begin{pmatrix}-\theta_r \hat{e}_{2N}\\ -\theta_{-r}\hat{f}_{2N}\end{pmatrix}_{\beta\alpha}, \qquad (\theta_r = \theta_{-r}).$$

Then the transformed infinite-dimensional fermion operator $\psi_{Nr+\alpha}(\hat{g}_\theta)$ is transformed into

$$\begin{aligned}
\psi_{Nr+\alpha}(\hat{g}_\theta) &= U(\hat{g}_\theta)\psi_{Nr+\alpha}U^{-1}(\hat{g}_\theta) = e^{\frac{1}{2}\{\widehat{K}_+(\theta)-\widehat{K}_-(\theta)\}}\psi_{Nr+\alpha}e^{-\frac{1}{2}\{\widehat{K}_+(\theta)-\widehat{K}_-(\theta)\}} \\
&= \sum_{s\in\mathbb{Z}} \psi_{N(r-s)+\beta}(g_{\theta_s})_{\beta\alpha}, \qquad g_{\theta_s} = e^{\frac{1}{2}(\hat{e}_{2N}-\hat{f}_{2N})(-\theta_s)},
\end{aligned} \tag{5.11}$$

The whole transformation of the infinite-dimensional fermion operator $\psi_{Nr+\alpha}(\hat{g})$ is given as

$$\begin{aligned}
\psi_{Nr+\alpha}(\hat{g}) &= U(\hat{g}_\psi)U(\hat{g}_\theta)U(\hat{g}_\varphi)\psi_{Nr+\alpha}U^{-1}(\hat{g}_\varphi)U^{-1}(\hat{g}_\theta)U^{-1}(\hat{g}_\psi) \\
&= U(\hat{g}_\psi)\sum_{s\in\mathbb{Z}} U(\hat{g}_\theta)\psi_{N(r-s)+\beta}U^{-1}(\hat{g}_\theta)U^{-1}(\hat{g}_\psi)(g_{\varphi_s})_{\beta\alpha} \\
&= \sum_{s,t\in\mathbb{Z}} U(\hat{g}_\psi)\psi_{N(r-s-t)+\gamma}U^{-1}(\hat{g}_\psi)(g_{\theta_t})_{\gamma\beta}(g_{\varphi_s})_{\beta\alpha} \\
&= \sum_{s,t,u\in\mathbb{Z}} \psi_{N(r-s-t-u)+\delta}(g_{\psi_u})_{\delta\gamma}(g_{\theta_t})_{\gamma\beta}(g_{\varphi_s})_{\beta\alpha} \\
&= \sum_{s,t,u\in\mathbb{Z}} \psi_{N(r-s-t-u)+\beta}(g_{u,t,s})_{\beta\alpha}.
\end{aligned} \tag{5.12}$$

The block matrix $g_{u,t,s}$ of an $SU(2N)_\infty$ transformation-matrix is given by (I.5). For simplicity, in the sum over u and t of (5.12), we pick up only the term $u = s$ and $t = -s$, then we have a simple transformation of the infinite-dimensional fermion operator $\psi_{Nr+\alpha}(\hat{g})$ as

$$\psi_{Nr+\alpha}(\hat{g}) = \sum_{s\in\mathbb{Z}} \psi_{N(r-s)+\beta}(g_s)_{\beta\alpha},$$

$$g_s = \begin{bmatrix} \cos\left(\frac{\theta_s}{2}\right) e^{-\frac{i}{2}(\psi_s+\varphi_s)} \cdot 1_N & -\sin\left(\frac{\theta_s}{2}\right) e^{-\frac{i}{2}(\psi_s-\varphi_s)} \cdot 1_N \\ \sin\left(\frac{\theta_s}{2}\right) e^{\frac{i}{2}(\psi_s-\varphi_s)} \cdot 1_N & \cos\left(\frac{\theta_s}{2}\right) e^{\frac{i}{2}(\psi_s+\varphi_s)} \cdot 1_N \end{bmatrix}.$$

5.6 Representation of infinite-dimensional HF Hamiltonian in terms of Schur polynomials

Using the τ rep of (5.4) and (5.5), an infinite-dimensional HF Hamiltonian for the LMG model is represented in terms of the Schur polynomials as

$$\begin{aligned}
\widehat{F}[\widehat{W}] &= -\sum_{r\in\mathbb{Z}} \left[\varepsilon\tfrac{1}{2}(\hat{h}_{2N})_{\alpha\beta} + \varepsilon\chi\tfrac{1}{2}\sin\theta e^{i\psi}(\hat{e}_{2N})_{\alpha\beta} + \varepsilon\chi\tfrac{1}{2}\sin\theta e^{-i\psi}(\hat{f}_{2N})_{\alpha\beta}\right] \tau\{e_{\alpha\beta}(r)\} \\
&= -\sum_{r\in\mathbb{Z}} \left[\varepsilon K_0(r) + \varepsilon\chi\tfrac{1}{2}\sin\theta e^{i\psi}K_+(r) + \varepsilon\chi\tfrac{1}{2}\sin\theta e^{-i\psi}K_-(r)\right] \\
&= \tfrac{\sqrt{N}}{2}\sum_{r\in\mathbb{Z}} \left[\varepsilon\tfrac{1}{\sqrt{N}}Y_{2r} - \varepsilon\chi\sin\theta\tfrac{1}{2}\left(e^{i\psi}+e^{-i\psi}\right)\tfrac{1}{\sqrt{N}}X_{2r+1} - \varepsilon\chi\sin\theta\tfrac{1}{2}\left(e^{i\psi}-e^{-i\psi}\right)\tfrac{1}{\sqrt{N}}Y_{2r+1}\right] \\
&= \tfrac{1}{2}\varepsilon\tfrac{1}{4}\left\{e^{2\sum_{m=0}^{\infty}x_{2m+1}} e^{-2\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{\partial}{\partial x_{2n+1}}} + e^{-2\sum_{m=0}^{\infty}x_{2m+1}} e^{2\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{\partial}{\partial x_{2n+1}}}\right\} \\
&\quad -\tfrac{1}{2}\varepsilon\chi\sin\theta\cdot\cos\psi\sum_{r\in\mathbb{Z}}\frac{\partial}{\partial x_{2r+1}} \\
&\quad -\tfrac{1}{2}\varepsilon\chi\sin\theta\cdot i\sin\psi\tfrac{1}{4}\left\{e^{2\sum_{m=0}^{\infty}x_{2m+1}} e^{-2\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{\partial}{\partial x_{2n+1}}} - e^{-2\sum_{m=0}^{\infty}x_{2m+1}} e^{2\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{\partial}{\partial x_{2n+1}}}\right\},
\end{aligned}$$

the second line of which has a form very similar to the Hamiltonian given by Mansfield [72]. The HF density matrix for the LMG model is also represented in terms of $K_0(r)$ and $K_\pm(r)$ as

$$\begin{aligned}
\widehat{W} - \tfrac{1}{2}\hat{I}_{2N} &= \sum_{r\in\mathbb{Z}} \left[\cos\theta\tfrac{1}{2}(\hat{h}_{2N})_{\alpha\beta} + \tfrac{1}{2}\sin\theta e^{-i\psi}(\hat{e}_{2N})_{\alpha\beta} + \tfrac{1}{2}\sin\theta e^{i\psi}(\hat{f}_{2N})_{\alpha\beta}\right]\tau\{e_{\alpha\beta}(r)\} \\
&= \sum_{r\in\mathbb{Z}} \left[\cos\theta K_0(r) + \tfrac{1}{2}\sin\theta e^{-i\psi}K_+(r) + \tfrac{1}{2}\sin\theta e^{i\psi}K_-(r)\right],
\end{aligned}$$

The infinite-dimensional HF operator for the LMG model $H_{F_\infty;\mathrm{HF}}(x, \widetilde{\partial}_x, \hat{g})$, corresponding to (3.60), is expressed in terms of $\widetilde{z}_{2N(s-r)+\alpha,2Ns+\beta}(x, \widetilde{\partial}_x)$ $(\alpha, \beta = 1, 2, \ldots, N, N+1, \ldots, 2N)$ as

$$\begin{aligned}
H_{F_\infty;\mathrm{HF}}(x, \widetilde{\partial}_x, \hat{g}) &= \sum_{r,s\in\mathbb{Z}} \left\{\widehat{F}[\widehat{W}](r)\right\}_{\alpha\beta} \widetilde{z}_{2N(s-r)+\alpha,2Ns+\beta}(x, \widetilde{\partial}_x) \\
&= \sum_{r,s\in\mathbb{Z}} \left\{-\varepsilon\tfrac{1}{2}\hat{h}(r) - \varepsilon\chi\tfrac{1}{2}\sin\theta e^{i\psi}\hat{e}(r) - \varepsilon\chi\tfrac{1}{2}\sin\theta e^{-i\psi}\hat{f}(r)\right\}_{\alpha\beta} \widetilde{z}_{2N(s-r)+\alpha,2Ns+\beta}(x, \widetilde{\partial}_x) \\
&= \sum_{r,s\in\mathbb{Z}} \Bigg[-\tfrac{1}{2}\varepsilon\sum_{\alpha=1}^{N}\widetilde{z}_{2N(s-r)+\alpha,2Ns+\alpha}(x, \widetilde{\partial}_x) + \tfrac{1}{2}\varepsilon\sum_{\alpha=N+1}^{2N}\widetilde{z}_{2N(s-r)+\alpha,2Ns+\alpha}(x, \widetilde{\partial}_x) \\
&\quad -\tfrac{1}{2}\varepsilon\chi\sin\theta e^{i\psi}\sum_{\alpha=1}^{N}\widetilde{z}_{2N(s-r)+\alpha,2Ns+\alpha+N}(x, \widetilde{\partial}_x) \\
&\quad -\tfrac{1}{2}\varepsilon\chi\sin\theta e^{-i\psi}\sum_{\alpha=1}^{N}\widetilde{z}_{2N(s-r)+\alpha+N,2Ns+\alpha}(x, \widetilde{\partial}_x)\Bigg] z^r,
\end{aligned} \tag{5.13}$$

in which the explicit form of $\widetilde{z}_{2N(s-r)+\alpha,2Ns+\beta}(x,\widetilde{\partial}_x)$ is given by (3.53). For example, the first term in the last line of (5.13), picking up only the first order in $S_k(-\widetilde{\partial}_x)$ and $S_k(\widetilde{\partial}_x)$, is given as

$$\begin{aligned}
\widetilde{z}_{2N(s-r)+\alpha,2Ns+\alpha+N}(x,\widetilde{\partial}_x) &= S_{2N(s-r)+\alpha+k+1-N}(x)S_{-2Ns-\alpha-k}(-x)S_1(-\widetilde{\partial}_x) \\
&+ S_{2N(s-r)+\alpha+k-N}(x)S_{-2Ns-\alpha-k+1}(-x)S_1(\widetilde{\partial}_x) + \cdots \\
&+ S_{2N(s-r)+\alpha+k}(x)S_{-2Ns-\alpha-k}(-x)S_N(-\widetilde{\partial}_x) \\
&+ S_{2N(s-r)+\alpha+k-N}(x)S_{-2Ns-\alpha-k+N}(-x)S_N(\widetilde{\partial}_x) + \cdots .
\end{aligned} \tag{5.14}$$

On the other hand, from (3.57), we have first leading terms of the τ-function, $\tau_{M=N}(x,g)$, for the LMG model with indices of the Plücker coordinates, $i_M = N, N+1, \ldots,$ $i_{M-1} = N-1, \ldots, i_1 = 1$ and etc., in the following form:

$$\begin{aligned}
\tau_N(x,g) &= v^{N,N-1,\ldots,1}_{N,N-1,\ldots,1}(g) \det \begin{vmatrix} S_0(x) & S_1(x) & S_2(x) & \cdots & S_{N-1}(x) \\ 0 & S_0(x) & S_1(x) & \cdots & S_{N-2}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & S_0(x) \end{vmatrix} \\
&+ v^{N,N-1,\ldots,1}_{N+1,N-1,\ldots,1}(g) \det \begin{vmatrix} S_1(x) & S_2(x) & S_3(x) & \cdots & S_N(x) \\ 0 & S_0(x) & S_1(x) & \cdots & S_{N-2}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & S_0(x) \end{vmatrix} + \cdots \\
&= v^{N,N-1,\ldots,1}_{N,N-1,\ldots,1}(g) + v^{N,N-1,\ldots,1}_{N+1,N-1,\ldots,1}(g)\left(S_1(x) + \cdots + S_N(x)\right) + \cdots \\
&= \left(\cos\tfrac{\theta}{2}\right)^N e^{-\frac{i}{2}N(\psi+\varphi)} + \left(\cos\tfrac{\theta}{2}\right)^{N-1} e^{-\frac{i}{2}(N-1)(\psi+\varphi)} \left(\sin\tfrac{\theta}{2}\right) e^{\frac{i}{2}(\psi-\varphi)} \\
&\times \left(S_1(x) + \cdots + S_N(x)\right) + \cdots .
\end{aligned} \tag{5.15}$$

The υ-dependent HF equation (3.59), $i\partial_\upsilon \tau_N\{x,\hat{g}(\upsilon)\} = H_{F_\infty;\mathrm{HF}}\{x,\widetilde{\partial}_x,\hat{g}(\upsilon)\}\tau_N\{x,\hat{g}(\upsilon)\}$, on the $\tau_N(x,\hat{g}(\upsilon))$ may be expected to give a new υ-dependent HF solution, where the υ-dependent HF operator is given by (5.13) and (5.14) and the τ-function is provided by (5.15).

5.7 Summary and discussions

In the preceding section, for the LMG model we have an approximate HF operator and a τ-function up to the first order in $S_k(-\widetilde{\partial}_x)$, $S_k(\widetilde{\partial}_x)$, $S_k(x)$ and $S_k(-x)$ which should satisfy the υ-dependent HF equation. Then, we meet inevitably with an interesting and exciting problem of solving the υ-dependent HF equation on the $\tau_N(x,g(\upsilon))$. After determining HF parameters θ and ψ self-consistently, a further study should be made to obtain a soliton solution derived from a υ-dependent Hirota's bilinear equation regarding υ as time t and relationship between a collective motion and the soliton solution. Such attractive problems have not been treated yet and just begin to open.

Here, we will recur to the representation of the infinite-dimensional LMG model. In the previous section, we already have obtained the expression for the LMG Hamiltonian (5.9) in which, however, the commutator term in the first term in the forth line of the equation brings an *anomaly* (an infinitely divergent result) for us, as shown below,

$$\sum_{m=0}^{\infty}\left[\frac{\partial}{\partial x_{2m+1}}, \sum_{n=0}^{\infty}(2n+1)x_{2n+1}\right] = \sum_{n=0}^{\infty}(2n+1) \to \infty \qquad (\textit{anomaly}).$$

For the present, we ought to discard this anomalous term to construct the *anomaly-free* infinite-dimensional LMG model.

Finally, we will point out the possibility of the extension of the present algebra $sl(2, C)$ to the affine Lie algebra $A_1^{(1)}$. Corresponding to an extension of the adopted simple LMG model to the so-called coupled LMG model [73, 74], we have a very interesting problem of constructing an $A_1^{(1)}$ LMG model for which the idea given in the paper [51] is considered to be very suggestive and useful. According to [51], we can choose the Chevalley basis for $A_1^{(1)}$ as follows:

$$\begin{aligned} h_1 &= [e_1, f_1], & h_2 &= [e_2, f_2], \\ e_1 &= \sum_{\nu \in \mathbb{Z}} \psi_{-1+2\nu} \psi^*_{2\nu}, & e_2 &= \sum_{\nu \in \mathbb{Z}} \psi_{2\nu} \psi^*_{1+2\nu}, \\ f_1 &= \sum_{\nu \in \mathbb{Z}} \psi_{2\nu} \psi^*_{-1+2\nu}, & f_2 &= \sum_{\nu \in \mathbb{Z}} \psi_{1+2\nu} \psi^*_{2\nu}, \end{aligned}$$

in which the total Hamiltonian of the coupled-LMG system is composed of two LMG model Hamitonians $H_1 = H_1\big(\widehat{K}_{1;0}, \widehat{K}_{1;\pm}\big)$ and $H_2 = H_2\big(\widehat{K}_{2;0}, \widehat{K}_{2;\pm}\big)$ and an interaction term $\widehat{K}_{1;+}\widehat{K}_{2;-} + \widehat{K}_{2;+}\widehat{K}_{1;-}$. We will discuss the above $A_1^{(1)}$ LMG model elsewhere using the present infinite-dimensional SCF method in τ-functional space on F_∞.

6 Summary and future problems

In Section 1, from algebro-geometric viewpoint, we have given a brief history of microscopic understanding of theoretical nuclear physics. It is summarized that we seek for an optimal coordinate-system describing dynamics on a group manifold based on a Lie algebra of fermion pairs. The TDHF/TDHB are nonlinear dynamics owing to their SCF characters. Seeking for collective coordinates in a fully parametrized dynamical system is exactly finding a symmetry of an evolution equation in nonlinear dynamics. In differential geometrical approaches for nonlinear problems, the integrability conditions are stated as the zero curvature of connection on the corresponding Lie groups of systems. Nonlinear evolution equations, e.g., the famous KdV and sine/sinh-Gordan equations and etc., come from the well-known Lax equation [70] which arises as the zero curvature [29]. These soliton equations describe motions of the tangent space of local gauge fields on a time t and a space x, which are Lie group/algebra-valued-equations arising from the integrability condition of gauge field with respect to t and x. In the TDHFT/TDHBT, the corresponding Lie groups are unitary transformation groups of their ortho-normal bases dependent on t but not on x.

In Section 2, along the Lax form for integrable systems, we have studied essential *curvature equations* to extract collective submanifolds out of the full TDHF/TDHB manifold and shown the following:

(i) Expectation values of the zero curvatures for a state function become a set of equations of motion, imposing weak orthogonal conditions among infinitesimal generators, i.e., equations for *tangent vector fields* on the group submanifold. Those of non-zero curvatures become gradients of a potential arising from a residual Hamiltonian along collective variables. These quantities are expected to give a criterion how the collective submanifold is truncated well.

(ii) The zero-curvature equation in QPF is nothing but the FRPAEQ imposed by the weak orthogonal conditions and has a simple geometrical interpretation: Relative vector fields made of the SCF Hamiltonian around each point on an integral curve constitute solutions for the FRPA around the same point which is in turn a fixed point in QPF. It means the FRPA is a natural extension of the usual RPA for small-amplitude fluctuations around a ground state to RPA at any point on the collective submanifold. The enveloping curve, made of a solution of the FRPA at each point on an integral curve, becomes another integral curve. The integrability condition is the infinitesimal condition to transfer a solution to another solution for the evolution equation. Then the usual RPAEQ becomes nothing but a method of determining an infinitesimal transformation of symmetry if fluctuating fields are composed of only normal modes.

In Section 3, to go beyond a perturbative method with respect to collective variables to extract large-amplitude collective motions, we have studied an algebro-geometric relation between SCFM and τ-FM, method of constructing integrable equations (Hirota's equations) in soliton theory. At the beginning, descriptions of dynamical fermion systems in both the methods had looked very different manners at first glance. In abstract fermion Fock spaces, each solution space of dynamics in both the methods is the corresponding Grassmannian. There is, however, a difference between finite-dimensional and infinite-dimensional fermion systems. In spite of such a difference, we have aimed at closely connecting the concept of mean-field potential with gauge of fermions and at making a role of *loop group* clear and consequently we have shown the relation between both the methods:

(i) The Plücker relation on the coset variable becomes analogous to the Hirota's bilinear form. The SCFM has been mainly devoted to the construction of boson-coordinate systems rather than to the construction of soliton solution by the τ-FM. It turns out that both the methods are equivalent with each other from the viewpoint of the Plücker relation or the bilinear identity equation defining Grassmannian.

(ii) The infinite-dimensional fermion operators are introduced through Laurent expansion of the finite-dimensional fermions with respect to degrees of freedom of fermions related to a υ dependent mean-field potential. Inversely, the mean-field potential is attributed to gauges of cooperating infinite-dimensional fermions. The construction of fermion operators can be contained in that of a Clifford algebra. This fact permits us to introduce an affine KM algebra. It means that the usual perturbative method with respect to collective variables with time periodicity has implicitly stood on a Gr_∞. Then we rebuilt the υ-HFT with the use of the affine KM algebra and map it to the corresponding τ-functional space. As a result, the υ-HFT becomes *a gauge theory of fermions* and the collective motion appears as the motion of fermion gauges with a common factor. The physical concept of *quasi-particle and vacuum* in the SCFM on S^1 is connected with *the Plücker relations*. Extracting sub-group orbits consisting of loop paths out of the Gr_M is just the formation of the Hirota's bilinear equation for the reduced KP hierarchy to $su(N)$ ($\subset sl(N)$). The present theory gives the manifest structure of gauge theory of fermions inherent in SCFM and provides a *new algebraic tool for microscopic understanding* of the fermion many-body system.

(iii) Through the investigation of physical meanings for the infinite-dimensional shift operators and the conditions of reduction to $sl(N)$ in τ-FM from the *loop* group viewpoint, it is induced that there is the close connection between *collective variables* and *spectral parameter* in soliton theory and that the algebraic mechanism bringing the physical concept of particle and collective motions arises from the reduction from $u(N)$ to $su(N)$ for the υ-dependent HF Hamiltonian.

(iv) It must be stressed that though the υ-HFT describes a dynamics on *real fermion-harmonic oscillators*, the soliton theory does on *complex fermion-harmonic oscillators*. This suggests us an important task to extend the υ-HFT on real space $\widehat{su}(N)$ to that on complex space $\widehat{sl}(N)$. It gives us a deeper understanding of the concept of quasi-particle energies and boson ones, in other words, independent particles and mean-field potential, in a microscopic treatment. Recently Wiegmann et al. have developed an approach in which the theory of classical integrable systems is applied to studies of 1D-fermion systems and the so-called orthogonality catastrophe in a Fermi gas. They have introduced a *boundary condition changing operator* [75] but have made no map $\sigma_M : :\psi_i\psi_j^* :\mapsto \widetilde{z}_{ij}(x, \widetilde{\partial}_x)$ (3.53) contrary to the present υ-HFT.

In Section 4, we have given a geometrical aspect of RPAEQ [50, 69] and an explicit expression for the RPAEQ with a normal mode on F_∞. We also have argued about the relation between a *loop* collective path and a FRPAEQ. Consequently, the usual perturbative method is shown to be involved in the present method which aims for constructing TDHFT on the affine KM algebra. It turns out that the collective submanifold is interpreted as a rotator on a curved surface in the Gr_∞. The present theory may lead to multi-circles occurring multiple parameterized collective motions. If we could arrive successfully at such a final goal, the present work may give us important clues for description of large-amplitude collective motions in nuclei and molecules and for construction of multi-dimensional soliton equations [61, 66] since the collective motions usually occur in multi-dimensional *loop* space.

In Section 5, as an illustration we have attempted to make a υ-HFT approach to an infinite-dimensional matrix model extended from the finite-dimensional $su(2)$ LMG model [42]. For this aim, we have given an affine KM algebra $\widehat{sl}(2,C)$ (complexification of $\widehat{su}(2)$) to which the LMG generators subject and their τ representations and the σ_K mappings for them. Further we have introduced infinite-dimensional "particle" and "hole" operators and operators $\widehat{K}_0$ and $\widehat{K}_\pm$ defined by the infinite-dimensional "particle-hole" pair operators. Using these operators, we have constructed the infinite-dimensional Heisenberg subalgebra of the affine KM algebra $\widehat{sl}(2,C)$. Thus the LMG Hamiltonian and its HF Hamiltonian have been represented in terms of the Heisenberg basic-elements whose representations are isomorphic to those in the corresponding boson space. They have been expressed in terms of infinite numbers of the variables x_k and the derivatives ∂_{x_k} through the Schur polynomials $S_k(x)$. Further we have obtained an approximate HF operator and a τ-function up to the first order in $S_k(-\widetilde{\partial}_x)$, $S_k(\widetilde{\partial}_x)$, $S_k(x)$ and $S_k(-x)$.

In Appendices, we have given the infinite-dimensional representation of $SU(2N)_\infty$ transformation of the "particle" and "hole" operators. The expression for τ rep of $(\mathrm{Y}_{-(2i+1)} + \mathrm{Y}_{(2i+1)})$, i.e., $g_{\mathrm{Y}_{-(2i+1)}+\mathrm{Y}_{(2i+1)}}(z)$ has been first given in terms of the Bessel functions. We have also shown an explicit expression for Plüker coordinate and calculated a quantity, $\det(1_N + p^\dagger p)$, in terms of the Schur polynomials.

Finally intimate relation of SCFT to soliton theory has been shown to come from ways of constructing a closed system of solution spaces. The ordinary SCFM has been almost devoted to approach cooperative phenomena in finite fermion systems. We must contrive construction of the optimal coordinate-system on the group manifold. For this purpose the relation between the boson expansion method for finite fermion systems and the τ-FM for infinite ones should be intensively investigated to clarify algebro-geometric structures of integrable systems. Such algebro-geometric approach will make a bridge between finite fermion systems and infinite ones. Various physical concepts and mathematical methods will work well also in the infinite ones. The SCFM based on global symmetry should be much improved noticing local symmetry of the infinite ones and then may open a new area in vigorous pursuit of wider fields of physics.

We have many future problems in connection with the above discussions, which are itemized as follows:

(i) To study the relation between the quantity of non-zero curvature and the collectivity: It is interesting to study the relation using the simple LMG model, which leads to an investigation of the effective condition for the collective submanifold extracted by the zero-curvature equation. Temporarily digressing from the integrability condition, adopting the Bethe anzatz (BA) we have obtained exact solutions for the LMG model solving the BA equation [76]. Contrary to Pan and Draayer's work [63] and our previous works [77], we do not use any bosonization nor infinite-dimensional techniques and hence have no restrictions on interaction-strengths of LMG Hamiltonian. Considering the advantage of the integrability condition, the famous Gaudin model plays an important role to solve effectively the BA equation [78]. From the *loop group* viewpoint, as shown by Sklyanin, with the use of the exactly-solvable Gaudin model obeying the Gaudin algebra, an exponential generating function of correlators is obtained from the Gauss decomposition for $sl(2,C)$ *loop algebra*, which gives correlators including the Richardson–Gaudin determinant formula for the Bethe eigen-function [79]. A generalization of the Gaudin algebra is given by Ortiz et al. [80]. These works may have an intimate relation with our recent work [35] and the present work.

(ii) To clarify the explicit relation between spectral parameter and collective variable and the physical concept of the geometrical connection: The spectral parameter of the iso-spectral equation in soliton theory and the collective variable in SCFM, though showing different aspects at a glance, work as scaling parameters on S^1. The former relates to a scaling by analytical continuation of S^1, i.e., z. The latter makes roles of deformation parameters of *loop paths* in Gr_M.

(iii) To study the relation between weak boson operators and boson mapping operators, i.e., the shift operators in τ-FM: The generators for collective variables in F_∞ can never take exact boson commutation relations because of the finite-dimensional matrices.

(iv) To study a relation hidden behind gauge of state functions and construction of fermion pairs: In the usual algebraic treatment of fermion many-body systems, we assign an abstract number to each of the set of quantal numbers and let their fermions make the Lie algebras ($u(N)$, $so(2N)$, $so(2N+1)$ and etc.). For the pair-constructions we have an interpretation as classifications of Laurent spectra in the infinite-dimensional fermions, although we did not manifestly state. On the other hand, as well known, electron spin can be described as a geometrical phase of gauge with the help of Möbius band. Then we inversely start from fermions with the abstract numbers and through any way we could return to the original fermions with the physical quantal numbers. We think it not so wrong to attempt to understand quantal numbers as the geometrical attribute of the Grassmannian made of the abstract fermions.

(v) To establish mathematical tools to obtain subgroup-orbits of loop paths in Gr_M, basing on the Plücker relations on S^1, i.e., soliton equations. This problem is most fundamental to solve the new theory: For this aim we must know sub-group orbits or corresponding sub-Lie algebras and establish mathematical tools to extract them out of Gr_M. In this concern, we are intensively interested in the algebraic mechanism for spontaneous decision of a fixed point and a collective submanifold around the point.

(vi) To study a relation between nonlinear superposition principle in soliton theory and generator coordinate method (GCM) in SCFM: The GCM may provide a superposition principle on a nonlinear space [81, 16]. Standing on the viewpoint of local symmetry of infinite fermion systems behind global symmetry of finite ones, we might reconstructed the GCM and nonlinear superposition methods using the infinite-dimensional shift operators. What relation does exist between the construction of exact solutions based on the idea of the imbricate series in soliton equations [82] and resonating mean-field theories [83]?

(vii) To study why soliton solutions for classical wave equations show fermion-like behaviours in quantum dynamics and about what symmetries are hidden in soliton equations. To both the questions suggested by Tajiri et al., we cannot give a satisfactory answer yet within the present framework. Because both the methods are *a priori* based on the fermion system from the outset. That is to say: SCFM describes a quasi classical dynamics on Grassmannian (S-det orbit) which is induced owing to the anti-commutative property of fermions. On the other hand, τ-FM also uses the fermions to explain Grassmannian of solution space which reflects the fermion-like behaviours of soliton solutions. Therefore we should study further why extraction of soliton equations out of classical wave equations brings out Grassmannian. In the reductive perturbation method [84], soliton equations appear as the symmetry space in a classical wave equation with respect to a transformation and a scaling transformation with a common parameter for independent and dependent variables. We can see a relation between the power exponents of the parameter and the *degree* of shift operators. At any point on extracting way of soliton equations from classical wave equations did we introduce the “anti-commutative character”, in other words, did we introduce the structure of Grassmannian?

A Coset variables

Following [16], let us introduce the triangular matrix functions $S(\zeta)$, $C(\zeta)$ and $\tilde{C}(\zeta)$ defined as

$$S(\zeta) = (S_{ia}(\zeta)) = \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)!}\zeta(\zeta^\dagger\zeta)^k,$$
$$C(\zeta) = (C_{ab}(\zeta)) = 1_M + \sum_{k=1}^{\infty}(-1)^k \frac{1}{(2k)!}(\zeta^\dagger\zeta)^k = C^\dagger(\zeta),$$
$$\tilde{C}(\zeta) = (\tilde{C}_{ij}(\zeta)) = 1_{N-M} + \sum_{k=1}^{\infty}(-1)^k \frac{1}{(2k)!}(\zeta\zeta^\dagger)^k = \tilde{C}^\dagger(\zeta),$$

which have the properties analogous to the usual triangular functions

$$C^2(\zeta) + S^\dagger(\zeta)S(\zeta) = 1_M, \qquad \tilde{C}^2(\zeta) + S(\zeta)S^\dagger(\zeta) = 1_{N-M}, \qquad S(\zeta)C(\zeta) = \tilde{C}(\zeta)S(\zeta).$$

Then the matrix p in (3.3) is defined as $p = (p_{ia}) = S(\zeta)C^{-1}(\zeta)$. The matrix g in (3.3) is decomposed as $g = g_\zeta g_w$ using the matrices given below

$$g_\zeta = e^{\gamma'} = \begin{bmatrix} C(\zeta) & -S(\zeta)^\dagger \\ S(\zeta) & \bar{C}(\zeta) \end{bmatrix}, \qquad \gamma' = \begin{bmatrix} 0 & -\zeta^\dagger \\ \zeta & 0 \end{bmatrix}, \qquad g_w = e^{\gamma''} = \begin{bmatrix} w & 0 \\ 0 & \bar{w} \end{bmatrix},$$
$$w = e^\eta, \qquad \bar{w} = e^{\bar{\eta}}, \qquad \gamma'' = \begin{bmatrix} \eta & 0 \\ 0 & \bar{\eta} \end{bmatrix}, \qquad \eta^\dagger = -\eta, \qquad \bar{\eta}^\dagger = -\bar{\eta},$$

where ζ is a $(N-M)\times M$ matrix (ζ_{ia}) and η and $\bar{\eta}$ are $M\times M$ and $(N-M)\times(N-M)$ anti-hermitian matrices (η_{ab}) and $(\bar{\eta}_{ij})$, respectively. The indices i and a denote unoccupied $(M+1 \sim N)$ states and occupied $(1\sim M)$ states, respectively.

B Properties of the differential operator e_{ia} acting on $\Phi_{M,M}$

Let us introduce a differential operator and a function corresponding to the so-called particle-hole operator and the free particle vacuum in the physical fermion space, respectively as

$$e^*_{ia} \stackrel{d}{=} -\left(p_{ib}p_{ja}\frac{\partial}{\partial p_{jb}} + \frac{\partial}{\partial p^*_{ia}} + \frac{i}{2}p_{ia}\frac{\partial}{\partial\tau}\right),$$
$$\Phi_{M,M}(p,p^*,\tau) = [\det(1+p^\dagger p)]^{-\frac{1}{2}}e^{i\tau}. \tag{B.1}$$

By using the formula for the differential of a determinant, we can easily calculate as

$$\frac{\partial}{\partial p_{jb}}[\det(1+p^\dagger p)]^{-\frac{1}{2}} = -\tfrac{1}{2}p^*_{jc}[(1+p^\dagger p)^{-1}]^{\mathrm{T}}_{cb}[\det(1+p^\dagger p)]^{-\frac{1}{2}},$$
$$\frac{\partial}{\partial p^*_{ia}}[\det(1+p^\dagger p)]^{-\frac{1}{2}} = -\tfrac{1}{2}p_{id}[(1+p^\dagger p)^{-1}]_{da}[\det(1+p^\dagger p)]^{-\frac{1}{2}}, \tag{B.2}$$

where a^{-1} denotes an inverse matrix of a. Then from equations (B.1) and (B.2) we get

$$e^*_{ia}\Phi_{M,M} = \left\{\tfrac{1}{2}p_{ib}p_{ja}\cdot p^*_{jc}[(1+p^\dagger p)^{-1}]^{\mathrm{T}}_{cb} + \tfrac{1}{2}p_{id}[(1+p^\dagger p)^{-1}]_{da} + \tfrac{1}{2}p_{ia}\right\}\Phi_{M,M}$$
$$= \tfrac{1}{2}\left[\{p(1+p^\dagger p)^{-1}(1+p^\dagger p)\}_{ia} + \tfrac{1}{2}p_{ia}\right]\Phi_{M,M} = p_{ia}\Phi_{M,M},$$
$$e^*_{ia}p_{jb} = -p_{ib}p_{ja} + p_{jb}e^*_{ia}.$$

Thus we can prove

$$e^*_{ia}\Phi_{M,M} = p_{ia}\Phi_{M,M}, \qquad [e^*_{ia}, p_{jb}] = -p_{ib}p_{ja}.$$

These are just the relation given in the first line of (3.11). The other relations in (3.11) can be proved in a similar manner [37].

C Affine Kac–Moody algebra

According to Kac and Raina [49], let $gl(N)$ be the Lie algebra of all $N \times N$ matrices with complex entries acting in $\mathbb{C}^N$ and let $\mathbb{C}[z, z^{-1}]$ be the ring of Laurent polynomials in indeterminate z and z^{-1}. The *loop* algebra [36] $\widetilde{gl}(N)(\supset \widetilde{u}(N))$ is defined as $gl(N)(\mathbb{C}[z, z^{-1}])$ $(\supset u(N)(\mathbb{C}[z, z^{-1}]))$, i.e., as the complex Lie algebra of $N \times N$ matrices with Laurent polynomials as entries. An element of $\widetilde{gl}(N)$ is given in the form

$$a(z) = \sum_{r \in \mathbb{Z}} z^r a_r \qquad (a_r \in gl(N)). \tag{C.1}$$

As was pointed out by Kac and Raina, we regard the fermion pair- and single-operators as

$$\begin{aligned} &c_\alpha^\dagger c_\beta \mapsto e_{\alpha\beta} \quad (1 \leq \alpha,\ \beta \leq N), \\ &c_\alpha^\dagger \mapsto u_\alpha \quad (1 \leq \alpha \leq N), \qquad c_\alpha \mapsto u_\alpha^{\mathrm{T}} \quad (1 \leq \alpha \leq N), \end{aligned}$$

where the matrix elements $e_{\alpha\beta}$ are equal to 1 in the (α, β) entry and 0 elsewhere. The matrix elements $e_{\alpha\beta}$ form a basis of $gl(N)$. The components of $N \times 1$ column vectors u_α are equal to 1 in the α-th row and 0 elsewhere. They span the vector space $\mathbb{C}^N$ in which the $gl(N)$ acts. The symbol T means the transpose of a vector or a matrix. The matrices $e_{\alpha\beta}(r) \stackrel{d}{=} z^r e_{\alpha\beta}$ constitute a basis of $\widetilde{gl}(N)$. The *loop* algebra $\widetilde{gl}(N)$ acts in the vector space $\mathbb{C}[z, z^{-1}]^N$ consisting of $N \times 1$ column vectors with the Laurent polynomials in z and z^{-1} as entries. The Lie bracket on $\widetilde{gl}(N)$ is the commutator

$$[e_{\alpha\beta}(r), e_{\gamma\delta}(s)] = \delta_{\beta\gamma} e_{\alpha\delta}(r + s) - \delta_{\alpha\delta} e_{\gamma\beta}(r + s). \tag{C.2}$$

The vectors defined as

$$\nu_{Nr+\alpha} = z^{-r} u_\alpha, \qquad \nu^*_{Nr+\alpha} = u_\alpha^{\mathrm{T}} z^r,$$

also form a basis of the vector space $\mathbb{C}[z, z^{-1}]^N$ indexed by $\mathbb{Z}$ and its dual space. The $\{\nu_i \,|\, i = Nr + \alpha \in \mathbb{Z}\}$ is given by the column vector with 1 as the i-th row and 0 elsewhere. Thus it is possible to identify $\mathbb{C}[z, z^{-1}]^N$ with $\mathbb{C}^\infty$. The relation $e_{\alpha\beta}(r)\nu_{Ns+\beta} = \nu_{N(s-r)+\alpha}$ is easily derived.

For $a(z) \in \widetilde{gl}(N)$ we denote the corresponding matrix in $\overline{a}_\infty$ by $\tau\{a(z)\}$. Then we deduce a matrix representation for $\tau\{e_{\alpha\beta}(r)\}$ in $\overline{a}_\infty$ as

$$\tau\{e_{\alpha\beta}(r)\} = \sum_{s \in \mathbb{Z}} E_{N(s-r)+\alpha, Ns+\beta}. \tag{C.3}$$

E_{ij} $(i, j \in \mathbb{Z})$ have 1 as the (i, j) entry and 0 elsewhere and form a gl_∞. Suppose a bigger algebra $\overline{a}_\infty$

$$\overline{a}_\infty = \{(a_{ij}) \,|\, i, j \in \mathbb{Z},\ a_{ij} = 0 \text{ for } |i - j| \gg \mathbb{N}\}. \tag{C.4}$$

There exists such an $\mathbb{N}$ satisfying the above condition. The corresponding matrix of the $a(z)$ in (C.1) in $\overline{a}_\infty$ has an infinite N periodic sequence of block form

$$\tau\{a(z)\} = \begin{bmatrix} \ddots & \ddots & \ddots & & \ddots & & \\ & a_{-1} & a_0 & a_1 & & \ddots & \\ \ddots & & a_{-1} & a_0 & a_1 & & \ddots \\ & \ddots & & a_{-1} & a_0 & a_1 & \\ & & \ddots & & \ddots & \ddots & \ddots \end{bmatrix}. \tag{C.5}$$

We regard (C.5) as a representation of the matrix $\overline{a}_\infty$ in which elements on each diagonal parallel to the principal diagonal form a periodic sequence with period N. Let $X(k) = z^k X$ $(X \in gl(N))$ be an element of $\widetilde{gl}(N)$. Define an antilinear anti-involution ω on $\widetilde{gl}(N)$ by $\omega[X(k)] = z^{-k}X^\dagger$. Then in the $\overline{a}_\infty$ we get $\tau\{\omega[X(k)]\} = \tau^\dagger\{X(k)\}$.

Using the fundamental idea of the Dirac theory [40], we define the vacuum state in which the state labeled by the "Laurent spectrum" with positive energy is empty but all the negative energy states labeled by the "Laurent spectra" are occupied. Denoting an exterior product of vectors as $\wedge$, a perfect vacuum Ψ_0 and a reference vacuum Ψ_M are expressed as

$$\Psi_0 = \nu_0 \wedge \nu_{-1} \wedge \nu_{-2} \wedge \cdots , \qquad \Psi_M = \nu_M \wedge \nu_{M-1} \wedge \nu_{M-2} \wedge \cdots .$$

Let the space $V = \oplus_{i\in\mathbb{Z}}\mathbb{C}\nu_i$ and its dual $V^* = \oplus_{j\in\mathbb{Z}}\mathbb{C}\nu_j^*$ be an infinite-dimensional complex space with a basis $\{\nu_i, \nu_j^* \,|\, i, j \in \mathbb{Z}\}$ giving a linear functional ν_j^* on V by $\nu_j^*(\nu_i) = \delta_{ij}$ $(i, j \in \mathbb{Z})$. Each $\nu \in V$ and $\nu^* \in V^*$ defines a *wedging operator* $\hat{\nu}$ and a *contracting operator* $\check{\nu}^*$ on the F_∞ as

$$\begin{aligned}
&\hat{\nu}(\nu_{i_1} \wedge \nu_{i_2} \wedge \cdots) = \nu \wedge \nu_{i_1} \wedge \nu_{i_2} \wedge \cdots ,\\
&\check{\nu}^*(\nu_{i_1} \wedge \nu_{i_2} \wedge \cdots) = \nu^*(\nu_{i_1})\nu_{i_2} \wedge \nu_{i_3} \wedge \nu_{i_4} \wedge \cdots - \nu^*(\nu_{i_2})\nu_{i_1} \wedge \nu_{i_3} \wedge \nu_{i_4} \wedge \cdots\\
&\qquad\qquad + \nu^*(\nu_{i_3})\nu_{i_1} \wedge \nu_{i_2} \wedge \nu_{i_4} \wedge \cdots .
\end{aligned} \tag{C.6}$$

Then the operators $\{\hat{\nu}_i,\ \check{\nu}_j^* \,|\, i, j \in \mathbb{Z}\}$ generate a Clifford algebra

$$\{\check{\nu}_i^*, \hat{\nu}_j\} = \delta_{ij}, \qquad \{\check{\nu}_i^*, \check{\nu}_j^*\} = \{\hat{\nu}_i, \hat{\nu}_j\} = 0. \tag{C.7}$$

Thus the anti-commutation relations lead us to the identification of the new fermion annhilation-creation operators (3.23) with the present operators at a pointwise z with $|z| = 1$ as

$$\psi_{Nr+\alpha}^* \mapsto \check{\nu}_{Nr+\alpha}^*, \qquad \psi_{Nr+\alpha} \mapsto \hat{\nu}_{Nr+\alpha}. \tag{C.8}$$

Using the above identification, the corresponding perfect vacuum and reference vacuum, it can be shown that

$$\begin{aligned}
&\Psi_0 \mapsto |\mathrm{Vac}\rangle, \qquad \Psi_M \mapsto |M\rangle = \psi_M \cdots \psi_1 |\mathrm{Vac}\rangle,\\
&\psi_{Nr+\alpha}|\mathrm{Vac}\rangle = 0, \qquad \langle \mathrm{Vac}|\psi_{Nr+\alpha}^* = 0 \qquad (r \le -1),\\
&\psi_{Nr+\alpha}^*|\mathrm{Vac}\rangle = 0, \qquad \langle \mathrm{Vac}|\psi_{Nr+\alpha} = 0 \qquad (r \ge 0).
\end{aligned}$$

From (C.3), (C.6), (C.8) and the Clifford algebra (C.7) we obtain representations

$$\tau\{e_{\alpha\beta}(r)\} = \sum_{s\in\mathbb{Z}} E_{N(s-r)+\alpha,Ns+\beta} \simeq \sum_{s\in\mathbb{Z}} \hat{\nu}_{N(s-r)+\alpha}\check{\nu}_{Ns+\beta}^* \simeq \sum_{s\in\mathbb{Z}} \psi_{N(s-r)+\alpha}\psi_{Ns+\beta}^*. \tag{C.9}$$

Construction of the SCFM on the infinite-dimensional Fock space is an interesting illustrative problem in order to clarify an algebraic structure among the original fermionic field, the vacuum field defined in the SCF potential and the bosonic field associated with Laurent spectra.

It is well known that the operator of (C.9) acting in the F_∞ has in general an "anomaly". To avoid the "anomaly", we had better use either of the normal-ordered products given below

$$\begin{aligned}
&: E_{Nr+\alpha,Ns+\beta} :\stackrel{d}{=} E_{Nr+\alpha,Ns+\beta} - \delta_{\alpha\beta}\delta_{rs} \qquad (s < 0),\\
&: \hat{\nu}_{Nr+\alpha}\check{\nu}_{Ns+\beta}^* :\stackrel{d}{=} \hat{\nu}_{Nr+\alpha}\check{\nu}_{Ns+\beta}^* - \delta_{\alpha\beta}\delta_{rs} \qquad (s < 0),\\
&: \psi_{Nr+\alpha}\psi_{Ns+\beta}^* :\stackrel{d}{=} \psi_{Nr+\alpha}\psi_{Ns+\beta}^* - \delta_{\alpha\beta}\delta_{rs} \qquad (s < 0).
\end{aligned} \tag{C.10}$$

We define important shift operators Λ_j and Λ_{-j} ($j \in \mathbb{Z}_+$) in the soliton theory on a group as

$$\Lambda_j = \tau\left(\begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \ddots & 1 & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 1 \\ z & \cdots & \cdots & \cdots & 0 \end{bmatrix}^j\right),$$

$$\Lambda_{-j} = \tau\left(\begin{bmatrix} 0 & \cdots & \cdots & \cdots & z^{-1} \\ 1 & \ddots & & & \vdots \\ \vdots & 1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 & 0 \end{bmatrix}^j\right) = \Lambda_j^\dagger, \tag{C.11}$$

which are images under τ as

$$\Lambda_j = \tau\left\{\left[\sum_{\alpha=1}^{N-1} e_{\alpha,\alpha+1} + z e_{N,1}\right]^j\right\} \simeq \tau\left\{\left[\sum_{\alpha=1}^{N-1} c_\alpha^\dagger c_{\alpha+1} + c_n^\dagger c_1 z\right]^j\right\}.$$

Through an easy calculation with the use of (C.9) and (C.10), the 2-*cocycle* α on $\bar{a}_\infty$ (C.4) induces the following 2-*cocycle* on a pair of basis elements of $\widetilde{gl}_n$:

$$\alpha(\tau\{e_{\alpha\beta}(k)\}, \tau\{e_{\gamma\delta}(l)\}) = \delta_{\alpha\delta}\delta_{\beta\gamma}\delta_{k+l,0} \cdot k, \tag{C.12}$$

where the Kac–Peterson 2-*cocycle* α [85, 48, 54] is defined as

$$\alpha(E_{ij},\ E_{kl}) = \delta_{jk}\delta_{il}\ (i \le 0) - \delta_{li}\delta_{kj}\ (j \le 0) = \begin{cases} 1, & \text{for } j = k,\ j \ge 1,\ i = l,\ i \le 0, \\ -1, & \text{for } i = l,\ i \ge 1,\ j = k,\ j \le 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have for the shift operators (C.11)

$$\alpha(\Lambda_k, \Lambda_l) = \delta_{k+l,0} \cdot k.$$

For any elements $a(z)$ and $b(z)$ in the $\widetilde{gl}(N)$ the formula (C.12) is written as

$$\alpha(\tau\{a(z)\}, \tau\{b(z)\}) = \mathrm{Res}_0 \mathrm{Tr}\, a'(z) b(z),$$

where $a'(z)$ is the derivative of a with respect to z and Res_0 is the residue at $z = 0$, i.e., the coefficient of z^{-1}. Investigation of the highest weight representation of $\widetilde{gl}(N)$ leads to its central extension $\widehat{gl}(N) = \widetilde{gl}_N + \mathbb{C} \cdot c$ in which general elements $a(z)$ and $b(z)$ and center $\mathbb{C} \cdot c$ satisfy the KM brackets

$$[a(z), c]_{\mathrm{KM}} = 0,$$
$$[a(z), b(z)]_{\mathrm{KM}} = [a(z), b(z)] + \left\{\mathrm{Res}_0 \mathrm{Tr}\, a'(z) b(z)\right\} \cdot c. \tag{C.13}$$

The Lie algebra $\widehat{gl}(N)$ is called the affine KM algebra associated with the Lie algebra $gl(N)$. For simplicity consider the level one case, $c|M\rangle = 1 \cdot |M\rangle$. Using the one-level formula it is possible to rewrite (C.13) as

$$X_a = \widehat{X}_a + \mathbb{C} \cdot c,$$

$$\widehat{X}_a \stackrel{d}{=} \sum_{r=-\mathbb{N}}^{\mathbb{N}} \sum_{s\in\mathbb{Z}} (a_r)_{\alpha\beta} : \psi_{N(s-r)+\alpha}\psi^*_{Ns+\beta} :, \qquad (a_r)_{\alpha\beta} \equiv (a_r)_{N(s-r)+\alpha, Ns+\beta},$$
$$[X_a, c]_{\mathrm{KM}} = 0, \qquad [X_a, X_b]_{\mathrm{KM}} = \widehat{X}_{[a,b]} + \alpha(a,b)\cdot c.$$

The matrices a and b are any elements of (C.5). The $[a,b]$ denotes the Lie bracket of their matrices. The τ-rep of $a(z)$ is given as

$$\tau\{a(z)\} = \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} (a_r)_{N(s-r)+\alpha, Ns+\beta}\psi_{N(s-r)+\alpha}\psi^*_{Ns+\beta},$$

from which the commutator and the 2-*cocycle* α between $\tau\{a(z)\}$ and $\tau\{b(z)\}$ are calculated as

$$[\tau\{a(z)\}, \tau\{b(z)\}] = \sum_{r,s\in\mathbb{Z}}\sum_{t\in\mathbb{Z}} ([a_r, a_s])_{N(t-r-s)+\alpha, Ns+\beta}\psi_{N(s-r)+\alpha}\psi^*_{Nt+\beta},$$
$$\alpha(\tau\{a(z)\}, \tau\{b(z)\}) = \sum_{r\in\mathbb{Z}} r\,\mathrm{Tr}(a_r b_{-r}).$$

D Schur polynomials and τ-function

The Schur polynomials $S_k(x)$ belonging to $\mathbb{C}(x_1, x_2, \dots)$ are defined by the generating function

$$\exp\sum_{k\geq 1}^{\infty} x_k p^k = \sum_{k\geq 0} S_k(x) p^k. \tag{D.1}$$

The Schur polynomials and their associated recursion formulas play important roles to evaluate matrix elements between two number-projected Hartree–Bogoliubov states [86]. The Schur polynomial is related to a symmetric function h_k $\sum_{k\geq 0} h_k p^k = \Pi_i^N (1-\epsilon_i p)^{-1}$. Then the Schur polynomial $S_k(x)$ is written as

$$S_k(x) = h_k(\epsilon_1, \epsilon_2, \dots, \epsilon_N), \qquad x_j = \tfrac{1}{j}\left(\epsilon_1^j + \epsilon_2^j + \cdots + \epsilon_N^j\right)$$

and is given explicitly as

$$S_0(x) = 1, \qquad S_1(x) = x_1, \qquad S_2(x) = x_2 + \tfrac{1}{2}x_1^2, \qquad S_3(x) = x_3 + x_1x_2 + \tfrac{1}{6}x_1^3,$$
$$S_4(x) = x_4 + x_1x_3 + \tfrac{1}{2}x_2^2 + \tfrac{1}{2}x_1^2x_2 + \tfrac{1}{24}x_1^4, \qquad \dots.$$

We here construct an explicit form of the polynomials in the bosonic Fock space which under the σ_M corresponds to the finite monomials of the $F^{(M)}$. It is given by

$$\sigma_M; \nu_{i_M}\wedge\nu_{i_{M-1}}\wedge\nu_{i_{M-2}}\wedge\cdots\wedge\nu_{i_1} \mapsto S_{i_M-M, i_{M-1}-(M-1), i_{M-2}-(M-2),\dots,i_1-1}(x), \tag{D.2}$$

where $i_M > i_{M-1} > \cdots > i_1$. The Schur polynomial $S_\lambda(x) = S_{\lambda_1,\lambda_2,\lambda_3,\dots,\lambda_k}(x)$ is given as

$$S_{\lambda_1,\lambda_2,\lambda_3,\dots,\lambda_k}(x) = \det\begin{vmatrix} S_{\lambda_1}(x) & S_{\lambda_1+1}(x) & S_{\lambda_1+2}(x) & \cdots & S_{\lambda_1+k-1}(x) \\ S_{\lambda_2-1}(x) & S_{\lambda_2}(x) & S_{\lambda_2+1}(x) & \cdots & S_{\lambda_2+k-2}(x) \\ S_{\lambda_3-2}(x) & S_{\lambda_3-1}(x) & S_{\lambda_3}(x) & \cdots & S_{\lambda_3+k-3}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ S_{\lambda_k+1-k}(x) & S_{\lambda_k+2-k}(x) & S_{\lambda_k+3-k}(x) & \cdots & S_{\lambda_k}(x) \end{vmatrix},$$

where λ denotes a partition $\mathrm{Par}\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0\}$ [48, 49]. As for the contravariant hermitian form (3.52) in $B^{(M)}$ they form an orthogonal basis $\langle S_\lambda | S_\mu\rangle = \delta_{\lambda\mu}$. For the special partition $\mathrm{Par}\,\lambda = 1^M \equiv \{\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1, \lambda_4 = 1, \dots, \lambda_M = 1\}$, i.e., the completely anti-symmetric Young diagram,

$$S_{1^M}(x) = \det \begin{vmatrix} S_1(x) & S_2(x) & S_3(x) & S_4(x) & \cdots & S_M(x) \\ 1 & S_1(x) & S_2(x) & S_3(x) & \cdots & S_{M-1}(x) \\ 0 & 1 & S_1(x) & S_2(x) & \cdots & S_{M-2}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & S_1(x) \end{vmatrix} = (-1)^M S_M(-x).$$

A group orbit of the highest weight vector $|M\rangle$ under the action $U(g)$ is mapped to a space of τ-function $\tau_M(x,g) = \langle M|e^{H(x)}U(g)|M\rangle$. Using (D.2), the Schur-polynomial expression for τ-function is given by (3.56). Noting the relation $\langle S_\lambda|S_\mu\rangle = S_\lambda(\widetilde{\partial}_x)S_\mu(x)|_{x=0} = \delta_{\lambda\mu}$, $v^{M,M-1,\cdots,1}_{i_M,i_{M-1},\dots,i_1}(g)$ is obtained from $\tau_M(x,g)$ as

$$v^{M,M-1,\ dots,1}_{i_M,i_{M-1},\dots,i_1}(g) = v^{M,M-1,\dots,1}_{i_M,i_{M-1},\dots,i_1}\{g(x)\}|_{x=0} = S_{i_M-M,i_{M-1}-(M-1),\dots,i_1}(\widetilde{\partial}_x)\tau_M(x,g)|_{x=0},$$

$$v\{g(x)\} \equiv e^{H(x)}v(g)e^{-H(x)}, \qquad g_{ij}(x) = \sum_{k\geq i}^{\infty} S_{k-i}(x)g_{kj}(x).$$

The coefficient $v^{M,M-1,\dots,1}_{i_M,i_{M-1},\dots,i_1}(g)$ is called the Plücker coordinate [87]. In the soliton theory on a group the Plücker relations among $v^{M,M-1,\dots,1}_{i_M,i_{M-1},\dots,i_1}\{g(x)\}$ correspond to the Hirota's forms in the KP hierarchy.

E Hirota's bilinear equation

According to Kac [48], a vector $|\tau_M\rangle \in F^{(M)}$ belongs to the group orbit of the highest weight vector (the vacuum) if and only if it satisfies the bilinear identity equation

$$\sum_{i\in\mathbb{Z}} \psi_i|\tau_M\rangle \otimes \psi_i^*|\tau_M\rangle = 0 \quad \Longleftrightarrow \quad |\tau_M\rangle = U(\hat{g})|M\rangle, \qquad \hat{g} \in GL(\infty).$$

This identity is cast into an infinite set of nonlinear differential equations for $\tau_M(x) \stackrel{d}{=} \langle M|e^{H(x)}|\tau_M\rangle$

$$0 = \frac{1}{2\pi i}\oint \frac{dp}{p}\langle M+1|e^{H(x')}\Psi(p)|\tau_M\rangle \otimes \langle M-1|e^{H(x'')}\Psi^*(p)|\tau_M\rangle \tag{E.1}$$

$$= \frac{1}{2\pi i}\oint dp \exp\left\{\sum_{j\geq 1} p^j(x'_j - x''_j)\right\} \exp\left\{-\sum_{j\geq 1}\frac{p^{-j}}{j}\left(\frac{\partial}{\partial x'_j} - \frac{\partial}{\partial x''_j}\right)\right\} \tau_M(x')\cdot\tau_M(x'').$$

It is described by the Hirota's bilinear differential operator as

$$P(D)f\cdot g \stackrel{d}{=} P\left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \dots\right)(f(x+y)\cdot g(x-y))|_{y=0},$$

where $P(D)$ is a polynomial in $D = (D_1, D_2, \dots)$. Note that $Pf\cdot f \equiv 0$ if and only if $P(D) = -P(-D)$. Defining new variables x and y by the relations $x' = x - y$ and $x'' = x + y$, with the help of (D.1) and the notation $\widetilde{D} = (D_1, \frac{1}{2}D_2, \dots)$, (E.1) is brought to the form

$$\sum_{j\geq 0} S_j(-2y)S_{j+1}(\widetilde{D})\exp\left(\sum_{s\geq 1} y_s D_s\right)\tau_M(x)\cdot\tau_M(x) = 0. \tag{E.2}$$

If we expand (E.2) into a multiple Taylor series of variables $y_1, y_2, \dots$ and make each coefficient of this series vanishing, we get an infinite set of nonlinear partial differential equation for the KP hierarchy.

F Calculation of commutators and 2-cocycles among operators $\boldsymbol{K}$

Commutators and 2-*cocycles* among operators $K_0(r)$ and $K_\pm(r)$ are calculated using (C.2) and (C.12) as follows:

$$
\begin{aligned}
[K_0(r), K_\pm(s)] &= \left[\left(\tfrac{1}{2}\hat{h}_{2N}\right)_{\alpha\beta} e_{\alpha\beta}(r), \left(\begin{array}{c}\hat{e}_{2N}\\ \hat{f}_{2N}\end{array}\right)_{\gamma\delta} e_{\gamma\delta}(s)\right]\\
&= \left[\left(\tfrac{1}{2}\hat{h}_{2N}\right)_{\alpha\beta} \times \left(\begin{array}{c}\hat{e}_{2N}\\ \hat{f}_{2N}\end{array}\right)_{\gamma\delta}\right] \{\delta_{\beta\gamma} e_{\alpha\delta}(r+s) - \delta_{\alpha\delta} e_{\gamma\beta}(r+s)\}\\
&= \left(\tfrac{1}{2}\hat{h}_{2N} \begin{array}{c}\hat{e}_{2N}\\ \hat{f}_{2N}\end{array}\right)_{\alpha\delta} e_{\alpha\delta}(r+s) - \left(\begin{array}{c}\hat{e}_{2N}\\ \hat{f}_{2N}\end{array} \tfrac{1}{2}\hat{h}_{2N}\right)_{\gamma\beta} e_{\gamma\beta}(r+s)\\
&= \left(\left[\tfrac{1}{2}\hat{h}_{2N}, \begin{array}{c}\hat{e}_{2N}\\ \hat{f}_{2N}\end{array}\right]\right)_{\alpha\beta} e_{\alpha\beta}(r+s) = \left(\begin{array}{c}\hat{e}_{2N}\\ -\hat{f}_{2N}\end{array}\right)_{\alpha\beta} e_{\alpha\beta}(r+s) = \pm K_\pm(r+s),
\end{aligned}
$$

$$
\begin{aligned}
\alpha\{K_0(r), K_\pm(s)\} &= \alpha\{\tau\{K_0(r)\}, \tau\{K_\pm(s)\}\}\\
&= \alpha\left\{\left(\tfrac{1}{2}\hat{h}_{2N}\right)_{\alpha\beta} \tau\{e_{\alpha\beta}(r)\}, \left(\begin{array}{c}\hat{e}_{2N}\\ \hat{f}_{2N}\end{array}\right)_{\gamma\delta} \tau\{e_{\gamma\delta}(s)\}\right\}\\
&= \left(\tfrac{1}{2}\hat{h}_{2N}\right)_{\alpha\beta} \left(\begin{array}{c}\hat{e}_{2N}\\ \hat{f}_{2N}\end{array}\right)_{\gamma\delta} \delta_{\alpha\delta}\delta_{\beta\gamma}\delta_{r+s,0}\cdot r\\
&= \tfrac{1}{2}\mathrm{Tr}\left(\hat{h}_{2N} \begin{array}{c}\hat{e}_{2N}\\ \hat{f}_{2N}\end{array}\right)\delta_{r+s,0}\cdot r = 0,
\end{aligned}
$$

$$
\begin{aligned}
[K_+(r), K_-(s)] &= \left[(\hat{e}_{2N})_{\alpha\beta} e_{\alpha\beta}(r), (\hat{f}_{2N})_{\gamma\delta} e_{\gamma\delta}(s)\right]\\
&= \left[(\hat{e}_{2N})_{\alpha\beta} \times (\hat{f}_{2N})_{\gamma\delta}\right] \{\delta_{\beta\gamma} e_{\alpha\delta}(r+s) - \delta_{\alpha\delta} e_{\gamma\beta}(r+s)\}\\
&= (\hat{e}_{2N}\hat{f}_{2N})_{\alpha\delta} e_{\alpha\delta}(r+s) - (\hat{f}_{2N}\hat{e}_{2N})_{\gamma\beta} e_{\gamma\beta}(r+s)\\
&= \left([\hat{e}_{2N}, \hat{f}_{2N}]\right)_{\alpha\beta} e_{\alpha\beta}(r+s) = 2\left(\tfrac{1}{2}\hat{h}_{2N}\right)_{\alpha\beta} e_{\alpha\beta}(r+s) = 2K_0(r+s),
\end{aligned}
$$

$$
\begin{aligned}
\alpha\{K_+(r), K_-(s)\} &= \alpha\{\tau\{K_+(r)\}, \tau\{K_-(s)\}\}\\
&= \alpha\left\{(\hat{e}_{2N})_{\alpha\beta}\tau\{e_{\alpha\beta}(r)\}, (\hat{f}_{2N})_{\gamma\delta}\tau\{e_{\gamma\delta}(s)\}\right\}\\
&= (\hat{e}_{2N})_{\alpha\beta}(\hat{f}_{2N})_{\gamma\delta}\delta_{\alpha\delta}\delta_{\beta\gamma}\delta_{r+s,0}\cdot r = \mathrm{Tr}\left(\hat{e}_{2N}\hat{f}_{2N}\right)\delta_{r+s,0}\cdot r = Nr\delta_{r+s,0}.
\end{aligned}
$$

Commutators and 2-*cocycles* among operators X_k and Y_k $(k = 2r+1)$ are calculated as follows:

$$
\begin{aligned}
&[X_k, X_l] = [Y_k, Y_l] = 0,\\
&\alpha\left\{\begin{pmatrix}X_k\\ Y_k\end{pmatrix}, \begin{pmatrix}X_l\\ Y_l\end{pmatrix}\right\} = \alpha\left\{\tau\begin{pmatrix}X_k\\ Y_k\end{pmatrix}, \tau\begin{pmatrix}X_l\\ Y_l\end{pmatrix}\right\} (k = 2r+1,\ l = 2s+1)\\
&\quad = \alpha\left\{(\hat{e}_{2N})_{\alpha\beta}\tau\{e_{\alpha\beta}(r)\} \pm (\hat{f}_{2N})_{\alpha\beta}\tau\{e_{\alpha\beta}(r+1)\},\right.\\
&\qquad \left.(\hat{e}_{2N})_{\gamma\delta}\tau\{e_{\gamma\delta}(s)\} \pm (\hat{f}_{2N})_{\gamma\delta}\tau\{e_{\gamma\delta}(s+1)\}\right\}\\
&\quad = (\hat{e}_{2N})_{\alpha\beta}(\hat{e}_{2N})_{\gamma\delta}\delta_{\alpha\delta}\delta_{\beta\gamma}\delta_{r+s,0}\cdot r\\
&\qquad \pm (\hat{e}_{2N})_{\alpha\beta}(\hat{f}_{2N})_{\gamma\delta}\delta_{\alpha\delta}\delta_{\beta\gamma}\delta_{r+s+1,0}\cdot r \pm (\hat{f}_{2N})_{\alpha\beta}(\hat{e}_{2N})_{\gamma\delta}\delta_{\alpha\delta}\delta_{\beta\gamma}\delta_{r+1+s,0}\cdot(r+1)\\
&\qquad + (\hat{f}_{2N})_{\alpha\beta}(\hat{f}_{2N})_{\gamma\delta}\delta_{\alpha\delta}\delta_{\beta\gamma}\delta_{r+1+s+1,0}\cdot(r+1)\\
&\quad = \pm N(2r+1)\cdot\delta_{r+s+1,0} = \left\{\begin{array}{l} N\cdot k\delta_{k+l,0},\\ -N\cdot k\delta_{k+l,0},\end{array}\right.
\end{aligned}
$$

$$
\begin{aligned}
[X_k, Y_l] &= \left[(\hat{e}_{2N})_{\alpha\beta} e_{\alpha\beta}(r) + (\hat{f}_{2N})_{\alpha\beta} e_{\alpha\beta}(r+1), (\hat{e}_{2N})_{\gamma\delta} e_{\gamma\delta}(s) - (\hat{f}_{2N})_{\gamma\delta} e_{\gamma\delta}(s+1)\right] \\
&= \left[(\hat{e}_{2N})_{\alpha\beta} \times (\hat{e}_{2N})_{\gamma\delta}\right] \{\delta_{\beta\gamma} e_{\alpha\delta}(r+s) - \delta_{\alpha\delta} e_{\gamma\beta}(r+s)\} \\
&\quad - \left[(\hat{e}_{2N})_{\alpha\beta} \times (\hat{f}_{2N})_{\gamma\delta}\right] \{\delta_{\beta\gamma} e_{\alpha\delta}(r+s+1) - \delta_{\alpha\delta} e_{\gamma\beta}(r+s+1)\} \\
&\quad + \left[(\hat{f}_{2N})_{\alpha\beta} \times (\hat{e}_{2N})_{\gamma\delta}\right] \{\delta_{\beta\gamma} e_{\alpha\delta}(r+1+s) - \delta_{\alpha\delta} e_{\gamma\beta}(r+1+s)\} \\
&\quad - \left[(\hat{f}_{2N})_{\alpha\beta} \times (\hat{f}_{2N})_{\gamma\delta}\right] \{\delta_{\beta\gamma} e_{\alpha\delta}(r+1+s+1) - \delta_{\alpha\delta} e_{\gamma\beta}(r+1+s+1)\} \\
&= (\hat{e}_{2N}\hat{e}_{2N})_{\alpha\delta} e_{\alpha\delta}(r+s) - (\hat{e}_{2N}\hat{e}_{2N})_{\gamma\beta} e_{\gamma\beta}(r+s) \\
&\quad - (\hat{e}_{2N}\hat{f}_{2N})_{\alpha\delta} e_{\alpha\delta}(r+s+1) + (\hat{f}_{2N}\hat{e}_{2N})_{\gamma\beta} e_{\gamma\beta}(r+s+1) \\
&\quad + (\hat{f}_{2N}\hat{e}_{2N})_{\alpha\delta} e_{\alpha\delta}(r+1+s) - (\hat{e}_{2N}\hat{f}_{2N})_{\gamma\beta} e_{\gamma\beta}(r+1+s) \\
&\quad - (\hat{f}_{2N}\hat{f}_{2N})_{\alpha\delta} e_{\alpha\delta}(r+1+s+1) + (\hat{f}_{2N}\hat{f}_{2N})_{\gamma\beta} e_{\gamma\beta}(r+1+s+1) \\
&= -\left(\left[\hat{e}_{2N}, \hat{f}_{2N}\right]\right)_{\alpha\beta} e_{\alpha\beta}(r+s+1) + \left(\left[\hat{f}_{2N}, \hat{e}_{2N}\right]\right)_{\alpha\beta} e_{\alpha\beta}(r+s+1) \\
&= 2(-\hat{h}_{2N})_{\alpha\beta} e_{\alpha\beta}(r+s+1) = 2Y_{2r+1+2s+1} = 2Y_{k+l}, \\
[X_k, Y_{2s}] &= \left[(\hat{e}_{2N})_{\alpha\beta} e_{\alpha\beta}(r) + (\hat{f}_{2N})_{\alpha\beta} e_{\alpha\beta}(r+1), (-\hat{h}_{2N})_{\gamma\delta} e_{\gamma\delta}(s)\right] \ (Y_{2r} \equiv (-\hat{h}_{2N})_{\alpha\beta} e_{\alpha\beta}(r)) \\
&= \left[(\hat{e}_{2N})_{\alpha\beta}, (-\hat{h}_{2N})_{\gamma\delta}\right] \{\delta_{\beta\gamma} e_{\alpha\delta}(r+s) - \delta_{\alpha\delta} e_{\gamma\beta}(r+s)\} \\
&\quad + \left[(\hat{f}_{2N})_{\alpha\beta}, (-\hat{h}_{2N})_{\gamma\delta}\right] \{\delta_{\beta\gamma} e_{\alpha\delta}(r+1+s) - \delta_{\alpha\delta} e_{\gamma\beta}(r+1+s)\} \\
&= -(\hat{e}_{2N}\hat{h}_{2N})_{\alpha\delta} e_{\alpha\delta}(r+s) + (\hat{h}_{2N}\hat{e}_{2N})_{\gamma\beta} e_{\gamma\beta}(r+s) \\
&\quad - (\hat{f}_{2N}\hat{h}_{2N})_{\alpha\delta} e_{\alpha\delta}(r+s+1) + (\hat{h}_{2N}\hat{f}_{2N})_{\gamma\beta} e_{\gamma\beta}(r+s+1) \\
&= \left(\left[\hat{h}_{2N}, \hat{e}\right]\right)_{\alpha\beta} e_{\alpha\beta}(r+s) + \left(\left[\hat{h}, \hat{f}_{2N}\right]\right)_{\alpha\beta} e_{\alpha\beta}(r+s+1) \\
&= 2\left\{(\hat{e}_{2N})_{\alpha\beta} e_{\alpha\beta}(r+s) - (\hat{f}_{2N})_{\alpha\beta} e_{\alpha\beta}(r+s+1)\right\} = 2Y_{2(r+s)+1} = 2Y_{k+2s}, \\
\alpha(X_k, Y_{2s}) &= \alpha\left\{\tau(X_k), \tau(Y_{2s})\right\} \\
&= \alpha\left\{(\hat{e}_{2N})_{\alpha\beta} \tau\left\{e_{\alpha\beta}(r)\right\} + (\hat{f}_{2N})_{\alpha\beta} \tau\left\{e_{\alpha\beta}(r+1)\right\}, (-\hat{h}_{2N})_{\gamma\delta} \tau\left\{e_{\gamma\delta}(s)\right\}\right\} \\
&= (\hat{e}_{2N})_{\alpha\beta} (-\hat{h}_{2N})_{\gamma\delta} \delta_{\alpha\delta} \delta_{\beta\gamma} \delta_{r+s,0} \cdot r \\
&\quad + (\hat{f}_{2N})_{\alpha\beta} (-\hat{h}_{2N})_{\gamma\delta} \delta_{\alpha\delta} \delta_{\beta\gamma} \delta_{r+1+s,0} \cdot (r+1) = 0, \\
[Y_k, Y_l] &= \left[(\hat{e}_{2N})_{\alpha\beta} e_{\alpha\beta}(r) - (\hat{f}_{2N})_{\alpha\beta} e_{\alpha\beta}(r+1), (-\hat{h}_{2N})_{\gamma\delta} e_{\gamma\delta}(s)\right] \ (k = 2r+1, \ l = 2s) \\
&= \left[(\hat{e}_{2N})_{\alpha\beta} \times (-\hat{h}_{2N})_{\gamma\delta}\right] \{\delta_{\beta\gamma} e_{\alpha\delta}(r+s) - \delta_{\alpha\delta} e_{\gamma\beta}(r+s)\} \\
&\quad - \left[(\hat{f}_{2N})_{\alpha\beta} \times (-\hat{h}_{2N})_{\gamma\delta}\right] \{\delta_{\beta\gamma} e_{\alpha\delta}(r+1+s) - \delta_{\alpha\delta} e_{\gamma\beta}(r+1+s)\} \\
&= \left(\left[\hat{h}_{2N}, \hat{e}_{2N}\right]\right)_{\alpha\beta} e_{\alpha\beta}(r+s) - \left(\left[\hat{h}_{2N}, \hat{f}_{2N}\right]\right)_{\alpha\beta} e_{\alpha\beta}(r+s+1) \\
&= 2\left\{(\hat{e}_{2N})_{\alpha\beta} e_{\alpha\beta}(r+s) + (\hat{f}_{2N})_{\alpha\beta} e_{\alpha\beta}(r+s+1)\right\} = 2X_{k+l},
\end{aligned}
$$

$$
\alpha\left(\frac{1}{\sqrt{N}} Y_{2r}, \frac{1}{\sqrt{N}} Y_{2s}\right) = 2r\delta_{r+s,0}, \qquad \left[\frac{1}{\sqrt{N}} Y_{2r}, \frac{1}{\sqrt{N}} Y_{2s}\right]_{\mathrm{KM}} = 2r \cdot \delta_{r+s,0} \cdot c,
$$

$$
\sigma_K : \ \ \frac{1}{\sqrt{N}} Y_{2r} \to \frac{\partial}{\partial y_{2r}}, \qquad \sigma_K : \ \ \frac{1}{\sqrt{N}} Y_{-2r} \to 2r y_{2r},
$$

$$
\alpha(X_k, Y_{2s}) = 0, \qquad \left[\frac{1}{\sqrt{N}} X_k, \frac{1}{\sqrt{N}} Y_{2s}\right]_{\mathrm{KM}} = \frac{2}{\sqrt{N}} \frac{1}{\sqrt{N}} Y_{k+2s} \cdot c,
$$

$$
\alpha\left(\frac{1}{\sqrt{N}} Y_k, \frac{1}{\sqrt{N}} Y_l\right) = -k\delta_{k+l,0} = 0 \ (k \neq -l), \qquad \left[\frac{1}{\sqrt{N}} Y_k, \frac{1}{\sqrt{N}} Y_l\right]_{\mathrm{KM}} = \frac{2}{\sqrt{N}} \frac{1}{\sqrt{N}} X_{k+l} \cdot c.
$$

G Sum-rules for $2(Y_i + Y_{-i})$

First we give the vertex operator for $\Gamma(p)$ in terms of the Schur polynomials

$$
\begin{aligned}
\Gamma(p) &= \exp\left\{ \sum_{m_{\mathrm{odd}}\geq 1} p^m (2x_m) \right\} \exp\left\{ \sum_{n_{\mathrm{odd}}\geq 1} \left(\frac{1}{p}\right)^n (-2\widetilde{\partial}_n) \right\} \\
&= \sum_{l\in\mathbb{Z}} \sum_{n=0}^{\infty} S_{n-l}(2x) S_n(-2\widetilde{\partial}) p^{-l},
\end{aligned}
$$

According to Kac [48], we have

$$
\begin{aligned}
&\sum_{l\in\mathbb{Z}} p^{-l} Y_l \to \tfrac{1}{2}(\Gamma(p) - 1), \\
&2Y_l + \delta_{l0} = \sum_{n=0}^{\infty} S_{n-l}(2x) S_n(-2\widetilde{\partial}) \qquad (x_n = \widetilde{\partial}_n = 0 \text{ for even } n).
\end{aligned} \tag{G.1}
$$

Then using (G.1), the sum-rules for $2(Y_i + Y_{-i})$ are derived as follows:

$$
\begin{aligned}
2Y_0 &= S_1(2x)S_1(-2\widetilde{\partial}) + S_2(2x)S_2(-2\widetilde{\partial}) + S_3(2x)S_3(-2\widetilde{\partial}) \\
&\quad + S_4(2x)S_4(-2\widetilde{\partial}) + S_5(2x)S_5(-2\widetilde{\partial}) + S_6(2x)S_6(-2\widetilde{\partial}) + \cdots, \\
2(Y_{-2} + Y_2) &= \sum_{n=0}^{\infty} \left(S_{n+2}(2x) + S_{n-2}(2x)\right) S_n(-2\widetilde{\partial}) \\
&= S_2(2x) + S_2(-2\widetilde{\partial}) + S_3(2x)S_1(-2\widetilde{\partial}) + S_1(2x)S_3(-2\widetilde{\partial}) + S_4(2x)S_2(-2\widetilde{\partial}) \\
&\quad + S_2(2x)S_4(-2\widetilde{\partial}) + S_5(2x)S_3(-2\widetilde{\partial}) + S_6(2x)S_4(-2\widetilde{\partial}) + \cdots, \\
2(Y_{-4} + Y_4) &= \sum_{n=0}^{\infty} \left(S_{n+4}(2x) + S_{n-4}(2x)\right) S_n(-2\widetilde{\partial}) \\
&= S_4(2x) + S_4(-2\widetilde{\partial}) + S_5(2x)S_1(-2\widetilde{\partial}) + S_6(2x)S_2(-2\widetilde{\partial}) \\
&\quad + S_7(2x)S_3(-2\widetilde{\partial}) + S_8(2x)S_4(-2\widetilde{\partial}) + \cdots, \\
2(Y_{-1} + Y_1) &= \sum_{n=0}^{\infty} \left(S_{n+1}(2x) + S_{n-1}(2x)\right) S_n(-2\widetilde{\partial}) \\
&= S_1(2x) + S_1(-2\widetilde{\partial}) + S_2(2x)S_1(-2\widetilde{\partial}) + S_1(2x)S_2(-2\widetilde{\partial}) \\
&\quad + S_3(2x)S_2(-2\widetilde{\partial}) + S_2(2x)S_3(-2\widetilde{\partial}) + S_4(2x)S_3(-2\widetilde{\partial}) + \cdots, \\
2(Y_{-3} + Y_3) &= \sum_{n=0}^{\infty} \left(S_{n+3}(2x) + S_{n-3}(2x)\right) S_n(-2\widetilde{\partial}) \\
&= S_3(2x) + S_3(-2\widetilde{\partial}) + S_4(2x)S_1(-2\widetilde{\partial}) + S_1(2x)S_4(-2\widetilde{\partial}) \\
&\quad + S_5(2x)S_2(-2\widetilde{\partial}) + S_6(2x)S_3(-2\widetilde{\partial}) + S_7(2x)S_4(-2\widetilde{\partial}) + \cdots, \\
2(Y_{-5} + Y_5) &= \sum_{n=0}^{\infty} \left(S_{n+5}(2x) + S_{n-5}(2x)\right) S_n(-2\widetilde{\partial}) \\
&= S_5(2x) + S_6(2x)S_1(-2\widetilde{\partial}) + S_7(2x)S_2(-2\widetilde{\partial}) \\
&\quad + S_8(2x)S_3(-2\widetilde{\partial}) + S_9(2x)S_4(-2\widetilde{\partial}) + \cdots,
\end{aligned}
$$

....

H Expression for $g_{\mathrm{Y}_{-(2i+1)}+\mathrm{Y}_{(2i+1)}}(z)$ in terms of Bessel functions

Using the formula for $(Y_{-(2i+1)}+Y_{(2i+1)})$,

$$\begin{aligned}
Y_{-(2i+1)}+Y_{(2i+1)} &= (\hat{e}_{2N})_{\alpha\beta}e_{\alpha\beta}(-i-1)-(\hat{f}_{2N})_{\alpha\beta}e_{\alpha\beta}(-i) \\
&\quad +(\hat{e}_{2N})_{\alpha\beta}e_{\alpha\beta}(i)-(\hat{f}_{2N})_{\alpha\beta}e_{\alpha\beta}(i+1), \\
\hat{e}_{2N}z^{-i-1}-\hat{f}_{2N}z^{-i}+\hat{e}_{2N}z^{i}-\hat{f}_{2N}z^{i+1} &= \hat{e}_{2N}(z^{-i-1}+z^{i})-\hat{f}_{2N}(z^{-i}+z^{i+1}) \\
&= z^{-i}(z^{-1}\hat{e}_{2N}-\hat{f}_{2N})(1+z^{2i+1}),
\end{aligned} \tag{H.1}$$

we express straightforwardly the $g_{\mathrm{Y}_{-(2i+1)}+\mathrm{Y}_{(2i+1)}}(z)$, τ rep of (H.1), in terms of the Bessel functions as,

$$\begin{aligned}
& g_{\mathrm{Y}_{-(2i+1)}+\mathrm{Y}_{(2i+1)}}(z) = \tau\left[\left\{e^{\frac{\theta}{4}z^{-i}(z^{-1}\hat{e}_{2N}-\hat{f}_{2N})(1+z^{2i+1})}\right\}_{\alpha\beta}e_{\alpha\beta}\right] \\
&= \tau\left[\left\{\left(\tfrac{\theta}{4}\right)^0\begin{bmatrix}1_N & 0\\ 0 & 1_N\end{bmatrix}+\tfrac{1}{2!}\left(\tfrac{\theta}{4}\right)^2\begin{bmatrix}-z^{-(2i+1)}(1+z^{2i+1})^2\cdot 1_N & 0\\ 0 & -z^{-(2i+1)}(1+z^{2i+1})^2\cdot 1_N\end{bmatrix}\right.\right. \\
&+\tfrac{1}{4!}\left(\tfrac{\theta}{4}\right)^4\begin{bmatrix}(z^{-(2i+1)})^2(1+z^{2i+1})^4\cdot 1_N & 0\\ 0 & (z^{-(2i+1)})^2(1+z^{2i+1})^4\cdot 1_N\end{bmatrix}+\cdots \\
&+\left(\tfrac{\theta}{4}\right)^1 z^{-i}(z^{-1}\hat{e}_{2N}-\hat{f}_{2N})(1+z^{2i+1})+\tfrac{1}{3!}\left(\tfrac{\theta}{4}\right)^3(-z^{-3i-1})(z^{-1}\hat{e}_{2N}-\hat{f}_{2N})(1+z^{2i+1})^3 \\
&\left.\left.+\tfrac{1}{5!}\left(\tfrac{\theta}{4}\right)^5 z^{-5i-2}(z^{-1}\hat{e}_{2N}-\hat{f}_{2N})(1+z^{2i+1})^5+\cdots\right\}_{\alpha\beta}e_{\alpha\beta}\right] \\
&= \tau\left[\left[\left\{\left(\tfrac{\theta}{4}\right)^0 1+\tfrac{1}{2!}\left(\tfrac{\theta}{4}\right)^2(-z^{-(2i+1)})(1+z^{2i+1})^2+\tfrac{1}{4!}\left(\tfrac{\theta}{4}\right)^4(z^{-(2i+1)})^2(1+z^{2i+1})^4+\cdots\right\}\hat{I}_{2N}\right.\right. \\
&+z^{i}\left\{\left(\tfrac{\theta}{4}\right)^1 z^{-(2i+1)}(1+z^{2i+1})+\tfrac{1}{3!}\left(\tfrac{\theta}{4}\right)^3(-z^{-(2i+1)})^2(1+z^{2i+1})^3\right. \\
&\left.+\tfrac{1}{5!}\left(\tfrac{\theta}{4}\right)^5(z^{-(2i+1)})^3(1+z^{2i+1})^5+\cdots\right\}\hat{e}_{2N} \\
&-z^{i+1}\left\{\left(\tfrac{\theta}{4}\right)^1 z^{-(2i+1)}(1+z^{2i+1})+\tfrac{1}{3!}\left(\tfrac{\theta}{4}\right)^3(-z^{-(2i+1)})^2(1+z^{2i+1})^3\right. \\
&\left.\left.\left.+\tfrac{1}{5!}\left(\tfrac{\theta}{4}\right)^5(z^{-(2i+1)})^3(1+z^{2i+1})^5+\cdots\right\}\hat{f}_{2N}\right]_{\alpha\beta}e_{\alpha\beta}\right] \\
&= \sum_{r\geq 0}(-1)^r J_{2r}\left(\tfrac{\theta}{2}\right)(\hat{\mathrm{I}}_{2N})_{\alpha\beta}\tau\left\{e_{\alpha\beta}((2i+1)r)+e_{\alpha\beta}(-(2i+1)r)\right\}-J_0\left(\tfrac{\theta}{2}\right)(\hat{\mathrm{I}}_{2N})_{\alpha\beta}\tau\left\{e_{\alpha\beta}(0)\right\} \\
&+\sum_{r\geq 0}(-1)^r J_{2r+1}\left(\tfrac{\theta}{2}\right)(\hat{e}_{2N})_{\alpha\beta}\tau\left\{e_{\alpha\beta}((2i+1)r+i)+e_{\alpha\beta}(-(2i+1)r-(i+1))\right\} \\
&-\sum_{r\geq 0}(-1)^r J_{2r+1}\left(\tfrac{\theta}{2}\right)(\hat{f}_{2N})_{\alpha\beta}\tau\left\{e_{\alpha\beta}((2i+1)r+i+1)+e_{\alpha\beta}(-(2i+1)r-i)\right\},
\end{aligned}$$

where we have used the relations

$$\begin{aligned}
&\sum_{r\geq 0}(-1)^r J_{2r}\left(\tfrac{\theta}{2}\right)\left\{z^{(2i+1)r}+z^{-(2i+1)r}\right\}=\sum_{r\in\mathbb{Z}}(-1)^r J_{2r}\left(\tfrac{\theta}{2}\right)z^{(2i+1)r}, \\
&\sum_{r\geq 0}(-1)^r J_{2r+1}\left(\tfrac{\theta}{2}\right)\left\{z^{(2i+1)r+i}+z^{-(2i+1)r-(i+1)}\right\}=\sum_{r\in\mathbb{Z}}(-1)^r J_{2r+1}\left(\tfrac{\theta}{2}\right)z^{(2i+1)r+i}, \\
&\sum_{r\geq 0}(-1)^r J_{2r+1}\left(\tfrac{\theta}{2}\right)\left\{z^{(2i+1)r+i+1}+z^{-(2i+1)r-i}\right\}=\sum_{r\in\mathbb{Z}}(-1)^r J_{2r+1}\left(\tfrac{\theta}{2}\right)z^{-(2i+1)r-i}.
\end{aligned}$$

Finally we can express the $g_{\mathrm{Y}_{-(2i+1)}+\mathrm{Y}_{(2i+1)}}(z)$ as

$$g_{\mathrm{Y}_{-(2i+1)}+\mathrm{Y}_{(2i+1)}}(z)=\tau\left[\left[\sum_{r\in\mathbb{Z}}(-1)^r J_{2r}\left(\tfrac{\theta}{2}\right)z^{(2i+1)r}\hat{I}_{2N}\right.\right.$$

$$
\begin{aligned}
&+\sum_{r\in\mathbb{Z}}(-1)^r J_{2r+1}\left(\tfrac{\theta}{2}\right) z^{(2i+1)r+i}\hat{e}_{2N} - \sum_{r\in\mathbb{Z}}(-1)^r J_{2r+1}\left(\tfrac{\theta}{2}\right) z^{-(2i+1)r-i}\hat{f}_{2N}\Bigg]_{\alpha\beta} e_{\alpha\beta}\Bigg] \\
&= \sum_{r\in\mathbb{Z}}(-1)^r J_{2r}\left(\tfrac{\theta}{2}\right)(\hat{I}_{2N})_{\alpha\beta}\tau\left\{e_{\alpha\beta}((2i+1)r)\right\} \\
&+\sum_{r\in\mathbb{Z}}(-1)^r J_{2r+1}\left(\tfrac{\theta}{2}\right)\left[(\hat{e}_{2N})_{\alpha\beta}\tau\left\{e_{\alpha\beta}((2i+1)r+i)\right\} - (\hat{f}_{2N})_{\alpha\beta}\tau\left\{e_{\alpha\beta}(-(2i+1)r-i)\right\}\right].
\end{aligned}
$$

This is the first expression which we give in terms of the Bessel functions.

I Properties of $SU(2N)_\infty$ transformation matrix

In (5.10) the $2N$-dimensional matrix g_{φ_s} is expressed as

$$
\begin{aligned}
g_{\varphi_s} &= \hat{1}_{2N} + i\left(-\frac{\varphi_s}{2}\right)\hat{h}_{2N} + \frac{i^2}{2!}\left(-\frac{\varphi_s}{2}\right)^2\hat{h}_{2N}^2 + \frac{i^3}{3!}\left(-\frac{\varphi_s}{2}\right)^3\hat{h}_{2N}^3 + \frac{i^4}{4!}\left(-\frac{\varphi_s}{2}\right)^4\hat{h}_{2N}^4 + \cdots \\
&= \left\{1 - \frac{1}{2!}\left(-\frac{\varphi_s}{2}\right)^2 + \frac{1}{4!}\left(-\frac{\varphi_s}{2}\right)^4 + \cdots\right\}\hat{1}_{2N} \\
&\quad + i\left\{\left(-\frac{\varphi_s}{2}\right) - \frac{1}{3!}\left(-\frac{\varphi_s}{2}\right)^3 + \frac{1}{5!}\left(-\frac{\varphi_s}{2}\right)^5 + \cdots\right\}\hat{h}_{2N} \\
&= \cos\left(-\frac{\varphi_s}{2}\right)\hat{1}_{2N} + i\sin\left(-\frac{\varphi_s}{2}\right)\hat{h}_{2N} = \begin{bmatrix} e^{-i\frac{\varphi_s}{2}}\cdot 1_N & 0 \\ 0 & e^{i\frac{\varphi_s}{2}}\cdot 1_N \end{bmatrix}.
\end{aligned} \tag{I.1}
$$

Adjoint actions $\widehat{K}_\pm(\theta)$ for $\psi_{Nr+\alpha}$ are computed as

$$
\begin{aligned}
\left[\widehat{K}_+(\theta), \psi_{Nr+\alpha}\right] &= \sum_{s\in\mathbb{Z}}\psi_{N(s-r)+\beta}\left(-\theta_r\hat{e}_{2N}\right)_{\beta\alpha}, \\
\left[\widehat{K}_-(\theta), \psi_{Nr+\alpha}\right] &= \sum_{s\in\mathbb{Z}}\psi_{N(s-r)+\beta}\big(-\theta_{-r}\hat{f}_{2N}\big)_{\beta\alpha},
\end{aligned} \tag{I.2}
$$

from which we obtain

$$
\left[\widehat{K}_+(\theta) - \widehat{K}_-(\theta), \psi_{Nr+\alpha}\right] = \sum_{s\in\mathbb{Z}}\psi_{N(s-r)+\beta}\left(-\theta_r\right)\left(\hat{e}_{2N} - \hat{f}_{2N}\right)_{\beta\alpha} \qquad (\theta_{-r} = \theta_r).
$$

We also have the relations

$$
\begin{aligned}
&(\hat{e}_{2N} - \hat{f}_{2N})^2 = -\hat{I}_{2N}, \qquad (\hat{e}_{2N} - \hat{f}_{2N})^3 = -(\hat{e}_{2N} - \hat{f}_{2N}), \\
&(\hat{e}_{2N} - \hat{f}_{2N})^4 = \hat{I}_{2N}, \qquad \ldots .
\end{aligned} \tag{I.3}
$$

Using (I.2) and (I.3), the transformed $\psi_{Nr+\alpha}(\hat{g}_\theta)$ is shown to be (5.11). The $2N$-dimensional matrix g_{θ_s} is expressed as

$$
\begin{aligned}
g_{\theta_s} &= \hat{1}_{2N} + \left(-\frac{\theta_s}{2}\right)(\hat{e}_{2N} - \hat{f}_{2N}) + \frac{1}{2!}\left(-\frac{\theta_s}{2}\right)^2(\hat{e}_{2N} - \hat{f}_{2N})^2 + \frac{1}{3!}\left(-\frac{\theta_s}{2}\right)^3(\hat{e}_{2N} - \hat{f}_{2N})^3 \\
&\quad + \frac{1}{4!}\left(-\frac{\theta_s}{2}\right)^4(\hat{e}_{2N} - \hat{f}_{2N})^4 + \frac{1}{5!}\left(-\frac{\theta_s}{2}\right)^5(\hat{e}_{2N} - \hat{f}_{2N})^5 + \cdots \\
&= \left\{1 - \frac{1}{2!}(-\frac{\theta_s}{2})^2 + \frac{1}{4!}(-\frac{\theta_s}{2})^4 + \cdots\right\}\hat{1}_{2N}
\end{aligned} \tag{I.4}
$$

$$+\left\{\left(-\frac{\theta_s}{2}\right)-\frac{1}{3!}\left(-\frac{\theta_s}{2}\right)^3+\frac{1}{5!}\left(-\frac{\theta_s}{2}\right)^5+\cdots\right\}(\hat{e}_{2N}-\hat{f}_{2N})$$

$$=\cos\left(-\frac{\theta_s}{2}\right)\cdot\hat{1}_{2N}+\sin\left(-\frac{\theta_s}{2}\right)\cdot(\hat{e}_{2N}-\hat{f}_{2N})=\begin{bmatrix}\cos\left(\frac{\theta_s}{2}\right)\cdot\hat{1}_N & -\sin\left(\frac{\theta_s}{2}\right)\cdot\hat{1}_N\\ \sin\left(\frac{\theta_s}{2}\right)\cdot\hat{1}_N & \cos\left(\frac{\theta_s}{2}\right)\cdot\hat{1}_N\end{bmatrix}.$$

Using (I.1) and (I.4), we obtain an explicit expression for the block matrix $g_{u,t,s}$ of an $SU(2N)_\infty$ transformation-matrix appeared in (5.12) as

$$g_{u,t,s}=\begin{bmatrix}\cos\left(\frac{\theta_t}{2}\right)e^{-\frac{i}{2}(\psi_u+\varphi_s)}\cdot 1_N & -\sin\left(\frac{\theta_t}{2}\right)e^{-\frac{i}{2}(\psi_u-\varphi_s)}\cdot 1_N\\ \sin\left(\frac{\theta_t}{2}\right)e^{\frac{i}{2}(\psi_u-\varphi_s)}\cdot 1_N & \cos\left(\frac{\theta_t}{2}\right)e^{\frac{i}{2}(\psi_u+\varphi_s)}\cdot 1_N\end{bmatrix}\quad(\theta_{-t}=\theta_t).\tag{I.5}$$

The transformation (5.12) is rewritten under the change of index of the block matrix as follows:

$$\begin{aligned}\psi_{Nr+\alpha}(\hat{g})&=\sum_{s\in\mathbb{Z}}\left(\sum_{t,u\in\mathbb{Z}}\psi_{N(r-s-t-u)+\beta}\right)(g_{u,t,s})_{\beta\alpha}\quad(g_{u,-t,s}=g_{u,t,s})\\ &\overset{s\to s-t-u}{=}\sum_{s\in\mathbb{Z}}\psi_{N(r-s)+\beta}\left(\sum_{t,u\in\mathbb{Z}}g_{u,t,s-t-u}\right)_{\beta\alpha}\\ &=\sum_{s\in\mathbb{Z}}\psi_{N(r-s)+\beta}\,(g_s)_{\beta\alpha}\qquad\left(g_s\equiv\sum_{t,u\in\mathbb{Z}}g_{u,t,s-t-u}\right).\end{aligned}$$

Noting the definition $(\hat{g})_{Nr+\alpha,Ns+\beta}\equiv(g_{s-r})_{\alpha\beta}$ in the second equation of (3.28) and again changing the index of the block matrix, the above transformation is further rewritten as

$$\begin{aligned}\psi_{Nr+\alpha}(\hat{g})&=\sum_{s\in\mathbb{Z}}\psi_{Ns+\beta}(\hat{g})_{Ns+\beta,Nr+\alpha}=\sum_{s\in\mathbb{Z}}\psi_{Ns+\beta}(g_{r-s})_{\beta\alpha}\overset{s\to r-s}{=}\sum_{s\in\mathbb{Z}}\psi_{N(r-s)+\beta}(g_s)_{\beta\alpha}\\ &=\sum_{s\in\mathbb{Z}}\psi_{N(r-s)+\beta}\left(\sum_{t,u\in\mathbb{Z}}g_{u,t,s-u-t}\right)_{\beta\alpha}\overset{t\to t-u}{=}\sum_{s\in\mathbb{Z}}\psi_{N(r-s)+\beta}\left(\sum_{t,u\in\mathbb{Z}}g_{u,t-u,s-t}\right)_{\beta\alpha}.\end{aligned}$$

Then the matrix element of $\hat{g}$ is also represented as

$$(\hat{g})_{Nr+\alpha,Ns+\beta}=\left(\sum_{t,u\in\mathbb{Z}}g_{u,t,s-t-u-r}\right)_{\alpha\beta}=\left(\sum_{t,u\in\mathbb{Z}}g_{u,t-u,s-t-r}\right)_{\alpha\beta}.$$

J Explicit expression for Plücker coordinate and calculation of $\det(1_N+p^\dagger p)$ in terms of Schur polynomials for LMG model

From (3.8), the coset variable p is expressed in terms of Plücker coordinates as

$$p=(p_{ia})=\left([S(\zeta)C^{-1}(\zeta)]_{ia}\right)=\left(\frac{v^{1,\dots,a,\dots,M}_{1,\dots,i,\dots,M}(g_\zeta)}{v^{1,\dots,M}_{1,\dots,M}(g_\zeta)}\right).$$

Using explicit expression for $SU(2N)$ matrix of the Thouless transformation (5.2), the matrix representations of the Plücker coordinates $v^{1,\dots,a,\dots,M}_{1,\dots,i,\dots,M}(g_\zeta)$ and $v^{1,\dots,M}_{1,\dots,M}(g_\zeta)$ are given simply as

$$v^{1,\dots,a,\dots,M}_{1,\dots,i,\dots,M}(g_\zeta)=\sin\tfrac{\theta}{2}\cdot 1_N,\qquad v^{1,\dots,M}_{1,\dots,M}(g_\zeta)=\cos\tfrac{\theta}{2}\cdot 1_N\qquad(M=N).$$

Then we have $p = \tan\frac{\theta}{2} \cdot 1_N$, which is identical with the expression given in (5.3) except the phase $e^{i\psi}$. Following [37], using the famous formula

$$\det(1+X) = \exp\{\mathrm{Tr}\,\ln(1+X)\} = \exp\left\{\sum_{l=1}^{\infty}(-1)^{l-1}\mathrm{Tr}\,(X^l)/l\right\}$$

and the Schur polynomials given in Appendix D, we have an expression for $\det(1+p^\dagger p)$ in (3.12) as

$$\det(1+p^\dagger p) = \sum_{l=0}^{\infty} S_l(\chi), \qquad \chi_l \equiv (-1)^{l-1}\,\mathrm{Tr}[(p^\dagger p)^l]/l,$$
$$\chi_l = S_l(\chi) = 0 \qquad (l \geq M+1). \tag{J.1}$$

Using the formula

$$[\det(1+X)]^{-\frac{1}{2}} = \exp\{\mathrm{Tr}\,\ln(1+X)^{-\frac{1}{2}}\} = \exp\left\{\sum_{l=1}^{\infty}(-1)^{l}\,\mathrm{Tr}\,(X^l)/(2l)\right\},$$

the vacuum function $\Phi_{M,M}(p,p^*,\tau)$ (3.12) is also expressed in terms of the Schur polynomials as

$$\Phi_{M,M}(p,p^*,\tau) = \sum_{l=0}^{\infty} S_l(\xi)\cdot e^{-iN\tau}, \qquad \xi_l \equiv (-1)^l\,\mathrm{Tr}\,[(p^\dagger p)]^l/(2l),$$
$$\xi_l = S_l(\xi) = 0 \qquad (l \geq M+1). \tag{J.2}$$

Rowe et al. have showed the number-projected $SO(2N)$ wave function satisfies some recursion relations. They express it with the aid of the relations in a form of determinant [88] which is well known as the completely anti-symmetric Schur function in the theory of group characters [89]. In the present $U(N)$ case, equation (J.1) is also given by a determinant form

$$\varphi_l(z) = \frac{1}{l!}\det\begin{vmatrix} \chi_1 & 1 & 0 & 0 & \cdots & 0 \\ 2\chi_2 & \chi_1 & 2 & 0 & \cdots & 0 \\ 3\chi_3 & 2\chi_2 & \chi_1 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \cdots & l-1 \\ l\chi_l & (l-1)\chi_{l-1} & (l-2)\chi_{l-2} & (l-3)\chi_{l-3} & \cdots & \chi_1 \end{vmatrix} = (-1)^l S_l(-\chi),$$

which is exactly the same form as that given in [86]. The Schur function $\varphi_l(\chi)$ satisfies the recursion relation and the differential formula

$$\varphi_l(\chi) = \frac{1}{l}\left\{\chi_1 - \sum_{l'=1}^{l-1}(l'+1)\chi_{l'+1}\frac{\partial}{\partial\chi_{l'}}\right\}\varphi_{l-1}(\chi), \qquad \frac{\partial}{\partial\chi_{l'}}\varphi_l(\chi) = (-1)^{l'+1}\varphi_{l-l'}(\chi). \tag{J.3}$$

By using the second equation of (J.3), we can rewrite the above recursion relation as

$$\varphi_l(\chi) = \frac{1}{l}\sum_{l'=1}^{l}(-1)^{l'+1}l'\chi_{l'}\varphi_{l-l'}(\chi) \qquad (\varphi_0 = 1).$$

With the aid of the second equation in (J.1) with $p = \tan\frac{\theta}{2}\cdot 1_N$ and the explicit form of the Schur polynomials given in Appendix D, the χ_l and $S_l(\chi)$ take simple forms, respectively, as

$$\chi_l = (-1)^{l-1}\frac{1}{l}\left(\tan^2\tfrac{\theta}{2}\right)^l\cdot N, \qquad S_l(\chi) = \frac{N!}{l!(N-l)!}1^{N-l}\left(\tan^2\tfrac{\theta}{2}\right)^l. \tag{J.4}$$

Then using the first equation in (J.1) and (J.4), we have

$$\det(1+p^\dagger p) = \sum_{l=0}^{\infty} S_l(\chi) = \sum_{l=0}^{N} \frac{N!}{l!(N-l)!} 1^{N-l} \left(\tan^2 \tfrac{\theta}{2}\right)^l = \left(1+\tan^2 \tfrac{\theta}{2}\right)^N = \left(\cos \tfrac{\theta}{2}\right)^{-2N},$$

which coincides with the result by the direct calculation of $\det(1+p^\dagger p)$. The vacuum function (J.2) is obtained as

$$\Phi_{M,M}(p,p^*,\tau) = \left(\cos \tfrac{\theta}{2}\right)^N e^{-iN\tau}, \qquad e_{ai}\Phi_{M,M}(p,p^*,\tau) = 0 \qquad ((3.10) \text{ and } (3.11)).$$

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