On the Spectrum of a Discrete Non-Hermitian Quantum System*

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Received October 28, 2008, in final form January 13, 2009; Published online January 19, 2009
doi:10.3842/SIGMA.2009.007

Abstract. In this paper, we develop spectral analysis of a discrete non-Hermitian quantum system that is a discrete counterpart of some continuous quantum systems on a complex contour. In particular, simple conditions for discreteness of the spectrum are established.

Key words: difference operator; non-Hermiticity; spectrum; eigenvalue; eigenvector; completely continuous operator

2000 Mathematics Subject Classification: 39A70; 81Q10

1 Introduction

Non-Hermitian Hamiltonians and complex extension of quantum mechanics have recently received a lot of attention (see review [1]). This field of mathematical physics came into flower due to the eigenvalue problem

$$-\psi''(x) + ix^3\psi(x) = E\psi(x), \qquad -\infty < x < \infty, \tag{1.1}$$

where $\psi(x)$ is a desired solution, $i = \sqrt{-1}$ is the imaginary unit, E is a complex parameter ("energy" or spectral parameter).

A complex value E_0 of the parameter E is called an *eigenvalue* of equation (1.1) if this equation with $E = E_0$ has a nontrivial (non-identically zero) solution $\psi_0(x)$ that belongs to the Hilbert space $L^2(-\infty,\infty)$ of complex-valued functions f defined on $(-\infty,\infty)$ such that

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx < \infty$$

with the inner (scalar) product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g}(x) dx \tag{1.2}$$

in which the bar over a function denotes the complex conjugate. The function $\psi_0(x)$ is called an *eigenfunction* of equation (1.1), corresponding to the eigenvalue E_0.

If we define the operator $S : D \subset L^2(-\infty,\infty) \to L^2(-\infty,\infty)$ with the domain D consisting of the functions $f \in L^2(-\infty,\infty)$ that are differentiable, with the derivative f' absolutely continuous on each finite subinterval of $(-\infty,\infty)$ and such that

$$-f'' + ix^3 f \in L^2(-\infty,\infty),$$

*This paper is a contribution to the Proceedings of the VIIth Workshop "Quantum Physics with Non-Hermitian Operators" (June 29 – July 11, 2008, Benasque, Spain). The full collection is available at http://www.emis.de/journals/SIGMA/PHHQP2008.html by letting

$$Sf = -f'' + ix^3 f \qquad \text{for} \quad f \in D, \tag{1.3}$$

then eigenvalue problem (1.1) can be written as $S\psi = E\psi$, $\psi \in D$, $\psi \neq 0$. Consequently, eigenvalues of equation (1.1) are eigenvalues of the operator S.

Because of the fact that the operator S defined by (1.3) has the complex non-real coefficient (potential) equal to ix^3, this operator is not Hermitian with respect to the inner product (1.2), that is, we cannot state that

$$\langle Sf, g\rangle = \langle f, Sg\rangle \qquad \text{for all} \ \ f, g \in D.$$

Therefore it is not obvious that the eigenvalues of S may be real (remember that the eigenvalues of any Hermitian operator are real and the eigenvectors corresponding to the distinct eigenvalues are orthogonal).

Around 1992 Bessis and Zinn-Justin had noticed on the basis of numerical work that some of the eigenvalues of equation (1.1) seemed to be real and positive and they conjectured (not in print) that for equation (1.1) the eigenvalues are all real and positive.

In 1998, Bender and Boettcher [2] generalized the BZJ conjecture, namely, they conjectured (again on the basis of numerical analysis) that the eigenvalues of the equation

$$-\psi''(x) - (ix)^m \psi(x) = E\psi(x), \qquad -\infty < x < \infty, \tag{1.4}$$

are all real and positive provided $m \geq 2$. Note that in equation (1.4) m is an arbitrary positive real number and equation (1.1) corresponds to the choice $m = 3$ in (1.4).

Bender and Boettcher assumed that the reason for reality of eigenvalues of (1.4) (in particular of (1.1)) must be certain symmetry property of this equation, namely, the so-called PT-symmetry of it (for more details see [1]).

P (parity) and T (time reversal) operations are defined by

$$Pf(x) = f(-x) \qquad \text{and} \qquad Tf(x) = \overline{f}(x),$$

respectively. A Hamiltonian

$$H = -\frac{d^2}{dx^2} + V(x) \tag{1.5}$$

with complex potential $V(x)$ is called PT-symmetric if it commutes with the composite operation PT:

$$[H, PT] = HPT - PTH = 0.$$

It is easily seen that PT-symmetricity of H given by (1.5) is equivalent to the condition

$$\overline{V}(-x) = V(x).$$

The potentials ix^3 and $-(ix)^m$ of equations (1.1) and (1.4), respectively, are PT-symmetric (note that ix^3 is not P-symmetric and T-symmetric, separately).

The first rigorous proof of reality and positivity of the eigenvalues of equation (1.4) was given in 2001 by Dorey, Dunning, and Tateo [3] (see also [4]).

Note that for $0 < m < 2$ the spectrum of (1.4) considered in $L^2(-\infty, \infty)$ is also discrete, however in this case only a finite number of eigenvalues are real and positive, and remaining eigenvalues (they are of infinite number) are non-real. Besides, if $m = 4k$ $(k = 1, 2, \ldots)$ then for any complex value of E all solutions of equation (1.4) belong to $L^2(-\infty, \infty)$ (the so-called Weyl's limit-circle case holds) and, therefore, all complex values of E are eigenvalues of equation (1.4) and hence the spectrum is not discrete for $m = 4k$ $(k = 1, 2, \ldots)$ if we consider the problem in $L^2(-\infty, \infty)$. In order to have in the case $m \geq 4$ a problem with discrete spectrum Bender and Boettcher made in [2] an important observation that the equation can be considered on an appropriately chosen complex contour. Namely, it is sufficient to consider the equation

$$-\psi''(z) - (iz)^m \psi(z) = E\psi(z), \qquad z \in \Gamma \tag{1.6}$$

together with the condition that

$$|\psi(z)| \to 0 \ \text{ exponentially as } z \text{ moves off to infinity along } \Gamma, \tag{1.7}$$

where Γ is a contour of the form

$$\Gamma = \{z = x - i\,|x| \tan \delta : -\infty < x < \infty\}$$

with

$$\delta = \frac{\pi(m-2)}{2(m+2)}.$$

Note that the contour Γ forms an angle in the lower complex z-plane, of value $\pi - 2\delta$ with the vertex at the origin and symmetric with respect to the imaginary axis.

Next, Mostafazadeh showed in [5] that problem (1.6), (1.7) is equivalent to finding solutions in $L^2(-\infty, \infty)$ of the problem

$$-\psi''(x) + |x|^m \psi(x) = E\rho(x)\psi(x), \qquad x \in (-\infty, 0) \cup (0, \infty), \tag{1.8}$$

$$\psi(0^-) = \psi(0^+), \qquad \psi'(0^-) = e^{2i\delta}\psi'(0^+), \tag{1.9}$$

where

$$\rho(x) = \begin{cases} e^{2i\delta} & \text{if} \quad x < 0, \\ e^{-2i\delta} & \text{if} \quad x > 0. \end{cases} \tag{1.10}$$

The main distinguishing features of problem (1.8), (1.9) are that it involves a complex-valued coefficient function $\rho(x)$ of the form (1.10) and that transition conditions (impulse conditions) of the form (1.9) are presented which also involve a complex coefficient. Such a problem is non-Hermitian with respect to the usual inner product (1.2) of space $L^2(-\infty, \infty)$.

Our aim in this paper is to construct and investigate a discrete version of problem (1.8), (1.9). Discrete equations (difference equations) form a reach field, both interesting and useful [6, 7]. Discrete equations arise when differential equations are solved approximately by discretization. On the other hand they often arise independently as mathematical models of many practical events. Discrete equations can easily been algorithmized to solve them on computers. There is only a small body of work concerning discrete non-Hermitian quantum systems. Some examples are [8, 9, 10, 11, 12, 13, 14, 15]. Note that in [15] the author considered a finite discrete interval version of the infinite discrete interval problem (1.12), (1.13) formulated below, and found conditions that ensure the reality of the eigenvalues. In the finite discrete interval case with the zero boundary conditions the problem is reduced to the eigenvalue problem for a finite-dimensional tridiagonal matrix.

Let $\mathbb{Z}$ denote the set of all integers. For any $l, m \in \mathbb{Z}$ with $l \leq m$, $[l, m]$ will denote the *discrete interval* being the set $\{l, l+1, \ldots, m\}$. Semi-infinite intervals of the form $(-\infty, l]$ and $[l, \infty)$ will denote the discrete sets $\{\ldots, l-2, l-1, l\}$ and $\{l, l+1, l+2, \ldots\}$, respectively. Throughout the paper all intervals will be discrete intervals. Let us set

$$\mathbb{Z}_0 = \mathbb{Z} \setminus \{0, 1\} = \{\ldots, -3, -2, -1\} \cup \{2, 3, 4, \ldots\} = (-\infty, -1] \cup [2, \infty). \tag{1.11}$$

We offer a discrete version of problem (1.8), (1.9) to be

$$-\Delta^2 y_{n-1} + q_n y_n = \lambda \rho_n y_n, \qquad n \in \mathbb{Z}_0, \tag{1.12}$$

$$y_{-1} = y_1, \qquad \Delta y_{-1} = e^{2i\delta} \Delta y_1, \tag{1.13}$$

where $y = (y_n)_{n \in \mathbb{Z}}$ is a desired solution, Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$ so that $\Delta^2 y_{n-1} = y_{n+1} - 2y_n + y_{n-1}$, the coefficients q_n are real numbers given for $n \in \mathbb{Z}_0$, $\delta \in [0, \pi/2)$, and ρ_n are given for $n \in \mathbb{Z}_0$ by

$$\rho_n = \begin{cases} e^{2i\delta} & \text{if} \quad n \leq -1, \\ e^{-2i\delta} & \text{if} \quad n \geq 2. \end{cases} \tag{1.14}$$

One of the main results of the present paper is that if

$$q_n \geq c > 0 \qquad \text{for} \quad n \in \mathbb{Z}_0 \tag{1.15}$$

and

$$\lim_{|n| \to \infty} q_n = \infty, \tag{1.16}$$

then the spectrum of problem (1.12), (1.13) is discrete.

The paper is organized as follows. In Section 2, we choose a suitable Hilbert space and define the main linear operators L, M, and $A = M^{-1}L$ related to problem (1.12), (1.13). Using these operators we introduce the concept of the spectrum for problem (1.12), (1.13). In Section 3, we demonstrate non-Hermiticity of the operators L and A. In Section 4, we present general properties of solutions of equations of type (1.12), (1.13). In Section 5, we construct two special solutions of problem (1.12), (1.13) under the condition (1.15). Using these solutions we show in Section 6 that the operator L is invertible and we describe the structure of the inverse operator L^{-1}. Finally, in Section 7, we show that the operator L^{-1} is completely continuous if, in addition, the condition (1.16) is satisfied. This fact yields the discreteness of the spectrum of problem (1.12), (1.13).

2 The concept of the spectrum for problem (1.12), (1.13)

In order to introduce the concept of the spectrum for problem (1.12), (1.13), define the Hilbert space l_0^2 of complex sequences $y = (y_n)_{n \in \mathbb{Z}_0}$ such that

$$\sum_{n \in \mathbb{Z}_0} |y_n|^2 < \infty$$

with the inner product and norm

$$\langle y, z \rangle = \sum_{n \in \mathbb{Z}_0} y_n \overline{z}_n, \qquad \|y\| = \sqrt{\langle y, y \rangle} = \left\{ \sum_{n \in \mathbb{Z}_0} |y_n|^2 \right\}^{1/2},$$

where $\mathbb{Z}_0$ is defined by (1.11) and the bar over a complex number denotes the complex conjugate.

Now we try to rewrite problem (1.12), (1.13) in the form of an equivalent vector equation in l_0^2 using appropriate operators. Denote by D the linear set of all vectors $y = (y_n)_{n \in \mathbb{Z}_0} \in l_0^2$ such that $(q_n y_n)_{n \in \mathbb{Z}_0} \in l_0^2$. Taking this set as the domain of L, $L : D \subset l_0^2 \to l_0^2$ is defined by

$$(Ly)_n = -\Delta^2 y_{n-1} + q_n y_n \qquad \text{for} \quad n \in \mathbb{Z}_0,$$

where y_0 and y_1 are defined from the equations

$$y_{-1} = y_1, \qquad \Delta y_{-1} = e^{2i\delta}\Delta y_1.$$

Note that y_0 and y_1 are needed when we evaluate $(Ly)_n$ for $n = -1$ and $n = 2$, respectively:

$$(Ly)_{-1} = -\Delta^2 y_{-2} + q_{-1}y_{-1} = (y_0 - 2y_{-1} + y_{-2}) + q_{-1}y_{-1},$$
$$(Ly)_2 = -\Delta^2 y_1 + q_2 y_2 = (y_3 - 2y_2 + y_1) + q_2 y_2$$

and (1.13) gives for y_1, y_0 the expressions

$$y_1 = y_{-1},$$
$$y_0 = y_{-1} + e^{2i\delta}(y_2 - y_1) = \left(1 - e^{2i\delta}\right)y_{-1} + e^{2i\delta}y_2.$$

Next, define the operator $M : l_0^2 \to l_0^2$ by

$$(My)_n = \rho_n y_n \qquad \text{for} \quad n \in \mathbb{Z}_0,$$

where ρ_n is given by (1.14). Obviously, the adjoint M^* of M is defined by

$$(M^* y)_n = \overline{\rho}_n y_n, \qquad n \in \mathbb{Z}_0,$$

and since $|\rho_n| = 1$, we get that M is a unitary operator:

$$MM^* = M^*M = I,$$

where the asterisk denotes the adjoint operator and I is the identity operator.

Therefore problem (1.12), (1.13) can be written as

$$Ly = \lambda My, \qquad y \in D, \qquad \text{or} \qquad M^{-1}Ly = \lambda y, \qquad y \in D.$$

This motivates to introduce the following definition.

Definition 1. By the spectrum of problem (1.12), (1.13) is meant the spectrum of the operator $A = M^{-1}L$ with the domain D in the space l_0^2.

Remember that (see [16]) if A is a linear operator with a domain dense in a Hilbert space, then a complex number λ is called a *regular point* of the operator A if the inverse $(A - \lambda I)^{-1}$ exists and represents a bounded operator defined on the whole space. All other points of the complex plane comprise the *spectrum* of the operator A. Obviously the eigenvalues λ of an operator belong to its spectrum, since the operator $(A - \lambda I)^{-1}$ does not exist for such points (the operator $A - \lambda I$ is not one-to-one). The set of all eigenvalues is called the *point spectrum* of the operator. The spectrum of the operator A is said to be *discrete* if it consists of a denumerable (i.e., at most countable) set of eigenvalues with no finite point of accumulation.

A linear operator acting in a Hilbert space and defined on the whole space is called *completely continuous* if it maps bounded sets into relatively compact sets (a set is called *relatively compact* if every infinite subset of this set has a limit point in the space, that may not belong to the set). Any completely continuous operator is bounded and hence its spectrum is a compact subset of the complex plane. As is well known [16], every nonzero point of the spectrum of a completely continuous operator is an eigenvalue of finite multiplicity (that is, to each eigenvalue there correspond only a finite number of linearly independent eigenvectors); the set of eigenvalues is at most countable and can have only one accumulation point $\lambda = 0$. It follows that if a linear operator A with a domain dense in a Hilbert space is invertible and its inverse A^{-1} is completely continuous, then the spectrum of A is discrete.

In this paper we show that the operator L is invertible under the condition (1.15) and that its inverse L^{-1} is a completely continuous operator if, in addition, the condition (1.16) is satisfied. This implies that the operator $A = M^{-1}L$ is invertible and $A^{-1} = L^{-1}M$ is a completely continuous operator. Hence the spectrum of the operator A is discrete.

3 Non-Hermiticity of the operators L and A

Let (f_k) be a given complex sequence, where $k \in \mathbb{Z}$. The forward and backward difference operators Δ and ∇ are defined by

$$\Delta f_k = f_{k+1} - f_k \qquad \text{and} \qquad \nabla f_k = f_k - f_{k-1},$$

respectively. We easily see that

$$\begin{aligned}
&\nabla f_k = \Delta f_{k-1},\\
&\Delta^2 f_k = \Delta(\Delta f_k) = f_{k+2} - 2f_{k+1} + f_k,\\
&\nabla^2 f_k = \nabla(\nabla f_k) = f_k - 2f_{k-1} + f_{k-2},\\
&\Delta\nabla f_k = f_{k+1} - 2f_k + f_{k-1} = \nabla\Delta f_k = \Delta^2 f_{k-1} = \nabla^2 f_{k+1}.
\end{aligned}$$

For any integers $a, b \in \mathbb{Z}$ with $a < b$ we have the summation by parts formulas

$$\sum_{k=a}^{b} (\Delta f_k) g_k = f_{k+1} g_k \big|_{a-1}^{b} - \sum_{k=a}^{b} f_k (\nabla g_k) = f_{b+1} g_b - f_a g_{a-1} - \sum_{k=a}^{b} f_k (\nabla g_k),$$

$$\sum_{k=a}^{b} (\nabla f_k) g_k = f_k g_{k+1} \big|_{a-1}^{b} - \sum_{k=a}^{b} f_k (\Delta g_k) = f_b g_{b+1} - f_{a-1} g_a - \sum_{k=a}^{b} f_k (\Delta g_k), \tag{3.1}$$

$$\sum_{k=a}^{b} (\Delta\nabla f_k) g_k = (\Delta f_k) g_k \big|_{a-1}^{b} - \sum_{k=a}^{b} (\nabla f_k)(\nabla g_k), \tag{3.2}$$

$$\sum_{k=a}^{b} (\Delta\nabla f_k) g_k = (\Delta f_k) g_{k+1} \big|_{a-1}^{b} - \sum_{k=a}^{b} (\Delta f_k)(\Delta g_k),$$

$$\begin{aligned}
\sum_{k=a}^{b} [(\Delta\nabla f_k) g_k - f_k (\Delta\nabla g_k)] &= [(\Delta f_k) g_k - f_k (\Delta g_k)]_{a-1}^{b}\\
&= [(\Delta f_b) g_b - f_b (\Delta g_b)] - [(\Delta f_{a-1}) g_{a-1} - f_{a-1} (\Delta g_{a-1})].
\end{aligned} \tag{3.3}$$

Theorem 1. *Let $\delta \in [0, \pi/2)$. If $\delta = 0$, then the operator L is Hermitian:*

$$\langle Ly, z\rangle = \langle y, Lz\rangle \qquad \text{for all } \ y, z \in D.$$

But if $\delta \neq 0$, then the operator L is not Hermitian.

Proof. Using formula (3.3) and equation

$$(Ly)_n = -\Delta^2 y_{n-1} + q_n y_n = -\Delta\nabla y_n + q_n y_n \qquad \text{for } \ n \in \mathbb{Z}_0,$$

where y_0 and y_1 are defined from the equations

$$y_{-1} = y_1, \qquad \Delta y_{-1} = e^{2i\delta} \Delta y_1, \tag{3.4}$$

and taking into account that for any $y = (y_n)_{n\in\mathbb{Z}_0} \in l_0^2$ we have $y_n \to 0$, $\Delta y_n \to 0$ as $|n| \to \infty$, we get for all $y, z \in D$,

$$\begin{aligned}
\langle Ly, z\rangle - \langle y, Lz\rangle &= -\sum_{n\in\mathbb{Z}_0} [(\Delta\nabla y_n)\overline{z}_n - y_n(\Delta\nabla \overline{z}_n)]\\
&= -\sum_{-\infty}^{n=-1} [(\Delta\nabla y_n)\overline{z}_n - y_n(\Delta\nabla \overline{z}_n)] - \sum_{n=2}^{\infty} [(\Delta\nabla y_n)\overline{z}_n - y_n(\Delta\nabla \overline{z}_n)]
\end{aligned}$$

$$
\begin{aligned}
&= -[(\Delta y_{-1})\overline{z}_{-1} - y_{-1}(\Delta \overline{z}_{-1})] + [(\Delta y_1)\overline{z}_1 - y_1(\Delta \overline{z}_1)] \\
&= -[(\Delta y_{-1})\overline{z}_{-1} - y_{-1}(\Delta \overline{z}_{-1})] + \left[e^{-2i\delta}(\Delta y_{-1})\overline{z}_{-1} - y_{-1}e^{2i\delta}(\Delta \overline{z}_{-1})\right] \\
&= \left(e^{-2i\delta} - 1\right)(\Delta y_{-1})\overline{z}_{-1} - \left(e^{2i\delta} - 1\right)y_{-1}(\Delta \overline{z}_{-1}).
\end{aligned}
$$

Thus,

$$
\langle Ly, z\rangle - \langle y, Lz\rangle = \left(e^{-2i\delta} - 1\right)(\Delta y_{-1})\overline{z}_{-1} - \left(e^{2i\delta} - 1\right)y_{-1}(\Delta \overline{z}_{-1}), \tag{3.5}
$$

for all $y, z \in D$. Formula (3.5) shows that if $\delta = 0$, then the operator L is Hermitian:

$$
\langle Ly, z\rangle = \langle y, Lz\rangle \qquad \text{for all } \ y, z \in D.
$$

The same formula shows that if $\delta \neq 0$ (recall that $\delta \in [0, \pi/2)$), then the operator L is not Hermitian:

$$
\langle Ly, z\rangle \neq \langle y, Lz\rangle \qquad \text{for some } \ y, z \in D.
$$

The theorem is proved. ■

Theorem 2. *Let $\delta \in [0, \pi/2)$. If $\delta = 0$, then the operator $A = M^{-1}L$ is Hermitian:*

$$
\langle Ay, z\rangle = \langle y, Az\rangle \qquad \textit{for all } \ y, z \in D.
$$

But if $\delta \neq 0$, then the operator A is not Hermitian.

Proof. We have for any $y, z \in D$,

$$
\begin{aligned}
\langle Ay, z\rangle - \langle y, Az\rangle &= \left\langle M^{-1}Ly, z\right\rangle - \left\langle y, M^{-1}Lz\right\rangle = \langle Ly, Mz\rangle - \langle My, Lz\rangle \\
&= \sum_{n\in\mathbb{Z}_0} [-(\Delta\nabla y_n) + q_n y_n]\overline{\rho_n z_n} - \sum_{n\in\mathbb{Z}_0} \rho_n y_n[-(\Delta\nabla \overline{z}_n) + q_n \overline{z}_n] \\
&= -\sum_{n\in\mathbb{Z}_0} [(\Delta\nabla y_n)\overline{\rho_n z_n} - \rho_n y_n(\Delta\nabla \overline{z}_n)] + \sum_{n\in\mathbb{Z}_0} (\overline{\rho}_n - \rho_n) q_n y_n \overline{z}_n.
\end{aligned}
$$

Next, from

$$
\rho_n = \begin{cases} e^{2i\delta} & \text{if } \ n \leq -1, \\ e^{-2i\delta} & \text{if } \ n \geq 2, \end{cases} \qquad \text{and} \qquad \overline{\rho}_n = \begin{cases} e^{-2i\delta} & \text{if } \ n \leq -1, \\ e^{2i\delta} & \text{if } \ n \geq 2, \end{cases}
$$

we find

$$
\overline{\rho}_n - \rho_n = \begin{cases} -2i\sin 2\delta & \text{if } \ n \leq -1, \\ 2i\sin 2\delta & \text{if } \ n \geq 2, \end{cases}
$$

so that

$$
\sum_{n\in\mathbb{Z}_0} (\overline{\rho}_n - \rho_n) q_n y_n \overline{z}_n = -2i\sin 2\delta \sum_{-\infty}^{n=-1} q_n y_n \overline{z}_n + 2i\sin 2\delta \sum_{n=2}^{\infty} q_n y_n \overline{z}_n.
$$

Besides, using formula (3.2) and equations in (3.4), we obtain

$$
\begin{aligned}
-\sum_{n\in\mathbb{Z}_0} [(\Delta\nabla y_n)\overline{\rho_n z_n} - \rho_n y_n(\Delta\nabla \overline{z}_n)] &= -e^{-2i\delta} \sum_{-\infty}^{n=-1} (\Delta\nabla y_n)\overline{z}_n - e^{2i\delta} \sum_{n=2}^{\infty} (\Delta\nabla y_n)\overline{z}_n \\
&\quad + e^{2i\delta} \sum_{-\infty}^{n=-1} y_n(\Delta\nabla \overline{z}_n) + e^{-2i\delta} \sum_{n=2}^{\infty} y_n(\Delta\nabla \overline{z}_n)
\end{aligned}
$$

$$
\begin{aligned}
&= -e^{-2i\delta}(\Delta y_{-1})\overline{z}_{-1} + e^{-2i\delta}\sum_{-\infty}^{n=-1}(\nabla y_n)(\nabla\overline{z}_n) + e^{2i\delta}(\Delta y_1)\overline{z}_1 + e^{2i\delta}\sum_{n=2}^{\infty}(\nabla y_n)(\nabla\overline{z}_n)\\
&\quad + e^{2i\delta}y_{-1}(\Delta\overline{z}_{-1}) - e^{2i\delta}\sum_{-\infty}^{n=-1}(\nabla y_n)(\nabla\overline{z}_n) - e^{-2i\delta}y_1(\Delta\overline{z}_1) - e^{-2i\delta}\sum_{n=2}^{\infty}(\nabla y_n)(\nabla\overline{z}_n)\\
&= \left(1-e^{-2i\delta}\right)(\Delta y_{-1})\overline{z}_{-1} - \left(1-e^{2i\delta}\right)y_{-1}(\Delta\overline{z}_{-1})\\
&\quad - 2i\sin 2\delta\sum_{-\infty}^{n=-1}(\nabla y_n)(\nabla\overline{z}_n) + 2i\sin 2\delta\sum_{n=2}^{\infty}(\nabla y_n)(\nabla\overline{z}_n).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\langle Ay, z\rangle - \langle y, Az\rangle &= \left(1-e^{-2i\delta}\right)(\Delta y_{-1})\overline{z}_{-1} - \left(1-e^{2i\delta}\right)y_{-1}(\Delta\overline{z}_{-1})\\
&\quad - 2i\sin 2\delta\sum_{-\infty}^{n=-1}[(\nabla y_n)(\nabla\overline{z}_n) + q_n y_n\overline{z}_n] + 2i\sin 2\delta\sum_{n=2}^{\infty}[(\nabla y_n)(\nabla\overline{z}_n) + q_n y_n\overline{z}_n]. \qquad (3.6)
\end{aligned}
$$

Formula (3.6) shows that if $\delta = 0$, then the operator A is Hermitian and if $\delta \neq 0$, then A is not Hermitian. ■

Remark 1. In the case $\delta = 0$ we have $A = L$.

4 Second order linear difference equations with impulse

Consider the second order linear homogeneous difference equation with impulse

$$-\Delta^2 y_{n-1} + p_n y_n = 0, \qquad n \in \mathbb{Z}_0 = \mathbb{Z}\setminus\{0,1\} = (-\infty,-1]\cup[2,\infty), \qquad (4.1)$$

$$y_{-1} = d_1 y_1, \qquad \Delta y_{-1} = d_2\Delta y_1, \qquad (4.2)$$

where $y = (y_n)$ with $n \in \mathbb{Z}$ is a desired solution, the coefficients p_n are complex numbers given for $n \in \mathbb{Z}_0$; d_1, d_2 presented in the "impulse conditions" (transition conditions) in (4.2) are nonzero complex numbers.

Using the definition of Δ-derivative we can rewrite problem (4.1), (4.2) in the form

$$-y_{n-1} + \widetilde{p}_n y_n - y_{n+1} = 0, \qquad n \in (-\infty,-1]\cup[2,\infty), \qquad (4.3)$$

$$y_{-1} = d_1 y_1, \qquad y_0 - y_{-1} = d_2(y_2 - y_1), \qquad (4.4)$$

where

$$\widetilde{p}_n = p_n + 2, \qquad n \in (-\infty,-1]\cup[2,\infty).$$

Theorem 3. *Let n_0 be a fixed point in $\mathbb{Z}$ and c_0, c_1 be given complex numbers. Then problem (4.1), (4.2) has a unique solution (y_n), $n \in \mathbb{Z}$, such that*

$$y_{n_0} = c_0, \qquad \Delta y_{n_0} = c_1, \qquad \text{that is,} \qquad y_{n_0} = c_0, \qquad y_{n_0+1} = c_0 + c_1 = c_1'. \qquad (4.5)$$

Proof. First assume that $n_0 \in (-\infty,-1]$. We can rewrite equation (4.3) in the form

$$y_{n-1} = \widetilde{p}_n y_n - y_{n+1} = 0, \qquad n \in (-\infty,-1]\cup[2,\infty) \qquad (4.6)$$

as well as in the form

$$y_{n+1} = \widetilde{p}_n y_n - y_{n-1} = 0, \qquad n \in (-\infty,-1]\cup[2,\infty). \qquad (4.7)$$

Using the initial conditions (4.5) we find, recurrently (step by step), y_n for $n \le n_0 + 1$ uniquely from (4.6) and for $n_0 + 2 \le n \le -1$ uniquely from (4.7). Then we find y_1 and y_2 from the transition conditions (4.4) and then we find y_n for $n \ge 3$ uniquely from (4.7).

In the case $n_0 \in [1, \infty)$ we are reasoning similarly; using equations (4.6), (4.7) we first find y_n uniquely for $n \ge 1$ and then using the transition conditions (4.4) we pass to the interval $(-\infty, -1]$.

Finally, if $n_0 = 0$, then we find y_0 and y_1 uniquely from the initial conditions (4.5) with $n_0 = 0$. Then we find y_{-1} and y_2 from the transition conditions (4.4). Next, solving equation (4.6) at first on $(-\infty, -1]$ we find y_n uniquely for $n \in (-\infty, -2]$ and then solving (4.7) on $[2, \infty)$ we find y_n uniquely for $n \in [3, \infty)$. ■

Definition 2. For two sequences $y = (y_n)$ and $z = (z_n)$ with $n \in \mathbb{Z}$, we define their Wronskian by

$$W_n(y, z) = y_n \Delta z_n - (\Delta y_n) z_n = y_n z_{n+1} - y_{n+1} z_n, \qquad n \in \mathbb{Z}.$$

Theorem 4. *The Wronskian of any two solutions y and z of problem* (4.1), (4.2) *is constant on each of the intervals $(-\infty, -1]$ and $[1, \infty)$:*

$$W_n(y, z) = \begin{cases} \omega^- & \text{if} \quad n \in (-\infty, -1], \\ \omega^+ & \text{if} \quad n \in [1, \infty). \end{cases} \tag{4.8}$$

In addition,

$$\omega^- = d_1 d_2 \omega^+ \tag{4.9}$$

and

$$W_0(y, z) = -d_2 \omega^+. \tag{4.10}$$

Proof. Suppose that $y = (y_n)$ and $z = (z_n)$, where $n \in \mathbb{Z}$, are solutions of (4.1), (4.2). Let us compute the Δ-derivative of $W_n(y, z)$. Using the product rule for Δ-derivative

$$\Delta(f_n g_n) = (\Delta f_n) g_n + f_{n+1} \Delta g_n = f_n \Delta g_n + (\Delta f_n) g_{n+1},$$

we have

$$\begin{aligned} \Delta W_n(y, z) &= \Delta\left[y_n \Delta z_n - (\Delta y_n) z_n\right] = (\Delta y_n)\Delta z_n + y_{n+1}\Delta^2 z_n - (\Delta y_n)\Delta z_n - (\Delta^2 y_n) z_{n+1} \\ &= y_{n+1}\Delta^2 z_n - (\Delta^2 y_n) z_{n+1}. \end{aligned}$$

Further, since y_n and z_n are solutions of (4.1), (4.2),

$$\begin{aligned} &\Delta^2 y_n = p_{n+1} y_{n+1}, \qquad n \in (-\infty, -2] \cup [1, \infty), \\ &\Delta^2 z_n = p_{n+1} z_{n+1}, \qquad n \in (-\infty, -2] \cup [1, \infty). \end{aligned}$$

Therefore

$$\Delta W_n(y, z) = 0 \qquad \text{for} \;\; n \in (-\infty, -2] \cup [1, \infty).$$

The latter implies that $W_n(y, z)$ is constant on $(-\infty, -1]$ and on $[1, \infty)$. Thus we have (4.8), where ω^- and ω^+ are some constants (depending on the solutions y and z).

Next using (4.8) and the impulse conditions in (4.2) for y_n and z_n, we have

$$\begin{aligned} \omega^- &= W_{-1}(y, z) = y_{-1}\Delta z_{-1} - (\Delta y_{-1}) z_{-1} = d_1 d_2 [y_1 \Delta z_1 - (\Delta y_1) z_1] \\ &= d_1 d_2 W_1(y, z) = d_1 d_2 \omega^+, \end{aligned}$$

so that (4.9) is established.

Finally, from the impulse conditions in (4.2) we find that

$$y_0 = (d_1 - d_2)y_1 + d_2 y_2.$$

Substituting this expression for y_0 and z_0 into

$$W_0(y,z) = y_0 z_1 - y_1 z_0,$$

we get

$$W_0(y,z) = -d_2 W_1(y,z) = -d_2\omega^+.$$

Therefore (4.10) is also proved. ∎

Corollary 1. *If y and z are two solutions of* (4.1), (4.2), *then either $W_n(y,z) = 0$ for all $n \in \mathbb{Z}$ or $W_n(y,z) \neq 0$ for all $n \in \mathbb{Z}$.*

By using Theorem 3, the following two theorems can be proved in exactly the same way when equation (4.1) does not include any impulse conditions [6].

Theorem 5. *Any two solutions of* (4.1), (4.2) *are linearly independent if and only if their Wronskian is not zero.*

Theorem 6. *Problem* (4.1), (4.2) *has two linearly independent solutions and every solution of* (4.1), (4.2) *is a linear combination of these solutions.*

We say that $y = (y_n)$ and $z = (z_n)$, where $n \in \mathbb{Z}$, form a *fundamental set* (or *fundamental system*) of solutions for (4.1), (4.2) provided that they are solutions of (4.1), (4.2) and their Wronskian is not zero.

Let us consider the nonhomogeneous equation

$$-\Delta^2 y_{n-1} + p_n y_n = h_n, \qquad n \in (-\infty,-1] \cup [2,\infty), \tag{4.11}$$

with the impulse conditions

$$y_{-1} = d_1 y_1, \qquad \Delta y_{-1} = d_2 \Delta y_1, \tag{4.12}$$

where h_n is a complex sequence defined for $n \in (-\infty,-1] \cup [2,\infty)$. We will extend h_n to the values $n = 0$ and $n = 1$ by setting

$$h_0 = h_1 = 0. \tag{4.13}$$

Theorem 7. *Suppose that $u = (u_n)$ and $v = (v_n)$ form a fundamental set of solutions of the homogeneous problem* (4.1), (4.2). *Then a general solution of the corresponding nonhomogeneous problem* (4.11), (4.12) *is given by*

$$y_n = c_1 u_n + c_2 v_n + x_n, \qquad n \in \mathbb{Z},$$

where c_1, c_2 are arbitrary constants and

$$x_n = \begin{cases} -\displaystyle\sum_{s=n}^{0} \frac{u_n v_s - u_s v_n}{W_s(u,v)} h_s & \text{if } n \le 0, \\ \displaystyle\sum_{s=1}^{n} \frac{u_n v_s - u_s v_n}{W_s(u,v)} h_s & \text{if } n \ge 1. \end{cases} \tag{4.14}$$

Proof. Taking into account (4.13) it is not difficult to verify that the sequence x_n defined by (4.14) is a particular solution of (4.11), (4.12), namely, x_n satisfies equation (4.11) and the conditions

$$x_{-1} = \Delta x_{-1} = 0, \qquad x_1 = \Delta x_1 = 0.$$

This implies that the statement of the theorem is true. ∎

5 Two special solutions

Consider the homogeneous problem

$$-\Delta^2 y_{n-1} + q_n y_n = 0, \qquad n \in \mathbb{Z}_0 = \mathbb{Z}\backslash\{0,1\}, \tag{5.1}$$

$$y_{-1} = y_1, \qquad \Delta y_{-1} = e^{2i\delta}\Delta y_1, \tag{5.2}$$

where $\delta \in [0, \pi/2)$ is a fixed real number and

$$q_n \geq c > 0 \qquad \text{for} \qquad n \in \mathbb{Z}_0. \tag{5.3}$$

In this section we show that under the condition (5.3) problem (5.1), (5.2) has two linearly independent solutions $\psi = (\psi_n)$ and $\chi = (\chi_n)$, where $n \in \mathbb{Z}$, such that

$$\sum_{n=0}^{\infty} |\psi_n|^2 < \infty \qquad \text{and} \qquad \sum_{-\infty}^{n=0} |\chi_n|^2 < \infty. \tag{5.4}$$

These solutions will allow us to find the inverse L^{-1} of the operator L introduced above in Section 2 and investigate the properties of L^{-1}.

First we derive two simple useful formulas related to the nonhomogeneous problem

$$-\Delta^2 y_{n-1} + q_n y_n = f_n, \qquad n \in \mathbb{Z}_0, \tag{5.5}$$

$$y_{-1} = y_1, \qquad \Delta y_{-1} = e^{2i\delta}\Delta y_1, \tag{5.6}$$

where (q_n) is a real sequence with $n \in \mathbb{Z}_0$, and $\delta \in [0, \pi/2)$; (f_n) is a complex sequence with $n \in \mathbb{Z}_0$.

Lemma 1. *Let $y = (y_n)$ with $n \in \mathbb{Z}$ be a solution of problem* (5.5), (5.6) *and a, b be any integers such that $a \leq -1$ and $b \geq 2$. Then the following formulas hold:*

$$\sum_{n=2}^{b} \left(|\Delta y_n|^2 + q_n |y_n|^2 \right) = (\Delta y_n)\overline{y}_{n+1}\Big|_1^b + \sum_{n=2}^{b} f_n \overline{y}_n, \tag{5.7}$$

$$\sum_{n=a}^{-1} \left(|\Delta y_n|^2 + q_n |y_n|^2 \right) = (\Delta y_n)\overline{y}_{n+1}\Big|_{a-1}^{-1} + \sum_{n=a}^{-1} f_n \overline{y}_n. \tag{5.8}$$

Proof. To prove (5.7), multiply equation (5.5) by $\overline{y}_n$ and sum from $n = 2$ to $n = b$:

$$-\sum_{n=2}^{b} (\Delta^2 y_{n-1})\overline{y}_n + \sum_{n=2}^{b} q_n |y_n|^2 = \sum_{n=2}^{b} f_n \overline{y}_n.$$

Next, applying the summation by parts formula (3.1) we get that

$$-\sum_{n=2}^{b} (\Delta^2 y_{n-1})\overline{y}_n = -\sum_{n=2}^{b} (\nabla\Delta y_n)\overline{y}_n = -(\Delta y_n)\overline{y}_{n+1}\Big|_1^b + \sum_{n=2}^{b} |\Delta y_n|^2.$$

Therefore the formula (5.7) follows.

The formula (5.8) can be proved in a similar way. ■

Theorem 8. *Under the condition* (5.3) *problem* (5.1), (5.2) *has two linearly independent solutions $\psi = (\psi_n)$ and $\chi = (\chi_n)$ with $n \in \mathbb{Z}$, possessing the properties stated in* (5.4).

Proof. Denote by $\varphi = (\varphi_n)$ and $\theta = (\theta_n)$, where $n \in \mathbb{Z}$, solutions of problem (5.1), (5.2) satisfying the initial conditions

$$\varphi_1 = 1, \qquad \Delta\varphi_1 = -1, \tag{5.9}$$

$$\theta_1 = 1, \qquad \Delta\theta_1 = 0. \tag{5.10}$$

Such solutions exist and are unique by Theorem 3. It follows from (5.9), (5.10) that $\varphi_2 = 0$, $\theta_2 = 1$. According to Theorem 4 we find that

$$W_0(\varphi,\theta) = -e^{2i\delta}, \qquad W_n(\varphi,\theta) = \begin{cases} e^{2i\delta} & \text{if } \ n \le -1, \\ 1 & \text{if } \ n \ge 1. \end{cases} \tag{5.11}$$

Therefore $W_n(\varphi,\theta) \neq 0$ and by Theorem 5 the solutions φ and θ are linearly independent.

We seek the desired solution $\psi = (\psi_n)$ of problem (5.1), (5.2) in the form

$$\psi_n = \varphi_n + v\theta_n, \qquad n \in \mathbb{Z}, \tag{5.12}$$

where v is a complex constant which we will choose.

Take an arbitrary integer $b \ge 2$. Applying (5.7) to

$$-\Delta^2\psi_{n-1} + q_n\psi_n = 0, \qquad n \in \mathbb{Z}_0,$$
$$\psi_{-1} = \psi_1, \qquad \Delta\psi_{-1} = e^{2i\delta}\Delta\psi_1,$$

we get

$$\sum_{n=2}^{b} \left(|\Delta\psi_n|^2 + q_n |\psi_n|^2 \right) = (\Delta\psi_n)\overline{\psi}_{n+1}\Big|_1^b.$$

Since $\Delta\psi_1 = -1$ and $\psi_2 = v$, by (5.12) and (5.9), (5.10), hence

$$\sum_{n=2}^{b} \left(|\Delta\psi_n|^2 + q_n |\psi_n|^2 \right) = (\Delta\psi_b)\overline{\psi}_{b+1} + \overline{v}.$$

Multiply the latter equality by $e^{i\delta}$ and take then the real part of both sides to get

$$(\cos\delta)\sum_{n=2}^{b} \left(|\Delta\psi_n|^2 + q_n |\psi_n|^2 \right) = \operatorname{Re}\big\{e^{i\delta}(\Delta\psi_b)\overline{\psi}_{b+1}\big\} + \operatorname{Re}\big(ve^{-i\delta}\big). \tag{5.13}$$

Now we choose v so that to have

$$\operatorname{Re}\big\{e^{i\delta}(\Delta\psi_b)\overline{\psi}_{b+1}\big\} = 0. \tag{5.14}$$

Since

$$\operatorname{Re}\big\{e^{i\delta}(\Delta\psi_b)\overline{\psi}_{b+1}\big\} = |\psi_{b+1}|^2 \operatorname{Re}\left\{ e^{i\delta}\frac{\Delta\psi_b}{\psi_{b+1}} \right\},$$

it is sufficient for (5.14) to have

$$\operatorname{Re}\left\{ e^{i\delta}\frac{\Delta\psi_b}{\psi_{b+1}} \right\} = 0. \tag{5.15}$$

Note that ψ_b cannot be zero for any two successive values of b (otherwise ψ_n would be identically zero by the uniqueness theorem for solution, that is not true since φ and θ are linearly independent). Therefore $\psi_b \neq 0$ for infinitely many values of b.

Under the condition (5.15) the equation (5.13) becomes

$$(\cos\delta)\sum_{n=2}^{b}\left(|\Delta\psi_n|^2+q_n|\psi_n|^2\right)=\mathrm{Re}\big(ve^{-i\delta}\big).$$

The condition (5.15) can be written as

$$e^{i\delta}\frac{\Delta\varphi_b+v\Delta\theta_b}{\varphi_{b+1}+v\theta_{b+1}}=\beta, \tag{5.16}$$

where β is a pure imaginary number ($\beta=it$, $t\in\mathbb{R}$). Note that

$$\varphi_{b+1}\Delta\theta_b-(\Delta\varphi_b)\theta_{b+1}=-\varphi_{b+1}\theta_b+\varphi_b\theta_{b+1}=W_b(\varphi,\theta)=1\neq 0 \tag{5.17}$$

by (5.11). Therefore (5.16) defines a linear-fractional transformation of the complex v-plane onto the complex β-plane. Solving (5.16) for v, we get

$$v(\beta)=\frac{\varphi_{b+1}\beta-e^{i\delta}\Delta\varphi_b}{-\theta_{b+1}\beta+e^{i\delta}\Delta\theta_b}. \tag{5.18}$$

Thus, condition (5.15) will be satisfied if we choose v by (5.18) for pure imaginary values of β. On the other hand, when β runs in (5.18) the imaginary axis, $v(\beta)$ describes a circle C_b in the v-plane. The center of the circle is symmetric point of the point at infinity with respect to the circle. Since

$$v(\beta')=\infty,\qquad \text{where}\qquad \beta'=e^{i\delta}\frac{\Delta\theta_b}{\theta_{b+1}},$$

the point

$$\beta_0=-\overline{\beta'}=-e^{-i\delta}\frac{\Delta\overline{\theta}_b}{\overline{\theta}_{b+1}}$$

which is symmetric point of the point β' with respect to the imaginary axis of the β-plane, is mapped onto the center of C_b. So the center of C_b is located at the point

$$v(\beta_0)=-\frac{e^{-i\delta}\varphi_{b+1}\Delta\overline{\theta}_b+e^{i\delta}(\Delta\varphi_b)\overline{\theta}_{b+1}}{e^{-i\delta}\theta_{b+1}\Delta\overline{\theta}_b+e^{i\delta}\overline{\theta}_{b+1}\Delta\theta_b}. \tag{5.19}$$

Note that the denominator in (5.19) is different from zero. This fact follows from the equality

$$\begin{aligned} e^{-i\delta}\theta_{b+1}\Delta\overline{\theta}_b+e^{i\delta}\overline{\theta}_{b+1}\Delta\theta_b&=2\mathrm{Re}\big\{e^{i\delta}(\Delta\theta_n)\overline{\theta}_{n+1}\big|_1^b\big\}\\ &=(2\cos\delta)\sum_{n=2}^{b}\left(|\Delta\theta_n|^2+q_n|\theta_n|^2\right), \end{aligned}\tag{5.20}$$

which can be derived as (5.13) taking into account (5.10). The radius R_b of the circle C_b is equal to the distance between the center $v(\beta_0)$ of C_b and the point $v(0)$ on the circle. Calculating the difference $v(\beta_0)-v(0)$ by using (5.17)–(5.20) we easily find that

$$R_b=\frac{1}{(2\cos\delta)\sum_{n=2}^{b}\left(|\Delta\theta_n|^2+q_n|\theta_n|^2\right)}.$$

Further, since

$$\operatorname{Re}(e^{i\delta}\overline{\theta}_{b+1}\Delta\theta_b) = -\left|\theta_{b+1}\right|^2 \operatorname{Re}\beta_0,$$

we get from (5.20)

$$(\cos\delta)\sum_{n=2}^{b}\left(\left|\Delta\theta_n\right|^2 + q_n\left|\theta_n\right|^2\right) = -\left|\theta_{b+1}\right|^2 \operatorname{Re}\beta_0.$$

Therefore $\operatorname{Re}\beta_0 < 0$. This means that the left half-plane of the β-plane is mapped onto the interior of the circle C_b. Consequently, $v(\beta)$ belongs to the interior of the circle C_b if and only if $\operatorname{Re}\beta < 0$. This inequality is equivalent by (5.13), (5.16) to

$$(\cos\delta)\sum_{n=2}^{b}\left(\left|\Delta\psi_n\right|^2 + q_n\left|\psi_n\right|^2\right) < \operatorname{Re}\bigl(ve^{-i\delta}\bigr). \tag{5.21}$$

Thus, v belongs to the interior of the circle C_b if and only if the inequality (5.21) holds and v lies on the circle C_b if and only if

$$(\cos\delta)\sum_{n=2}^{b}\left(\left|\Delta\psi_n\right|^2 + q_n\left|\psi_n\right|^2\right) = \operatorname{Re}\bigl(ve^{-i\delta}\bigr).$$

Now let $b_2 > b_1$. Then if v is inside or on C_{b_2}

$$(\cos\delta)\sum_{n=2}^{b_1}\left(\left|\Delta\psi_n\right|^2 + q_n\left|\psi_n\right|^2\right) < (\cos\delta)\sum_{n=2}^{b_2}\left(\left|\Delta\psi_n\right|^2 + q_n\left|\psi_n\right|^2\right) \le \operatorname{Re}\bigl(ve^{-i\delta}\bigr)$$

and therefore v is inside C_{b_1}. This means C_{b_1} contains C_{b_2} in its interior if $b_2 > b_1$. It follows that, as $b \to \infty$, the circles C_b converge either to a limit-circle or to a limit-point. If $\widehat{v}$ is the limit-point or any point on the limit-circle, then $\widehat{v}$ is inside any C_b. Hence

$$(\cos\delta)\sum_{n=2}^{b}\left(\left|\Delta\psi_n\right|^2 + q_n\left|\psi_n\right|^2\right) < \operatorname{Re}\bigl(\widehat{v}e^{-i\delta}\bigr),$$

where

$$\psi_n = \varphi_n + \widehat{v}\theta_n, \qquad n \in \mathbb{Z}, \tag{5.22}$$

and letting $b \to \infty$ we get

$$(\cos\delta)\sum_{n=2}^{\infty}\left(\left|\Delta\psi_n\right|^2 + q_n\left|\psi_n\right|^2\right) \le \operatorname{Re}\bigl(\widehat{v}e^{-i\delta}\bigr). \tag{5.23}$$

It also follows that

$$\operatorname{Re}\bigl(\widehat{v}e^{-i\delta}\bigr) > 0. \tag{5.24}$$

Since $q_n \ge c > 0$, (5.23) implies that for the solution $\psi = (\psi_n)$ defined by (5.22) we have (5.4). Thus, the statement of the theorem concerning the solution $\psi = (\psi_n)$ is proved.

Let us now show existence of the solution $\chi = (\chi_n)$ satisfying (5.4). We seek the desired solution $\theta = (\theta_n)$ of problem (5.1), (5.2) in the form

$$\chi_n = \varphi_n + u\theta_n, \qquad n \in \mathbb{Z},$$

where u is a complex constant to be determined.

Take an arbitrary integer $a \le -1$. Applying (5.8) to the equations

$$-\Delta^2\chi_{n-1} + q_n\chi_n = 0, \qquad n \in \mathbb{Z}_0,$$
$$\chi_{-1} = \chi_1, \qquad \Delta\chi_{-1} = e^{2i\delta}\Delta\chi_1,$$

we get

$$\sum_{n=a}^{-1}\left(|\Delta\chi_n|^2 + q_n|\chi_n|^2\right) = (\Delta\chi_n)\overline{\chi}_{n+1}\Big|_{a-1}^{-1}.$$

Since

$$\Delta\chi_{-1} = -e^{2i\delta}, \qquad \chi_0 = 1 - e^{2i\delta} + u,$$

we have

$$\sum_{n=a}^{-1}\left(|\Delta\chi_n|^2 + q_n|\chi_n|^2\right) = -e^{2i\delta} + 1 - \overline{u}e^{2i\delta} - (\Delta\chi_{a-1})\overline{\chi}_a. \tag{5.25}$$

Multiply both sides of (5.25) by $e^{-i\delta}$ and take then the real part of both sides to get

$$(\cos\delta)\sum_{n=a}^{-1}\left(|\Delta\chi_n|^2 + q_n|\chi_n|^2\right) = -\mathrm{Re}\big(ue^{-i\delta}\big) - \mathrm{Re}\big\{e^{-i\delta}(\Delta\chi_{a-1})\overline{\chi}_a\big\}. \tag{5.26}$$

Now we choose u so that to have

$$\mathrm{Re}\big\{e^{-i\delta}(\Delta\chi_{a-1})\overline{\chi}_a\big\} = 0. \tag{5.27}$$

Since

$$\mathrm{Re}\big\{e^{-i\delta}(\Delta\chi_{a-1})\overline{\chi}_a\big\} = |\chi_a|^2\,\mathrm{Re}\left\{e^{-i\delta}\frac{\Delta\chi_{a-1}}{\chi_a}\right\},$$

it is sufficient for (5.27) to have

$$\mathrm{Re}\left\{e^{-i\delta}\frac{\Delta\chi_{a-1}}{\chi_a}\right\} = 0. \tag{5.28}$$

Then (5.26) becomes

$$(\cos\delta)\sum_{n=a}^{-1}\left(|\Delta\chi_n|^2 + q_n|\chi_n|^2\right) = -\mathrm{Re}\big(ue^{-i\delta}\big).$$

The condition (5.28) can be written as

$$e^{-i\delta}\frac{\Delta\varphi_{a-1} + u\Delta\theta_{a-1}}{\varphi_a + u\theta_a} = \alpha, \tag{5.29}$$

where α is a pure imaginary number ($\alpha = it$, $t \in \mathbb{R}$). Note that

$$\varphi_a\Delta\theta_{a-1} - (\Delta\varphi_{a-1})\theta_a = W_{a-1}(\varphi,\theta) = e^{2i\delta} \ne 0 \tag{5.30}$$

by (5.11). Therefore (5.29) defines a linear-fractional transformation of the complex u-plane onto the complex α-plane. Solving (5.29) for u, we get

$$u(\alpha) = \frac{\varphi_a \alpha - e^{-i\delta} \Delta \varphi_{a-1}}{-\theta_a \alpha + e^{-i\delta} \Delta \theta_{a-1}}. \tag{5.31}$$

Thus, condition (5.28) will be satisfied if we choose u by (5.31) for pure imaginary values of α. On the other hand, when α runs in (5.31) the imaginary axis, $u(\alpha)$ describes a circle K_a in the u-plane. The center of the circle is symmetric point of the point at infinity with respect to the circle. Since

$$u(\alpha') = \infty, \qquad \text{where} \qquad \alpha' = e^{-i\delta} \frac{\Delta \theta_{a-1}}{\theta_a},$$

the point

$$\alpha_0 = -\overline{\alpha'} = -e^{i\delta} \frac{\Delta \overline{\theta}_{a-1}}{\overline{\theta}_a}$$

which is symmetric point of the point α' with respect to the imaginary axis of the α-plane, is mapped onto the center of K_a. So the center of K_a is located at the point

$$u(\alpha_0) = -\frac{e^{i\delta} \varphi_a \Delta \overline{\theta}_{a-1} + e^{-i\delta} (\Delta \varphi_{a-1}) \overline{\theta}_a}{e^{i\delta} \theta_a \Delta \overline{\theta}_{a-1} + e^{-i\delta} \overline{\theta}_a \Delta \theta_{a-1}}. \tag{5.32}$$

Note that the denominator in (5.32) is different from zero. This fact follows from the equality

$$\begin{aligned} e^{-i\delta} \theta_a \Delta \overline{\theta}_{a-1} + e^{i\delta} \overline{\theta}_a \Delta \theta_{a-1} &= 2\mathrm{Re}\big\{ e^{i\delta} (\Delta \theta_{a-1}) \overline{\theta}_a \big\} \\ &= -(2\cos\delta) \sum_{n=a}^{-1} \big(|\Delta \theta_n|^2 + q_n |\theta_n|^2 \big) \end{aligned} \tag{5.33}$$

which can be derived as (5.26) taking into account $\Delta\theta_{-1} = 0$, $\theta_0 = 1$. Calculating the difference $u(\alpha_0) - u(0)$ we easily find the radius $R_a = |u(\alpha_0) - u(0)|$ of the circle K_a, using (5.30)–(5.33),

$$R_a = \frac{1}{(2\cos\delta) \sum\limits_{n=a}^{-1} \big(|\Delta \theta_n|^2 + q_n |\theta_n|^2 \big)}.$$

Further, since

$$\mathrm{Re}\big(e^{-i\delta} \overline{\theta}_a \Delta \theta_{a-1} \big) = -|\theta_a|^2 \,\mathrm{Re}\,\alpha_0,$$

we get from (5.33)

$$(\cos\delta) \sum_{n=a}^{-1} \big(|\Delta \theta_n|^2 + q_n |\theta_n|^2 \big) = |\theta_a|^2 \,\mathrm{Re}\,\alpha_0.$$

Therefore $\mathrm{Re}\,\alpha_0 > 0$. This means that the right half-plane of the α-plane is mapped onto the interior of the circle K_a. Consequently, $u(\alpha)$ lies inside the circle K_a if and only if $\mathrm{Re}\,\alpha > 0$. This inequality is equivalent by (5.26), (5.29) to

$$(\cos\delta) \sum_{n=a}^{-1} \big(|\Delta \chi_n|^2 + q_n |\chi_n|^2 \big) < -\mathrm{Re}\big(u e^{-i\delta} \big). \tag{5.34}$$

Thus, u lies inside the circle K_a if and only if the inequality (5.34) holds and u lies on the circle K_a if and only if

$$(\cos\delta)\sum_{n=a}^{-1}\left(|\Delta\chi_n|^2+q_n|\chi_n|^2\right)=-\mathrm{Re}\big(ue^{-i\delta}\big).$$

Now let $a_2<a_1$. Then if u is inside or on K_{a_2}

$$(\cos\delta)\sum_{n=a_1}^{-1}\left(|\Delta\chi_n|^2+q_n|\chi_n|^2\right)<(\cos\delta)\sum_{n=a_2}^{-1}\left(|\Delta\chi_n|^2+q_n|\chi_n|^2\right)\le-\mathrm{Re}\big(ue^{-i\delta}\big)$$

and therefore u is inside K_{a_1}. This means K_{a_1} contains K_{a_2} in its interior if $a_2<a_1$. It follows that, as $a\to-\infty$, the circles K_a converge either to a limit-circle or to a limit-point. If $\widehat{u}$ is the limit-point or any point on the limit-circle, then $\widehat{u}$ is inside any K_a. Hence

$$(\cos\delta)\sum_{n=a}^{-1}\left(|\Delta\chi_n|^2+q_n|\chi_n|^2\right)<-\mathrm{Re}\big(\widehat{u}e^{-i\delta}\big),$$

where

$$\chi_n=\varphi_n+\widehat{u}\theta_n,\qquad n\in\mathbb{Z},\tag{5.35}$$

and letting $a\to-\infty$ we get

$$(\cos\delta)\sum_{-\infty}^{n=-1}\left(|\Delta\chi_n|^2+q_n|\chi_n|^2\right)\le-\mathrm{Re}\big(\widehat{u}e^{-i\delta}\big).\tag{5.36}$$

It also follows that

$$\mathrm{Re}\big(\widehat{u}e^{-i\delta}\big)<0.\tag{5.37}$$

Since $q_n\ge c>0$, (5.36) implies that for the solution $\chi=(\chi_n)$ defined by (5.35) we have (5.4). Thus, the statement of the theorem concerning the solution $\chi=(\chi_n)$ is also proved.

Finally, let us show that the solutions $\psi=(\psi_n)$ and $\chi=(\chi_n)$ defined by (5.22) and (5.35), respectively, are linearly independent. We have

$$W_n(\psi,\chi)=W_n(\varphi+\widehat{v}\theta,\varphi+\widehat{u}\theta)=(\widehat{u}-\widehat{v})W_n(\varphi,\theta).\tag{5.38}$$

Next, $W_n(\varphi,\theta)\ne0$ by (5.11) and $\widehat{u}\ne\widehat{v}$ by (5.24), (5.37). Therefore $W_n(\psi,\chi)\ne0$ and hence ψ and χ are linearly independent by Theorem 5. ■

6 The inverse operator L^{-1}

The following lemma will play crucial role in this and next sections.

Lemma 2. *Let us set*

$$\sigma_n=\begin{cases}e^{-i\delta} & \textit{for}\quad n\le-1,\\ e^{i\delta} & \textit{for}\quad n\ge0.\end{cases}\tag{6.1}$$

Under the conditions of Lemma 1 *the following formula holds:*

$$(\cos\delta)\left(\sum_{n=a}^{-1}+\sum_{n=2}^{b}\right)\left(|\Delta y_n|^2+q_n|y_n|^2\right)$$
$$=\mathrm{Re}\left\{\sigma_n(\Delta y_n)\overline{y}_{n+1}\big|_{a-1}^{b}+\left(\sum_{n=a}^{-1}+\sum_{n=2}^{b}\right)\sigma_nf_n\overline{y}_n\right\}.\tag{6.2}$$

Proof. To prove (6.2) we multiply (5.8) by $e^{-i\delta}$ and (5.7) by $e^{i\delta}$ and add together to get

$$\left(e^{-i\delta}\sum_{n=a}^{-1}+e^{i\delta}\sum_{n=2}^{b}\right)\left(|\Delta y_n|^2+q_n|y_n|^2\right)$$
$$=e^{-i\delta}(\Delta y_n)\overline{y}_{n+1}\Big|_{a-1}^{-1}+e^{i\delta}(\Delta y_n)\overline{y}_{n+1}\Big|_1^b+\left(e^{-i\delta}\sum_{n=a}^{-1}+e^{i\delta}\sum_{n=2}^{b}\right)f_n\overline{y}_n. \tag{6.3}$$

Next, using the conditions (5.6) we have

$$e^{-i\delta}(\Delta y_{-1})\overline{y}_0-e^{i\delta}(\Delta y_1)\overline{y}_2=e^{i\delta}(\Delta y_1)\overline{y}_0-e^{i\delta}(\Delta y_1)\overline{y}_2$$
$$=e^{i\delta}(\Delta y_1)(\overline{y}_0-\overline{y}_2)=e^{i\delta}(\Delta y_1)(\overline{y}_0-\overline{y}_1+\overline{y}_1-\overline{y}_2)$$
$$=e^{i\delta}(\Delta y_1)(\overline{y}_0-\overline{y}_{-1}+\overline{y}_1-\overline{y}_2)=e^{i\delta}(\Delta y_1)(\overline{\Delta y_{-1}}-\overline{\Delta y_1})$$
$$=-e^{i\delta}|\Delta y_1|^2+e^{i\delta}(\Delta y_1)e^{-2i\delta}\overline{\Delta y_1}=-e^{i\delta}|\Delta y_1|^2+e^{-i\delta}|\Delta y_1|^2=-2i(\sin\delta)|\Delta y_1|^2.$$

Therefore taking (6.1) into account we can rewrite (6.3) in the form

$$\left(\sum_{n=a}^{-1}+\sum_{n=2}^{b}\right)\sigma_n\left(|\Delta y_n|^2+q_n|y_n|^2\right)$$
$$=-2i(\sin\delta)|\Delta y_1|^2+\sigma_n(\Delta y_n)\overline{y}_{n+1}\Big|_{a-1}^{b}+\left(\sum_{n=a}^{-1}+\sum_{n=2}^{b}\right)\sigma_n f_n\overline{y}_n. \tag{6.4}$$

Taking in (6.4) the real parts of both sides and taking into account that $\operatorname{Re}\sigma_n=\cos\delta$ for all n, we obtain (6.2). ■

Let $L: D\subset l_0^2\to l_0^2$ be the operator defined above in Section 2. Further, let $\psi=(\psi_n)$ and $\chi=(\chi_n)$, where $n\in\mathbb{Z}$, be solutions of problem (5.1), (5.2), constructed in Theorem 8. Let us introduce the discrete Green function

$$G_{nk}=\frac{1}{W_k(\psi,\chi)}\begin{cases}\chi_k\psi_n & \text{if } k\le n,\\ \chi_n\psi_k & \text{if } k\ge n,\end{cases}$$

of discrete variables $k,n\in\mathbb{Z}$. Note that by (5.38) and (5.11), we have

$$W_0(\psi,\chi)=-(\widehat{u}-\widehat{v})e^{2i\delta},\qquad W_k(\psi,\chi)=\begin{cases}(\widehat{u}-\widehat{v})e^{2i\delta} & \text{if } k\le -1,\\ \widehat{u}-\widehat{v} & \text{if } k\ge 1,\end{cases} \tag{6.5}$$

and, besides,

$$\widehat{u}\ne\widehat{v}$$

by (5.24) and (5.37).

Theorem 9. *Under the condition* (5.3) *the inverse operator L^{-1} exists and is a bounded operator defined on the whole space l_0^2. Next, for every $f=(f_n)\in l_0^2$*

$$\left(L^{-1}f\right)_n=\sum_{k\in\mathbb{Z}_0}G_{nk}f_k,\qquad n\in\mathbb{Z}_0, \tag{6.6}$$

and

$$\left\|L^{-1}f\right\|\le\frac{1}{c\cos\delta}\|f\|\qquad \textit{for all } f\in l_0^2, \tag{6.7}$$

where c is a constant from condition (5.3) *and δ is from* (5.2), *$\|\cdot\|$ denotes the norm of space l_0^2.*

Proof. Let us show that

$$\ker L = \{y \in D : Ly = 0\}$$

consists only of the zero element. Indeed, if $y \in D$ and $Ly = 0$, then $(y)_{n\in\mathbb{Z}_0}$ satisfies the equation

$$-\Delta^2 y_{n-1} + q_n y_n = 0, \qquad n \in \mathbb{Z}_0, \tag{6.8}$$

in which y_0 and y_1 (these values arise in (6.8) for $n = -1$ and $n = 2$, respectively) are defined from the equations

$$y_{-1} = y_1, \qquad \Delta y_{-1} = e^{2i\delta}\Delta y_1. \tag{6.9}$$

Since χ and ψ form a fundamental system of solutions of (6.8), (6.9), we can write

$$y_n = C_1\psi_n + C_2\chi_n, \qquad n \in \mathbb{Z},$$

with some constants C_1 and C_2. Hence

$$W_n(y,\psi) = C_1 W_n(\psi,\psi) + C_2 W_n(\chi,\psi), \qquad n \in \mathbb{Z}. \tag{6.10}$$

Next, since $y \in l_0^2$, we have $y_n \to 0$ as $|n| \to \infty$ and by (5.4) we have $\psi_n \to 0$ as $n \to \infty$ Hence $W_n(y,\psi) \to 0$ as $n \to \infty$. Besides $W_n(\psi,\psi) = 0$ for all n and $W_n(\chi,\psi)$ is equal to a nonzero constant for $n \geq 1$ by (6.5). Therefore taking the limit in (6.10) as $n \to \infty$ we get that $C_2 = 0$. It can similarly be shown, by considering $W_n(y,\chi)$, that $C_1 = 0$. Thus $y = 0$.

It follows that the inverse operator L^{-1} exists. Now take an arbitrary $f = (f_n)_{n\in\mathbb{Z}_0} \in l_0^2$ and extend the sequence $(f_n)_{n\in\mathbb{Z}_0}$ to the values $n = 0$ and $n = 1$ by setting

$$f_0 = f_1 = 0.$$

Let us put

$$g_n = \sum_{k\in\mathbb{Z}_0} G_{nk} f_k = \sum_{k\in\mathbb{Z}} G_{nk} f_k = \psi_n \sum_{-\infty}^{k=n} \frac{\chi_k f_k}{W_k(\psi,\chi)} + \chi_n \sum_{k=n+1}^{\infty} \frac{\psi_k f_k}{W_k(\psi,\chi)}, \qquad n \in \mathbb{Z}. \tag{6.11}$$

Then it is easy to check that this sequence (g_n), where $n \in \mathbb{Z}$, satisfies the equations

$$-\Delta^2 g_{n-1} + q_n g_n = f_n, \qquad n \in \mathbb{Z}_0, \tag{6.12}$$

$$g_{-1} = g_1, \qquad \Delta g_{-1} = e^{2i\delta}\Delta g_1. \tag{6.13}$$

We want to show that $g = (g_n)_{n\in\mathbb{Z}_0} \in l_0^2$ and that

$$\|g\| \leq \frac{1}{c\cos\delta}\|f\|. \tag{6.14}$$

For this purpose we take the sequences of integers a_m and b_m defined for any positive integer m, such that

$$a_m < 0 < b_m \quad \text{and} \quad a_m \to -\infty, \quad b_m \to \infty \quad \text{as} \quad m \to \infty.$$

Next, for each m we define the sequence $(f_n^{(m)})_{n\in\mathbb{Z}}$ by

$$f_n^{(m)} = f_n \quad \text{if} \quad a_m \leq n \leq b_m, \tag{6.15}$$

$$f_n^{(m)} = 0 \quad \text{if} \quad n < a_m \quad \text{or} \quad n > b_m, \tag{6.16}$$

and put

$$g_n^{(m)} = \sum_{k\in\mathbb{Z}_0} G_{nk} f_k^{(m)} = \sum_{k\in\mathbb{Z}} G_{nk} f_k^{(m)} = \psi_n \sum_{-\infty}^{k=n} \frac{\chi_k f_k^{(m)}}{W_k(\psi,\chi)} + \chi_n \sum_{k=n+1}^{\infty} \frac{\psi_k f_k^{(m)}}{W_k(\psi,\chi)}, \qquad n\in\mathbb{Z}.$$

It follows that

$$g_n^{(m)} = \begin{cases} \chi_n \displaystyle\sum_{k=a_m}^{b_m} \frac{\psi_k f_k}{W_k(\psi,\chi)} & \text{if} \quad n < a_m, \\ \psi_n \displaystyle\sum_{k=a_m}^{b_m} \frac{\chi_k f_k}{W_k(\psi,\chi)} & \text{if} \quad n > b_m. \end{cases} \tag{6.17}$$

We have also that, for each m,

$$-\Delta^2 g_{n-1}^{(m)} + q_n g_n^{(m)} = f_n^{(m)}, \qquad n\in\mathbb{Z}_0, \tag{6.18}$$

$$g_{-1}^{(m)} = g_1^{(m)}, \qquad \Delta g_{-1}^{(m)} = e^{2i\delta}\Delta g_1^{(m)}. \tag{6.19}$$

Fix m and take a positive integer N such that

$$-N < a_m \qquad \text{and} \qquad b_m < N.$$

Then applying Lemma 2 to (6.18), (6.19) we can write

$$\begin{aligned} &(\cos\delta)\left(\sum_{n=-N}^{-1} + \sum_{n=2}^{N}\right)\left(\left|\Delta g_n^{(m)}\right|^2 + q_n\left|g_n^{(m)}\right|^2\right) \\ &\quad = \operatorname{Re}\left\{\sigma_n(\Delta g_n^{(m)})\overline{g}_{n+1}^{(m)}\Big|_{-N-1}^{N} + \left(\sum_{n=-N}^{-1} + \sum_{n=2}^{N}\right)\sigma_n f_n^{(m)}\overline{g}_n^{(m)}\right\}. \end{aligned} \tag{6.20}$$

It follows from (6.17) by (5.4) that

$$\sum_{n\in\mathbb{Z}} \left|g_n^{(m)}\right|^2 < \infty.$$

Therefore the sums on the right-hand side of (6.20) are convergent as $N\to\infty$ and besides

$$\operatorname{Re}\left\{\sigma_n(\Delta g_n^{(m)})\overline{g}_{n+1}^{(m)}\Big|_{-N-1}^{N}\right\} \to 0 \qquad \text{as} \quad N\to\infty.$$

(note that $|\sigma_n| = 1$ for all n by (6.1)). Now taking the limit in (6.20) as $N\to\infty$, we get

$$(\cos\delta)\sum_{n\in\mathbb{Z}_0}\left(\left|\Delta g_n^{(m)}\right|^2 + q_n\left|g_n^{(m)}\right|^2\right) = \operatorname{Re}\sum_{n\in\mathbb{Z}_0}\sigma_n f_n^{(m)}\overline{g}_n^{(m)}. \tag{6.21}$$

Using the condition (5.3) we get from (6.21) that

$$\begin{aligned} \sum_{n\in\mathbb{Z}_0}\left|g_n^{(m)}\right|^2 &\le \frac{1}{c\cos\delta}\operatorname{Re}\sum_{n\in\mathbb{Z}_0}\sigma_n f_n^{(m)}\overline{g}_n^{(m)} \le \frac{1}{c\cos\delta}\left|\sum_{n\in\mathbb{Z}_0}\sigma_n f_n^{(m)}\overline{g}_n^{(m)}\right| \\ &\le \frac{1}{c\cos\delta}\sum_{n\in\mathbb{Z}_0}\left|f_n^{(m)}\overline{g}_n^{(m)}\right| \le \frac{1}{c\cos\delta}\left\{\sum_{n\in\mathbb{Z}_0}\left|f_n^{(m)}\right|^2\right\}^{1/2}\left\{\sum_{n\in\mathbb{Z}_0}\left|g_n^{(m)}\right|^2\right\}^{1/2}. \end{aligned}$$

Hence

$$\left\{\sum_{n\in\mathbb{Z}_0}\left|g_n^{(m)}\right|^2\right\}^{1/2}\le\frac{1}{c\cos\delta}\left\{\sum_{n\in\mathbb{Z}_0}\left|f_n^{(m)}\right|^2\right\}^{1/2},$$

that is,

$$\left\|g^{(m)}\right\|\le\frac{1}{c\cos\delta}\left\|f^{(m)}\right\|. \tag{6.22}$$

Writing (6.18), (6.19) for $m=m_1$ and $m=m_2$, and subtracting the obtained equations side-by-side, we get

$$-\Delta^2\left[g_{n-1}^{(m_1)}-g_{n-1}^{(m_2)}\right]+q_n\left[g_n^{(m_1)}-g_n^{(m_2)}\right]=f_n^{(m_1)}-f_n^{(m_2)},\qquad n\in\mathbb{Z}_0,$$
$$g_{-1}^{(m_1)}-g_{-1}^{(m_2)}=g_1^{(m_1)}-g_1^{(m_2)},\qquad \Delta\left[g_{-1}^{(m_1)}-g_{-1}^{(m_2)}\right]=e^{2i\delta}\Delta\left[g_1^{(m_1)}-g_1^{(m_2)}\right].$$

Hence, repeating the same reasonings as above, we get

$$\left\|g^{(m_1)}-g^{(m_2)}\right\|\le\frac{1}{c\cos\delta}\left\|f^{(m_1)}-f^{(m_2)}\right\|.$$

It follows that $g^{(m)}$ converges in l_0^2 to an element $\widetilde{g}$ as $m\to\infty$. On the other hand, it can be seen from (6.11), (6.17) taking into account (6.15), (6.16) that

$$g_n^{(m)}\to g_n\qquad\text{as}\quad m\to\infty,$$

for each n. Consequently, $\widetilde{g}=g$ and hence $g\in l_0^2$. Passing in (6.22) to the limit as $m\to\infty$, we get (6.14).

Next, from (6.12) we have

$$q_ng_n=f_n+g_{n-1}-2g_n+g_{n+1},\qquad n\in\mathbb{Z}_0.$$

Hence $(q_ng_n)_{n\in\mathbb{Z}_0}\in l_0^2$. Therefore $g\in D$, where D is the domain of the operator L. If we define an operator $B:l_0^2\to l_0^2$ by the formula $Bf=g$, where $f=(f_n)_{n\in\mathbb{Z}_0}\in l_0^2$ and $g=(g_n)_{n\in\mathbb{Z}_0}$ with g_n defined by (6.11), then we get by (6.12), (6.13) that $LBf=f$. Therefore B is the inverse of the operator L: $B=L^{-1}$, so that $g=L^{-1}f$ and from (6.11) and (6.14) we get (6.6) and (6.7), respectively. ■

7 Completely continuity of the operator L^{-1}

In this section we will show that the operator L^{-1} is completely continuous, that is, it is continuous and maps bounded sets into relatively compact sets.

Theorem 10. *Let*

$$q_n\ge c>0\qquad\textit{for}\quad n\in\mathbb{Z}_0, \tag{7.1}$$

and

$$\lim_{|n|\to\infty}q_n=\infty. \tag{7.2}$$

Then the operator L^{-1} is completely continuous.

Proof. The operator L^{-1} is continuous in virtue of (6.7) that holds under the condition (7.1). In order to show that L^{-1} maps bounded sets into relatively compact sets consider any bounded set X in l_0^2,

$$X = \left\{ f \in l_0^2 : \|f\| \le d \right\},$$

and prove that $L^{-1}(X) = Y$ is relatively compact in l_0^2. To this end, we use the following known (see [17]) criterion for the relative compactness in l_0^2 : *A set $Y \subset l_0^2$ is relatively compact if and only if Y is bounded and for every $\varepsilon > 0$ there exists a positive integer n_0 (depending only on ε) such that*

$$\sum_{|n|>n_0} |y_n|^2 \le \varepsilon \qquad \textit{for all} \quad y \in Y.$$

Take an arbitrary $f \in X$ and set

$$L^{-1} f = y.$$

Then $Ly = f$ or explicitly

$$-\Delta^2 y_{n-1} + q_n y_n = f_n, \qquad n \in \mathbb{Z}_0, \tag{7.3}$$

where y_0 and y_1 are defined from the equations

$$y_{-1} = y_1, \qquad \Delta y_{-1} = e^{2i\delta} \Delta y_1. \tag{7.4}$$

Note that y_0 and y_1 are needed when we write out equation (7.3) for $n = -1$ and $n = 2$, respectively.

Applying Lemma 2 to (7.3), (7.4), we get that for any integers $a \le -1$ and $b \ge 2$,

$$\begin{aligned}(\cos\delta) &\left(\sum_{n=a}^{-1} + \sum_{n=2}^{b} \right) \left(|\Delta y_n|^2 + q_n |y_n|^2 \right) \\ &= \operatorname{Re} \left\{ \sigma_n (\Delta y_n) \overline{y}_{n+1} \Big|_{a-1}^{b} + \left(\sum_{n=a}^{-1} + \sum_{n=2}^{b} \right) \sigma_n f_n \overline{y}_n \right\},\end{aligned} \tag{7.5}$$

where σ_n is defined by (6.1).

Since $f, y \in l_0^2$ and $|\sigma_n| = 1$, the sums on the right-hand side of (7.5) are convergent as $a \to -\infty$, $b \to \infty$. Also from $y \in l_0^2$ it follows that $y_n \to 0$ as $|n| \to \infty$ so that

$$\sigma_n (\Delta y_n) \overline{y}_{n+1} \Big|_{a-1}^{b} \to 0 \qquad \text{as} \qquad a \to -\infty, \qquad b \to \infty.$$

Consequently, we arrive at the equality

$$(\cos\delta) \sum_{n \in \mathbb{Z}_0} \left(|\Delta y_n|^2 + q_n |y_n|^2 \right) = \operatorname{Re} \sum_{n \in \mathbb{Z}_0} \sigma_n f_n \overline{y}_n.$$

Hence

$$(\cos\delta) \sum_{n \in \mathbb{Z}_0} q_n |y_n|^2 \le \operatorname{Re} \sum_{n \in \mathbb{Z}_0} \sigma_n f_n \overline{y}_n \tag{7.6}$$

and therefore using (7.1) and $\|f\| \le d$, we get

$$\|y\| \le \frac{d}{c \cos\delta} \qquad \text{for all} \quad y \in Y. \tag{7.7}$$

This means that the set $Y = L^{-1}(X)$ is bounded.

From (7.6) we also have, using (7.7),

$$\sum_{n\in\mathbb{Z}_0} q_n |y_n|^2 \leq \frac{d^2}{c\cos^2\delta} \qquad \text{for all} \quad y \in Y. \tag{7.8}$$

Take now an arbitrary $\varepsilon > 0$. By condition (7.2) we can choose a positive integer n_0 such that

$$q_n \geq \frac{d^2}{\varepsilon c\cos^2\delta} \qquad \text{for} \quad |n| > n_0.$$

Then we get from (7.8) that

$$\sum_{|n|>n_0} |y_n|^2 \leq \varepsilon \qquad \text{for all} \quad y \in Y.$$

Thus the completely continuity of the operator L^{-1} is proved. ■

Corollary 2. *The operator $A = M^{-1}L$ is invertible and its inverse $A^{-1} = L^{-1}M$ is a completely continuous operator to be a product of completely continuous operator with bounded operator. Therefore the spectrum of the operator A is discrete.*

8 Conclusions

In this paper we have explored a new class of discrete non-Hermitian quantum systems. The concept of the spectrum for the considered discrete system is introduced and discreteness of the spectrum is proved under some simple conditions.

As a tool for the investigation we have established main statements for second order linear difference equations with impulse conditions (transition conditions). We have chosen a suitable (infinite-dimensional) Hilbert space and defined the main linear operator A so that the spectrum of the problem in question coincides with the spectrum of A. Next, we have constructed the inverse A^{-1} of the operator A by using an appropriate discrete Green function. Finally, we have shown that the inverse operator A^{-1} is completely continuous. This implies, in particular, discreteness of the spectrum of A.

Acknowledgements

This work was supported by Grant 106T549 from the Scientific and Technological Research Council of Turkey (TUBITAK). The author thanks Mesude Saglam and Gusein Guseinov for useful discussions.

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