Some Orthogonal Polynomials in Four Variables*

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Abstract. The symmetric group on 4 letters has the reflection group D_3 as an isomorphic image. This fact follows from the coincidence of the root systems A_3 and D_3. The isomorphism is used to construct an orthogonal basis of polynomials of 4 variables with 2 parameters. There is an associated quantum Calogero–Sutherland model of 4 identical particles on the line.

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1 Introduction

The symmetric group on N letters acts naturally on $\mathbb{R}^N$ (for $N = 2, 3, \dots$) but not irreducibly, because the vector $(1, 1, \dots, 1)$ is fixed. However the important basis consisting of nonsymmetric Jack polynomials is defined for N variables and does not behave well under restriction to the orthogonal complement of $(1, 1, \dots, 1)$, in general. In this paper we consider the one exception to this situation, occurring when $N = 4$. In this case there is a coordinate system, essentially the 4×4 Hadamard matrix, which allows a different basis of polynomials, derived from the type-B nonsymmetric Jack polynomials for the subgroup D_3 of the octahedral group B_3. We will construct an orthogonal basis for the L^2-space of the measure

$$\prod_{1\le i<j\le 4} |x_i - x_j|^{2\kappa} |x_1 + x_2 + x_3 + x_4|^{2\kappa'} \exp\left(-\frac{1}{2}\sum_{i=1}^{4} x_i^2\right) dx$$

on $\mathbb{R}^4$, with $\kappa, \kappa' > 0$.

We will use the following notations: $\mathbb{N}_0$ denotes the set of nonnegative integers; $\mathbb{N}_0^N$ is the set of compositions (or multi-indices), if $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$ then $|\alpha| := \sum_{i=1}^N \alpha_i$ and the length of α is $\ell(\alpha) := \max\{i : \alpha_i > 0\}$. Let $\mathbb{N}_0^{N,+}$ denote the subset of partitions, that is, $\lambda \in \mathbb{N}_0^N$ and $\lambda_i \ge \lambda_{i+1}$ for $1 \le i < N$. For $\alpha \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$ let $x^\alpha = \prod_{i=1}^N x_i^{\alpha_i}$, a monomial of degree $|\alpha|$; the space of polynomials is $\mathcal{P} = \operatorname{span}_{\mathbb{R}}\left\{x^\alpha : \alpha \in \mathbb{N}_0^N\right\}$. For $x, y \in \mathbb{R}^N$ the inner product is $\langle x, y\rangle := \sum_{i=1}^N x_i y_i$, and $|x| := \langle x, x\rangle^{1/2}$; also $x^\perp := \{y : \langle x, y\rangle = 0\}$. The cardinality of a set E is denoted by $\#E$.

Consider the elements of S_N as permutations on $\{1, 2, \dots, N\}$. For $x \in \mathbb{R}^N$ and $w \in \mathcal{S}_N$ let $(xw)_i := x_{w(i)}$ for $1 \le i \le N$ and extend this action to polynomials by $(wf)(x) = f(xw)$. Monomials transform to monomials: $w(x^\alpha) := x^{w\alpha}$ where $(w\alpha)_i := \alpha_{w^{-1}(i)}$ for $\alpha \in \mathbb{N}_0^N$. (Consider x as a row vector, α as a column vector, and w as a permutation matrix, with 1's at the

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$(w(j), j)$ entries.) For $1 \le i \le N$ and $f \in \mathcal{P}$ the Dunkl operators are

$$\mathcal{D}_i f(x) := \frac{\partial}{\partial x_i} f(x) + \kappa \sum_{j \ne i} \frac{f(x) - f(x(i,j))}{x_i - x_j},$$

and

$$\mathcal{U}_i f(x) := \mathcal{D}_i (x_i f(x)) - \kappa \sum_{j=1}^{i-1} (j,i) f(x).$$

Then $\mathcal{U}_i \mathcal{U}_j = \mathcal{U}_j \mathcal{U}_i$ for $1 \le i, j \le N$ and these operators are self-adjoint for the following pairing

$$\langle f, g \rangle_\kappa := f(\mathcal{D}_1, \ldots, \mathcal{D}_N) g(x)|_{x=0}.$$

This satisfies $\langle f, g \rangle_\kappa = \langle g, f \rangle_\kappa = \langle wf, wg \rangle_\kappa$ for $f, g \in \mathcal{P}$ and $w \in \mathcal{S}_N$; furthermore $\langle f, f \rangle_\kappa > 0$ when $f \ne 0$ and $\kappa \ge 0$. The operators $\mathcal{U}_i$ have the very useful property of acting as triangular matrices on the monomial basis furnished with a certain partial order. However the good properties depend completely on the use of $\mathbb{R}^N$ even though the group $\mathcal{S}_N$ acts irreducibly on $(1, 1, \ldots, 1)^\perp$. We suggest that an underlying necessity for the existence of an analog of $\{\mathcal{U}_i\}$ for any reflection group W is the existence of a W-orbit in which any two points are orthogonal or antipodal (as in the analysis of the hyperoctahedral group B_N). This generally does not hold for the action of $\mathcal{S}_N$ on $(1, \ldots, 1)^\perp$. We consider the exceptional case $N = 4$ and exploit the isomorphism between $\mathcal{S}_4$ and the group of type D_3, that is, the subgroup of B_3 whose simple roots are $(1, -1, 0)$, $(0, 1, -1)$, $(0, 1, 1)$. We map these root vectors to the simple roots $(0, 1, -1, 0)$, $(0, 0, 1, -1)$, $(1, -1, 0, 0)$ of $\mathcal{S}_4$, in the same order. This leads to the linear isometry

$$\begin{aligned}
y_1 &= \tfrac{1}{2}(x_1 + x_2 - x_3 - x_4), \\
y_2 &= \tfrac{1}{2}(x_1 - x_2 + x_3 - x_4), \\
y_3 &= \tfrac{1}{2}(x_1 - x_2 - x_3 + x_4), \\
y_0 &= \tfrac{1}{2}(x_1 + x_2 + x_3 + x_4).
\end{aligned} \tag{1}$$

Consider the group D_3 acting on (y_1, y_2, y_3) and use the type-B_3 Dunkl operators with the parameter $\kappa' = 0$ (associated with the class of sign-changes $y_i \mapsto -y_i$ which are not in D_3). Let σ_{ij}, τ_{ij} denote the reflections in $y_i - y_j = 0$, $y_i + y_j = 0$ respectively. Then for $i = 1, 2, 3$ let

$$\begin{aligned}
\mathcal{D}_i^B f(y) &= \frac{\partial}{\partial y_i} f(y) + \kappa \sum_{j=1, j \ne i}^{3} \left(\frac{f(y) - f(y\sigma_{ij})}{y_i - y_j} + \frac{f(y) - f(y\tau_{ij})}{y_i + y_j} \right), \\
\mathcal{U}_i^B f(y) &= \mathcal{D}_i^B (y_i f(y)) - \kappa \sum_{1 \le j < i} (\sigma_{ij} + \tau_{ij}) f(y).
\end{aligned}$$

The operators $\{\mathcal{U}_i^B\}$ commute pairwise and are self-adjoint for the usual inner product. The simultaneous eigenvectors are expressed in terms of nonsymmetric Jack polynomials with argument (y_1^2, y_2^2, y_3^2). In the sequel we consider polynomials with arguments x or y with the convention that y is given in terms of x by equation (1).

2 Nonsymmetric Jack polynomials

Nonsymmetric Jack polynomials (NSJP) are the simultaneous eigenfunctions of $\{\mathcal{U}_i\}_{i=1}^N$. We consider the formulae for arbitrary N since there is really no simplification for $N = 3$.

Definition 1. For $\alpha \in \mathbb{N}_0^N$, let α^+ denote the unique partition such that $\alpha^+ = w\alpha$ for some $w \in S_N$. For $\alpha, \beta \in \mathbb{N}_0^N$ the partial order $\alpha \succ \beta$ (α dominates β) means that $\alpha \neq \beta$ and $\sum_{i=1}^{j} \alpha_i \geq \sum_{i=1}^{j} \beta_i$ for $1 \leq j \leq N$; $\alpha \rhd \beta$ means that $|\alpha| = |\beta|$ and either $\alpha^+ \succ \beta^+$ or $\alpha^+ = \beta^+$ and $\alpha \succ \beta$.

For example $(2,6,4) \rhd (5,4,3) \rhd (3,4,5)$. When acting on the monomial basis $\left\{x^\alpha : \alpha \in \mathbb{N}_0^N, |\alpha| = n\right\}$ for $n \in \mathbb{N}_0$ the operators $\mathcal{U}_i$ have on-diagonal coefficients given by the following functions on $\mathbb{N}_0^N$.

Definition 2. For $\alpha \in \mathbb{N}_0^N$ and $1 \leq i \leq N$ let

$$\begin{aligned} r(\alpha, i) &:= \#\{j : \alpha_j > \alpha_i\} + \#\{j : 1 \leq j \leq i, \alpha_j = \alpha_i\}, \\ \xi_i(\alpha) &:= (N - r(\alpha, i))\kappa + \alpha_i + 1. \end{aligned}$$

Clearly for a fixed $\alpha \in \mathbb{N}_0^N$ the values $\{r(\alpha, i) : 1 \leq i \leq N\}$ consist of all of $\{1, \ldots, N\}$; let w be the inverse function of $i \mapsto r(\alpha, i)$ so that $w \in \mathcal{S}_N$, $r(\alpha, w(i)) = i$ and $\alpha^+ = w\alpha$ (note that $\alpha \in \mathbb{N}_0^{N,+}$ if and only if $r(\alpha, i) = i$ for all i). Then

$$\mathcal{U}_i x^\alpha = \xi_i(\alpha) x^\alpha + q_{\alpha,i}(x)$$

where $q_{\alpha,i}(x)$ is a sum of terms $\pm\kappa x^\beta$ with $\alpha \rhd \beta$.

Definition 3. For $\alpha \in \mathbb{N}_0^N$, let ζ_α denote the x-monic simultaneous eigenfunction (NSJP), that is, $\mathcal{U}_i \zeta_\alpha = \xi_i(\alpha) \zeta_\alpha$ for $1 \leq i \leq N$ and

$$\zeta_\alpha = x^\alpha + \sum_{\alpha \rhd \beta} A_{\beta\alpha} x^\beta,$$

with coefficients $A_{\beta\alpha} \in \mathbb{Q}(\kappa)$, rational functions of κ.

There are norm formulae for the pairing $\langle \cdot, \cdot \rangle_\kappa$. Suppose $\alpha \in \mathbb{N}_0^N$ and $\ell(\alpha) = m$; the *Ferrers diagram* of α is the set $\{(i,j) : 1 \leq i \leq m, 0 \leq j \leq \alpha_i\}$. For each node (i,j) with $1 \leq j \leq \alpha_i$ there are two special subsets of the Ferrers diagram, the *arm* $\{(i,l) : j < l \leq \alpha_i\}$ and the *leg* $\{(l,j) : l > i, j \leq \alpha_l \leq \alpha_i\} \cup \{(l, j-1) : l < i, j-1 \leq \alpha_l < \alpha_i\}$. The node itself, the arm and the leg make up the *hook*. (For the case of partitions the nodes $(i,0)$ are customarily omitted from the Ferrers diagram.) The cardinality of the leg is called the leg-length, formalized by the following:

Definition 4. For $\alpha \in \mathbb{N}_0^N$, $1 \leq i \leq \ell(\alpha)$ and $1 \leq j \leq \alpha_i$ the leg-length is

$$L(\alpha; i, j) := \#\{l : l > i, j \leq \alpha_l \leq \alpha_i\} + \#\{l : l < i, j \leq \alpha_l + 1 \leq \alpha_i\}.$$

For $t \in \mathbb{Q}(\kappa)$ the *hook-length* and the hook-length product for α are given by

$$\begin{aligned} h(\alpha, t; i, j) &:= (\alpha_i - j + t + \kappa L(\alpha; i, j)), \\ h(\alpha, t) &:= \prod_{i=1}^{\ell(\alpha)} \prod_{j=1}^{\alpha_i} h(\alpha, t; i, j), \end{aligned}$$

and for $\lambda \in \mathbb{N}_0^{N,+}$ and $t \in \mathbb{Q}(\kappa)$ the generalized Pochhammer symbol is

$$(t)_\lambda := \prod_{i=1}^{N} \prod_{j=0}^{\lambda_i - 1} (t - (i-1)\kappa + j).$$

(The product over j is an ordinary Pochhammer symbol.)

Proposition 1. *For $\alpha, \beta \in \mathbb{N}_0^N$, the following orthogonality and norm formula holds:*

$$\langle \zeta_\alpha, \zeta_\beta \rangle_\kappa = \delta_{\alpha\beta} \left(N\kappa + 1\right)_{\alpha^+} \frac{h(\alpha, 1)}{h(\alpha, \kappa+1)}.$$

Details can be found in the book by Xu and the author [2, Chapter 8], the concept of leg-length and its use in the norm formula is due to Knop and Sahi [3]. The (symmetric) Jack polynomial with leading term x^λ for $\lambda \in \mathbb{N}_0^{N,+}$ is obtained by symmetrizing ζ_λ. The coefficients involve, for $\alpha \in \mathbb{N}_0^N$, $\varepsilon = \pm 1$:

$$\mathcal{E}_\varepsilon(\alpha) := \prod_{i<j, \alpha_i < \alpha_j} \left(1 + \frac{\varepsilon\kappa}{(r(\alpha, i) - r(\alpha, j))\kappa + \alpha_j - \alpha_i}\right),$$

in fact, [1, Lemma 3.10],

$$h(\alpha, \kappa+1) = \mathcal{E}_1(\alpha) h\left(\alpha^+, \kappa+1\right),$$
$$h\left(\alpha^+, 1\right) = h(\alpha, 1) \mathcal{E}_{-1}(\alpha),$$

for $\alpha \in \mathbb{N}_0^N$. Then

$$j_\lambda = \sum_{\alpha^+ = \lambda} \mathcal{E}_{-1}(\alpha) \zeta_\alpha,$$
$$\langle j_\lambda, j_\lambda \rangle_\kappa = \#\left\{\alpha : \alpha^+ = \lambda\right\} \frac{(N\kappa+1)_\lambda h(\lambda, 1)}{\mathcal{E}_1(\lambda^R) h(\lambda, \kappa+1)},$$

where $\lambda_i^R = \lambda_{N+1-i}$ for $1 \le i \le N$ (the reverse of λ). Note $\{\alpha : \alpha^+ = \lambda\} = \{w\lambda : w \in \mathcal{S}_N\}$.

3 The groups $\mathcal{S}_4$ and D_3

By using the $x \leftrightarrow y$ correspondence (equation (1)) we obtain operators which behave well on $(1, \ldots, 1)^\perp$. Here are the lists of reflections in corresponding order:

$$[\sigma_{12}, \tau_{12}, \sigma_{13}, \tau_{13}, \sigma_{23}, \tau_{23}],$$
$$[(23), (14), (24), (13), (34), (12)].$$

The following orthonormal basis is used in the directional derivatives:

$$v_0 = \tfrac{1}{2}(1, 1, 1, 1),$$
$$v_1 = \tfrac{1}{2}(1, 1, -1, -1),$$
$$v_2 = \tfrac{1}{2}(1, -1, 1, -1),$$
$$v_3 = \tfrac{1}{2}(1, -1, -1, 1).$$

That is, $y_i = \langle x, v_i \rangle$ and $\frac{\partial}{\partial y_i} = \sum_{j=1}^4 (v_i)_j \frac{\partial}{\partial x_j}$ for $0 \le i \le 3$. Note that $\{\pm v_1, \pm v_2, \pm v_3\}$ is an octahedron and an $\mathcal{S}_4$-orbit. Then

$$\mathcal{D}_1^B f(x) = \sum_{j=1}^4 (v_1)_j \frac{\partial f(x)}{\partial x_j} + \kappa \left(\frac{1-(23)}{x_2 - x_3} + \frac{1-(14)}{x_1 - x_4} + \frac{1-(24)}{x_2 - x_4} + \frac{1-(13)}{x_1 - x_3}\right) f(x),$$

and similar expressions hold for $\mathcal{D}_2^B$, $\mathcal{D}_3^B$. Furthermore

$$\begin{aligned}
\mathcal{U}_1^B f(x) &= \mathcal{D}_1^B (\langle v_1, x\rangle f(x)),\\
\mathcal{U}_2^B f(x) &= \mathcal{D}_2^B (\langle v_2, x\rangle f(x)) - \kappa((14)+(23)) f(x),\\
\mathcal{U}_3^B f(x) &= \mathcal{D}_3^B (\langle v_3, x\rangle f(x)) - \kappa((12)+(13)+(24)+(34)) f(x).
\end{aligned}$$

For a subset $E \subset \{1,2,3\}$ let $y_E = \prod_{i\in E} y_i$, also let $E_0 = \varnothing$ and $E_k = \{1,\dots,k\}$ for $k = 1,2,3$. The simultaneous eigenfunctions are of the form $y_E f(y^2)$ where $y^2 := (y_1^2, y_2^2, y_3^2)$ and when $E = E_k$ with $0 \le k \le 3$ they are directly expressed as NSJP's (for $\mathbb{R}^3$). The following is the specialization to $\kappa' = 0$ of the type-B result from [2, Corollary 9.3.3, p. 342].

Proposition 2. *Suppose $\alpha \in \mathbb{N}_0^3$ and $k = 0,1,2,3$, then for $1 \le i \le k$*

$$\mathcal{U}_i^B y_{E_k} \zeta_\alpha(y^2) = 2\xi_i(\alpha) y_{E_k} \zeta_\alpha(y^2),$$

and for $k < i \le 3$

$$\mathcal{U}_i^B y_{E_k} \zeta_\alpha(y^2) = (2\xi_i(\alpha) - 1) y_{E_k} \zeta_\alpha(y^2).$$

The polynomial $y_{E_k}\zeta_\alpha(y^2)$ is labeled by $\beta \in \mathbb{N}_0^3$ where $\beta_i = 2\alpha_i + 1$ for $1 \le i \le k$ and $\beta_i = 2\alpha_i$ for $k < i \le 3$. The difference $\beta - \alpha \in \mathbb{N}_0^3$ and appears in the norm formula (the result for the pairing $(f,g) \mapsto f(\mathcal{D}_1^B, \mathcal{D}_2^B, \mathcal{D}_3^B) g(y)|_{y=0}$ applies because of the isomorphism).

Proposition 3. *Suppose $\beta \in \mathbb{N}_0^3$ and β_i is odd for $1 \le i \le k$ and is even otherwise, then for $\alpha_i = \left\lfloor \frac{\beta_i}{2} \right\rfloor$, $1 \le i \le 3$*

$$\left\langle y_{E_k}\zeta_\alpha(y^2), y_{E_k}\zeta_\alpha(y^2)\right\rangle_\kappa = 2^{|\beta|}(3\kappa+1)_{\alpha^+}\left(2\kappa + \frac{1}{2}\right)_{(\beta-\alpha)^+} \frac{h(\alpha,1)}{h(\alpha,\kappa+1)}.$$

(The formulae in [2, Chapter 9] are given for the p-monic polynomials, here we use the x-monic type, see [2, pp. 323–324]). There is an evaluation formula for $\zeta_\alpha(1,1,1)$ which provides the value at $x = (2,0,0,0)$, corresponding to $y = (1,1,1,1)$. Indeed for $\alpha \in \mathbb{N}_0^3$ (see [2, p. 324])

$$\zeta_\alpha(1,1,1) = \frac{(3\kappa+1)_{\alpha^+}}{h(\alpha,\kappa+1)}.$$

For any point $(\pm 2,0,0,0)w$ with $w \in \mathcal{S}_4$ the corresponding y satisfies $y_i = \pm 1$ for $1 \le i \le 3$, so that $y^2 = (1,1,1)$. For any other subset $E \subset \{1,2,3\}$ with $\#E = k$ let $w \in \mathcal{S}_3$ be such that $w(i) \in E$ for $1 \le i \le k$, $1 \le i < j \le k$ or $k < i < j \le 3$ implies $w(i) < w(j)$ (that is, w preserves order on $\{1,\dots,k\}$ and on $\{k+1,\dots,3\}$). Here is the list of sets with corresponding permutations $(w(i))_{i=1}^3$:

$$\begin{aligned}
E &= \{2\}, & w &= (2,1,3),\\
E &= \{3\}, & w &= (3,1,2),\\
E &= \{1,3\}, & w &= (1,3,2),\\
E &= \{2,3\}, & w &= (2,3,1).
\end{aligned}$$

Then (letting w act on y) $wy_{E_k} = y_E$ and for $\alpha \in \mathbb{N}_0^3$ the polynomial $w(y_{E_k}\zeta_\alpha(y^2))$ is a simultaneous eigenfunction and

$$\begin{aligned}
\mathcal{U}_{w(i)}^B w y_{E_k}\zeta_\alpha(y^2) &= 2\xi_i(\alpha) w y_{E_k}\zeta_\alpha(y^2), & 1 \le i \le k,\\
\mathcal{U}_{w(i)}^B w y_{E_k}\zeta_\alpha(y^2) &= (2\xi_i(\alpha) - 1) w y_{E_k}\zeta_\alpha(y^2), & k < i \le 3.
\end{aligned}$$

Define β as before ($\beta_i = 2\alpha_i + 1$ for $1 \le i \le k$ and $\beta_i = 2\alpha_i$ for $k < i \le 3$) then the label for the polynomial $wy_{E_k}\zeta_\alpha\left(y^2\right)$ is $w\beta$ (recall $(w\beta)_i = \beta_{w^{-1}(i)}$). Denote

$$p_{w\beta}(y) := wy_{E_k}\zeta_\alpha\left(y^2\right).$$

This defines a polynomial p_γ for any $\gamma \in \mathbb{N}_0^3$. The norm of $wy_{E_k}\zeta_\alpha\left(y^2\right)$ is the same as that of $y_{E_k}\zeta_\alpha\left(y^2\right)$ since any $w \in \mathcal{S}_3$ acts as an isometry for $\langle\cdot,\cdot\rangle_\kappa$. Suppose $E, E' \subset \{1,2,3\}$ and $E \ne E'$ and $f, g \in \mathcal{P}^{(3)}$ then $\left\langle y_E f\left(y^2\right), y_{E'} g\left(y^2\right)\right\rangle_\kappa = 0$. The root system D_3 is an orbit of the subgroup of diagonal elements of B_3 (isomorphic to $\mathbb{Z}_2^3$). Denote the sign change $y_i \longmapsto -y_i$ by σ_i for $1 \le i \le 3$. From the B_3 results we have $\sigma_i \mathcal{D}_j^B = \mathcal{D}_j^B \sigma_i$ for $1 \le i, j \le 3$ and this implies $\left\langle y_E f\left(y^2\right), y_{E'} g\left(y^2\right)\right\rangle_\kappa = \left\langle \sigma_i y_E f\left(y^2\right), \sigma_i y_{E'} g\left(y^2\right)\right\rangle_\kappa = -\left\langle y_E f\left(y^2\right), y_{E'} g\left(y^2\right)\right\rangle_\kappa$ for any $i \in (E\backslash E') \cup (E'\backslash E)$ (the symmetric difference). Thus $\left\{p_\gamma : \gamma \in \mathbb{N}_0^3\right\}$ is an orthogonal basis for $\langle\cdot,\cdot\rangle_\kappa$.

We consider the $\mathcal{S}_4$-invariant polynomials: they are generated by y_0, $\sum_{i=1}^3 y_i^2$, $y_1y_2y_3$, $\sum_{i=1}^3 y_i^4$. Any invariant is a sum of terms of the form $y_0^n (y_1y_2y_3)^s f\left(y^2\right)$ where $n \in \mathbb{N}_0$, $s = 0$ or 1, and f is a symmetric polynomial in three variables. For now consider only polynomials in $\{y_1, y_2, y_3\}$. Let $\lambda \in \mathbb{N}_0^{3,+}$, then there are two corresponding simultaneous eigenfunctions of $\sum_{i=1}^3 \left(\mathcal{U}_i^B\right)^n$ (it suffices to take $n = 1, 2, 3$ to generate the commutative algebra of $\mathcal{S}_4$-invariant operators). From [2, Theorem 8.5.10] let

$$\begin{aligned}
&A_\lambda = \#\left\{\alpha : \alpha^+ = \lambda\right\} \frac{(3\kappa+1)_\lambda h(\lambda,1)}{\mathcal{E}_1\left(\lambda^R\right) h(\lambda,\kappa+1)}, \qquad (2)\\
&F_\lambda^0(x) = j_\lambda\left(y^2\right),\\
&\left\langle F_\lambda^0, F_\lambda^0\right\rangle_\kappa = 2^{2|\lambda|}\left(2\kappa + \frac{1}{2}\right)_\lambda A_\lambda,\\
&F_\lambda^1(x) = y_1y_2y_3 j_\lambda\left(y^2\right),\\
&\left\langle F_\lambda^1, F_\lambda^1\right\rangle_\kappa = 2^{2|\lambda|}\left(2\kappa + \frac{1}{2}\right)_{(\lambda_1+1,\lambda_2+1,\lambda_3+1)} A_\lambda.
\end{aligned}$$

The polynomials $\left\{F_\lambda^0, F_\lambda^1 : \lambda \in \mathbb{N}_0^{3,+}\right\}$ are pairwise orthogonal.

Up to now we have mostly ignored the fourth dimension, namely, the coordinate y_0. The reflection σ_0 along v_0 (given by $x\sigma_0 = x - \left(\sum_{i=1}^4 x_i\right) v_0$) commutes with the $\mathcal{S}_4$-action. We introduce another parameter κ' and let

$$\begin{aligned}
\mathcal{D}_0 f(x) &= \frac{1}{2}\sum_{i=1}^4 \frac{\partial}{\partial x_i} f(x) + \frac{\kappa'}{\langle x, v_0\rangle}\left(f(x) - f(x\sigma_0)\right),\\
\mathcal{D}_i' f(x) &= \mathcal{D}_i f(x) + \frac{\kappa'}{2\langle x, v_0\rangle}\left(f(x) - f(x\sigma_0)\right).
\end{aligned}$$

The operators $\{\mathcal{D}_i' : 1 \le i \le 4\}$ are the Dunkl operators for the group $W = \mathcal{S}_4 \times \mathbb{Z}_2$ (the reflection group generated by $\{(1,2), (2,3), (3,4), \sigma_0\}$). Then $\mathcal{D}_0 y_0^{2n} = 2n y_0^{2n-1}$ and $\mathcal{D}_0 y_0^{2n+1} = (2n+1+2\kappa') y_0^{2n}$. We define the extended pairing for polynomials

$$\langle f(x), g(x)\rangle_{\kappa,\kappa'} = f\left(\mathcal{D}_1', \ldots, \mathcal{D}_4'\right) g(x)|_{x=0};$$

in terms of y

$$\begin{aligned}
&\langle f_0(y_0) f_1(y_1,y_2,y_3), g_0(y_0) g_1(y_1,y_2,y_3)\rangle_{\kappa,\kappa'}\\
&\quad = f_0(\mathcal{D}_0) g_0(y_0)|_{y_0=0} \times f_1\left(\mathcal{D}_1^B, \ldots\right) g_1(y_1,y_2,y_3)|_{y=0}\\
&\quad = f_0(\mathcal{D}_0) g_0(y_0)|_{y_0=0} \times \langle f_1, g_1\rangle_\kappa.
\end{aligned}$$

It is easily shown by induction that for $n \in \mathbb{N}_0$

$$\begin{aligned}
\left\langle y_0^{2n}, y_0^{2n} \right\rangle_{\kappa,\kappa'} &= 2^{2n} n! \left(\kappa' + \frac{1}{2} \right)_n , \\
\left\langle y_0^{2n+1}, y_0^{2n+1} \right\rangle_{\kappa,\kappa'} &= 2^{2n+1} n! \left(\kappa' + \frac{1}{2} \right)_{n+1} .
\end{aligned}$$

The direct product structure implies that $\left\{ p_{(\gamma_1,\gamma_2,\gamma_3)}(y)\, y_0^{\gamma_4} : \gamma \in \mathbb{N}_0^4 \right\}$ is an orthogonal basis for $\langle \cdot, \cdot \rangle_{\kappa,\kappa'}$.

4 Hermite polynomials

The pairing $\langle \cdot, \cdot \rangle_{\kappa,\kappa'}$ is related to a measure on $\mathbb{R}^4$: let $\kappa, \kappa' \geq 0$ and

$$\begin{aligned}
dm(x) &:= (2\pi)^{-2} \exp\left(-\frac{1}{2} \, |x|^2 \right) dx, \qquad x \in \mathbb{R}^4, \\
h(x) &:= \prod_{1 \leq i < j \leq 4} |x_i - x_j|^{\kappa} \, |y_0|^{\kappa'} , \\
c_{\kappa,\kappa'}^{-1} &:= \int_{\mathbb{R}^4} h(x)^2 \, dm(x) , \\
d\mu_{\kappa,\kappa'}(x) &:= c_{\kappa,\kappa'} h(x)^2 \, dm(x) .
\end{aligned}$$

In fact

$$c_{\kappa,\kappa'}^{-1} = 2^{\kappa'} \frac{\Gamma\left(\kappa' + \frac{1}{2}\right) \Gamma(2\kappa + 1) \Gamma(3\kappa + 1) \Gamma(4\kappa + 1)}{\Gamma\left(\frac{1}{2}\right) \Gamma(\kappa + 1)^3} .$$

The integral is a special case of the general formula (any suitably integrable function f on $\mathbb{R}$):

$$\begin{aligned}
&(2\pi)^{-N/2} \int_{\mathbb{R}^N} \prod_{1 \leq i < j \leq N} |x_i - x_j|^{2\kappa} f\left(\sum_{i=1}^{N} x_i \right) \exp\left(-\frac{1}{2} \, |x|^2 \right) dx \\
&\qquad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\left(t\sqrt{N} \right) e^{-t^2/2} dt \cdot \prod_{j=2}^{N} \frac{\Gamma(j\kappa + 1)}{\Gamma(\kappa + 1)} ;
\end{aligned}$$

this follows from the Macdonald–Mehta–Selberg integral for $\mathcal{S}_N$ and the use of an orthogonal coordinate system for $\mathbb{R}^N$ in which $\sum_{i=1}^{N} x_i / \sqrt{N}$ is one of the coordinates. The Laplacian is $\Delta_h := \sum_{i=1}^{4} (\mathcal{D}_i')^2 = \sum_{i=1}^{3} \left(\mathcal{D}_i^B\right)^2 + \mathcal{D}_0^2$. Also set $\Delta_B := \sum_{i=1}^{3} \left(\mathcal{D}_i^B\right)^2$. Then for $f, g \in \mathcal{P}$ [2, Theorem 5.2.7]

$$\langle f, g \rangle_{\kappa,\kappa'} = \int_{\mathbb{R}^4} \left(e^{-\Delta_h/2} f(x) \right) \left(e^{-\Delta_h/2} g(x) \right) d\mu_{\kappa,\kappa'}(x) .$$

The orthogonal basis elements $p_\gamma(y)\, y_0^n$ ($\gamma \in \mathbb{N}_0^3, n \in \mathbb{N}_0$) are transformed to orthogonal polynomials in $L^2\left(\mathbb{R}^4, \mu_{\kappa,\kappa'}\right)$ under the action of $e^{-\Delta_h/2} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{2}\right)^n \Delta_h^n$ (only finitely many terms are nonzero when acting on a polynomial). We have

$$e^{-\Delta_h/2} \left(p_\gamma(y)\, y_0^n \right) = \left(e^{-\Delta_B/2} p_\gamma(y) \right) \left(e^{-\mathcal{D}_0^2/2} y_0^n \right) .$$

Then for $n \in \mathbb{N}_0$

$$e^{-\mathcal{D}_0^2/2} y_0^{2n} = (-2)^n\, n! L_n^{\kappa' - \frac{1}{2}} \left(\frac{y_0^2}{2} \right),$$
$$e^{-\mathcal{D}_0^2/2} y_0^{2n+1} = (-2)^n\, n! y_0 L_n^{\kappa' + \frac{1}{2}} \left(\frac{y_0^2}{2} \right).$$

Recall the Laguerre polynomials $\{L_n^a(t) : n \in \mathbb{N}_0\}$ are the orthogonal polynomials for the measure $t^a e^{-t} dt$ on $\{t : t \geq 0\}$ with $a > -1$, and

$$L_n^a(t) = \frac{(a+1)_n}{n!} \sum_{i=0}^{n} \frac{(-n)_i}{(a+1)_i} \frac{t^i}{i!}.$$

The result of applying $e^{-\Delta_B/2}$ to a polynomial $x_{E_k} \zeta_\alpha(y^2)$ is a complicated expression involving some generalized binomial coefficients (see [2, Proposition 9.4.5]). For the symmetric cases $j_\lambda(y^2)$ and $y_1 y_2 y_3 j_\lambda(y^2)$, $\lambda \in \mathbb{N}_0^{3,+}$ these coefficients were investigated by Lassalle [4] and Okounkov and Olshanski [5, equation (3.2)]; in the latter paper there is an explicit formula.

Finally we can use our orthogonal basis to analyze a modification of the type-A quantum Calogero–Sutherland model with four particles on a line and harmonic confinement. By rescaling, the Hamiltonian (with exchange terms) can be written as:

$$\mathcal{H} = -\Delta + \frac{|x|^2}{4} + 2\kappa \sum_{1 \leq i < j \leq 4} \frac{\kappa - (i,j)}{(x_i - x_j)^2} + \frac{4\kappa'(\kappa' - \sigma_0)}{(x_1 + x_2 + x_3 + x_4)^2}.$$

When this is applied to a W-invariant the reflections (i,j) and σ_0 are replaced by the scalar 1. We combine the type-B results from [2, Section 9.6.5] (setting $\kappa' = 0$ in the formulae) with simple $\mathbb{Z}_2$ calculations. The nonnormalized base state is

$$\psi_0(x) := \prod_{1 \leq i < j \leq 4} |x_i - x_j|^\kappa |y_0|^{\kappa'} \exp\left(-\frac{1}{4} |x|^2 \right).$$

Then

$$\psi_0^{-1} \mathcal{H} \psi_0 = -\Delta_B - \mathcal{D}_0^2 + \sum_{i=0}^{3} y_i \frac{\partial}{\partial y_i} + 6\kappa + \kappa' + 2.$$

This operator has polynomial eigenfunctions and the eigenvalues are the energy levels of the associated states. From [2, Section 9.6.5] we have

$$e^{-\Delta_B/2} \sum_{i=1}^{3} \mathcal{U}_i^B e^{\Delta_B/2} = -\Delta_B + \sum_{i=1}^{3} y_i \frac{\partial}{\partial y_i} + 6\kappa + 3,$$

and by direct calculations

$$\mathcal{D}_0^2 = \frac{\partial^2}{\partial y_0^2} + \frac{2\kappa'}{y_0} \frac{\partial}{\partial y_0} - \kappa' \frac{1 - \sigma_0}{y_0^2},$$
$$e^{-\mathcal{D}_0^2/2} (\mathcal{D}_0 y_0 - \kappa' \sigma_0) e^{\mathcal{D}_0^2/2} = -\mathcal{D}_0^2 + y_0 \frac{\partial}{\partial y_0} + \kappa' + 1.$$

Combine these results:

$$\psi_0^{-1} \mathcal{H} \psi_0 = e^{-\Delta_h/2} \left(\sum_{i=1}^{3} \mathcal{U}_i^B + \mathcal{D}_0 y_0 - \kappa' \sigma_0 - 2 \right) e^{\Delta_h/2}.$$

Thus $\left(e^{-\Delta_h/2}\left(p_\gamma\left(y\right)y_0^n\right)\right)\psi_0$ is an eigenfunction of $\mathcal{H}$ for each $\gamma\in\mathbb{N}_0^3$, $n\in\mathbb{N}_0$. It suffices to consider $y_{E_k}\zeta_\alpha\left(y^2\right)y_0^n$. We have

$$\begin{aligned}&\left(\mathcal{D}_0 y_0-\kappa'\sigma_0\right)y_0^{2n}=\left(\left(2n+1+2\kappa'\right)-\kappa'\right)y_0^{2n},\\&\left(\mathcal{D}_0 y_0-\kappa'\sigma_0\right)y_0^{2n+1}=\left(\left(2n+2\right)+\kappa'\right)y_0^{2n},\\&\left(\mathcal{D}_0 y_0-\kappa'\sigma_0\right)y_0^{n}=\left(n+1+\kappa'\right)y_0^{n}.\end{aligned}$$

Furthermore $\sum_{i=1}^3\mathcal{U}_i^B\left(y_{E_k}\zeta_\alpha\left(y^2\right)\right)=\left(2\sum_{i=1}^3\xi_i\left(\alpha\right)-\left(3-k\right)\right)y_{E_k}\zeta_\alpha\left(y^2\right)$; the eigenvalue is $\left(2\left|\alpha\right|+k\right)+6\kappa+3=\left|\beta\right|+6\kappa+3$ (where $\beta_i=2\alpha_i+1$ for $1\le i\le k$ and $\beta_i=2\alpha_i$ for $k<i\le 3$). The energy level for $\left(e^{-\Delta_h/2}\left(p_\beta\left(y\right)y_0^n\right)\right)\psi_0$ is $\left|\beta\right|+n+6\kappa+\kappa'+2$. Observe the degeneracy of the energy levels; only the total degree $\left|\beta\right|+n$ appears. The (nonnormalized) W-invariant eigenfunctions are ($\lambda\in\mathbb{N}_0^3$)

$$\begin{aligned}&\left(e^{-\Delta_B/2}\left(j_\lambda\left(y^2\right)\right)L_n^{\kappa'-1/2}\left(\frac{y_0^2}{2}\right)\right)\psi_0\left(x\right),\\&\left(e^{-\Delta_B/2}\left(y_1y_2y_3j_\lambda\left(y^2\right)\right)L_n^{\kappa'-1/2}\left(\frac{y_0^2}{2}\right)\right)\psi_0\left(x\right).\end{aligned}$$

The L^2-norms can be found by using equation (2).

In conclusion, we have found an unusual basis for polynomials which allowed an extra parameter in the action of $\mathcal{S}_4$ on $\mathbb{R}^4$. This exploited the fact that $v_0^\perp$ has an orthogonal basis which together with its antipodes forms an $\mathcal{S}_4$-orbit. The pairing $\langle\cdot,\cdot\rangle_\kappa$ has an analog for each reflection group and weight function. We are left with the interesting problem of how to construct orthogonal bases for groups not of type A or B.

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