Symmetry, Integrability and Geometry: Methods and Applications SIGMA 4 (2008), 087, 24 pages

The $N = 1$ Triplet Vertex Operator Superalgebras: Twisted Sector*

Dražen ADAMOVIĆ [†] *and Antun MILAS* [‡]

[†] *Department of Mathematics, University of Zagreb, Croatia*
E-mail: *adamovic@math.hr*
URL: http://web.math.hr/~adamovic

[‡] *Department of Mathematics and Statistics, University at Albany (SUNY), Albany, NY 12222, USA*
E-mail: *amilas@math.albany.edu*
URL: http://www.albany.edu/~am815139

Received August 31, 2008, in final form December 05, 2008; Published online December 13, 2008
Original article is available at http://www.emis.de/journals/SIGMA/2008/087/

Abstract. We classify irreducible σ-twisted modules for the $N = 1$ super triplet vertex operator superalgebra $\mathcal{SW}(m)$ introduced recently [Adamović D., Milas A., *Comm. Math. Phys.*, to appear, arXiv:0712.0379]. Irreducible graded dimensions of σ-twisted modules are also determined. These results, combined with our previous work in the untwisted case, show that the $SL(2,\mathbb{Z})$-closure of the space spanned by irreducible characters, irreducible supercharacters and σ-twisted irreducible characters is $(9m + 3)$-dimensional. We present strong evidence that this is also the (full) space of generalized characters for $\mathcal{SW}(m)$. We are also able to relate irreducible $\mathcal{SW}(m)$ characters to characters for the triplet vertex algebra $\mathcal{W}(2m+1)$, studied in [Adamović D., Milas A., *Adv. Math.* **217** (2008), 2664–2699, arXiv:0707.1857].

Key words: vertex operator superalgebras; Ramond twisted representations

2000 Mathematics Subject Classification: 17B69; 17B67; 17B68; 81R10

1 Introduction

Many constructions and results in vertex algebra theory are easily extendable to the setup of vertex superalgebras by simply adding adjective "super". Still, there are results that deviate from this "super-principle" and new ideas are needed compared to the non-super case. For example, modular invariance for vertex operator superalgebras requires inclusion of supercharacters of (untwisted) modules, and more importantly, the characters of σ-twisted modules [13], where σ is the canonical parity automorphism. Since the construction and classification of σ-twisted modules is more or less independent of the untwisted construction, many aspects of the theory need to be reworked for the twisted modules (e.g., twisted Zhu's algebra [29]). In fact, even for the free fermion vertex operator superalgebra, construction of σ-twisted modules is far from being trivial (see [13] for details).

Present work is a natural continuation of our very recent paper [5], where we introduced a new family of C_2-cofinite $N = 1$ vertex operator superalgebra that we call the *supertriplet* family $\mathcal{SW}(m)$, $m \geq 1$. In this installment we focus on σ-twisted $\mathcal{SW}(m)$-modules and their irreducible characters. The σ-twisted $\mathcal{SW}(m)$-modules are usually called modules in the *Ramond*

*This paper is a contribution to the Special Issue on Kac–Moody Algebras and Applications. The full collection is available at http://www.emis.de/journals/SIGMA/Kac-Moody_algebras.html

sector, while untwisted (ordinary) modules are referred to as modules in the *Neveu–Schwarz sector*, in parallel with two distinct $N = 1$ superconformal algebras.

Of course, Ramond sector has been studied for various (mostly rational) vertex operators superalgebras (e.g., [24, 10, 27], etc.). We should also point out that Ramond twisted representation of certain vertex $\mathcal{W}$-algebras were recently investigated in [8] and [23].

Here are our main results:

Theorem 1.1. *The twisted Zhu's algebra $A_\sigma(\mathcal{SW}(m))$ is finite-dimensional with precisely $2m+1$, non-isomorphic, $\mathbb{Z}_2$-graded irreducible modules. Consequently, $\mathcal{SW}(m)$ has $2m+1$, non-isomorphic, $\mathbb{Z}_2$-graded irreducible σ-twisted modules.*

By using embedding structure of $N = 1$ Feigin–Fuchs modules [20] we can also easily compute the characters of σ-twisted $\mathcal{SW}(m)$-modules. Equipped with these formulas, results from [5] about untwisted modules and supercharacters, and transformation formulas for classical Jacobi theta functions we are able to prove:

Theorem 1.2. *The $SL(2,\mathbb{Z})$-closure of the space of characters, (untwisted) supercharacters and σ-twisted $\mathcal{SW}(m)$ characters is $(9m+3)$-dimensional.*

Conjecturally, this is also the space of certain *generalized characters* studied in [28] (strictly speaking pseudotraces are yet-to-be defined in the setup of σ-twisted modules). A closely related conjecture is

Conjecture 1.1. *Let $Z(A)$ denote the center of associative algebra A and $T(A) = A/[A,A]$ the trace group of A. Then*

$$\dim(Z(A(\mathcal{SW}(m)))) + \dim(Z(A_\sigma(\mathcal{SW}(m)))) = 9m + 3,$$

and

$$\dim(T(A(\mathcal{SW}(m)))) + \dim(T(A_\sigma(\mathcal{SW}(m)))) = 9m + 3,$$

where $A(\mathcal{SW}(m))$ is the untwisted Zhu's associative algebra of $\mathcal{SW}(m)$ (cf. [5]).

The conjecture would follow if we knew more about structure of logarithmic $\mathcal{SW}(m)$-modules.

Here is a short outline of the paper. In Section 2 we recall the standard results about the σ-twisted Zhu's algebra, the free fermion vertex superalgebra F, and the construction of the σ-twisted F-module(s). In Section 4 we focus on σ-twisted modules for the super singlet vertex algebra introduced also in [5]. Sections 5 and 6 deal with the construction and classification of σ-twisted $\mathcal{SW}(m)$-modules. Finally, in Section 7 we derive characters formulas for irreducible σ-twisted $\mathcal{SW}(m)$-modules, discuss modular invariance, and relate our characters with the characters for the triplet vertex algebra $\mathcal{W}(2m+1)$ (see for instance [4] and [15]).

Needless to say, this paper is largely continuation of [4] and especially [5]. Thus the reader is strongly encouraged to consult [5], where we studied $\mathcal{SW}(m)$ in great details.

2 Preliminaries

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vertex operator superalgebra, where as usual $V_{\bar{0}}$ is the even and $V_{\bar{1}}$ is the odd subspace (cf. [13, 17, 21]). Every vertex operator superalgebra has the canonical parity automorphism σ, where $\sigma_{V_{\bar{0}}} = 1$ and $\sigma_{V_{\bar{1}}} = -1$. This leads to the notion of σ-twisted V-modules, well recorded in the literature (see for example [14] and [29]). As in the untwisted case, a large part of representation theory of σ-twisted V-modules can be analyzed via the σ-twisted Zhu's algebra whose construction we recall here.

Within the same setup, consider the subspace $O(V) \subset V$, spanned by elements of the form

$$\mathrm{Res}_x \frac{(1+x)^{\deg(u)}}{x^2} Y(u,x)v,$$

where $u \in V$ is homogeneous. It can be easily shown that

$$\mathrm{Res}_x \frac{(1+x)^{\deg(u)}}{x^n} Y(u,x)v \in O(V) \qquad \text{for} \quad n \geq 2.$$

Then, the vector space $A_\sigma(V) = V/O(V)$ is equipped with an associative algebra structure via

$$u * v = \mathrm{Res}_x \frac{(1+x)^{\deg(\mathrm{u})}}{x} Y(u,x)v$$

(see [29]). An important difference between the untwisted associative algebra $A(V)$ and $A_\sigma(V)$ is that the latter is $\mathbb{Z}_2$-graded, so

$$A_\sigma(V) = A_\sigma^0(V) \oplus A_\sigma^1(V).$$

It is also not clear that $A_\sigma(V) \neq 0$, while $A(V)$ is always nontrivial. We shall often use $[a] \in A_\sigma(V)$ for the image of $a \in V$ under the natural map $V \longrightarrow A_\sigma(V)$.

The result we need is [29]:

Theorem 2.1. *There is a one-to-one correspondence between irreducible $\mathbb{Z}_{\geq 0}$-gradable σ-twisted V-modules and irreducible $A_\sigma(V)$-modules.*

In the theorem there is no reference to graded $A_\sigma(V)$-modules. For practical purposes we shall need a slightly different version of the above theorem, because some modules are more natural if considered as $\mathbb{Z}_2$-graded modules (shorthand, *graded* modules).

Theorem 2.2. *There is a one-to-one correspondence between graded irreducible $\mathbb{Z}_{\geq 0}$-gradable σ-twisted V-module and graded irreducible $A_\sigma(V)$-module.*

Proof. The proof mimics the non-graded case, so we shall omit details. As in the non-graded case, by applying Zhu's theory, from an irreducible graded $A_\sigma(V)$-module U we construct a σ-twisted graded V-module $L(U)$ (but of course in the process of getting $L(U)$ will be moding out by the maximal *graded* submodule). On the other hand, if M is an irreducible graded V-module, then the top component $\Omega(M)$ is clearly a graded $A_\sigma(V)$-module. Suppose that $\Omega(M)$ is not graded irreducible, therefore there is a graded submodule $\Omega(M')$. Then we let $M' = \mathcal{U}(V[\sigma])\Omega'(M)$, where $\mathcal{U}(V[\sigma])$ is the enveloping algebra of the Lie algebra $V[\sigma]$. But M' is a proper graded submodule of M, so we get a contradiction. ■

Our goals are to describe the structure of $A_\sigma(\overline{SM(1)})$, where $\overline{SM(1)}$ is the super singlet vertex algebra [5], and to discuss $A_\sigma(\mathcal{SW}(m))$ (in fact, we have a very precise conjecture about the structure of $A_\sigma(\mathcal{SW}(m))$). Let us recall (see [5] for details) that both $\overline{SM(1)}$ and $\mathcal{SW}(m)$ are $N = 1$ superconformal vertex operator superalgebras, with the superconformal vector τ. In other words, if we let $Y(\tau, x) = \sum_{n \in \mathbb{Z}+1/2} G(n)x^{-n-3/2}$, then $G(n)$ and $L(m)$ close the $N = 1$ Neveu–Schwarz superalgebra.

It is not hard to see that the following hold:

$$\begin{aligned}
&(L(-m-2) + 2L(-m-1) + L(-m))v \in O(V), \qquad m \geq 2,\\
&(L(0) + L(-1))v \in O(V),\\
&L(-m)v \equiv (-1)^m((m-1)(L(-2) + L(-1)) + L(0))v \mod O(V), \qquad m \geq 2,\\
&\sum_{n \geq 0} \binom{\frac{3}{2}}{n} G(-3/2 - i + n)v \in O(V), \qquad i \geq 1,
\end{aligned} \tag{2.1}$$

where in all formulas v is a vector in $N = 1$ vertex operator superalgebra V.

To illustrate how to use twisted Zhu's algebra, let us classify $\mathbb{Z}$-graded σ-twisted modules for the (neutral) free fermion vertex operator superalgebra F, used in [5]. The next result is known so we will not provide its proof here.

Proposition 2.1. *We have $A_\sigma(F) \cong \mathbb{C}[x]/\big(x^2 - \frac{1}{2}\big)$ where $x = [\phi(-1/2)\mathbf{1}]$, an odd generator in the associative algebra.*

This result in particular implies there are precisely two irreducible σ-twisted F-modules: $M^\pm$. These two modules can be constructed explicitly. As vector spaces

$$M^\pm = \oplus_{n\geq 0}\Lambda^n(\phi(-1), \phi(-2), \dots),$$

where Λ^* is the exterior algebra, which is also a $\mathbb{Z}_{\geq 0}$-graded module for the Clifford algebra K spanned by $\phi(n)$, $n \in \mathbb{Z}$ with anti-bracket relations $\{\phi(m), \phi(n)\} = \delta_{m+n,0}$. The only difference between M^+ and M^- is in the action of $\phi(0)$ on the one-dimensional top subspace $M^\pm(0)$. More precisely we have $\phi(0)|_{M^\pm(0)} = \pm\frac{1}{\sqrt{2}}$. However, notice that $M^\pm$ are not $\mathbb{Z}_2$-graded thus it is more natural to examine graded $A_\sigma(F)$-modules. There is a unique such module (up to parity switch), spanned by $\mathbf{1}^{\mathfrak{R}}$ and $\phi(0)\mathbf{1}^{\mathfrak{R}}$. Thus

$$M = \oplus_{n\geq 0}\Lambda^n(\phi(0), \phi(-1), \phi(-2), \dots) = M^+ \oplus M^-.$$

Then $\mathbf{1}^{\mathfrak{R}}$ is a cyclic vector in M, i.e., $M = K.\mathbf{1}^{\mathfrak{R}}$. Moreover, $M^\pm = K.\mathbf{1}^\pm$, where

$$\mathbf{1}^\pm = \mathbf{1}^{\mathfrak{R}} \pm \sqrt{2}\phi(0)\mathbf{1}^{\mathfrak{R}}.$$

Next we describe the twisted vertex operators

$$\overline{Y}: \quad F \otimes M \longrightarrow M[[x^{1/2}, x^{-1/2}]].$$

Details are spelled out in [13], here we only give the explicit formula. Define first

$$Y\big(\phi(-n - \tfrac{1}{2})\mathbf{1}, x\big) = \frac{1}{n!}\left(\frac{d}{dx}\right)^n \left(\sum_{m\in\mathbb{Z}} \phi(m)x^{-m-1/2}\right),$$

acting on M and use (fermionic!) normal ordering $:\ :$ to define

$$Y\big(\phi(-n_1 - \tfrac{1}{2}) \cdots \phi(-n_k - \tfrac{1}{2})\mathbf{1}, x\big) = {:}Y\big(\phi(-n_1 - \tfrac{1}{2}), x\big) \cdots Y\big(\phi(-n_k - \tfrac{1}{2}), x\big){:},$$

and extend Y by linearity on all of F (see [13] and [14] for details, especially about normal ordering). Then, we let

$$\overline{Y}(v, x) := Y(e^{\Delta_x}v, x),$$

where

$$\Delta_x = \frac{1}{2}\sum_{m,n\geq 0} C_{m,n}\phi\big(m + \tfrac{1}{2}\big)\phi\big(n + \tfrac{1}{2}\big)x^{-m-n-1},$$

$$C_{m,n} = \frac{1}{2}\frac{m-n}{m+n+1}\binom{-1/2}{m}\binom{-1/2}{n}.$$

Let us fix the Virasoro generator

$$\omega_s = \tfrac{1}{2}\phi\big(-\tfrac{3}{2}\big)\phi\big(-\tfrac{1}{2}\big)\mathbf{1}.$$

Then $e^{\Delta_x}\omega_s = \omega_s + \frac{1}{16}x^{-2}\mathbf{1}$.

The following lemma will be important in the rest of the paper.

Lemma 2.1. *Let*

$$G(x_1,x_2)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}C_{m,n}x_1^m x_2^n,$$

where $C_{m,n}$ are as above. Then

$$G(x_1,x_2)=\frac{1}{2}\left(\frac{(1+x_1)^{1/2}(1+x_2)^{-1/2}-1}{x_2-x_1}+\frac{(1+x_1)^{-1/2}(1+x_2)^{1/2}-1}{x_2-x_1}\right)\in\mathbb{Q}[[x_1,x_2]].$$

Proof. The lemma follows directly from the identity

$$(x_2-x_1)G(x_1,x_2)=\tfrac{1}{2}(1+x_1)^{1/2}(1+x_2)^{-1/2}+\tfrac{1}{2}(1+x_1)^{-1/2}(1+x_2)^{1/2}-1.$$

which can be easily checked by expanding both sides as power series in x_1 and x_2. ∎

3 Highest weight representations of the Ramond algebra

The $N=1$ Ramond algebra $\mathfrak{R}$ is the infinite-dimensional Lie superalgebra

$$\mathfrak{R}=\bigoplus_{n\in\mathbb{Z}}\mathbb{C}L(n)\bigoplus\bigoplus_{m\in\mathbb{Z}}\mathbb{C}G(m)\bigoplus\mathbb{C}C$$

with commutation relations $(m,n\in\mathbb{Z})$:

$$[L(m),L(n)]=(m-n)L(m+n)+\delta_{m+n,0}\frac{m^3-m}{12}C,$$
$$[G(m),L(n)]=\left(m-\frac{n}{2}\right)G(m+n),$$
$$\{G(m),G(n)\}=2L(m+n)+\frac{1}{3}(m^2-1/4)\delta_{m+n,0}C,$$
$$[L(m),C]=0,\qquad [G(m),C]=0.$$

The representation theory of the $N=1$ Ramond algebra has been intensively studied first in [22] and other papers (cf. [25, 12], etc.).

Assume that $(c,h)\in\mathbb{C}^2$ such that $24h\neq c$. Let $L^{\mathfrak{R}}(c,h)^{\pm}$ denote the irreducible highest weight $\mathfrak{R}$-module generated by the highest weight vector $v_{c,h}^{\pm}$ such that

$$G(n)v_{c,h}^{\pm}=\pm\sqrt{h-c/24}\,\delta_{n,0}v_{c,h}^{\pm},\qquad L(n)v_{c,h}^{\pm}=h\delta_{n,0}v_{c,h}^{\pm}\qquad(n\geq 0).$$

These modules can be considered as irreducible σ-twisted modules for the Neveu–Schwarz vertex operator superalgebra (cf. [24]). Since $L^{\mathfrak{R}}(c,h)^{\pm}$ are not $\mathbb{Z}_2$-graded (notice that $v_{c,h}^{\pm}$ are eigenvectors for $G(0)$), it is more useful to consider graded modules. It is not hard to show that the direct sum

$$L^{\mathfrak{R}}(c,h)=L^{\mathfrak{R}}(c,h)^{+}\oplus L^{\mathfrak{R}}(c,h)^{-}$$

is in fact a $\mathbb{Z}_2$-graded $\mathfrak{R}$-module. Indeed, for $\mathbb{Z}_2$-graded subspaces take

$$L^{\mathfrak{R}}(c,h)_0=U(\mathfrak{R})\left(v_{c,h}^{+}+\frac{1}{\sqrt{h-c/24}}v_{c,h}^{-}\right)$$

and

$$L^{\mathfrak{R}}(c,h)_1=U(\mathfrak{R})\left(v_{c,h}^{+}-\frac{1}{\sqrt{h-c/24}}v_{c,h}^{-}\right)$$

Since $L^{\mathfrak{R}}(c,h)$ does not contain non-trivial $\mathbb{Z}_2$-graded submodules, we shall say that this module is $\mathbb{Z}_2$-graded irreducible.

Remark 3.1. Details about construction of $L^{\mathfrak{R}}(c,h)$ and its relation to irreducible non-graded modules can be found in [12] and [19]. In Section 5 we shall present free fields realization of modules $L^{\mathfrak{R}}(c,h)$ and $L^{\mathfrak{R}}(c,h)^{\pm}$.

For $i,n,m \in \mathbb{Z}_{\geq 0}$, let

$$c_{2m+1,1} = \frac{3}{2} - \frac{12m^2}{2m+1}, \qquad h^{1,3} = 2m + \frac{1}{2},$$
$$h^{2i+2,2n+1} := \frac{(2i+2-(2n+1)(2m+1))^2 - 4m^2}{8(2m+1)} + \frac{1}{16}.$$

Let $L(c_{2m+1,1},h)$ be the irreducible highest weight module for the Neveu–Schwarz algebra with central charge c and highest weight h. Then $L(c_{2m+1,1},0)$ is a simple vertex operator superalgebra, $L^{\mathfrak{R}}(c,h)^{\pm}$ are irreducible σ-twisted $L(c_{2m+1,1},0)$-modules, and $L^{\mathfrak{R}}(c,h)$ is $\mathbb{Z}_2$-graded irreducible σ-twisted $L(c_{2m+1,1},0)$-module.

3.1 Intertwining operators among twisted modules

If V is a vertex operator superalgebra and W_1, W_2 and W_3 are three V-modules then we consider the space of intertwining operators $\binom{W_3}{W_1\ W_2}$. It is perhaps less standard to study intertwining operators between twisted V-modules, so we recall the definition here (see [29, p. 120]).

Definition 3.1. Let W_1, W_2 and W_3 be σ_i-twisted V-modules, respectively, where σ_i is a finite order automorphism of order ν_i, with common period T. An intertwining operator of type $\binom{W_3}{W_1\ W_2}$ is a linear map

$$\mathcal{Y}(\cdot,x): \quad W_1 \longrightarrow \mathrm{End}(W_2,W_3)\{x\},$$

such that

$$\mathcal{Y}(L(-1)w,x) = \frac{d}{dx}\mathcal{Y}(w,x), \qquad w \in W_1$$

and Jacobi identity holds

$$\frac{1}{Tx_0}\sum_{p=0}^{T-1}\delta\left(\left(\frac{x_1-x_2}{x_0}\right)^{-1/T}\omega_T^p\right)Y^{\nu_3}(\sigma_1^p v,x_1)\mathcal{Y}(w_1,x_2)$$
$$-(-1)^{ij}\frac{1}{Tx_0}\sum_{p=0}^{T-1}\delta\left(\left(\frac{-x_2+x_1}{x_0}\right)^{-1/T}\omega_T^p\right)\mathcal{Y}(w_1,x_2)Y^{\nu_2}(\sigma_1^p v,x_1)$$
$$= \frac{1}{Tx_2}\sum_{p=0}^{T-1}\delta\left(\left(\frac{x_1-x_0}{x_2}\right)^{1/T}\omega_T^p\right)\mathcal{Y}(Y^{\nu_1}(\sigma_2^p v,x_0)w_1,x_2),$$

where $v \in V_i$, $w_1 \in (W_1)_j$, $i,j \in \mathbb{Z}_2$ and ω_T is a primitive T-th root of unity.

We shall also need the following result on the fusion rules. The proof is completely analogous to that of Proposition 4.1 of [5] (see also [18]).

Proposition 3.1. *For every $i = 0,\dots,m-1$ and $n \geq 1$ we have: the space*

$$I\binom{L^{\mathfrak{R}}(c_{2m+1,1},h)^{\pm}}{L(c_{2m+1,1},h^{1,3})\ \ L^{\mathfrak{R}}(c_{2m+1,1},h^{2i+2,2n+1})^{\pm}}$$

is nontrivial only if $h \in \{h^{2i+2,2n-1}, h^{2i+2,2n+1}, h^{2i+2,2n+3}\}$, *and*

$$I\begin{pmatrix} & L^{\mathfrak{R}}(c_{2m+1,1}, h)^{\pm} & \\ L(c_{2m+1,1}, h^{1,3}) & & L^{\mathfrak{R}}(c_{2m+1,1}, h^{2i+2,1})^{\pm} \end{pmatrix}$$

is nontrivial only if $h = h^{2i+2,3}$.

Similarly, for every $i = 0, \ldots, m-1$ *and* $n \geq 2$ *we have: the space*

$$I\begin{pmatrix} & L^{\mathfrak{R}}(c_{2m+1,1}, h)^{\pm} & \\ L(c_{2m+1,1}, h^{1,3}) & & L^{\mathfrak{R}}(c_{2m+1,1}, h^{2i+2,-2n+1})^{\pm} \end{pmatrix}$$

is nontrivial only if $h \in \{h^{2i+2,-2n-1}, h^{2i+2,-2n+1}, h^{2i+2,-2n+3}\}$, *and*

$$I\begin{pmatrix} & L^{\mathfrak{R}}(c_{2m+1,1}, h)^{\pm} & \\ L(c_{2m+1,1}, h^{1,3}) & & L^{\mathfrak{R}}(c_{2m+1,1}, h^{2i+2,-1})^{\pm} \end{pmatrix}$$

is nontrivial only if $h \in \{h^{2i+2,-3}, h^{2i+2,-1}\}$.

We have analogous result for fusion rules of type

$$\begin{pmatrix} & L^{\mathfrak{R}}(c_{2m+1,1}, h) & \\ L(c_{2m+1,1}, h^{1,3}) & & L^{\mathfrak{R}}(c_{2m+1,1}, h^{2i+2,2n+1}) \end{pmatrix}.$$

Proof. The proof goes along the lines in [5] (cf. [26]), with some minor modifications due to twisting. In fact, in order to avoid the twisted version of Frenkel–Zhu's formula we can proceed in a more straightforward fashion. Because all modules in question are irreducible and because all intertwining operators we are interested in are of the form $\binom{W_3}{W_1\ W_2}$, where W_1 is untwisted (so $\nu_1 = 0$), the left-hand side in the Jacobi identity looks like the ordinary Jacobi identity for intertwining operators. Thus we can use commutator formula and the null vector conditions

$$\begin{aligned} &\big(-L(-1)G\big(-\tfrac{1}{2}\big) + (2m+1)G\big(-\tfrac{3}{2}\big)\big)v_{1,3} = 0, \\ &G\big(-\tfrac{1}{2}\big)\big(-L(-1)G\big(-\tfrac{1}{2}\big) + (2m+1)G\big(-\tfrac{3}{2}\big)\big)v_{1,3} = 0, \end{aligned}$$

which hold in $L(c_{2m+1,1}, h^{1,3})$ (here $v_{1,3}$ is the highest weight vector), to study the matrix coefficient

$$f(x) = \langle w', \mathcal{Y}(v_{1,3}, x)w\rangle,$$

where w and w' are highest weight vectors in appropriate modules. This leads to differential equations for $f(x)$, which can be solved. The general solution is a linear combination of power functions x^s, where s is a rational number. The rest follows by interpreting s in terms of conformal weights for the three modules involved in $\mathcal{Y}$. ■

4 $\boldsymbol{\sigma}$-twisted modules for the super singlet algebra $\overline{SM(1)}$

In this section σ_2 will denote the parity automorphism of F. Recall also the Heisenberg vertex operator algebra $M(1)$. Then, as in [5] we equip the space $M(1) \otimes F$ with a vertex superalgebra structure such that the total central charge is $\frac{3}{2}(1 - \frac{8m^2}{2m+1})$. We define the following superconformal and conformal vectors:

$$\begin{aligned} \tau &= \frac{1}{\sqrt{2m+1}}\left(\alpha(-1)\mathbf{1} \otimes \phi\big(-\tfrac{1}{2}\big)\mathbf{1} + 2m\mathbf{1} \otimes \phi\big(-\tfrac{3}{2}\big)\mathbf{1}\right), \\ \omega &= \frac{1}{2(2m+1)}(\alpha(-1)^2 + 2m\alpha(-2))\mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \omega^{(s)}. \end{aligned}$$

Recall also the singlet superalgebra $\overline{SM(1)}$ obtained as the kernel of the screening operator

$$\widetilde{Q} = \mathrm{Res}_x Y\big(e^{\frac{-\alpha}{2m+1}} \otimes \phi(-\tfrac{1}{2}), x\big)$$

acting from $M(1) \otimes F$ to $M(1) \otimes e^{-\frac{\alpha}{2m+1}} \otimes F$, where $\langle \alpha, \alpha \rangle = 2m+1$. The vertex operator superalgebra $\overline{SM(1)}$ is generated by $\tau = G(-\frac{3}{2})\mathbf{1}$ and $H = Qe^{-\alpha}$, where

$$Q = \mathrm{Res}_x Y\big(e^{\alpha} \otimes \phi(-\tfrac{1}{2}), x\big).$$

We will also use ω and $\widehat{H}$, where the latter is proportional to $G(-\frac{1}{2})H$ (for details see [5]).

Consider the automoprhism $\sigma = 1 \otimes \sigma_2$ of $M(1) \otimes F$, acting nontrivially on the second tensor factor. This automorphism plainly preserves $\overline{SM(1)}$, thus we can study σ-twisted $\overline{SM(1)}$-modules. It is clear that $M(1,\lambda) \otimes M$ is a graded σ-twisted $\overline{SM(1)}$-module, and $M(1,\lambda) \otimes M^{\pm}$ are σ-twisted $\overline{SM(1)}$-modules, where $M(1,\lambda)$ is as in [4] and $\lambda \in \mathbb{C}$.

We would like to classify irreducible $\overline{SM(1)}$-modules by virtue of Zhu's algebra. As usual, we denote by $[a] \in A_{\sigma}(\overline{SM(1)})$ the image of $a \in \overline{SM(1)}$.

Lemma 4.1. *Let $v_{\lambda}^{\pm} = v_{\lambda} \otimes \mathbf{1}^{\pm}$ be the highest weight vector in $M(1,\lambda) \otimes M^{\pm}$, where $\langle \alpha, \lambda \rangle = t$. Then the twisted generators act as*

$$\begin{aligned}
G(0) \cdot v_{\lambda}^{\pm} &= \pm \frac{t-m}{\sqrt{2(2m+1)}} v_{\lambda}^{\pm}, \\
L(0) \cdot v_{\lambda}^{\pm} &= \left(\frac{t(t-2m)}{2(2m+1)} + \frac{1}{16} \right) v_{\lambda}^{\pm}, \\
H(0) \cdot v_{\lambda}^{\pm} &= \pm \frac{1}{\sqrt{2}} \binom{t-1/2}{2m} v_{\lambda}^{\pm}, \\
\widehat{H}(0) \cdot v_{\lambda}^{\pm} &= \frac{m-t}{2m+1} \binom{t-1/2}{2m} v_{\lambda}^{\pm}.
\end{aligned}$$

Proof. Recall from [5] the formulas

$$\begin{aligned}
H &= S_{2m}(\alpha) \otimes \phi(-\tfrac{1}{2}) + S_{2m-1} \otimes \phi(-\tfrac{3}{2}) + \cdots + 1 \otimes \phi(-2m - \tfrac{1}{2}), \\
\widehat{H} &= \phi(1/2) S_{2m+1}(\alpha) \phi(-\tfrac{1}{2}) + w \\
&= S_{2m+1}(\alpha) + S_{2m-1}(\alpha) \otimes \phi(-\tfrac{3}{2})\phi(-\tfrac{1}{2}) + \cdots + \mathbf{1} \otimes \phi(-2m - \tfrac{1}{2})\phi(-\tfrac{1}{2}).
\end{aligned}$$

As in [5] we have

$$S_r(\alpha)(0) \cdot v_{\lambda}^{\pm} = \binom{t}{r} v_{\lambda}^{\pm}.$$

Then we get

$$H(0) \cdot v_{\lambda}^{\pm} = \pm \frac{1}{\sqrt{2}} \sum_{n=0}^{2m} \binom{t}{n} \binom{-1/2}{2m-n} v_{\lambda}^{\pm} = \pm \frac{1}{\sqrt{2}} \binom{t-1/2}{2m} v_{\lambda}^{\pm}.$$

The last formula is proven similarly by using e^{Δ_x} operator. ■

Here are the main result of this section.

Theorem 4.1. *The Zhu's algebra $A_\sigma(\overline{SM(1)})$ is an associative algebra generated by $[\tau]$, $[\omega]$, $[H]$ and $[\widehat{H}]$, where the following relations hold (here $[\omega]$ is central):*

$$
\begin{aligned}
&[\tau]^2 = [\omega] - \frac{c_{2m+1,1}}{24},\\
&[\tau] * [H] = [H] * [\tau] = \frac{-\sqrt{2m+1}}{2}[\widehat{H}],\\
&[H] * [H] = \frac{1}{2}\left(\frac{\sqrt{2(2m+1)}[\tau] + m - 1/2}{2m}\right)^2,\\
&[H] * [H] = C_m \prod_{i=0}^{m-1} \left([\omega] - h^{2i+2,1}\right)^2, \qquad C_m = \frac{2^{2m-1}(2m+1)^{2m}}{(2m)!^2},\\
&[\widehat{H}] * [\widehat{H}] = \frac{4}{2m+1} C_m \left([\omega] - \frac{c_{2m+1,1}}{24}\right) \prod_{i=0}^{m-1} \left([\omega] - h^{2i+2,1}\right)^2.
\end{aligned}
\tag{4.1}
$$

In particular, Zhu's algebra is commutative.

Proof. First notice (cf. [5]) that

$$G\left(-\tfrac{1}{2}\right)F = -\sqrt{2m+1}\phi\left(-\tfrac{1}{2}\right)F = -\sqrt{2m+1}\widehat{F}$$

and consequently

$$G\left(-\tfrac{1}{2}\right)H = -\sqrt{2m+1}\widehat{H}.$$

We also have the relation

$$\left((2m+1)G\left(-\tfrac{3}{2}\right) - L(-1)G\left(-\tfrac{1}{2}\right)\right)H = 0,$$

because $H = v_{1,3}$ is a highest weight vector. By using (2.1) we obtain

$$
\begin{aligned}
[\tau] * [H] &= \left[G\left(-\tfrac{3}{2}\right)H\right] + \tfrac{3}{2}\left[G\left(-\tfrac{1}{2}\right)H\right] = \left[\frac{1}{2m+1}L(-1)G\left(-\tfrac{1}{2}\right)H\right] + \tfrac{3}{2}\left[G\left(-\tfrac{1}{2}\right)H\right]\\
&= \tfrac{1}{2}\left[G\left(-\tfrac{1}{2}\right)H\right] = -\frac{\sqrt{2m+1}}{2}[\widehat{H}].
\end{aligned}
$$

On the other hand, by using skew-symmetry we also have

$$
\begin{aligned}
[H] * [\tau] &= -\left[\mathrm{Res}_x x^{-1} e^{xL(-1)} Y(\tau, -x)(1+x)^{2m+\frac{1}{2}} H\right]\\
&= -\left[\mathrm{Res}_x x^{-1} e^{-xL(0)} Y(\tau, -x)(1+x)^{2m+\frac{1}{2}} H\right]\\
&= -\left(\left[G\left(-\tfrac{3}{2}\right)H\right] - \left[\left(2m + \tfrac{1}{2} - (2m+1)\right)G\left(-\tfrac{1}{2}\right)H\right]\right)\\
&= -\left[G\left(-\tfrac{3}{2}\right)H\right] - \tfrac{1}{2}\left[G\left(-\tfrac{1}{2}\right)H\right] = \tfrac{1}{2}\left[G\left(-\tfrac{1}{2}\right)H\right].
\end{aligned}
$$

Combined, we obtain

$$[\tau] * [H] = [H] * [\tau]. \qquad \blacksquare$$

Notice that relation (4.1) can be written as

$$[H] * [H] = C_m \prod_{i=0}^{m-1} \left([\tau]^2 - \frac{(2i+1-2m)^2}{8(2m+1)}\right)^2, \qquad C_m \neq 0.$$

Let $\mathbb{C}[a, b]$ denote the $\mathbb{Z}_2$-graded complex commutative associative algebra generated by odd vectors a, b.

Theorem 4.2. *The associative algebra $A_\sigma(\overline{SM(1)})$ is isomorphic to the $\mathbb{Z}_2$-graded commutative associative algebra*

$$\mathbb{C}[a,b]/\langle H(a,b)\rangle,$$

where $\langle H(a,b)\rangle$ is (two-sided) ideal in $\mathbb{C}[a,b]$, generated by

$$H(a,b) = b^2 - C_m \prod_{i=0}^{m-1} \left(a^2 - \frac{(2i+1-2m)^2}{8(2m+1)}\right)^2.$$

Proof. The proof is similar to that of Theorem 6.1 of [2]. First we notice that we have a surjective homomorphism

$$\Phi: \ \mathbb{C}[a,b] \to A_\sigma(\overline{SM(1)}), \qquad a \mapsto [\tau], \qquad b \mapsto [H].$$

It is easy to see that $\operatorname{Ker}\Phi$ is a $\mathbb{Z}_2$-graded ideal. We shall now prove that $\operatorname{Ker}\Phi = \langle H(a,b)\rangle$. Evidently the generating element $H(a,b)$ is even, so Theorem 4.1 gives $\langle H(a,b)\rangle \subset \operatorname{Ker}\Phi$.

Assume now that $K(a,b) \in \operatorname{Ker}\Phi$. By using division algorithm we get

$$K(a,b) = A(a,b)H(a,b) + R(a,b),$$

where $A(a,b), R(a,b) \in \mathbb{C}[a,b]$ and $R(a,b)$ has degree at most 1 in b. Assume that $R(a,b) \neq 0$. Then $R(a,b) = A(a)b + B(a)$ for certain polynomials $A, B \in \mathbb{C}[x]$. We also notice that $R(a,b) \in \operatorname{Ker}\Phi$. Since $\operatorname{Ker}\Phi$ is $\mathbb{Z}_2$-graded ideal, we can assume that $R(a,b)$ is homogeneous. If $R(a,b)$ is an even element we have that A has odd degree and B has even degree, and therefore

$$\deg(B) - \deg(A) \qquad \text{is an odd natural number.} \tag{4.2}$$

The case when $R(a,b)$ is odd element again leads to formula (4.2).

As in [2] we now shall evaluate $R(a,b)$ on $A_\sigma(\overline{SM(1)})$-modules and get

$$A\left(\frac{t-m}{\sqrt{2(2m+1)}}\right)\binom{t-1/2}{2m} + \sqrt{2}B\left(\frac{t-m}{\sqrt{2(2m+1)}}\right) = 0 \qquad \forall\, t \in \mathbb{C}.$$

This implies that $\deg(B) - \deg(A) = 2m$. Contradiction. Therefore $R(a,b) = 0$ and $K(a,b) \in \langle H(a,b)\rangle$. The proof follows. ∎

Remark 4.1. By using the same arguments as in [2] and [5], we can conclude that every irreducible $\mathbb{Z}_{\geq 0}$-gradable σ-twisted $\overline{SM(1)}$-module is isomorphic to an irreducible subquotient of $M(1,\lambda) \otimes M^{\pm}$. By using the structure of twisted Zhu's algebra $A_\sigma(\overline{SM(1)})$ and the methods developed in [3], we can also construct logarithmic σ-twisted $\overline{SM(1)}$-modules.

5 The $N = 1$ Ramond module structure of twisted $V_L \otimes F$-modules

In this section we shall assume that the reader is familiar with basic results on twisted representations of lattice vertex superalgebras. Details can be found in [9, 10, 16] and [29].

We shall use the same notation as in [5]. Let $L = \mathbb{Z}\alpha$ be a rank one lattice with nondegenerate form given by $\langle \alpha, \alpha\rangle = 2m+1$, where $m \in \mathbb{Z}_{>0}$. Let V_L be the corresponding vertex superalgebra.

For $i \in \mathbb{Z}$, we set

$$\gamma_i = \frac{i}{2m+1}\alpha, \qquad \gamma_i^R = \frac{\alpha}{2(2m+1)} + \frac{i}{2m+1}\alpha.$$

Then $\sigma_1 = \exp[\frac{\pi i}{2m+1}\alpha(0)]$ is a canonical automorphism of order two of V_L. The set $\{V_{\gamma_i^R + L},\ i = 0, \dots, 2m\}$ provides all the irreducible σ_1-twisted V_L-modules.

Remark 5.1. It is important to notice that σ_1-twisted V_L-module $(V_{\gamma_i^R+L}, Y_{\gamma_i^R+L})$ can be constructed from untwisted module $(V_{\gamma_i+L}, Y_{\gamma_i+L})$ as follows (cf. [29]):

$$V_{\gamma_i^R+L} := V_{\gamma_i+L} \qquad \text{as vector space;}$$
$$Y_{\gamma_i^R+L}(\cdot, z) := Y_{\gamma_i+L}\big(\Delta\big(\tfrac{\alpha}{2(2m+1)}, z\big)\cdot, z\big),$$

where

$$\Delta(h, z) := z^{h(0)} \exp\left(\sum_{n=1}^{\infty} \frac{h(n)}{-n}(-z)^{-n}\right).$$

Let F be the fermionic vertex operator superalgebra with central charge 1/2 and σ_2 its parity map. Let M be the σ_2 twisted F-module (cf. [14]).

Then $\sigma = \sigma_1 \otimes \sigma_2$ is the parity automorphism of order two of the vertex superalgebra $V_L \otimes F$ and

$$V_{\gamma_i^R+L} \otimes M, \qquad i = 0, \ldots, 2m$$

are σ-twisted $V_L \otimes F$-modules.

Let (W, Y_W) be any σ-twisted $V_L \otimes F$-module. From the Jacobi identity for σ-twisted modules it follows that the coefficients of

$$Y_W(\tau, z) = \sum_{n\in\mathbb{Z}} G(n) z^{-n-\frac{3}{2}} \qquad \text{and} \qquad Y_W(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$$

define a representation of the $N = 1$ Ramond algebra.

Recall that

$$h^{2i+2,2n+1} = \frac{((2i+2) - (2n+1)(2m+1))^2 - 4m^2}{8(2m+1)} + \frac{1}{16}.$$

Proposition 5.1. *Assume that $n \in \mathbb{Z}$. Then $e^{\gamma_i^R - n\alpha} \otimes \mathbf{1}^{\mathfrak{R}}$ is a singular vector for the $N = 1$ Ramond algebra $\mathfrak{R}$ and*

$$\begin{aligned} U(\mathfrak{R})\big(e^{\gamma_i^R - n\alpha} \otimes \mathbf{1}^{\mathfrak{R}}\big) &\cong L^{\mathfrak{R}}\big(c_{2m+1,1}, h^{2i+2,2n+1}\big) \\ &= L^{\mathfrak{R}}\big(c_{2m+1,1}, h^{2i+2,2n+1}\big)^+ \oplus L^{\mathfrak{R}}\big(c_{2m+1,1}, h^{2i+2,2n+1}\big)^-. \end{aligned}$$

Moreover,

$$U(\mathfrak{R})\big(e^{\gamma_i^R - n\alpha} \otimes \mathbf{1}^{\pm}\big) \cong L^{\mathfrak{R}}\big(c_{2m+1,1}, h^{2i+2,2n+1}\big)^{\pm}.$$

As in [5] (see also [20] and [25]) we have the following result.

Lemma 5.1. *The (screening) operator*

$$Q = \mathrm{Res}_x Y_W\big(e^{\alpha} \otimes \phi\big(-\tfrac{1}{2}\big), x\big)$$

commutes with the action of $\mathfrak{R}$.

Remark 5.2. By using generalized (lattice) vertex algebras and their twisted representations one can also define the second screening operator $\widetilde{Q}$ acting between certain σ-twisted $V_L \otimes F$-modules such that

$$[\widetilde{Q}, \mathfrak{R}] = 0, \qquad [Q, \widetilde{Q}] = 0$$

(for details and some applications see [7]).

We shall first present results on the structure of σ-twisted $V_L \otimes F$-modules, viewed as modules for the $N=1$ Ramond algebra. Each $V_{L+\gamma_i^R} \otimes M$ is a direct sum of super Feigin–Fuchs modules via

$$V_{L+\gamma_i^R} \otimes M = \bigoplus_{n\in\mathbb{Z}} \big(M(1) \otimes e^{\gamma_i^R + n\alpha}\big) \otimes M.$$

Since operators Q^j, $j \in \mathbb{Z}_{>0}$, commute with the action of the Ramond algebra, they are actually (Lie superalgebra) intertwiners between super Feigin–Fuchs modules inside $V_{L+\gamma_i^R} \otimes M$.

To simplify the notation, we shall identify e^{β} with $e^{\beta} \otimes \mathbf{1}^{\mathfrak{R}}$ for every $\beta \in L + \gamma_i^R$.

Assume that $0 \le i \le m-1$. If $Q^j e^{\gamma_i^R - n\alpha}$ is nontrivial, it is a singular vector of weight

$$\operatorname{wt}\big(Q^j e^{\gamma_i^R - n\alpha}\big) = \operatorname{wt}\big(e^{\gamma_i^R - n\alpha}\big) = h^{2i+2,2n+1}.$$

Since $\operatorname{wt}\big(e^{\gamma_i^R + (j-n)\alpha}\big) > \operatorname{wt}\big(e^{\gamma_i^R - n\alpha}\big)$ if $j > 2n$, we conclude that

$$Q^j e^{\gamma_i^R - n\alpha} = 0 \qquad \text{for} \quad j > 2n.$$

One can similarly see that for $m \le i \le 2m$:

$$Q^j e^{\gamma_i^R - n\alpha} = 0 \qquad \text{for} \quad j > 2n+1.$$

The following lemma is useful for constructing singular vectors in $V_{L+\gamma_i^R} \otimes M$:

Lemma 5.2.

(1) $Q^{2n} e^{\gamma_i^R - n\alpha} \neq 0$ *for* $0 \le i \le m$.

(2) $Q^{2n+1} e^{\gamma_i^R - n\alpha} \neq 0$ *for* $m+1 \le i \le 2m$.

Proof. The proof uses the results on fusion rules from Proposition 3.1 and is completely analogous to that of Lemma 6.1 in [5]. ■

As in the Virasoro algebra case the $N=1$ Feigin–Fuchs modules are classified according to their embedding structure. For the purposes of our paper we shall focus only on modules of certain types (Type 4 and 5 in [20]). These modules are either semisimple (Type 5) or they become semisimple after quotienting with the maximal semisimple submodule (Type 4).

The following result follows directly from Lemma 5.2 and the structure theory of super Feigin–Fuchs modules, after some minor adjustments of parameters (cf. Type 4 embedding structure in [20]).

Theorem 5.1. *Assume that* $i \in \{0, \dots, m-1\}$.

(*i*) *As an* $\mathfrak{R}$*-module,* $V_{L+\gamma_i^R} \otimes M$ *is generated by the family of singular and cosingular vectors* $\widetilde{\operatorname{Sing}}_i \bigcup \widetilde{\operatorname{CSing}}_i$*, where*

$$\widetilde{\operatorname{Sing}}_i = \big\{u_i^{(j,n)} \mid j,n \in \mathbb{Z}_{\ge 0},\ 0 \le j \le 2n\big\};$$
$$\widetilde{\operatorname{CSing}}_i = \big\{w_i^{(j,n)} \mid n \in \mathbb{Z}_{>0}, 0 \le j \le 2n-1\big\}.$$

These vectors satisfy the following relations:

$$u_i^{(j,n)} = Q^j e^{\gamma_i^R - n\alpha}, \qquad Q^j w_i^{(j,n)} = e^{\gamma_i^R + n\alpha}.$$

The submodule generated by singular vectors $\widetilde{\operatorname{Sing}}_i$*, denoted by* $R\Lambda(i+1)$*, is isomorphic to*

$$\bigoplus_{n=0}^{\infty} (2n+1) L^{\mathfrak{R}}\big(c_{2m+1,1}, h^{2i+2,2n+1}\big).$$

(ii) *Let $R\Lambda(i+1)^{\pm}$ denote the submodule of $V_{L+\gamma_i^R} \otimes M^{\pm}$ generated by singular vectors*

$$Q^j\big(e^{\gamma_i^R - n\alpha} \otimes \mathbf{1}^{\pm}\big), \qquad n \in \mathbb{Z}_{\geq 0}, \qquad 0 \leq j \leq 2n.$$

Then

$$R\Lambda(i+1)^{\pm} = \bigoplus_{n=0}^{\infty} (2n+1) L^{\mathfrak{R}}\big(c_{2m+1,1}, h^{2i+2,2n+1}\big)^{\pm},$$

and

$$R\Lambda(i+1) = R\Lambda(i+1)^{+} \oplus R\Lambda(i+1)^{-}.$$

(iii) *For the quotient module we have*

$$R\Pi(m-i) := (V_{L+\gamma_i^R} \otimes M)/R\Lambda(i+1) \cong \bigoplus_{n=1}^{\infty} (2n) L^{\mathfrak{R}}\big(c_{2m+1,1}, h^{2i+2,-2n+1}\big).$$

Moreover, we have

$$R\Pi(m-i) = R\Pi(m-i)^{+} \oplus R\Pi(m-i)^{-},$$

where

$$R\Pi(m-i)^{\pm} = (V_{L+\gamma_i^R} \otimes M)/R\Lambda(i+1)^{\pm} = \bigoplus_{n=1}^{\infty} (2n) L^{\mathfrak{R}}\big(c_{2m+1,1}, h^{2i+2,-2n+1}\big)^{\pm}.$$

Theorem 5.2. *Assume that $i \in \{0, \dots, m-1\}$.*

(i) *As an $\mathfrak{R}$-module, $V_{L+\gamma_{m+i}^R} \otimes M$ is generated by the family of singular and cosingular vectors $\widetilde{\mathrm{Sing}}_i' \bigcup \widetilde{\mathrm{CSing}}_i'$, where*

$$\widetilde{\mathrm{Sing}}_i' = \big\{ u_i'^{(j,n)} \mid n \in \mathbb{Z}_{>0}, 0 \leq j \leq 2n-1 \big\};$$
$$\widetilde{\mathrm{CSing}}_i' = \big\{ w_i'^{(j,n)} \mid j, n \in \mathbb{Z}_{\geq 0},\ 0 \leq j \leq 2n \big\}.$$

These vectors satisfy the following relations:

$$u_i'^{(j,n)} = Q^j e^{\gamma_{m+i}^R - n\alpha}, \qquad Q^j w_i'^{(j,n)} = e^{\gamma_{m+i}^R + n\alpha}.$$

The submodule generated by singular vectors $\widetilde{\mathrm{Sing}}_i$ is isomorphic to

$$R\Pi(i+1) \cong \bigoplus_{n=1}^{\infty} (2n) L^{\mathfrak{R}}\big(c_{2m+1,1}, h^{2m-2i,-2n+1}\big).$$

(ii) *For the quotient module we have*

$$R\Lambda(m-i) \cong (V_{L+\gamma_i^R} \otimes M)/R\Pi(i+1) \cong \bigoplus_{n=0}^{\infty} (2n+1) L^{\mathfrak{R}}\big(c_{2m+1,1}, h^{2m-2i,2n+1}\big).$$

Theorem 5.3.

(*i*) *As an* $\mathfrak{R}$*-module,* $V_{L+\gamma^R_{2m}} \otimes M$ *is completely reducible and generated by the family of singular vectors*

$$\widetilde{\mathrm{Sing}}_{2m} = \left\{u^{(j,n)}_{2m} := Q^j e^{\gamma^R_{2m} - n\alpha} \mid n \in \mathbb{Z}_{>0},\ j \in \mathbb{Z}_{\geq 0},\ 0 \leq j \leq 2n-1\right\};$$

and it is isomorphic to

$$R\Pi(m+1) := V_{L+\gamma^R_{2m}} \otimes M \cong \bigoplus_{n=1}^{\infty} (2n) L^{\mathfrak{R}}\big(c_{2m+1,1}, h^{4m+2,-2n+1}\big).$$

(*ii*) *Let* $R\Pi(m+1)^{\pm}$ *be the submodule of* $V_{L+\gamma^R_{2m}} \otimes M^{\pm}$ *generated by the singular vectors*

$$Q^j\big(e^{\gamma^R_{2m} - n\alpha} \otimes \mathbf{1}^{\pm}\big), \qquad n \in \mathbb{Z}_{>0}, \qquad j \in \mathbb{Z}_{\geq 0}, \qquad 0 \leq j \leq 2n-1.$$

Then

$$R\Pi(m+1)^{\pm} = \bigoplus_{n=1}^{\infty} (2n) L^{\mathfrak{R}}\big(c_{2m+1,1}, h^{4m+2,-2n+1}\big)^{\pm},$$

and

$$R\Pi(m+1) = R\Pi(m+1)^{+} \oplus R\Pi(m+1)^{-}.$$

Remark 5.3. In this section we actually constructed explicitly all the non-trivial intertwining operators from Proposition 3.1.

6 The σ-twisted $\mathcal{SW}(m)$-modules

Since $\mathcal{SW}(m) \subset V_L \otimes F$ is σ-invariant, then every σ-twisted $V_L \otimes F$-module is also a σ-twisted module for the vertex operator superalgebra $\mathcal{SW}(m)$. In this section we shall consider σ-twisted $V_L \otimes F$-modules from Section 5 as σ-twisted $\mathcal{SW}(m)$-modules. In what follows we shall classify all the irreducible σ-twisted $\mathcal{SW}(m)$-modules by using Zhu's algebra $A_\sigma(\mathcal{SW}(m))$.

Following [4] and [5], we first notice the following important fact:

$$Q^2 e^{-2\alpha} \in O(\mathcal{SW}(m)). \tag{6.1}$$

Proposition 6.1. *Let* $v^{\pm}_{\lambda}$ *be the highest weight vector in* $M(1,\lambda) \otimes M^{\pm}$*. We have*

$$o\big(Q^2 e^{-2\alpha}\big) v^{\pm}_{\lambda} = F_m(t)\, v^{\pm}_{\lambda},$$

where $t = \langle \lambda, \alpha \rangle$*, and*

$$F_m(t) = A_m \binom{t + m + 1/2}{3m+1} \binom{t - 1/2}{3m+1}, \qquad \textit{where} \quad A_m = (-1)^m \frac{\binom{2m}{m}}{\binom{4m+1}{m}}.$$

Proof. First we notice that

$$Q^2 e^{-2\alpha} = w_1 + w_2,$$

where

$$w_1 = \sum_{i=0}^{\infty} e^{\alpha}_{-i-1} e^{\alpha}_{i} e^{-2\alpha}, \qquad w_2 = \sum_{i,j=0}^{\infty} e^{\alpha}_{i} e^{\alpha}_{j} e^{-2\alpha} \otimes \phi\big(-i - \tfrac{1}{2}\big)\phi\big(-j - \tfrac{1}{2}\big)\mathbf{1}.$$

The proof of Proposition 8.3 from [5] gives that

$$\begin{aligned} o(w_1)v_\lambda^\pm &= \mathrm{Res}_{x_1}\mathrm{Res}_{x_2}(x_2-x_1)^{2m}(1+x_1)^t(1+x_2)^t(x_1x_2)^{-4m-2}v_\lambda \\ &= A_m\binom{t+m}{3m+1}\binom{t}{3m+1}v_\lambda^\pm, \end{aligned}$$

so it remains to examine $o(w_2)$. Recall Lemma 2.1, so that

$$(x_2-x_1)G(x_1,x_2)=\tfrac{1}{2}\left((1+x_1)^{1/2}(1+x_2)^{-1/2}+(1+x_1)^{-1/2}(1+x_2)^{1/2}-2\right).$$

Now, we compute

$$\begin{aligned} o(w_2)\cdot v_\lambda^\pm &= \sum_{i,j\geq 0} o\left(e_i^\alpha e_j^\alpha e^{-2\alpha}\otimes\phi(-i-\tfrac{1}{2})\phi(-j-\tfrac{1}{2})\mathbf{1}\right)\cdot v_\lambda^\pm \\ &= \sum_{i,j\geq 0}\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}x_1^ix_2^j o\big(Y(e^\alpha,x_1)Y(e^\alpha,x_2)e^{-2\alpha}\big)o\big(\psi(-i-\tfrac{1}{2})\psi(-j-\tfrac{1}{2})\big)\cdot v_\lambda^\pm \\ &= \mathrm{Res}_{x_1}\mathrm{Res}_{x_2}\Bigg(\sum_{i,j\geq 0}x_1^ix_2^jc_{i,j}\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}(x_2-x_1)^{2m+1}(1+x_1)^t \\ &\qquad\qquad \times(1+x_2)^t(x_1x_2)^{-4m-2}v_\lambda^\pm\Bigg) \\ &= \mathrm{Res}_{x_1}\mathrm{Res}_{x_2}(G(x_1,x_2)(x_2-x_1)^{2m+1}(1+x_1)^t(1+x_2)^t)(x_1x_2)^{-4m-2}v_\lambda^\pm \\ &= -\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}(x_2-x_1)^{2m}(1+x_1)^t(1+x_2)^t(x_1x_2)^{-4m-2}v_\lambda^\pm \qquad (6.2) \\ &\quad+\tfrac{1}{2}\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}\left((x_2-x_1)^{2m}(x_1x_2)^{-4m-2}(1+x_1)^{t+1/2}(1+x_2)^{t-1/2}\right)v_\lambda^\pm \qquad (6.3) \\ &\quad+\tfrac{1}{2}\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}\left((x_2-x_1)^{2m}(x_1x_2)^{-4m-2}(1+x_1)^{t-1/2}(1+x_2)^{t+1/2}\right)v_\lambda^\pm. \qquad (6.4) \end{aligned}$$

Now, observe that the expression in (6.2) is precisely $-o(w_1)\cdot v_\lambda^\pm$, while (6.3) and (6.4) are equal. Consequently,

$$o(w_1+w_2)\cdot v_\lambda^\pm=\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}\left((x_2-x_1)^{2m}(1+x_1)^{t-1/2}(1+x_2)^{t+1/2}(x_1x_2)^{-4m-2}\right)v_\lambda^\pm.$$

Now, we expand the generalized rational function in the last formula and obtain

$$o\big(Q^2e^{-2\alpha}\big)v_\lambda^\pm=\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{t+1/2}{4m+1-k}\binom{t-1/2}{2m+1+k}v_\lambda^\pm.$$

The sum in the last formula can be evaluated as in [4] and [5]. We have

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{t+1/2}{4m+1-k}\binom{t-1/2}{2m+1+k}=A_m\binom{t+m+1/2}{3m+1}\binom{t-1/2}{3m+1},$$

where A_m is above. The proof follows. ∎

A direct consequence of Proposition 6.1 and relation (6.1) is the following important result:

Theorem 6.1. *In Zhu's algebra $A_\sigma(\mathcal{SW}(m))$ we have the following relation*

$$f_m^{\mathfrak{R}}([\omega])=0,$$

where

$$f_m^{\mathfrak{R}}(x)=\prod_{i=0}^{3m}\left(x-h^{2i+2,1}\right)=\left(\prod_{i=0}^{m-1}\left(x-h^{2i+2,1}\right)^2\right)\left(\prod_{i=2m}^{3m}\left(x-h^{2i+2,1}\right)\right).$$

In parallel with [5] we conjecture that $f_m^{\mathfrak{R}}(x)$ is in fact the minimal polynomial of $[\omega]$ in $A_\sigma(\mathcal{SW}(m))$.

We have the following irreducibility result. The proof is similar to that of Theorem 3.7 in [4].

Theorem 6.2.

(1) *For every $0 \leq i \leq m-1$, $R\Lambda(i+1)^{\pm}$ are $\mathbb{Z}_{\geq 0}$-gradable irreducible σ-twisted $\mathcal{SW}(m)$-modules and the top components $R\Lambda(i+1)^{\pm}(0)$ are 1-dimensional irreducible $A_\sigma(\mathcal{SW}(m))$-modules.*

(1′) *For every $0 \leq i \leq m-1$, $R\Lambda(i+1)$ is a graded irreducible σ-twisted $\mathcal{SW}(m)$-module, and its top component $R\Lambda(i+1)(0)$ is a graded irreducible $A_\sigma(\mathcal{SW}(m))$-module.*

(2) *For every $0 \leq j \leq m$, $R\Pi(j+1)^{\pm}$ are irreducible $\mathbb{Z}_{\geq 0}$-gradable σ-twisted $\mathcal{SW}(m)$-modules and the top components $S\Pi(j+1)^{\pm}(0)$ are irreducible 2-dimensional $A_\sigma(\mathcal{SW}(m))$-modules.*

(2′) *For every $0 \leq j \leq m$, $R\Pi(j+1)$ is a graded irreducible σ-twisted $\mathcal{SW}(m)$-module and its top component $S\Pi(j+1)(0)$ is a graded irreducible $A_\sigma(\mathcal{SW}(m))$-module.*

Corollary 6.1. *The minimal polynomial of $[\omega]$ is divisible by*

$$\prod_{i=0}^{m-1}(x-h^{2i+2,1})\left(\prod_{i=2m}^{3m}(x-h^{2i+2,1})\right).$$

Moreover, both $[\omega]$ and $[\tau]$ are units in $A_\sigma(\mathcal{SW}(m))$. Consequently, $\mathcal{SW}(m)$ has no supersymmetric sector (i.e., there is no σ-twisted $\mathcal{SW}(m)$-modules of highest weight $\frac{c_{2m+1,1}}{24}$).

By using similar arguments as in [4] and [5] we have the following result on the structure of Zhu's algebra $A_\sigma(\mathcal{SW}(m))$ (as in the proof of Theorem 4.1 we see that $[\tau]$ is central).

Proposition 6.2. *The associative algebra $A_\sigma(\mathcal{SW}(m))$ is generated by $[E]$, $[H]$, $[F]$, $[\tau]$, $[\widehat{E}]$, $[\widehat{H}]$, $[\widehat{F}]$ and $[\omega]$. The following relations hold:*

$$\begin{aligned}
&[\tau] \quad \text{and} \quad [\omega] \quad \text{are in the center of } A_\sigma(\mathcal{SW}(m)),\\
&[\tau]^2 = [\omega] - \frac{c_{2m+1,1}}{24},\\
&[\tau] * [X] = \frac{-\sqrt{2m+1}}{2}[\widehat{X}], \qquad \text{for} \quad X \in \{E, F, H\},\\
&[\widehat{H}] * [\widehat{F}] - [\widehat{F}] * [\widehat{H}] = -2q([\omega])[\widehat{F}],\\
&[\widehat{H}] * [\widehat{E}] - [\widehat{E}] * [\widehat{H}] = 2q([\omega])[\widehat{E}],\\
&[\widehat{E}] * [\widehat{F}] - [\widehat{F}] * [\widehat{E}] = -2q([\omega])[\widehat{H}],
\end{aligned}$$

where q is a certain polynomial.

Equipped with all these results we are now ready to classify irreducible σ-twisted $\mathcal{SW}(m)$-modules.

Remark 6.1. By using same arguments as in Proposition 5.6 of [1] we have that for C_2-cofinite SVOAs, every week (twisted) module is admissible (see also [11]). Thus, the classification of irreducible σ-twisted $\mathcal{SW}(m)$-modules reduces to classification of irreducible $\mathbb{Z}_{\geq 0}$-gradable modules.

Theorem 6.3.

(*i*) *The set*

$$\{R\Pi(i)^{\pm}(0), : 1 \leq i \leq m+1\} \cup \{R\Lambda(i)^{\pm}(0) : 1 \leq i \leq m\}$$

provides, up to isomorphism, all irreducible modules for Zhu's algebra $A_\sigma(\mathcal{SW}(m))$.

(*ii*) *The set*

$$\{R\Pi(i)(0), : 1 \le i \le m+1\} \cup \{R\Lambda(i)(0) : 1 \le i \le m\}$$

provides, up to isomorphism, all $\mathbb{Z}_2$*-graded irreducible modules for Zhu's algebra* $A_\sigma(\mathcal{SW}(m))$.

Proof. The proof is similar to those of Theorem 3.11 in [4] and Theorem 10.3 in [5]. Assume that U is an irreducible $A_\sigma(\mathcal{SW}(m))$-module. Relation $f_m^{\mathfrak{R}}([\omega]) = 0$ in $A_\sigma(\mathcal{SW}(m))$ implies that

$$L(0)|U = h^{2i+2,1}\,\mathrm{Id}, \qquad \text{for} \quad i \in \{0, \dots, m-1\} \cup \{2m, \dots, 3m\}.$$

Moreover, since $[\tau]$ is in the center of $A_\sigma(\mathcal{SW}(m))$, we conclude that $G(0)$ also acts on U as a scalar. Then relation $[\tau]^2 = [\omega] - c_{2m+1,1}/24$ implies that

$$G(0)|U = \frac{1/2 + i - m}{\sqrt{2(2m+1)}}\,\mathrm{Id} \qquad \text{or} \qquad G(0)|U = -\frac{1/2 + i - m}{\sqrt{2(2m+1)}}\,\mathrm{Id}.$$

Assume first that $i = 2m + j$ for $0 \le j \le m$. By combining Propositions 6.2 and Theorem 6.2 we have that $q(h^{2i+2,1}) \neq 0$. Define

$$e = \frac{1}{\sqrt{2}q(h^{2i+2,1})}[\widehat{E}], \qquad f = -\frac{1}{\sqrt{2}q(h^{2i+2,1})}[\widehat{F}], \qquad h = \frac{1}{q(h^{2i+2,1})}[\widehat{H}].$$

Therefore U carries the structure of an irreducible, $\mathfrak{sl}_2$-module with the property that $e^2 = f^2 = 0$ and $h \neq 0$ on U. This easily implies that U is a 2-dimensional irreducible $\mathfrak{sl}_2$-module. Moreover, as an $A_\sigma(\mathcal{SW}(m))$-module U is isomorphic to either $R\Pi(m+1-j)^+(0)$ or $R\Pi(m+1-j)^-(0)$.

In the case $0 \le i \le m-1$, as in [4] we prove that $U \cong R\Lambda(i+1)^+(0)$ or $U \cong R\Lambda(i+1)^-(0)$.

Let us now prove the second assertion. Let $N = N^0 \oplus N^1$ be the graded irreducible $A_\sigma(\mathcal{SW}(m))$-module. As above, we have that

$$L(0)|N = h^{2i+2,1}\,\mathrm{Id}, \qquad \text{for} \quad i \in \{0, \dots, m-1\} \cup \{2m, \dots, 3m\}.$$

Then $N = N^+ \oplus N^-$, where

$$N^\pm = \mathrm{span}_{\mathbb{C}} \left\{ v \pm \frac{\sqrt{2(2m+1)}}{1/2 + i - m} G(0)v \;|v \in N^0 \right\} \qquad \text{and} \qquad G(0)|N^\pm = \pm\frac{1/2 + i - m}{\sqrt{2(2m+1)}}\,\mathrm{Id}.$$

By using assertion (i), we easily get that $N^\pm = R\Lambda(i+1)^\pm(0)$ (if $0 \le i \le m-1$) and $N^\pm = R\Pi(3m+1-i)^\pm(0)$ (if $2m \le i \le 3m$). ∎

Theorem 6.4.

(*i*) *The set*

$$\{R\Pi(i)^\pm : 1 \le i \le m+1\} \cup \{R\Lambda(i)^\pm : 1 \le i \le m\}$$

provides, up to isomorphism, all irreducible σ*-twisted* $\mathcal{SW}(m)$*-modules.*

(*ii*) *The set*

$$\{R\Pi(i) : 1 \le i \le m+1\} \cup \{R\Lambda(i) : 1 \le i \le m\}$$

provides, up to isomorphism, all $\mathbb{Z}_2$*-graded irreducible* σ*-twisted* $\mathcal{SW}(m)$*-modules.*

So the vertex operator algebra $\mathcal{SW}(m)$ contains only finitely many irreducible modules. But one can easily see that modules $V_{L+\gamma_i^R} \otimes M$ and $V_{L+\gamma_{m+i}^R} \otimes M$ ($0 \le i \le m-1$) constructed in Theorems 5.1 and 5.2 are not completely reducible. Thus we have:

Corollary 6.2. *The vertex operator superalgebra $\mathcal{SW}(m)$ is not σ-rational, i.e., the category of σ-twisted $\mathcal{SW}(m)$-modules is not semisimple.*

Remark 6.2. In our forthcoming paper [7] we shall prove that $\mathcal{SW}(m)$ also contains logarithmic σ-twisted representations.

7 Modular properties of characters of σ-twisted $\mathcal{SW}(m)$-modules

We first introduce some basic modular forms needed for description of irreducible twisted $\mathcal{SW}(m)$ characters. The Dedekind η-function is usually defined as the infinite product

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n),$$

an automorphic form of weight $\frac{1}{2}$. As usual in all these formulas $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$. We also introduce

$$\mathfrak{f}(\tau) = q^{-1/48}\prod_{n=0}^{\infty}(1+q^{n+1/2}),$$

$$\mathfrak{f}_1(\tau) = q^{-1/48}\prod_{n=1}^{\infty}(1-q^{n-1/2}),$$

$$\mathfrak{f}_2(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1+q^{n}).$$

Let us recall Jacobi Θ-function

$$\Theta_{j,k}(\tau,z) = \sum_{n\in\mathbb{Z}} q^{k(n+\frac{j}{2k})^2} e^{2\pi i k z(n+\frac{j}{2k})},$$

where $j,k \in \frac{1}{2}\mathbb{Z}$. If $j \in \mathbb{Z}+\frac{1}{2}$ or $k \in \mathbb{N}+\frac{1}{2}$ it will be useful to use the formula

$$\Theta_{j,k}(\tau,z) = \Theta_{2j,4k}(\tau,z) + \Theta_{2j-4k,4k}(\tau,z).$$

Observe that

$$\Theta_{j+2k,k}(\tau,z) = \Theta_{j,k}(\tau,z).$$

We also let

$$\partial\Theta_{j,k}(\tau,z) := \frac{1}{\pi i}\frac{d}{dz}\Theta_{j,k}(\tau,z) = \sum_{n\in\mathbb{Z}}(2kn+j)q^{(2kn+j)^2/4k}.$$

Related Θ-functions needed for supercharacters are

$$G_{j,k}(\tau,z) = \sum_{n\in\mathbb{Z}}(-1)^n q^{k(n+\frac{j}{2k})^2} e^{2\pi i k z(n+\frac{j}{2k})},$$

where $j \in \mathbb{Z}$ and $k \in \frac{1}{2}\mathbb{N}$. It is easy to see that

$$G_{j,k}(\tau, z) = \Theta_{2j,4k}(\tau, z) - \Theta_{2j-4k,4k}(\tau, z).$$

Eventually, we shall let $z = 0$ so before we introduce

$$\Theta_{j,k}(\tau) := \Theta_{j,k}(\tau, 0), \qquad G_{j,k}(\tau) := G_{j,k}(\tau, 0).$$

Similarly, we define $\partial\Theta_{j,k}(\tau)$ and $\partial G_{j,k}(\tau)$.

The following formulas will be useful:

$$\begin{aligned}
\Theta_{j,k}\left(\frac{-1}{\tau}, \frac{z}{\tau}\right) &= \frac{\sqrt{-i\tau}}{\sqrt{2k}} e^{\frac{4\pi i k z^2}{\tau}} \left(\sum_{j'=0}^{4k-1} e^{-\pi i j' j/k} \Theta_{2j',4k}(\tau, z)\right), \\
\partial\Theta'_{j,k}\left(\frac{-1}{\tau}, \frac{z}{\tau}\right) &= \frac{\sqrt{-i\tau}}{\sqrt{2k}} e^{\frac{4\pi i k z^2}{\tau}} 8kz \left(\sum_{j'=0}^{4k-1} e^{-\pi i j' j/k} \Theta_{2j',4k}(\tau, z)\right) \\
&\quad + \tau \frac{\sqrt{-i\tau}}{\sqrt{2k}} e^{\frac{4\pi i k z^2}{\tau}} \left(\sum_{j'=0}^{4k-1} e^{-\pi i j' j/k} \partial\Theta_{2j',4k}(\tau, z)\right).
\end{aligned}$$

If we let now $z = 0$, we obtain

$$\Theta_{j,k}\left(\frac{-1}{\tau}\right) = \frac{\sqrt{-i\tau}}{\sqrt{2k}} \sum_{j'=0}^{4k-1} e^{\frac{-\pi i j j'}{k}} \Theta_{2j',4k}(\tau, 0), \tag{7.1}$$

and

$$\partial\Theta'_{j,k}\left(\frac{-1}{\tau}\right) = \tau \frac{\sqrt{-i\tau}}{\sqrt{2k}} \sum_{j'=0}^{4k-1} e^{\frac{-\pi i j j'}{k}} \partial\Theta_{2j',4k}(\tau, 0). \tag{7.2}$$

For $j \in \mathbb{Z}$ and $k \in \mathbb{N} + \frac{1}{2}$ we now have:

$$\begin{aligned}
&\Theta_{j,k}\left(\frac{-1}{\tau}\right) = \sqrt{\frac{-i\tau}{2k}} \sum_{j'=0}^{2k-1} e^{-i\pi j j'/k} \Theta_{j',k}(\tau), \\
&\Theta_{j,k}(\tau + 1) = e^{i\pi j^2/2k} G_{j,k}(\tau), \\
&(\partial\Theta)_{j,k}(\tau + 1) = e^{i\pi j^2/2k} (\partial G)_{j,k}(\tau), \\
&(\partial\Theta)_{j,k}\left(\frac{-1}{\tau}\right) = \tau\sqrt{-i\tau/2k} \sum_{j'=1}^{2k-1} e^{-i\pi j j'/k} (\partial\Theta)_{j',k}(\tau),
\end{aligned}$$

where we used (7.1) and (7.2), together with $\Theta_{j,k}(\tau) = \Theta_{j+2k,k}(\tau)$ and $\partial\Theta_{j,k}(\tau) = \partial\Theta_{j+2k,k}(\tau)$.

For $j \in \mathbb{Z} + \frac{1}{2}$ and $k \in \mathbb{N} + \frac{1}{2}$ the transformation formulas are slightly different:

$$\begin{aligned}
&\Theta_{j,k}\left(\frac{-1}{\tau}\right) = \sqrt{\frac{-i\tau}{2k}} \sum_{j'=0}^{2k-1} e^{-i\pi j j'/k} G_{j',k}(\tau), \\
&\Theta_{j,k}(\tau + 1) = e^{i\pi j^2/2k} \Theta_{j,k}(\tau), \\
&(\partial\Theta)_{j,k}(\tau + 1) = e^{i\pi j^2/2k} (\partial\Theta)_{j,k}(\tau), \\
&(\partial\Theta)_{j,k}\left(\frac{-1}{\tau}\right) = \tau\sqrt{-i\tau/2k} \sum_{j'=1}^{2k-1} e^{-i\pi j j'/k} (\partial G)_{j',k}(\tau).
\end{aligned}$$

For a (twisted) vertex operator algebra module W we define its graded-dimension or simply character

$$\chi_W(\tau) = \mathrm{tr}|_W q^{L(0)-c/24}.$$

If $V = L^{\mathfrak{ns}}(c_{2m+1,0}, 0)$ and $W = L^{\mathfrak{R}}(c_{2m+1,0}, h^{2i+2,2n+1})$, then (see [20], for instance)

$$\chi_{L^{\mathfrak{R}}(c_{2m+1,1},h^{2i+2,2n+1})}(\tau) = 2q^{\frac{m^2}{2(2m+1)}-\frac{1}{16}}\frac{\mathfrak{f}_2(\tau)}{\eta(\tau)}\left(q^{h^{2i+2,2n+1}} - q^{h^{2i+2,-2n-1}}\right). \tag{7.3}$$

By combining Theorems 5.1, 5.2 and 5.3, and formula (7.3) we obtain

Proposition 7.1. *For $i = 0, \dots, m-1$*

$$\chi_{R\Lambda(i+1)}(\tau) = 2\frac{\mathfrak{f}_2(\tau)}{\eta(\tau)}\left(\frac{2i+2}{2m+1}\Theta_{m-i-\frac{1}{2},\frac{2m+1}{2}}(\tau) + \frac{2}{2m+1}(\partial\Theta)_{m-i-\frac{1}{2},\frac{2m+1}{2}}(\tau)\right), \tag{7.4}$$

$$\chi_{R\Pi(m-i)}(\tau) = 2\frac{\mathfrak{f}_2(\tau)}{\eta(\tau)}\left(\frac{2m-2i-1}{2m+1}\Theta_{m-i-\frac{1}{2},\frac{2m+1}{2}}(\tau) - \frac{2}{2m+1}(\partial\Theta)_{m-i-\frac{1}{2},\frac{2m+1}{2}}(\tau)\right). \tag{7.5}$$

Also,

$$\chi_{R\Pi(m+1)}(\tau) = 2\frac{\mathfrak{f}_2(\tau)}{\eta(\tau)}\Theta_{m+\frac{1}{2},\frac{2m+1}{2}}(\tau).$$

As in [5], the characters of irreducible σ-twisted $\mathcal{SW}(m)$-modules can be described by using characters of irreducible modules for the triplet vertex algebra $\mathcal{W}(p)$, where $p = 2m+1$. Let $\Lambda(1), \dots, \Lambda(p)$, $\Pi(1), \dots, \Pi(p)$ be the irreducible $\mathcal{W}(p)$-module. We have the following result:

Proposition 7.2.

(*i*) *For $0 \le i \le m-1$, we have:*

$$\chi_{R\Lambda(i+1)}(\tau) = 2\frac{\chi_{\Lambda(2i+2)}(\tau/2)}{\mathfrak{f}(\tau)}.$$

(*ii*) *For $0 \le i \le m$, we have:*

$$\chi_{R\Pi(m+1-i)}(\tau) = 2\frac{\chi_{\Pi(2m-2i+1)}(\tau/2)}{\mathfrak{f}(\tau)}.$$

Now, we recall also formulas for irreducible $\mathcal{SW}(m)$ characters and supercharacters obtained in [5].

For $i = 0, \dots, m-1$

$$\chi_{S\Lambda(i+1)}(\tau) = \frac{\mathfrak{f}(\tau)}{\eta(\tau)}\left(\frac{2i+1}{2m+1}\Theta_{m-i,\frac{2m+1}{2}}(\tau) + \frac{2}{2m+1}(\partial\Theta)_{m-i,\frac{2m+1}{2}}(\tau)\right), \tag{7.6}$$

$$\chi_{S\Pi(m-i)}(\tau) = \frac{\mathfrak{f}(\tau)}{\eta(\tau)}\left(\frac{2m-2i}{2m+1}\Theta_{m-i,\frac{2m+1}{2}}(\tau) - \frac{2}{2m+1}(\partial\Theta)_{m-i,\frac{2m+1}{2}}(\tau)\right). \tag{7.7}$$

Also,

$$\chi_{S\Lambda(m+1)}(\tau) = \frac{\mathfrak{f}(\tau)}{\eta(\tau)}\Theta_{0,\frac{2m+1}{2}}(\tau). \tag{7.8}$$

For supercharacters we have: for $i=0,\dots,m-1$

$$\chi^F_{S\Lambda(i+1)}(\tau)=\frac{\mathfrak{f}_1(\tau)}{\eta(\tau)}\left(\frac{2i+1}{2m+1}G_{m-i,\frac{2m+1}{2}}(\tau)+\frac{2}{2m+1}(\partial G)_{m-i,\frac{2m+1}{2}}(\tau)\right), \tag{7.9}$$

$$\chi^F_{S\Pi(m-i)}(\tau)=\frac{\mathfrak{f}_1(\tau)}{\eta(\tau)}\left(\frac{2m-2i}{2m+1}G_{m-i,\frac{2m+1}{2}}(\tau)-\frac{2}{2m+1}(\partial G)_{m-i,\frac{2m+1}{2}}(\tau)\right). \tag{7.10}$$

Also,

$$\chi^F_{S\Lambda(m+1)}(\tau)=\frac{\mathfrak{f}_1(\tau)}{\eta(\tau)}G_{0,\frac{2m+1}{2}}(\tau). \tag{7.11}$$

These characters and supercharacters can be expressed by using characters of $W(2m+1)$-modules.

Proposition 7.3.

(*i*) *For $0\le i\le m$, we have*

$$\chi_{S\Lambda(i+1)}(\tau)=\frac{\chi_{\Lambda(2i+1)}(\frac{\tau}{2})}{\mathfrak{f}_2(\tau)},\qquad \chi^F_{S\Lambda(i+1)}(\tau)=\frac{\chi_{\Lambda(2i+1)}(\frac{\tau+1}{2})}{\mathfrak{f}_2(\tau)}.$$

(*ii*) *For $0\le i\le m-1$, we also have*

$$\chi_{S\Pi(m-i)}(\tau)=\frac{\chi_{\Pi(2m-2i)}(\frac{\tau}{2})}{\mathfrak{f}_2(\tau)},\qquad \chi^F_{S\Pi(m-i)}(\tau)=\frac{\chi_{\Pi(2m-2i)}(\frac{\tau+1}{2})}{\mathfrak{f}_2(\tau)}.$$

Here $\Lambda(i)$ and $\Pi(2m+2-i)$, $i=1,\dots,2m+1$, are irreducible $\mathcal{W}(2m+1)$-modules [4].

By combining transformation formulas for $\Theta_{j,k}(\tau)$, $\partial\Theta_{j,k}(\tau)$, formulas (7.4)–(7.8), (7.9)–(7.11) and Proposition 7.1 we obtain second main result of our paper.

Theorem 7.1. *The $SL(2,\mathbb{Z})$ closure, called $\mathcal{H}$, of the vector space determined by $\mathcal{SW}(m)$-characters, $\mathcal{SW}(m)$-supercharacters and σ-twisted $\mathcal{SW}(m)$-characters has the following basis:*

$$\begin{array}{lll}
\frac{\mathfrak{f}_1(\tau)}{\eta(\tau)}G_{0,\frac{2m+1}{2}}(\tau), & \frac{\mathfrak{f}(\tau)}{\eta(\tau)}\Theta_{0,\frac{2m+1}{2}}(\tau), & \frac{\mathfrak{f}_2(\tau)}{\eta(\tau)}\Theta_{m+\frac{1}{2},\frac{2m+1}{2}}(\tau),\\
\frac{\mathfrak{f}_1(\tau)}{\eta(\tau)}G_{m-i,\frac{2m+1}{2}}(\tau), & \frac{\mathfrak{f}(\tau)}{\eta(\tau)}\Theta_{m-i,\frac{2m+1}{2}}(\tau), & \frac{\mathfrak{f}_2(\tau)}{\eta(\tau)}\Theta_{m-i-\frac{1}{2},\frac{2m+1}{2}}(\tau),\\
\frac{\mathfrak{f}_1(\tau)}{\eta(\tau)}\partial G_{m-i,\frac{2m+1}{2}}(\tau), & \frac{\mathfrak{f}(\tau)}{\eta(\tau)}\partial\Theta_{m-i,\frac{2m+1}{2}}(\tau), & \frac{\mathfrak{f}_2(\tau)}{\eta(\tau)}\partial\Theta_{m-i-\frac{1}{2},\frac{2m+1}{2}}(\tau),\\
\tau\frac{\mathfrak{f}_1(\tau)}{\eta(\tau)}\partial G_{m-i,\frac{2m+1}{2}}(\tau), & \tau\frac{\mathfrak{f}(\tau)}{\eta(\tau)}\partial\Theta_{m-i,\frac{2m+1}{2}}(\tau), & \tau\frac{\mathfrak{f}_2(\tau)}{\eta(\tau)}\partial\Theta_{m-i-\frac{1}{2},\frac{2m+1}{2}}(\tau),
\end{array}$$

where $i=0,\dots,m-1$. In particular, the space is $9m+3$ dimensional.

7.1 Modular differential equations for σ-twisted $\mathcal{SW}(m)$ characters

Let us recall classical $SL(2,\mathbb{Z})$ Eisenstein series ($k\ge 1$):

$$G_{2k}(\tau)=\frac{-B_{2k}}{(2k)!}+\frac{2}{(2k-1)!}\sum_{n=1}^{\infty}\frac{n^{2k-1}q^n}{1-q^n},$$

and certain linear combination of level 2 Eisenstein series ($k \geq 1$):

$$G_{2k,1}(\tau) = \frac{B_{2k}}{(2k)!} + \frac{2}{(2k-1)!} \sum_{n \geq 1} \frac{n^{2k-1} q^n}{1+q^n},$$

$$G_{2k,0}(\tau) = \frac{B_{2k}(1/2)}{(2k)!} + \frac{2}{(2k-1)!} \sum_{n=1}^{\infty} \frac{(n-1/2)^{2k-1} q^{n-1/2}}{1+q^{n-1/2}},$$

where $B_{2k}(x)$ are the Bernoulli polynomials, and B_{2k} are the Bernoulli numbers.

A modular differential equation is an (ordinary) differential equation of the form:

$$\left(q\frac{d}{dq}\right)^k y(q) + \sum_{j=0}^{k-1} H_j(q) \left(d\frac{d}{dq}\right)^i y(q) = 0,$$

where $H_j(q)$ are polynomials in Eisenstein series G_{2i}, $i \geq 1$, such that the vector space of solutions is modular invariant.

It is known (cf. [30]) that C_2-cofiniteness condition leads to certain modular differential equation satisfied by irreducible characters $\mathrm{tr}_M q^{L(0)-c/24}$. In some instances the degree of this differential equation is bigger than the number of irreducible characters. So it is not clear what k should be in general. In the case of Virasoro minimal models, the degree of the modular differential equation is precisely the number of (linearly independent) irreducible characters.

If V is a vertex operator superalgebra one can also get modular differential equation satisfied by ordinary characters but with respect to the subgroup $\Gamma_\theta \subset SL(2,\mathbb{Z})$, where H_i are polynomials in G_{2i} and $G_{2i,0}$ (see [27] for the precise statement in the case of $N=1$ minimal models).

In [6] we proved that the C_2-cofiniteness for the super triplet vertex algebra $\mathcal{SW}(m)$ gives rise to a differential equation of order $3m+1$ satisfied by $2m+1$ irreducible characters (additional m solutions can be interpreted as certain *pseudotraces*). By applying arguments similar to those in [6] it is not hard to prove

Theorem 7.2. *The irreducible σ-twisted $\mathcal{SW}(m)$ characters satisfy the differential equation of the form*

$$\left(d\frac{d}{dq}\right)^{3m+1} y(q) + \sum_{j=0}^{3m} \tilde{H}_{j,0}(q) \left(q\frac{d}{dq}\right)^i y(q) = 0,$$

where $\tilde{H}_{j,0}(q)$ are certain polynomials in G_{2i} and $G_{2i,1}$.

As in the case of $\mathcal{W}(p)$-modules and ordinary $\mathcal{SW}(m)$-modules, we expect that additional m linearly independent solutions in Theorem 7.2 have interpretation in terms of σ-twisted pseudotraces (cf. Conjecture 8.2 below).

8 Conclusion

Here we gather a few more-or-less expected conjectures (especially in view of our earlier work [4] and [5]).

The first one is concerned about the structure of $A_\sigma(V)$. As in the case of $A(\mathcal{W}(p))$ and $A(\mathcal{SW}(m))$, from our analysis in Chapter 6, we can show that in fact

$$\dim(A_\sigma(\mathcal{SW}(m))) \geq 10m + 8.$$

But this is well below the conjectural dimension, because $10m + 8$-dimensional part cannot control possible logarithmic modules. Thus, as in the case of ordinary $\mathcal{SW}(m)$-modules, we expect to have $2m$ non-isomorphic (non-graded) logarithmic modules with two-dimensional top component. This then leads to the following conjecture:

Conjecture 8.1. *For every $m \in \mathbb{N}$,*

$$\dim(A_\sigma(\mathcal{SW}(m))) = 12m + 8.$$

If we assume the existence of m logarithmic modules so that Conjecture 8.1 holds true, then the following fact is expected.

Conjecture 8.2. *The vector space of (suitably defined) generalized σ-twisted characters is $3m+1$-dimensional.*

Thus, generalized $\mathcal{SW}(m)$ characters, supercharacters and σ-twisted characters together should give rise to a $9m + 3$-dimensional modular invariant space.

Acknowledgments

The second author was partially supported by NSF grant DMS-0802962.

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