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Liouville Theorem for Dunkl Polyharmonic Functions*

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Abstract. Assume that f is Dunkl polyharmonic in $\mathbb{R}^n$ (i.e. $(\Delta_h)^p f = 0$ for some integer p, where Δ_h is the Dunkl Laplacian associated to a root system R and to a multiplicity function κ, defined on R and invariant with respect to the finite Coxeter group). Necessary and successful condition that f is a polynomial of degree $\leq s$ for $s \geq 2p-2$ is proved. As a direct corollary, a Dunkl harmonic function bounded above or below is constant.

Key words: Liouville theorem; Dunkl Laplacian; polyharmonic functions

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1 Introduction

The classical Liouville theorem for harmonic functions states that a harmonic function on $\mathbb{R}^n$ must be a constant if it is bounded or nonnegative. Nicolesco [14] extended the Liouville theorem to polyharmonic functions with the Pizetti formula as a starting point (see also [8]). Kuran [9], Armitage [2], and Futamura, Kishi, and Mizuta [6] proved further extensions and showed that if f is a polyharmonic function on $\mathbb{R}^n$ and the growth of the positive part of f is suitably restricted, then f must be a polynomial. Their starting point is the Almansi decomposition theorem for polyharmonic functions. We also refer to [10, 11] for the extension of Liouville theorems for conformally invariant fully nonlinear equations. Recently, Gallardo and Godefroy [7] showed that if f is a bounded Dunkl harmonic function in $\mathbb{R}^n$, then it is a constant. However, their approach is not adaptable to Dunkl polyharmonic functions.

The purpose of this article is to establish the Liouville theorem for Dunkl polyharmonic functions. To achieve this, we shall resort to the Almansi decomposition for Dunkl polyharmonic functions [15]. As a direct corollary of our results, a Dunkl harmonic function bounded below or above is actually constant, which extends the corresponding result of Gallardo and Godefroy for the bounded case [7]. In the Dunkl ananlysis, the multiplicity function is usually restricted to be non-negative. We shall discuss in the final section the possible extension of our main result to the case when the multiplicity function is negative.

2 Dunkl polyharmonic functions

For a nonzero vector $v = (v_1, \dots, v_n) \in \mathbb{R}^n$, the reflection σ_v with respect to the hyperplane orthogonal to v is defined by

$$\sigma_v x := x - 2\frac{\langle x, v\rangle}{|v|^2} v, \qquad x = (x_1, \dots, x_m) \in \mathbb{R}^m,$$

where the symbol $\langle x, y\rangle$ denotes the usual Euclidean inner product and $|x|^2 = \langle x, x\rangle$.

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A root system R is a finite set of nonzero vectors in $\mathbb{R}^m$ such that $\sigma_v R = R$ and $R \cap \mathbb{R}v = \{\pm v\}$ for all $v \in R$.

The Coxeter group G (or the finite reflection group) generated by the root system R is the subgroup of the orthogonal group $O(n)$ generated by $\{\sigma_u : u \in R\}$.

The positive subsystem R_+ is a subset of R such that $R = R_+ \cup (-R_+)$, where R_+ and $-R_+$ are separated by some hyperplane through the origin.

A multiplicity function

$$\kappa : \ R \longrightarrow \mathbb{C}, \qquad v \longmapsto \kappa_v$$

is a G-invariant complex valued function defined on R, i.e. $\kappa_v = \kappa_{gv}$ for all $g \in G$.

Fix a positive subsystem R_+ of R and denote

$$\gamma = \gamma_\kappa := \sum_{v \in R_+} \kappa_v.$$

Let $\mathcal{D}_j$ be the Dunkl operator associated to the group G and to the multiplicity function κ, defined by

$$\mathcal{D}_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{v \in R_+} \kappa_v \frac{f(x) - f(\sigma_v x)}{\langle x, v \rangle} v_j. \tag{1}$$

We call $\Delta_h = \sum_{j=1}^{n} \mathcal{D}_j{}^2$ the Dunkl Laplacian. We always assume that $\kappa_v \geq 0$.

Let $d\sigma$ be the Lebesgue surface measure in the unit sphere and $h_k(x) = \prod_{v \in R_+} |\langle v, x \rangle|^{\kappa_v}$. Denote $f^+ = \frac{|f|+f}{2}$ and

$$M_1(r, f) = \int_{|y|=r} |f(y)| h_k^2(y) d\sigma(y).$$

The mean value property holds for Dunkl harmonic functions f, i.e.,

$$\int_{|x|=1} f(x) h_k^2(x) d\sigma(x) = c f(0) \tag{2}$$

for some constant $c > 0$. This property for polynomials is implicit in the orthogonality relation in [3]. Then one only needs a limiting argument for arbitrary Dunkl harmonic functions. See also [13, 12].

Our main theorem is as follows.

Theorem 1. *Assume that $s \in \mathbb{N} \cup \{0\}$, $p \in \mathbb{N}$, and $s \geq 2(p-1)$. Let $f \in C^{2p}(\mathbb{R}^n)$ and $\Delta_h^p f = 0$. Then f is a polynomial of degree $\leq s$ if and only if*

$$\liminf_{r \to \infty} \frac{M_1(r, f^+)}{r^{s+n-1+2\gamma}} \in [0, +\infty).$$

Moreover, when $s > 2(p-1)$, we have

(i) f is a polynomial of degree $< s$ if and only if

$$\liminf_{r \to \infty} \frac{M_1(r, f^+)}{r^{s+n-1+2\gamma}} = 0;$$

(ii) f is a polynomial of degree s if and only if

$$\liminf_{r \to \infty} \frac{M_1(r, f^+)}{r^{s+n-1+2\gamma}} \in (0, +\infty).$$

As a direct corollary, a Dunkl harmonic function on $\mathbb{R}^n$ must be constant if it is bounded below or above. Indeed, let f be Dunkl harmonic in $\mathbb{R}^n$ and set $s = 0$. If $f \geq 0$, then $f^+ = f$ and so that the mean value property shows that

$$\frac{M_1(r, f^+)}{r^{n-1+2\gamma}} = \int_{|x|=1} f(x) h_k^2(x) d\sigma(x) = cf(0) \in [0, +\infty).$$

Therefore, Theorem 1 shows that f is a polynomial of degree less than or equal to $s = 0$, which means that f must be a constant. If f is bounded above or below, multiplication by -1 if necessary makes it bounded below, adding a constant if necessary makes it positive. Thus the known result for positive functions shows that f is a constant.

3 Homogeneous expansions

The non-Dunkl case of the following homogeneous expansions is well-known (see [1, Corollary 5.34]). Its Dunkl version will be helpful in the proof of Theorem 1.

Denote by $\mathcal{H}_m^n(h_\kappa^2)$ the space of Dunkl harmonic polynomials of degree m in $\mathbb{R}^n$.

Lemma 1. *If u is a Dunkl harmonic function in $\mathbb{R}^n$, then there exist $p_m \in \mathcal{H}_m^n(h_\kappa^2)$, such that*

$$u(x) = \sum_{m=1}^{\infty} p_m(x), \qquad |x| < 1,$$

the series converging absolutely and uniformly on compact subsets of the unit ball.

Proof. The formula can be verified as in the classical case of Corollary 5.34 in [1] by using the formulae corresponding to the classical case in [5]. Indeed, notice that u is harmonic on the closed unit ball. Theorem 5.31 in [5] and the preceding statement show that

$$u(x) = \sum_{m=1}^{\infty} c_h' \int_{|y|=1} u(y) P_m(h_\kappa^2; x, y) h_\kappa^2(y) d\sigma(y), \qquad |x| < 1,$$

where

$$P_m(h_\kappa^2; x, y) = \sum_{0 \leq j \leq n/2} \frac{(\gamma_\kappa + \frac{n}{2})_m 2^{m-2j}}{(2 - m - \gamma_\kappa - n/2)_j j!} |x|^{2j} |y|^{2k} K_{m-2j}(x, y).$$

Let

$$p_m(x) = c_h' \int_{|y|=1} u(y) P_m(h_\kappa^2; x, y) h_\kappa^2(y) d\sigma(y), \qquad x \in \mathbb{R}^n.$$

Then $p_m \in \mathcal{H}_m^n(h_\kappa^2)$, since $P_m(h_\kappa^2; x, y)$ is the reproducing kernel of $\mathcal{H}_m^n(h_\kappa^2)$ (see [5, p. 131, p. 189]).

By Proposition 4.6.2(ii) in [5], we have

$$K_m(x, y) \leq \frac{1}{m!} |x|^m |y|^m,$$

so that

$$|p_m(x)| \leq C m^{2+\gamma+n/2} |x|^m \int_{|y|=1} |u(y)| h_\kappa^2(y) d\sigma(y), \qquad \forall\, x \in \mathbb{R}^n,$$

and thus the series $\sum_m p_m$ converges absolutely and uniformly to u on compact subsets of the unit ball. ■

4 Proof of the theorem

Proof. The necessity of Theorem 1 is clearly true. Now we prove that f is a polynomial of degree $\leq s$ whenever

$$\liminf_{r\to\infty} \frac{M_1(r,f^+)}{r^{s+n-1+2\gamma}} \in [0,\infty). \tag{3}$$

From the Almansi decomposition theorem for Dunkl polyharmonic functions [15], we have

$$f(x) = \sum_{m=0}^{p-1} |x|^{2m} \varphi_m(x),$$

φ_m being Dunkl harmonic functions. Therefore, the mean value property in (2) shows

$$\begin{aligned}
\int_{|y|=r} f(y)h_k^2(y)d\sigma(y) &= \int_{|y|=r} \sum_{m=0}^{p-1} |y|^{2m}\varphi_m(y)h_k^2(y)d\sigma(y)\\
&= \sum_{m=0}^{p-1} r^{2m} \int_{|y|=r} \varphi_m(y)h_k^2(y)d\sigma(y)\\
&= \sum_{m=0}^{p-1} r^{2m+n-1+2\gamma} c\varphi_m(0) = O(r^{2(p-1)+n-1+2\gamma}), \qquad r\to\infty.
\end{aligned}$$

Thus

$$\lim_{r\to\infty} \frac{1}{r^{j+n-1+2\gamma}} \int_{|y|=r} f(y)h_k^2(y)d\sigma(y) = \begin{cases} c\,\varphi_{p-1}(0), & j = 2(p-1),\\ 0, & j > 2(p-1). \end{cases}$$

Since $|f| = 2f^+ - f$, we have

$$\begin{aligned}
0 \leq \liminf_{r\to\infty} \frac{M_1(r,f)}{r^{s+n-1+2\gamma}} &= 2\liminf_{r\to\infty} \frac{M_1(r,f^+)}{r^{s+n-1+2\gamma}} - \lim_{r\to\infty} \frac{1}{r^{s+n-1+2\gamma}} \int_{|y|=r} f(y)h_k^2(y)d\sigma(y)\\
&= 2\liminf_{r\to\infty} \frac{M_1(r,f^+)}{r^{s+n-1+2\gamma}} - C_1 < +\infty,
\end{aligned} \tag{4}$$

where $C_1 = c\varphi_{p-1}(0)$ for $s = 2(p-1)$ and $C_1 = 0$ for $s > 2(p-1)$.

Since φ_m are Dunkl harmonic in $\mathbb{R}^n$, by Lemma 1 we can write

$$\varphi_m = \sum_{j=1}^{\infty} g_{m,j},$$

where $g_{m,j}$ are Dunkl harmonic homogeneous polynomials of degree j and the convergence of the series is uniform on compact subsets of the unit ball.

We claim that $g_{m,j} = 0$ when $2m + j > s$. With this claim, we have

$$f(x) = \sum_{m=0}^{p-1} |x|^{2m}\varphi_m(x) = \sum_{m=0}^{p-1} |x|^{2m} \sum_{j=0}^{s-2m} g_{m,j}, \tag{5}$$

so that $f(x)$ is a polynomial of degree no more than s.

We now prove the claim by contradiction. Suppose there is a g_{m_0,j_0} not identical 0 and $2m_0 + j_0 > s$, then

$$\begin{aligned}
&\liminf_{r\to\infty} \frac{1}{r^{s+j_0+n-1+2\gamma}} \left| \int_{|y|=r} f(y) g_{m_0,j_0}(y) h_k^2(y) d\sigma(y) \right| \\
&\quad \le \liminf_{r\to\infty} \frac{1}{r^{s+j_0+n-1+2\gamma}} \max_{|y|=r} |g_{m_0,j_0}(y)| \int_{|y|=r} |f(y)| h_k^2(y) d\sigma(y) \\
&\quad = \max_{|y|=1} |g_{m_0,j_0}(y)| \liminf_{r\to\infty} \frac{M_1(r,f)}{r^{s+n-1+2\gamma}} \in [0,\infty). \qquad (6)
\end{aligned}$$

On the other side, notice that $\{g_{m,j}\}_{j=0}^{\infty}$ are orthogonal in $L^2(S_{n-1}, h_k^2 d\sigma)$ (see [5]), where S_{n-1} is the unit sphere in $\mathbb{R}^n$, we have

$$\begin{aligned}
\int_{|y|=r} f(y) g_{m_0,j_0}(y) h_k^2(y) d\sigma(y) &= \int_{|y|=r} \sum_{m=0}^{p-1} |y|^{2m} \sum_{j=0}^{\infty} g_{m,j}(y) g_{m_0,j_0}(y) h_k^2(y) d\sigma(y) \\
&= \int_{|y|=r} \sum_{m=0}^{p-1} |y|^{2m} g_{m,j_0}(y) g_{m_0,j_0}(y) h_k^2(y) d\sigma(y).
\end{aligned}$$

On considering item $m = m_0$, the above integral is of the form $r^{2\gamma} P(r)$ with $P(r)$ a polynomial of r with degree at least $2m_0 + 2j_0 + n - 1$, which is strictly larger than $s + j_0 + n - 1$. This contradicts the fact that

$$\begin{aligned}
\int_{|y|=r} f(y) g_{m_0,j_0}(y) h_k^2(y) d\sigma(y) &= \int_{|y|=r} (2f^+(y) - |f(y)|) g_{m_0,j_0}(y) h_k^2(y) d\sigma(y) \\
&= O(r^{s+j_0+n-1+2\gamma}). \qquad (7)
\end{aligned}$$

The last step used (3) and (6).

We have proved that f is a polynomial of degree $\le s$ if and only if (3) holds. With this result, we know that the statements (i) and (ii) in Theorem 1 are equivalent.

It remains to prove the sufficiency of Theorem 1(i) in the case $s > 2p - 2$.

Let $s > 2p - 2$ and assume that f satisfies

$$\liminf_{r\to\infty} \frac{M_1(r, f^+)}{r^{s+n-1+2\gamma}} = 0.$$

In this case, from the proof of (4), we find

$$\liminf_{r\to\infty} \frac{M_1(r, f)}{r^{s+n-1+2\gamma}} = 0,$$

so that the same reasoning as in (7) shows that

$$\int_{|y|=r} f(y) g_{m,j}(y) h_k^2 d\sigma(y) = o(r^{s+n-1+j+2\gamma}), \qquad r \to \infty.$$

We now claim that $g_{m,j} = 0$ if $2m + j \ge s$.

Indeed, if not, then there exists $g_{m_0,j_0} \not\equiv 0$ such that $2m_0 + j_0 \ge s$. However,

$$\int_{|y|=r} f(y) g_{m_0,j_0}(y) h_k^2 d\sigma(y) = \int_{|y|=r} \sum_{m=0}^{p-1} |y|^{2m} \sum_{j=0}^{\infty} g_{m,j}(y) g_{m_0,j_0}(y) h_k^2(y) d\sigma(y),$$

and the above integral is of the form $r^{2\gamma} Q(r)$, where $Q(r)$ is a polynomial of r with degree no less than $2m_0 + 2j_0 + n - 1 \ge s + j_0 + n - 1$. Since $r^{2\gamma} Q(r)$ is not an $o(r^{s+j_0+n-1+2\gamma})$, we arrive at a contradiction.

From the claim and (5), we see that f is a polynomials of degree less than s. ■

4.1 Further remarks

We have shown that our main results hold for any non-negative multiplicity function κ. With the same approach, we can extend our main result to the case when

$$\kappa \in K^{\mathrm{reg}} \qquad \text{and} \qquad \mathfrak{Re}\,\kappa > -\frac{n}{2},$$

where K^{reg} is the regular parameter set (see [4, 17]). This is because in the proof we applied the Dunkl intertwining operator in Lemma 1 and the Almansi decomposition. The existence of the intertwining operator depends on the restriction that $\kappa \in K^{\mathrm{reg}}$, while the Almansi decomposition holds for any $\mathfrak{Re}\,\kappa > -\frac{n}{2}$ (see [15, 16]). We mention that in the rank-one case, i.e., $\dim \operatorname{span}_{\mathbb{R}} R = 1$, we have $K^{\mathrm{reg}} = \mathbb{C} \setminus \{-1/2 - m,\ m = 0, 1, 2, \ldots\}$. This means that in the rank-one case, Theorem 1 holds for $\mathfrak{Re}\,\kappa > -\frac{1}{2}$.

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