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On Miura Transformations and Volterra-Type Equations Associated with the Adler–Bobenko–Suris Equations

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Abstract. We construct Miura transformations mapping the scalar spectral problems of the integrable lattice equations belonging to the Adler–Bobenko–Suris (ABS) list into the discrete Schrödinger spectral problem associated with Volterra-type equations. We show that the ABS equations correspond to Bäcklund transformations for some particular cases of the discrete Krichever–Novikov equation found by Yamilov (YdKN equation). This enables us to construct new generalized symmetries for the ABS equations. The same can be said about the generalizations of the ABS equations introduced by Tongas, Tsoubelis and Xenitidis. All of them generate Bäcklund transformations for the YdKN equation. The higher order generalized symmetries we construct in the present paper confirm their integrability.

Key words: Miura transformations; generalized symmetries; ABS lattice equations

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1 Introduction

The discovery of new two-dimensional integrable partial difference equations (or $\mathbb{Z}^2$-lattice equations) is always a very challenging problem as, by proper continuous limits, many other results on differential-difference and partial differential equations may be obtained. Moreover many physical and biological applications involve discrete systems, see for instance [13, 25] and references therein.

The theory of nonlinear integrable differential equations got a boost when Gardner, Green, Kruskal and Miura introduced the Inverse Scattering Method for the solution of the Korteweg–de Vries equation. A summary of these results can be found in the Encyclopedia of Mathematical Physics [12]. A few techniques have been introduced to classify integrable partial differential equations. Let us just mention the classification scheme introduced by Shabat, using the formal symmetry approach (see [21] for a review). This approach has been successfully extended to the differential-difference case by Yamilov [30, 31, 20]. In the completely discrete case the situation turns out to be quite different. For instance, in the case of $\mathbb{Z}^2$-lattice equations the formal symmetry technique does not work. In this framework, the first exhaustive classifications of families of lattice equations have been presented in [1] by Adler and in [2, 3] by Adler, Bobenko and Suris.

In the present paper we shall consider the Adler–Bobenko–Suris (ABS) classification of $\mathbb{Z}^2$-lattice equations defined on the square lattice [2]. We refer to the papers [3, 24, 28, 17, 18, 27] for some recents results about these equations. Our main purpose is the analysis of their transformation properties. In fact, our aim is, on the one hand, to present new Miura transformations between the ABS equations and Volterra-type difference equations and on the other hand, to show that the ABS equations correspond to Bäcklund transformations for some particular cases of the discrete Krichever–Novikov equation found by Yamilov (YdKN equation) [30].

Section 2 is devoted to a short review of the integrable $\mathbb{Z}^2$-lattice equations derived in [2] and to present details on their matrix and scalar spectral problems. In Section 3, by transforming the obtained scalar spectral problems into the discrete Schrödinger spectral problem associated with the Volterra lattice we will be able to connect the ABS equation with Volterra-type equations. In Section 4 we prove that the ABS equations correspond to Bäcklund transformations for certain subcases of the YdKN equation. Using this result and a master symmetry of the YdKN equation, we construct new generalized symmetries for the ABS list. Then we discuss the integrability of a class of non-autonomous ABS equations and of a generalization of the ABS equations introduced by Tongas, Tsoubelis and Xenitidis in [28]. Section 5 is devoted to some concluding remarks.

2 A short review of the ABS equations

A two-dimensional partial difference equation is a functional relation among the values of a function $u : \mathbb{Z}^2 \to \mathbb{C}$ at different points of the lattices of indices n, m. It involves the independent variables n, m and the lattice parameters α, β

$$\mathcal{E}(n, m, u_{n,m}, u_{n+1,m}, u_{n,m+1}, \dots; \alpha, \beta) = 0.$$

For the dependent variable u we shall adopt the following notation throughout the paper

$$u = u_{0,0} = u_{n,m}, \qquad u_{k,l} = u_{n+k,m+l}, \qquad k, l \in \mathbb{Z}. \tag{1}$$

We consider here the ABS list of integrable lattice equations, namely those affine linear (i.e. polynomial of degree one in each argument) partial difference equations of the form

$$\mathcal{E}(u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}; \alpha, \beta) = 0, \tag{2}$$

whose integrability is based on the *consistency around a cube* [2, 3]. The function $\mathcal{E}$ depends explicitly on the values of u at the vertices of an elementary quadrilateral, i.e. $\partial_{u_{i,j}} \mathcal{E} \neq 0$, where $i, j = 0, 1$. The lattice parameters α, β may, in general, depend on the variables n, m, i.e. $\alpha = \alpha_n$, $\beta = \beta_m$. However, we shall discuss such non-autonomous extensions in Section 4.

The complete list of the ABS equations can be found in [2]. Their integrability holds by construction since the consistency around a cube furnishes their Lax pairs [2, 9, 22]. The ABS equations are given by the list H

$$\begin{aligned}
&(\mathrm{H1}) \quad (u_{0,0} - u_{1,1})(u_{1,0} - u_{0,1}) - \alpha + \beta = 0,\\
&(\mathrm{H2}) \quad (u_{0,0} - u_{1,1})(u_{1,0} - u_{0,1}) + (\beta - \alpha)(u_{0,0} + u_{1,0} + u_{0,1} + u_{1,1}) - \alpha^2 + \beta^2 = 0,\\
&(\mathrm{H3}) \quad \alpha(u_{0,0}u_{1,0} + u_{0,1}u_{1,1}) - \beta(u_{0,0}u_{0,1} + u_{1,0}u_{1,1}) + \delta(\alpha^2 - \beta^2) = 0,
\end{aligned}$$

and the list Q

$$\begin{aligned}
&(\mathrm{Q1}) \quad \alpha(u_{0,0} - u_{0,1})(u_{1,0} - u_{1,1}) - \beta(u_{0,0} - u_{1,0})(u_{0,1} - u_{1,1}) + \delta^2\alpha\beta(\alpha - \beta) = 0,\\
&(\mathrm{Q2}) \quad \alpha(u_{0,0} - u_{0,1})(u_{1,0} - u_{1,1}) - \beta(u_{0,0} - u_{1,0})(u_{0,1} - u_{1,1})
\end{aligned}$$

$$+\,\alpha\beta(\alpha-\beta)(u_{0,0}+u_{1,0}+u_{0,1}+u_{1,1})-\alpha\beta(\alpha-\beta)(\alpha^2-\alpha\beta+\beta^2)=0,$$

$$\begin{aligned}(\mathrm{Q3})\quad &(\beta^2-\alpha^2)(u_{0,0}u_{1,1}+u_{1,0}u_{0,1})+\beta(\alpha^2-1)(u_{0,0}u_{1,0}+u_{0,1}u_{1,1})\\ &-\alpha(\beta^2-1)(u_{0,0}u_{0,1}+u_{1,0}u_{1,1})-\frac{\delta^2(\alpha^2-\beta^2)(\alpha^2-1)(\beta^2-1)}{4\alpha\beta}=0,\end{aligned}$$

$$\begin{aligned}(\mathrm{Q4})\quad &a_0u_{0,0}u_{1,0}u_{0,1}u_{1,1}+a_1(u_{0,0}u_{1,0}u_{0,1}+u_{1,0}u_{0,1}u_{1,1}+u_{0,1}u_{1,1}u_{0,0}+u_{1,1}u_{0,0}u_{1,0})\\ &+a_2(u_{0,0}u_{1,1}+u_{1,0}u_{0,1})+\bar{a}_2(u_{0,0}u_{1,0}+u_{0,1}u_{1,1})+\widetilde{a}_2(u_{0,0}u_{0,1}+u_{1,0}u_{1,1})\\ &+a_3(u_{0,0}+u_{1,0}+u_{0,1}+u_{1,1})+a_4=0.\end{aligned}$$

The coefficients a_i's appearing in equation (Q4) are connected to α and β by the relations

$$\begin{aligned}&a_0=a+b,\qquad a_1=-a\beta-b\alpha,\qquad a_2=a\beta^2+b\alpha^2,\\ &\bar{a}_2=\frac{ab(a+b)}{2(\alpha-\beta)}+a\beta^2-\left(2\alpha^2-\frac{g_2}{4}\right)b,\qquad \widetilde{a}_2=\frac{ab(a+b)}{2(\beta-\alpha)}+b\alpha^2-\left(2\beta^2-\frac{g_2}{4}\right)a,\\ &a_3=\frac{g_3}{2}a_0-\frac{g_2}{4}a_1,\qquad a_4=\frac{g_2^2}{16}a_0-g_3a_1,\end{aligned}$$

with $a^2=r(\alpha)$, $b^2=r(\beta)$, $r(x)=4x^3-g_2x-g_3$.

Following [2] we remark that

- Equations (Q1)–(Q3) and (H1)–(H3) are all degenerate subcases of equation (Q4) [7].
- Parameter δ in equations (H3), (Q1) and (Q3) can be rescaled, so that one can assume without loss of generality that $\delta=0$ or $\delta=1$.
- The original ABS list contains two further equations (list A)

$$\begin{aligned}(\mathrm{A1})\quad &\alpha(u_{0,0}+u_{0,1})(u_{1,1}+u_{1,0})-\beta(u_{0,0}+u_{1,0})(u_{1,1}+u_{0,1})-\delta^2\alpha\beta(\alpha-\beta)=0,\\ (\mathrm{A2})\quad &(\beta^2-\alpha^2)(u_{0,0}u_{1,0}u_{0,1}u_{1,1}+1)+\beta(\alpha^2-1)(u_{0,0}u_{0,1}+u_{1,0}u_{1,1})\\ &\quad-\alpha(\beta^2-1)(u_{0,0}u_{1,0}+u_{0,1}u_{1,1})=0.\end{aligned}$$

  Equations (A1) and (A2) can be transformed by an extended group of Möbius transformations into equations (Q1) and (Q3) respectively. Indeed, any solution $u=u_{n,m}$ of (A1) is transformed into a solution $\widetilde{u}=\widetilde{u}_{n,m}$ of (Q1) by $u_{n,m}=(-1)^{n+m}\widetilde{u}_{n,m}$ and any solution $u=u_{n,m}$ of (A2) is transformed into a solution $\widetilde{u}=\widetilde{u}_{n,m}$ of (Q3) with $\delta=0$ by $u_{n,m}=(\widetilde{u}_{n,m})^{(-1)^{n+m}}$.

Some of the above equations were known before Adler, Bobenko and Suris presented their classification, see for instance [23, 14]. We finally recall that a more general classification of integrable lattice equations defined on the square has been recently carried out by Adler, Bobenko and Suris in [3]. But here we shall consider only the lists H and Q contained in [2].

2.1 Spectral problems of the ABS equations

The algorithmic procedure described in [2, 9, 22] produces a 2×2 matrix Lax pair for the ABS equations, thus ensuring their integrability. It may be written as

$$\Psi_{1,0}=L(u_{0,0},u_{1,0};\alpha,\lambda)\Psi_{0,0},\qquad \Psi_{0,1}=M(u_{0,0},u_{0,1};\beta,\lambda)\Psi_{0,0},\tag{3}$$

with $\Psi=(\psi(\lambda),\phi(\lambda))^T$, where the lattice parameter λ plays the role of the spectral parameter. We shall use the following notation

$$L(u_{0,0},u_{1,0};\alpha,\lambda)=\frac{1}{\ell}\begin{pmatrix}L_{11}&L_{12}\\L_{21}&L_{22}\end{pmatrix},\qquad M(u_{0,0},u_{0,1};\beta,\lambda)=\frac{1}{t}\begin{pmatrix}M_{11}&M_{12}\\M_{21}&M_{22}\end{pmatrix},$$

Table 1. Matrix L for the ABS equations (in equation (Q4) $a^2 = r(\alpha)$, $b^2 = r(\lambda)$, $r(x) = 4x^3 - g_2x - g_3$).

	L_{11}	L_{12}	L_{21}	L_{22}
H1	$u_{0,0} - u_{1,0}$	$(u_{0,0} - u_{1,0})^2 + \alpha - \lambda$	1	$u_{0,0} - u_{1,0}$
H2	$u_{0,0} - u_{1,0} + \alpha - \lambda$	$(u_{0,0} - u_{1,0})^2 + 2(\alpha - \lambda)(u_{0,0} + u_{1,0}) +$ $+ \alpha^2 - \lambda^2$	1	$u_{0,0} - u_{1,0} - \alpha + \lambda$
H3	$\lambda u_{0,0} - \alpha u_{1,0}$	$\lambda(u_{0,0}^2 + u_{1,0}^2) - 2\alpha u_{0,0}u_{1,0} + \delta(\lambda^2 - \alpha^2)$	α	$\alpha u_{0,0} - \lambda u_{1,0}$
Q1	$\lambda(u_{1,0} - u_{0,0})$	$-\lambda(u_{1,0} - u_{0,0})^2 + \delta\alpha\lambda(\alpha - \lambda)$	$-\alpha$	$\lambda(u_{1,0} - u_{0,0})$
Q2	$\lambda(u_{1,0} - u_{0,0}) +$ $+ \alpha\lambda(\alpha - \lambda)$	$-\lambda(u_{1,0} - u_{0,0})^2 +$ $+ 2\alpha\lambda(\alpha - \lambda)(u_{1,0} + u_{0,0}) -$ $- \alpha\lambda(\alpha - \lambda)(\alpha^2 - \alpha\lambda + \lambda^2)$	$-\alpha$	$\lambda(u_{1,0} - u_{0,0}) -$ $- \alpha\lambda(\alpha - \lambda)$
Q3	$\alpha(\lambda^2 - 1)u_{0,0} -$ $- (\lambda^2 - \alpha^2)u_{1,0}$	$-\lambda(\alpha^2 - 1)u_{0,0}u_{1,0} +$ $+ \delta(\alpha^2 - \lambda^2)(\alpha^2 - 1)(\lambda^2 - 1)/(4\alpha\lambda)$	$\lambda(\alpha^2 - 1)$	$(\lambda^2 - \alpha^2)u_{0,0} -$ $- \alpha(\lambda^2 - 1)u_{1,0}$
Q4	$-a_1u_{0,0}u_{1,0} -$ $- a_2u_{1,0} - \tilde{a}_2u_{0,0} - a_3$	$-\bar{a}_2u_{0,0}u_{1,0} - a_3(u_{0,0} + u_{1,0}) - a_4$	$a_0u_{0,0}u_{1,0} + \bar{a}_2 +$ $+ a_1(u_{0,0} + u_{1,0})$	$a_1u_{0,0}u_{1,0} + a_2u_{0,0} +$ $+ \tilde{a}_2u_{1,0} + a_3$

where $\ell = \ell_{0,0} = \ell(u_{0,0}, u_{1,0}; \alpha, \lambda)$, $t = t_{0,0} = t(u_{0,0}, u_{0,1}; \beta, \lambda)$, $L_{ij} = L_{ij}(u_{0,0}, u_{1,0}; \alpha, \lambda)$ and $M_{ij} = M_{ij}(u_{0,0}, u_{0,1}; \beta, \lambda)$, $i, j = 1, 2$. The matrix M can be obtained from L by replacing α with β and shifting along direction 2 instead of 1. In Table 1 we give the entries of the matrix L for the ABS equations.

Note that ℓ and t are computed by requiring that the compatibility condition between L and M produces the ABS equations (H1)–(H3) and (Q1)–(Q4). The factor ℓ can be written as

$$\ell_{0,0} = f(\alpha, \lambda)[\rho(u_{0,0}, u_{1,0}; \alpha)]^{1/2}, \tag{4}$$

where the functions $f = f(\alpha, \lambda)$ is an arbitrary normalization factor. The functions $f = f(\alpha, \lambda)$ and $\rho = \rho_{0,0} = \rho(u_{0,0}, u_{1,0}; \alpha)$ for equations (H1)–(H3) and (Q1)–(Q4) are given in Table 2. A formula similar to (4) holds also for the factor t.

The scalar Lax pairs for the ABS equations may be immediately computed from equation (3). Let us write the scalar equation just for the second component ϕ of the vector Ψ (the use of the first component would give similar results). For equations (H1)–(H3) and (Q1)–(Q3) it reads

$$(\rho_{1,0})^{1/2}\phi_{2,0} - (u_{2,0} - u_{0,0})\phi_{1,0} + (\rho_{0,0})^{1/2}\mu\phi_{0,0} = 0, \tag{5}$$

where the explicit expressions of $\mu = \mu(\alpha, \lambda)$ are given in Table 2. The corresponding scalar equation for equation (Q4) takes a different form and needs a separate analysis which will be done in a separate work.

3 Miura transformations for equations (H1)–(H3) and (Q1)–(Q3)

The aim of this Section is to show the existence of a Miura transformation mapping the scalar spectral problem (5) of equations (H1)–(H3) and (Q1)–(Q3) into the discrete Schrödinger spectral problem associated with the Volterra lattice [10]

$$\phi_{-1,0} + v_{0,0}\phi_{1,0} = p(\lambda)\phi_{0,0}, \tag{6}$$

where $v_{0,0}$ is the potential of the spectral problem and the function $p(\lambda)$ plays the role of the spectral parameter.

Table 2. Scalar spectral problems for the ABS equations (in equation (Q4) $c^2 = r(\lambda)$, $r(x) = 4x^3 - g_2x - g_3$).

	$f(\alpha,\lambda)$	$\rho(u_{0,0},u_{1,0};\alpha)$	$\mu(\alpha,\lambda)$
H1	-1	1	$\lambda-\alpha$
H2	-1	$u_{0,0}+u_{1,0}+\alpha$	$2(\lambda-\alpha)$
H3	$-\lambda$	$u_{0,0}u_{1,0}+\delta\alpha$	$\dfrac{\alpha^2-\lambda^2}{\alpha\lambda^2}$
Q1	λ	$(u_{1,0}-u_{0,0})^2-\delta^2\alpha^2$	$\dfrac{\lambda-\alpha}{\lambda}$
Q2	λ	$(u_{1,0}-u_{0,0})^2-2\alpha^2(u_{1,0}+u_{0,0})+\alpha^4$	$\dfrac{\lambda-\alpha}{\lambda}$
Q3	$\alpha(1-\lambda^2)$	$\alpha(u_{0,0}^2+u_{1,0}^2)-(\alpha^2+1)u_{0,0}u_{1,0}+\dfrac{\delta^2(\alpha^2-1)^2}{4\alpha}$	$\dfrac{\alpha^2-\lambda^2}{\alpha^2(1-\lambda^2)}$
Q4	$(\alpha-\lambda)c^{1/2}\times$ $\times\left[2a+c+\frac{1}{4}\left(\frac{a+c}{\alpha-\lambda}\right)^3-\frac{3\alpha(a+c)}{\alpha-\lambda}\right]^{1/2}$	$(u_{0,0}u_{1,0}+\alpha u_{0,0}+\alpha u_{1,0}+g_2/4)^2-$ $-(u_{0,0}+u_{1,0}+\alpha)(4\alpha u_{0,0}u_{1,0}-g_3)$	$-$

Suppose that a function $s_{0,0} = s(u_{0,0}, u_{1,0}, u_{0,1}, \dots)$ is given by the linear equation

$$\frac{s_{0,0}}{s_{1,0}} = \frac{u_{2,0}-u_{0,0}}{(\rho_{0,0})^{1/2}}. \tag{7}$$

By performing the transformation $\phi_{0,0} \mapsto \mu^{n/2}s_{0,0}\phi_{0,0}$, and taking into account equation (7), equation (5) is mapped into the scalar spectral problem (6) with

$$v_{0,0} = \frac{\rho_{0,0}}{(u_{1,0}-u_{-1,0})(u_{2,0}-u_{0,0})}, \qquad p(\lambda) = [\mu(\alpha,\lambda)]^{-1/2}. \tag{8}$$

From these results there follow some remarkable consequences: (i) There exists a Miura transformation between all equations of the set (H1)–(H3) and (Q1)–(Q3). Some results on this claim can be found in [7]; (ii) The Miura transformation (8) can be inverted by solving a linear difference equation. Therefore we can in principle use these remarks to find explicit solutions of the ABS equations in terms of the solutions of the Volterra equation.

The following statement holds.

Proposition 1. *The field u for equations* (H1)–(H3) *and* (Q1)–(Q3) *can be expressed in terms of the potential v of the spectral problem* (6) *through the solution of the following linear difference equations*

$$\mathrm{H1}: \quad u_{2,0} - \frac{(v_{0,0}+v_{-1,0})}{v_{0,0}}u_{0,0} + \frac{v_{-1,0}}{v_{0,0}}u_{-2,0} = 0, \tag{9}$$

$$\mathrm{H2}: \quad u_{2,0} - \frac{v_{0,0}+v_{-1,0}}{v_{0,0}}u_{0,0} + \frac{v_{-1,0}}{v_{0,0}}u_{-2,0} - \frac{1}{v_{0,0}} = 0, \tag{10}$$

$$\mathrm{H3}: \quad u_{2,0} - \frac{1+v_{0,0}+v_{-1,0}}{v_{0,0}}u_{0,0} + \frac{v_{-1,0}}{v_{0,0}}u_{-2,0} = 0, \tag{11}$$

$$\mathrm{Q1}: \quad u_{2,0} - \frac{1}{v_{0,0}}u_{1,0} + \frac{2-v_{0,0}-v_{-1,0}}{v_{0,0}}u_{0,0} - \frac{1}{v_{0,0}}u_{-1,0} + \frac{v_{-1,0}}{v_{0,0}}u_{-2,0} = 0, \tag{12}$$

$$\mathrm{Q2}: \quad u_{2,0} - \frac{1}{v_{0,0}}u_{1,0} + \frac{2-v_{0,0}-v_{-1,0}}{v_{0,0}}u_{0,0} - \frac{1}{v_{0,0}}u_{-1,0} + \frac{v_{-1,0}}{v_{0,0}}u_{-2,0} + \frac{2\alpha^2}{v_{0,0}} = 0, \tag{13}$$

$$\mathrm{Q3}: \quad u_{2,0} - \frac{\alpha}{v_{0,0}}u_{1,0} + \frac{\alpha^2+1-v_{0,0}-v_{-1,0}}{v_{0,0}}u_{0,0} - \frac{\alpha}{v_{0,0}}u_{-1,0} + \frac{v_{-1,0}}{v_{0,0}}u_{-2,0} = 0. \tag{14}$$

Proof. From equation (8) we get

$$v_{0,0}(u_{2,0}-u_{0,0})=\frac{\rho_{0,0}}{u_{1,0}-u_{-1,0}},\qquad v_{-1,0}(u_{0,0}-u_{-2,0})=\frac{\rho_{-1,0}}{u_{1,0}-u_{-1,0}}.$$

Subtracting these relations and taking into account that (see equation (A.11) in [28])

$$\partial_{u_{1,0}}\rho_{0,0}+\partial_{u_{-1,0}}\rho_{-1,0}=2\frac{\rho_{0,0}-\rho_{-1,0}}{u_{1,0}-u_{-1,0}},$$

one arrives at

$$v_{0,0}(u_{2,0}-u_{0,0})-v_{-1,0}(u_{0,0}-u_{-2,0})=\frac{1}{2}\left(\partial_{u_{1,0}}\rho_{0,0}+\partial_{u_{-1,0}}\rho_{-1,0}\right). \tag{15}$$

Writing equation (15) explicitly for equations (H1)–(H3) and (Q1)–(Q3) we obtain equations (9)–(14). ■

4 Generalized symmetries of the ABS equations

Lie symmetries of equation (2) are given by those continuous transformations which leave the equation invariant. We refer to [19, 31] for a review on symmetries of discrete equations.

From the infinitesimal point of view, Lie symmetries are obtained by requiring the infinitesimal invariant condition

$$\left(\mathrm{pr}\,\widehat{X}_{0,0}\right)\mathcal{E}\big|_{\mathcal{E}=0}=0, \tag{16}$$

where

$$\widehat{X}_{0,0}=F_{0,0}(u_{0,0},u_{1,0},u_{0,1},\dots)\partial_{u_{0,0}}. \tag{17}$$

By $\mathrm{pr}\,\widehat{X}_{0,0}$ we mean the prolongation of the infinitesimal generator $\widehat{X}_{0,0}$ to all points appearing in $\mathcal{E}=0$.

If $F_{0,0}=F_{0,0}(u_{0,0})$ then we get *point symmetries* and the procedure to construct them from equation (16) is purely algorithmic [19]. If $F_{0,0}=F_{0,0}(u_{0,0},u_{1,0},u_{0,1},\dots)$ the obtained symmetries are called *generalized symmetries*. In the case of nonlinear discrete equations, the Lie point symmetries are not very common, but, if the equation is integrable, it is possible to construct an infinite family of generalized symmetries.

In correspondence with the infinitesimal generator (17) we can in principle construct a group transformation by integrating the initial boundary problem

$$\frac{du_{0,0}(\varepsilon)}{d\varepsilon}=F_{0,0}(u_{0,0}(\varepsilon),u_{1,0}(\varepsilon),u_{0,1}(\varepsilon),\dots), \tag{18}$$

with $u_{0,0}(\varepsilon=0)=v_{0,0}$, where $\varepsilon\in\mathbb{R}$ is the continuous Lie group parameter and $v_{0,0}$ is a solution of equation (2). This can be done effectively only in the case of point symmetries as in the generalized case we have a nonlinear differential-difference equation for which we cannot find the general solution , but, at most, we can construct particular solutions.

Equation (16) is equivalent to the request that the ε-derivative of the equation $\mathcal{E}=0$, written for $u_{0,0}(\varepsilon)$, is identically satisfied on its solutions when the ε-evolution of $u_{0,0}(\varepsilon)$ is given by equation (18). This is also equivalent to say that the flows (in the group parameter space) given by equation (18) are compatible or commute with $\mathcal{E}=0$.

In the papers [24, 28] the three and five-point generalized symmetries have been found for all equations of the ABS list. We shall use these results to show that the ABS equations may be interpreted as Bäcklund transformations for the differential-difference YdKN equation [30]. This observation will allow us to provide an infinite class of generalized symmetries for the lattice equations belonging to the ABS list. We shall also discuss the non-autonomous case and the generalizations of the ABS equations considered in [28].

4.1 The ABS equations as Bäcklund transformations of the YdKN equation

In the following we show that the ABS equations may be seen as Bäcklund transformations of the YdKN equation. Moreover we prove that the symmetries of the ABS equations [24, 28] are subcases of the YdKN equation. For the sake of clarity we consider in a more detailed way just the case of equation (H3). Similar results can be obtained for the whole ABS list (see Proposition 2).

According to [24, 28] equation (H3) admits the compatible three-point generalized symmetries

$$\frac{du_{0,0}}{d\varepsilon} = \frac{u_{0,0}(u_{1,0} + u_{-1,0}) + 2\alpha\delta}{u_{1,0} - u_{-1,0}}, \tag{19}$$

$$\frac{du_{0,0}}{d\varepsilon} = \frac{u_{0,0}(u_{0,1} + u_{0,-1}) + 2\beta\delta}{u_{0,1} - u_{0,-1}}. \tag{20}$$

Notice that under the discrete map $n \leftrightarrow m$, $\alpha \leftrightarrow \beta$, equation (19) goes into equation (20), while equation (H3) is left invariant.

The compatibility between equation (H3) and equation (19) generates a Bäcklund transformation (see an explanation below) of any solution $u_{0,0}$ of equation (19) into its new solution

$$\widetilde{u}_{0,0} = u_{0,1}, \qquad \widetilde{u}_{1,0} = u_{1,1}. \tag{21}$$

Thus equation (H3) can be rewritten as a Bäcklund transformation for the differential-difference equation (19)

$$\alpha(u_{0,0}u_{1,0} + \widetilde{u}_{0,0}\widetilde{u}_{1,0}) - \beta(u_{0,0}\widetilde{u}_{0,0} + u_{1,0}\widetilde{u}_{1,0}) + \delta(\alpha^2 - \beta^2) = 0. \tag{22}$$

Moreover, the discrete symmetry $n \leftrightarrow m$, $\alpha \leftrightarrow \beta$ implies the existence of the Bäcklund transformation for equation (20)

$$\widehat{u}_{0,0} = u_{1,0}, \qquad \widehat{u}_{0,1} = u_{1,1}.$$

This interpretation of lattice equations as Bäcklund transformations has been discussed for the first time in the differential-difference case in [16]. Examples of Bäcklund transformations similar to equation (22) for Volterra-type equations can be found in [29, 11].

In [24, 28] generalized symmetries have been obtained for autonomous ABS equations, i.e. such that α, β are constants. We present here some results on the non-autonomous case when α and β depend on n and m. Similar results can be found in [24].

Let the lattice parameters in equation (2) be such that α is a constant and $\beta = \beta_0 = \beta_m$. Let us consider the following two forms of equation (2)

$$u_{1,1} = \xi(u_{0,0}, u_{1,0}, u_{0,1}; \alpha, \beta_0), \qquad u_{0,1} = \zeta(u_{0,0}, u_{1,0}, u_{1,1}; \alpha, \beta_0), \tag{23}$$

and a symmetry

$$\frac{du_{0,0}}{d\varepsilon} = f_{0,0} = f(u_{1,0}, u_{0,0}, u_{-1,0}; \alpha), \tag{24}$$

given by equation (19). We suppose that $u_{k,l}$ depends on ε in all equations and write down the compatibility condition between equation (23) and equation (24)

$$f_{1,1} = f_{0,0}\partial_{u_{0,0}}\xi + f_{1,0}\partial_{u_{1,0}}\xi + f_{0,1}\partial_{u_{0,1}}\xi. \tag{25}$$

As a consequence of equations (23), (24) the functions $f_{1,1}$, $f_{1,0}$ and $f_{0,1}$ may be expressed in terms of the fields $u_{k,0}$, $u_{0,l}$. Therefore, equation (25) depends explicitly only on the variables $u_{k,0}$, $u_{0,l}$, which can be considered here as independent variables for any fixed n, m. For all autonomous ABS equations, the compatibility condition (25) is satisfied identically for all values of these variables and of the constant parameter β. In the non-autonomous case, equation (25) depends only on β_0 and α. Therefore the compatibility condition is satisfied also for any m.

So, equation (19) is compatible with equation (H3) also in the case when α is constant, but $\beta = \beta_m$. In a similar way, one can prove that equation (20) is the generalized symmetry of equation (H3) if β is constant, but $\alpha = \alpha_n$.

Let us now discuss the interpretation of the ABS equations as Bäcklund transformations. Let $u_{0,0}$ be a solution of equation (24), and the function $\widetilde{u}_{0,0} = \widetilde{u}_{n,m}(\varepsilon)$ given by equation (21) be a solution of equation (23), which is compatible with equation (24). equation (23) can be rewritten as the ordinary difference equation

$$\widetilde{u}_{1,0} = \xi(u_{0,0}, u_{1,0}, \widetilde{u}_{0,0}; \alpha, \beta_0), \tag{26}$$

where α is constant, $\beta_0 = \beta_m$, m is fixed, $n \in \mathbb{Z}$. Differentiating equation (26) with respect to ε and using equation (24) together with the compatibility condition (25), one gets

$$\frac{d\widetilde{u}_{1,0}}{d\varepsilon} - \frac{d\widetilde{u}_{0,0}}{d\varepsilon}\partial_{\widetilde{u}_{0,0}}\xi = f_{0,0}\partial_{u_{0,0}}\xi + f_{1,0}\partial_{u_{1,0}}\xi = \widetilde{f}_{1,0} - \widetilde{f}_{0,0}\partial_{\widetilde{u}_{0,0}}\xi,$$

where

$$\widetilde{f}_{k,0} = f(\widetilde{u}_{k+1,0}, \widetilde{u}_{k,0}, \widetilde{u}_{k-1,0}; \alpha) = f_{k,1}, \qquad \widetilde{u}_{k,0} = u_{k,1}.$$

The resulting equation is expressed in the form

$$\Xi_{1,0} = \Xi_{0,0}\partial_{\widetilde{u}_{0,0}}\xi, \qquad \Xi_{k,0} = \frac{d\widetilde{u}_{k,0}}{d\varepsilon} - \widetilde{f}_{k,0}. \tag{27}$$

There is for the ABS equations a formal condition $\partial_{\widetilde{u}_{0,0}}\xi = \partial_{u_{0,1}}\xi \neq 0$. We suppose here that, for the functions $u_{0,0}$, $\widetilde{u}_{0,0}$ under consideration, $\partial_{\widetilde{u}_{0,0}}\xi \neq 0$ for all $n \in \mathbb{Z}$. The function $\widetilde{u}_{0,0}$ is defined by equation (26) up to an integration function $\mu_0 = \mu_m(\varepsilon)$. We require that μ_0 satisfies the first order ordinary differential equation given by $\Xi_{0,0}|_{n=0} = 0$. Then equation (27) implies that $\Xi_{0,0} = 0$ for all n, i.e. $\widetilde{u}_{0,0}$ is a solution of equation (24).

So, we start with a solution of a generalized symmetry of the form (24), define a function $\widetilde{u}_{0,0}$ by the difference equation (26) which is a form of corresponding ABS equation, then we specify the integration function μ_0 by the ordinary differential equation $\Xi_{0,0}|_{n=0} = 0$, and thus obtain a new solution of equation (24). This solution depends on an integration constant $\nu_0 = \nu_m$ and the parameter β_0. We can construct in this way the solutions $u_{0,2}, u_{0,3}, \ldots, u_{0,N}$, and the last of them will depend on $2N$ arbitrary constants $\nu_0, \beta_0, \nu_1, \beta_1, \ldots, \nu_{N-1}, \beta_{N-1}$. Using such Bäcklund transformation and starting with a simple initial solution, one can obtain, in principle, a multi-soliton solution. See [6, 8] for the construction of some examples of solutions.

The symmetries (19), (20) are Volterra-type equations, namely

$$\frac{du_0}{d\varepsilon} = f(u_1, u_0, u_{-1}), \tag{28}$$

where we have dropped one of the independent indexes n or m, since it does not vary. The Volterra equation corresponds to $f(u_1, u_0, u_{-1}) = u_0(u_1 - u_{-1})$. An exhaustive list of differential-difference integrable equations of the form (28) has been obtained in [30] (details can be found in [31]). All three-point generalized symmetries of the ABS equations, with no explicit dependence on n, m, have the same structure as equation (19) (see details in Section 4.4 below) and are particular cases of the YdKN equation

$$\frac{du_0}{d\varepsilon} = \frac{R(u_1, u_0, u_{-1})}{u_1 - u_{-1}}, \qquad R(u_1, u_0, u_{-1}) = R_0 = A_0 u_1 u_{-1} + B_0(u_1 + u_{-1}) + C_0, \tag{29}$$

where

$$A_0 = c_1u_0^2 + 2c_2u_0 + c_3, \qquad B_0 = c_2u_0^2 + c_4u_0 + c_5, \qquad C_0 = c_3u_0^2 + 2c_5u_0 + c_6,$$

and the c_i's are constants. equation (29) has been found by Yamilov in [30], discussed in [21, 4], and in most detailed form in [31]. Its continuous limit goes into the Krichever–Novikov equation [15]. This is the only integrable example of the form (28) which cannot be reduced, in general, to the Toda or Volterra equations by Miura-type transformations. Moreover, equation (29) is also related to the Landau–Lifshitz equation [26]. A generalization of equation (29) with nine arbitrary constant coefficients has been considered in [20].

By a straightforward computation we get the following result: all three-point generalized symmetries in the n-direction with no explicit dependence on n, m for the ABS equations are particular cases of the YdKN equation. For the various equations of the ABS classification the coefficients c_i, $1 \le i \le 6$, read

$$\begin{array}{lllllll}
\mathrm{H1}: & c_1 = 0, & c_2 = 0, & c_3 = 0, & c_4 = 0, & c_5 = 0, & c_6 = 1, \\
\mathrm{H2}: & c_1 = 0, & c_2 = 0, & c_3 = 0, & c_4 = 0, & c_5 = 1, & c_6 = 2\alpha, \\
\mathrm{H3}: & c_1 = 0, & c_2 = 0, & c_3 = 0, & c_4 = 1, & c_5 = 0, & c_6 = 2\alpha\delta, \\
\mathrm{Q1}: & c_1 = 0, & c_2 = 0, & c_3 = -1, & c_4 = 1, & c_5 = 0, & c_6 = \alpha^2\delta^2, \\
\mathrm{Q2}: & c_1 = 0, & c_2 = 0, & c_3 = 1, & c_4 = -1, & c_5 = -\alpha^2, & c_6 = \alpha^4, \\
\mathrm{Q3}: & c_1 = 0, & c_2 = 0, & c_3 = -4\alpha^2, & c_4 = 2\alpha(\alpha^2+1), & c_5 = 0, & c_6 = -(\alpha^2-1)^2\delta^2, \\
\mathrm{Q4}: & c_1 = 1, & c_2 = -\alpha, & c_3 = \alpha^2, & c_4 = \frac{g_2}{4} - \alpha^2, & c_5 = \frac{\alpha g_2}{4} + \frac{g_3}{2}, & c_6 = \frac{g_2^2}{16} + \alpha g_3.
\end{array}$$

Proposition 2. *The ABS equations* (H1)–(H3) *and* (Q1)–(Q4) *correspond to Bäcklund transformations of the particular cases of the YdKN equation* (29) *listed above. The same holds for the non-autonomous ABS equations, such that α is constant and $\beta = \beta_m$ or $\alpha = \alpha_n$ and β is constant. Equation* (29) *and the replacement $u_i \to u_{i,0}$ provide the three-point generalized symmetries in the n-direction of the ABS equations with a constant α and $\beta = \beta_m$, while equation* (29) *and the replacement $u_i \to u_{0,i}$, $\alpha \to \beta$ provide symmetries in the m-direction for the case $\alpha = \alpha_n$ and a constant β.*

The non-autonomous case is briefly discussed in [24] where they state that if α is not constant, then the ABS equations have no local three-point symmetries in the n-direction. We shall present three-, five- and many-point generalized symmetries in the m-direction for such equations in Subsection 4.3.

A relation between the ABS equations and differential-difference equations is discussed in [2, 5]. In [2] most of the ABS equations are interpreted as nonlinear superposition principles for differential-difference equations of the form

$$(\partial_x u_{n+1})(\partial_x u_n) = h(u_{n+1}, u_n; \alpha), \tag{30}$$

where h is a polynomial of u_{n+1}, u_n. Equations of the form (30) define Bäcklund transformations for subcases of the Krichever–Novikov equation

$$\partial_t u = \partial_{xxx} u - \frac{3}{2}\frac{(\partial_{xx}u)^2 - P(u)}{\partial_x u}, \tag{31}$$

where P is a fourth degree polynomial with arbitrary constant coefficients. In the case of equations (H1) and (H3) with $\delta = 0$, the corresponding differential-difference equations have a different form, and the resulting KdV-type equations differ from equation (31).

In [5] it is shown that the continuous limit of equation (Q4) goes into a subcase of the YdKN equation. It is stated that equation (Q4) defines a Bäcklund transformation for the same subcase. The same scheme holds for equations (Q1)–(Q3), but it is not clear if the resulting Volterra-type equations are of the form (29).

4.2 Miura transformations revised

It is possible to revise the Miura transformations constructed in Section 3 from the point of view of the generalized symmetries.

Let us introduce the following function

$$r_0 = r(u_0, u_{-1}) = A_0 u_{-1}^2 + 2B_0 u_{-1} + C_0 = R(u_{-1}, u_0, u_{-1}).$$

It can be checked that $r(u_0, u_{-1}) = r(u_{-1}, u_0)$ and, in terms of r_0 the right hand side of equation (29) reads

$$\frac{R_0}{u_1 - u_{-1}} = \frac{r_0}{u_1 - u_{-1}} + \frac{1}{2}\partial_{u_{-1}} r_0 = \frac{r_1}{u_1 - u_{-1}} - \frac{1}{2}\partial_{u_1} r_1. \tag{32}$$

All the ABS equations, up to equation (Q4), are such that $c_1 = c_2 = 0$, so that the polynomial R_0 is of second degree. In this case equation (29) may be transformed [31] into equation (28) with $f(u_1, u_0, u_{-1}) = u_0(u_1 - u_{-1})$ (Volterra equation) by the Miura transformation

$$\widetilde{u}_0 = -\frac{r_1}{(u_2 - u_0)(u_1 - u_{-1})}.$$

The above map brings any solution u_0 of equation (29) with $c_1 = c_2 = 0$ into a solution $\widetilde{u}_0$ of the Volterra equation. This is exactly the same Miura transformation we have already presented in Section 3. So, also at the level of the generalized symmetries, we may see that there is a deep relation between equations (H1)–(H3) and (Q1)–(Q3) and the Volterra equation. If equation (29) cannot be transformed to the case with $c_1 = c_2 = 0$, using a Möbius transformation, then it cannot be mapped into the Volterra equation by $\widetilde{u}_0 = G(u_0, u_1, u_{-1}, u_2, u_{-2}, \dots)$ [31]. Equation (Q4) is of this kind and thus is the only equation of the ABS list which cannot be related to the Volterra equation.

4.3 Master symmetries

Generalized symmetries of equation (29) will also be compatible with the ABS equations, which are, according to Proposition 2, their Bäcklund transformations. Such symmetries can be constructed, using the master symmetry of equation (29) presented in [4].

Let us rewrite equation (29) by using the equivalent n-dependent notation (see equation (1)), namely

$$\frac{du_n}{d\varepsilon_0} = f_n^{(0)} = \frac{R(u_{n+1}, u_n, u_{n-1})}{u_{n+1} - u_{n-1}}, \tag{33}$$

where ε_0 is the continuous symmetry parameter (previously denoted with ε). We shall denote with ε_i, $i \geq 1$, the parameters corresponding to higher generalized symmetries

$$\frac{du_n}{d\varepsilon_i} = f_n^{(i)}, \qquad \text{such that} \qquad \frac{df_n^{(j)}}{d\varepsilon_i} - \frac{df_n^{(i)}}{d\varepsilon_j} = 0, \qquad i, j \geq 0.$$

Let us introduce the master symmetry

$$\frac{du_n}{d\tau} = g_n, \qquad \text{such that} \qquad \frac{df_n^{(i)}}{d\tau} - \frac{dg_n}{d\varepsilon_i} = f_n^{(i+1)}, \qquad i \geq 0. \tag{34}$$

Once we know the master symmetry (34) we can construct explicitly the infinite hierarchy of generalized symmetries.

The master symmetry of equation (33) is given by

$$g_n = n f_n^{(0)}. \tag{35}$$

According to a general procedure described in [31] we need to introduce an explicit dependence on the parameter τ into the master symmetry (35) and into equation (33) itself. Let the coefficients c_i, appearing in the polynomials A_n, B_n, C_n, be functions of τ. This τ-dependence implies that r_n satisfies the following partial differential equation

$$2\partial_\tau r_n = r_n \partial_{u_n}\partial_{u_{n-1}} r_n - (\partial_{u_n} r_n)\left(\partial_{u_{n-1}} r_n\right). \tag{36}$$

On the left hand side of the above equation, we differentiate only the coefficients of r_n with respect to τ. The right hand side has the same form as r_n, but with different coefficients. Collecting the coefficients of the terms $u_n^i u_{n-1}^j$ for various powers i and j, we obtain a system of six ordinary differential equations for the six coefficients $c_i(\tau)$, whose initial conditions are $c_i(0) = c_i$. Generalized symmetries constructed by using equation (34) explicitly depend on τ. They remain generalized symmetries for any value of τ, as τ is just a parameter for them and for equation (33). So, going over to the initial conditions, we get generalized symmetries of equation (33) and of the corresponding ABS equations.

Let us derive, as an illustrative example, a formula for the symmetry $f_n^{(1)}$ from equation (34). From equations (33)–(35) it follows that

$$f_n^{(1)} = \partial_\tau f_n^{(0)} + f_{n+1}^{(0)} \partial_{u_{n+1}} f_n^{(0)} - f_{n-1}^{(0)} \partial_{u_{n-1}} f_n^{(0)}. \tag{37}$$

Using equations (32) and (36) one obtains

$$\partial_{u_{n+1}} f_n^{(0)} = -\frac{r_n}{(u_{n+1}-u_{n-1})^2}, \qquad \partial_{u_{n-1}} f_n^{(0)} = \frac{r_{n+1}}{(u_{n+1}-u_{n-1})^2},$$

and

$$\partial_\tau R_n = \mathcal{R}_n = \mathcal{R}(u_{n+1}, u_n, u_{n-1}) = \mathcal{A}_n\, u_{n+1}u_{n-1} + \frac{\mathcal{B}_n}{2}(u_{n+1}+u_{n-1}) + \mathcal{C}_n,$$

with

$$\mathcal{A}_n = B_n \partial_{u_n} A_n - A_n \partial_{u_n} B_n, \qquad \mathcal{B}_n = C_n \partial_{u_n} A_n - A_n \partial_{u_n} C_n,$$
$$\mathcal{C}_n = C_n \partial_{u_n} B_n - B_n \partial_{u_n} C_n.$$

From equation (37) we get the first generalized symmetry

$$\frac{du_n}{d\varepsilon_1} = f_n^{(1)} = \frac{\mathcal{R}_n}{u_{n+1}-u_{n-1}} - \frac{r_n f_{n+1}^{(0)} + r_{n+1} f_{n-1}^{(0)}}{(u_{n+1}-u_{n-1})^2}. \tag{38}$$

Up to our knowledge this formula is new. It provides five-point generalized symmetries in both n- and m-directions for the ABS equations. Examples of such five-point symmetries for equations (H1) and (Q1) with $\delta = 0$ can be found in [24, 27].

Let us clarify the construction of the symmetry $f_n^{(1)}$ for equations (H1)–(H3). In these cases the function r_n takes the form

$$r_n = 2c_4(\tau)u_n u_{n-1} + 2c_5(\tau)(u_n + u_{n-1}) + c_6(\tau),$$

and equation (36) is equivalent to the system

$$\partial_\tau c_4(\tau) = 0, \qquad \partial_\tau c_5(\tau) = 0, \qquad \partial_\tau c_6(\tau) = c_4(\tau)c_6(\tau) - 2c_5^2(\tau). \tag{39}$$

The initial conditions of system (39) are (see the list above Proposition 2)

$$\begin{aligned}
&\mathrm{H1}: \quad c_4(0)=0, \qquad c_5(0)=0, \qquad c_6(0)=1,\\
&\mathrm{H2}: \quad c_4(0)=0, \qquad c_5(0)=1, \qquad c_6(0)=2\alpha,\\
&\mathrm{H3}: \quad c_4(0)=1, \qquad c_5(0)=0, \qquad c_6(0)=2\alpha\delta,
\end{aligned}$$

and its solutions are given by

$$\begin{aligned}
&\mathrm{H1}: \quad c_4(\tau)=0, \qquad c_5(\tau)=0, \qquad c_6(\tau)=1,\\
&\mathrm{H2}: \quad c_4(\tau)=0, \qquad c_5(\tau)=1, \qquad c_6(\tau)=2(\alpha-\tau),\\
&\mathrm{H3}: \quad c_4(\tau)=1, \qquad c_5(\tau)=0, \qquad c_6(\tau)=2\alpha\delta e^{\tau}.
\end{aligned}$$

Note that the master symmetry with the above $c_i(\tau)$ generates τ-dependent symmetries for a τ-dependent equation, but by fixing τ we obtain τ-independent symmetries for a τ-independent equation. Let us remark that the τ-dependence is independent of the order of the symmetry and it may be used for the construction of all higher symmetries.

So, according to formula (38), we may construct the generalized symmetry $f_n^{(1)}$, in the case of the list H, from the following expressions

$$\begin{aligned}
&\mathrm{H1}: \quad f_n^{(0)}=\frac{1}{u_{n+1}-u_{n-1}}, && r_n=1, && \mathcal{R}_n=0,\\
&\mathrm{H2}: \quad f_n^{(0)}=\frac{u_{n+1}+u_{n-1}+2(u_n+\alpha)}{u_{n+1}-u_{n-1}}, && r_n=2(u_n+u_{n-1}+\alpha), && \mathcal{R}_n=-2,\\
&\mathrm{H3}: \quad f_n^{(0)}=\frac{u_n(u_{n+1}+u_{n-1})+2\alpha\delta}{u_{n+1}-u_{n-1}}, && r_n=2(u_nu_{n-1}+\alpha\delta), && \mathcal{R}_n=2\alpha\delta.
\end{aligned}$$

It is possible to verify that the symmetries (38) with $f_n^{(0)}$, r_n, $\mathcal{R}_n$ given above are compatible with both equations (33) and (H1)–(H3).

By using the master symmetry constructed above we can construct infinite hierarchies of many-point generalized symmetries of the ABS equations in both directions. In the non-autonomous cases (see Proposition 2) we provide one hierarchy in the n- or m-direction. The master symmetry and formula (38) will also be useful in the case of the generalizations of the ABS equations presented in the next Subsection. It should be remarked that in [24] the authors constructed master symmetries for all autonomous and non-autonomous ABS equations, which are of a different kind with respect to the ones presented here.

4.4 Generalizations of the ABS equations

Here we discuss the generalization of the ABS equations introduced by Tongas, Tsoubelis and Xenitidis (TTX) in [28]. The TTX equations are autonomous lattice equations of the form (2) which possess only two of the four main properties of the ABS equations: they are affine linear and possess the symmetries of the square.

In terms of the polynomial $\mathcal{E}$, see equation (2), one generates the following function h

$$h(u_{0,0},u_{1,0};\alpha,\beta)=\mathcal{E}\partial_{u_{0,1}}\partial_{u_{1,1}}\mathcal{E}-\left(\partial_{u_{0,1}}\mathcal{E}\right)\left(\partial_{u_{1,1}}\mathcal{E}\right),$$

which is a biquadratic and symmetric polynomial in its first two arguments. It has been proved in [28] that the TTX equations admit three-point generalized symmetries in the n-direction of the form

$$\frac{du_{0,0}}{d\varepsilon}=\frac{h}{u_{1,0}-u_{-1,0}}-\frac{1}{2}\partial_{u_{1,0}}h. \tag{40}$$

Of course, there is a similar symmetry in the m-direction. Comparing equations (29), (32) and (40), we see that the symmetry (40) is nothing but the YdKN equation in its general form. This shows that all TTX equations can also be considered as Bäcklund transformations for the YdKN equation. However, they probably describe the general picture for Bäcklund transformations of the YdKN equation, which have the form (2). The general formula (38) and the master symmetry discussed in the previous Subsection, provide five- and many-point generalized symmetries of the TTX equations in both directions, thus confirming their integrability.

5 Concluding remarks

In this paper we have considered some further properties of the ABS equations. In particular we have shown that equations (H1)–(H3) and (Q1)–(Q3) can be transformed into equations associated with the spectral problem of the Volterra equation. Therefore all known results for the solution of the Volterra equation can be used to construct solutions of the ABS equations. Moreover, all equations of the ABS list, except equation (Q4), can be transformed among themselves by Miura transformations.

The situation of equation (Q4) is somehow different. It is shown that this equation can be thought as a Bäcklund transformation for a subcase of the Yamilov discretization of the Krichever–Novikov equation. But it cannot be related by a Miura transformation to a Volterra-type equation and this explains the complicate form of its scalar spectral problem. The master symmetry constructed for the YdKN equation can, however, be used also in this case to construct generalized symmetries.

It turns out that a generalizations of the ABS equations introduced by Tongas, Tsoubelis and Xenitidis are Bäcklund transformations for the YdKN equation.

Further generalizations of the TTX and ABS equations can be probably obtained by a proper explicit dependence on the point of the lattice not only in the lattice parameters α and β, but also in the $\mathbb{Z}^2$-lattice equation itself. The existence of an n-dependent generalization of the YdKN equation, introduced in [20], could help in solving this problem. Such a generalization is integrable in the sense that it has a master symmetry [4] similar to the one presented here.

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