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Inversion Formulas for the Dunkl Intertwining Operator and Its Dual on Spaces of Functions and Distributions*

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Abstract. In this paper we prove inversion formulas for the Dunkl intertwining operator V_k and for its dual ${}^tV_k$ and we deduce the expression of the representing distributions of the inverse operators V_k^{-1} and ${}^tV_k^{-1}$, and we give some applications.

Key words: inversion formulas; Dunkl intertwining operator; dual Dunkl intertwining operator

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1 Introduction

We consider the differential-difference operators T_j, $j = 1, 2, \dots, d$, on $\mathbb{R}^d$ associated to a root system R and a multiplicity function k, introduced by C.F. Dunkl in [3] and called the Dunkl operators in the literature. These operators are very important in pure mathematics and in physics. They provide a useful tool in the study of special functions related to root systems [4, 6, 2]. Moreover the commutative algebra generated by these operators has been used in the study of certain exactly solvable models of quantum mechanics, namely the Calogero–Sutherland–Moser models, which deal with systems of identical particles in a one dimensional space (see [8, 11, 12]).

C.F. Dunkl proved in [4] that there exists a unique isomorphism V_k from the space of homogeneous polynomials $\mathcal{P}_n$ on $\mathbb{R}^d$ of degree n onto itself satisfying the transmutation relations

$$T_j V_k = V_k \frac{\partial}{\partial x_j}, \qquad j = 1, 2, \dots, d, \tag{1.1}$$

and

$$V_k(1) = 1. \tag{1.2}$$

This operator is called the Dunkl intertwining operator. It has been extended to an isomorphism from $\mathcal{E}(\mathbb{R}^d)$ (the space of C^∞-functions on $\mathbb{R}^d$) onto itself satisfying the relations (1.1) and (1.2) (see [15]).

The operator V_k possesses the integral representation

$$\forall\, x \in \mathbb{R}^d, \qquad V_k(f)(x) = \int_{\mathbb{R}^d} f(y) d\mu_x(y), \qquad f \in \mathcal{E}(\mathbb{R}^d), \tag{1.3}$$

where μ_x is a probability measure on $\mathbb{R}^d$ with support in the closed ball $B(0, \|x\|)$ of center 0 and radius $\|x\|$ (see [14, 15]).

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We have shown in [15] that for each $x \in \mathbb{R}^d$, there exists a unique distribution η_x in $\mathcal{E}'(\mathbb{R}^d)$ (the space of distributions on $\mathbb{R}^d$ of compact support) with support in $B(0, \|x\|)$ such that

$$V_k^{-1}(f)(x) = \langle \eta_x, f\rangle, \qquad f \in \mathcal{E}(\mathbb{R}^d). \tag{1.4}$$

We have studied also in [15] the transposed operator ${}^tV_k$ of the operator V_k, satisfying for f in $\mathcal{S}(\mathbb{R}^d)$ (the space of C^∞-functions on $\mathbb{R}^d$ which are rapidly decreasing together with their derivatives) and g in $\mathcal{E}(\mathbb{R}^d)$, the relation

$$\int_{\mathbb{R}^d} {}^tV_k(f)(y)g(y)dy = \int_{\mathbb{R}^d} V_k(g)(x)f(x)\omega_k(x)dx,$$

where ω_k is a positive weight function on $\mathbb{R}^d$ which will be defined in the following section. It has the integral representation

$$\forall\, y \in \mathbb{R}^d, \qquad {}^tV_k(f)(y) = \int_{\mathbb{R}^d} f(x)d\nu_y(x), \tag{1.5}$$

where ν_y is a positive measure on $\mathbb{R}^d$ with support in the set $\{x \in \mathbb{R}^d;\ \|x\| \geq \|y\|\}$. This operator is called the dual Dunkl intertwining operator.

We have proved in [15] that the operator ${}^tV_k$ is an isomorphism from $\mathcal{D}(\mathbb{R}^d)$ (the space of C^∞-functions on $\mathbb{R}^d$ with compact support) (resp. $\mathcal{S}(\mathbb{R}^d)$) onto itself, satisfying the transmutation relations

$$\forall\, y \in \mathbb{R}^d, \qquad {}^tV_k(T_jf)(y) = \frac{\partial}{\partial y_j}{}^tV_k(f)(y), \qquad j = 1, 2, \ldots, d.$$

Moreover for each $y \in \mathbb{R}^d$, there exists a unique distribution Z_y in $\mathcal{S}'(\mathbb{R}^d)$ (the space of tempered distributions on $\mathbb{R}^d$) with support in the set $\{x \in \mathbb{R}^d; \|x\| \geq \|y\|\}$ such that

$${}^tV_k^{-1}(f)(y) = \langle Z_y, f\rangle, \qquad f \in \mathcal{S}(\mathbb{R}^d). \tag{1.6}$$

Using the operator V_k, C.F. Dunkl has defined in [5] the Dunkl kernel K by

$$\forall\, x \in \mathbb{R}^d, \qquad \forall\, z \in \mathbb{C}^d, \qquad K(x, -iz) = V_k(e^{-i\langle \cdot, z\rangle})(x). \tag{1.7}$$

Using this kernel C.F. Dunkl has introduced in [5] a Fourier transform $\mathcal{F}_D$ called the Dunkl transform.

In this paper we establish the following inversion formulas for the operators V_k and ${}^tV_k$:

$$\begin{aligned} &\forall\, x \in \mathbb{R}^d, \qquad V_k^{-1}(f)(x) = P\,{}^tV_k(f)(x), \qquad f \in \mathcal{S}(\mathbb{R}^d), \\ &\forall\, x \in \mathbb{R}^d, \qquad {}^tV_k^{-1}(f)(x) = V_k(P(f))(x), \qquad f \in \mathcal{S}(\mathbb{R}^d), \end{aligned} \tag{1.8}$$

where P is a pseudo-differential operator on $\mathbb{R}^d$.

When the multiplicity function takes integer values, the formula (1.8) can also be written in the form

$$\forall\, x \in \mathbb{R}^d, \qquad V_k^{-1}(f)(x) = {}^tV_k(Q(f))(x), \qquad f \in \mathcal{S}(\mathbb{R}^d),$$

where Q is a differential-difference operator.

Also we give another expression of the operator ${}^tV_k^{-1}$ on the space $\mathcal{E}'(\mathbb{R}^d)$. From these relations we deduce the expressions of the representing distributions η_x and Z_x of the inverse operators V_k^{-1} and ${}^tV_k^{-1}$ by using the representing measures μ_x and ν_x of V_k and ${}^tV_k$. They are given by the following formulas

$$\begin{aligned} &\forall\, x \in \mathbb{R}^d, \qquad \eta_x = {}^tQ(\nu_x), \\ &\forall\, x \in \mathbb{R}^d, \qquad Z_x = {}^tP(\mu_x), \end{aligned}$$

where ${}^tP$ and ${}^tQ$ are the transposed operators of P and Q respectively.

The contents of the paper are as follows. In Section 2 we recall some basic facts from Dunkl's theory, and describe the Dunkl operators and the Dunkl kernel. We define in Section 3 the Dunkl transform introduced in [5] by C.F. Dunkl, and we give the main theorems proved for this transform, which will be used in this paper. We study in Section 4 the Dunkl convolution product and the Dunkl transform of distributions which will be useful in the sequel, and when the multiplicity function takes integer values, we give another proof of the geometrical form of Paley–Wiener–Schwartz theorem for the Dunkl transform. We prove in Section 5 some inversion formulas for the Dunkl intertwining operator V_k and its dual ${}^tV_k$ on spaces of functions and distributions. Section 6 is devoted to proving under the condition that the multiplicity function takes integer values an inversion formula for the Dunkl intertwining operator V_k, and we deduce the expression of the representing distributions of the inverse operators V_k^{-1} and ${}^tV_k^{-1}$. In Section 7 we give some applications of the preceding inversion formulas.

2 The eigenfunction of the Dunkl operators

In this section we collect some notations and results on the Dunkl operators and the Dunkl kernel (see [3, 4, 5, 7, 9, 10]).

2.1 Reflection groups, root systems and multiplicity functions

We consider $\mathbb{R}^d$ with the Euclidean scalar product $\langle\cdot,\cdot\rangle$ and $\|x\| = \sqrt{\langle x, x\rangle}$. On $\mathbb{C}^d$, $\|\cdot\|$ denotes also the standard Hermitian norm, while $\langle z, w\rangle = \sum_{j=1}^d z_j\overline{w_j}$.

For $\alpha \in \mathbb{R}^d\backslash\{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α, i.e.

$$\sigma_\alpha(x) = x - \left(\frac{2\langle\alpha, x\rangle}{\|\alpha\|^2}\right)\alpha.$$

A finite set $R \subset \mathbb{R}^d\backslash\{0\}$ is called a root system if $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. For a given root system R the reflections σ_α, $\alpha \in R$, generate a finite group $W \subset O(d)$, the reflection group associated with R. All reflections in W correspond to suitable pairs of roots. For a given $\beta \in \mathbb{R}^d\backslash \cup_{\alpha\in R} H_\alpha$, we fix the positive subsystem $R_+ = \{\alpha \in R; \langle\alpha, \beta\rangle > 0\}$, then for each $\alpha \in R$ either $\alpha \in R_+$ or $-\alpha \in R_+$.

A function $k : R \to \mathbb{C}$ on a root system R is called a multiplicity function if it is invariant under the action of the associated reflection group W. If one regards k as a function on the corresponding reflections, this means that k is constant on the conjugacy classes of reflections in W. For abbreviation, we introduce the index

$$\gamma = \gamma(R) = \sum_{\alpha\in R_+} k(\alpha).$$

Moreover, let ω_k denotes the weight function

$$\omega_k(x) = \prod_{\alpha\in R_+} |\langle\alpha, x\rangle|^{2k(\alpha)},$$

which is W-invariant and homogeneous of degree 2γ.

For $d = 1$ and $W = \mathbb{Z}_2$, the multiplicity function k is a single parameter denoted γ and

$$\forall\, x \in \mathbb{R}, \qquad \omega_k(x) = |x|^{2\gamma}.$$

We introduce the Mehta-type constant

$$c_k = \left(\int_{\mathbb{R}^d} e^{-\|x\|^2}\omega_k(x)dx\right)^{-1},$$

which is known for all Coxeter groups W (see [3, 6]).

2.2 The Dunkl operators and the Dunkl kernel

The Dunkl operators T_j, $j = 1, \dots, d$, on $\mathbb{R}^d$, associated with the finite reflection group W and the multiplicity function k, are given for a function f of class C^1 on $\mathbb{R}^d$ by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle}.$$

In the case $k \equiv 0$, the T_j, $j = 1, 2, \dots, d$, reduce to the corresponding partial derivatives. In this paper, we will assume throughout that $k \geq 0$ and $\gamma > 0$.

For f of class C^1 on $\mathbb{R}^d$ with compact support and g of class C^1 on $\mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} T_j f(x) g(x) \omega_k(x) dx = -\int_{\mathbb{R}^d} f(x) T_j g(x) \omega_k(x) dx, \qquad j = 1, 2, \dots, d. \tag{2.1}$$

For $y \in \mathbb{R}^d$, the system

$$\begin{aligned} &T_j u(x, y) = y_j u(x, y), \qquad j = 1, 2, \dots, d, \\ &u(0, y) = 1, \end{aligned} \tag{2.2}$$

admits a unique analytic solution on $\mathbb{R}^d$, denoted by $K(x, y)$ and called the Dunkl kernel.

This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$.

Example 2.1. From [5], if $d = 1$ and $W = \mathbb{Z}_2$, the Dunkl kernel is given by

$$K(z, t) = j_{\gamma-1/2}(izt) + \frac{zt}{2\gamma + 1} j_{\gamma+1/2}(izt), \qquad z, t \in \mathbb{C},$$

where for $\alpha \geq -1/2$, j_α is the normalized Bessel function defined by

$$j_\alpha(u) = 2^\alpha \Gamma(\alpha + 1) \frac{J_\alpha(u)}{u^\alpha} = \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \frac{(-1)^n (u/2)^{2n}}{n! \Gamma(n + \alpha + 1)}, \qquad u \in \mathbb{C},$$

with J_α being the Bessel function of first kind and index α (see [16]).

The Dunkl kernel possesses the following properties.

(i) For $z, t \in \mathbb{C}^d$, we have $K(z, t) = K(t, z)$, $K(z, 0) = 1$, and $K(\lambda z, t) = K(z, \lambda t)$ for all $\lambda \in \mathbb{C}$.

(ii) For all $\nu \in \mathbb{Z}_+^d$, $x \in \mathbb{R}^d$, and $z \in \mathbb{C}^d$ we have

$$|D_z^\nu K(x, z)| \leq \|x\|^{|\nu|} \exp\left[\max_{w \in W} \langle wx, \operatorname{Re} z\rangle\right]. \tag{2.3}$$

with

$$D_z^\nu = \frac{\partial^{|\nu|}}{\partial z_1^{\nu_1} \cdots \partial z_d^{\nu_d}} \qquad \text{and} \qquad |\nu| = \nu_1 + \cdots + \nu_d.$$

In particular

$$\begin{aligned} &|D_z^\nu K(x, z)| \leq \|x\|^{|\nu|} \exp[\|x\| \|\operatorname{Re} z\|], \\ &|K(x, z)| \leq \exp[\|x\| \|\operatorname{Re} z\|], \end{aligned} \tag{2.4}$$

and for all $x, y \in \mathbb{R}^d$

$$|K(ix, y)| \leq 1, \tag{2.5}$$

(iii) For all $x, y \in \mathbb{R}^d$ and $w \in W$ we have

$$K(-ix, y) = \overline{K(ix, y)} \qquad \text{and} \qquad K(wx, wy) = K(x, y).$$

(iv) The function $K(x, z)$ admits for all $x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$ the following Laplace type integral representation

$$K(x, z) = \int_{\mathbb{R}^d} e^{\langle y, z\rangle} d\mu_x(y), \tag{2.6}$$

where μ_x is the measure given by the relation (1.3) (see [14]).

Remark 2.1. When $d = 1$ and $W = \mathbb{Z}_2$, the relation (2.6) is of the form

$$K(x, z) = \frac{\Gamma(\gamma + 1/2)}{\sqrt{\pi}\Gamma(\gamma)} |x|^{-2\gamma} \int_{-|x|}^{|x|} (|x| - y)^{\gamma-1}(|x| + y)^{\gamma} e^{yz} dy.$$

Then in this case the measure μ_x is given for all $x \in \mathbb{R}\backslash\{0\}$ by $d\mu_x(y) = \mathcal{K}(x, y)dy$ with

$$\mathcal{K}(x, y) = \frac{\Gamma(\gamma + 1/2)}{\sqrt{\pi}\Gamma(\gamma)} |x|^{-2\gamma}(|x| - y)^{\gamma-1}(|x| + y)^{\gamma} 1_{]-|x|,|x|[}(y),$$

where $1_{]-|x|,|x|[}$ is the characteristic function of the interval $]-|x|, |x|[$.

3 The Dunkl transform

In this section we define the Dunkl transform and we give the main results satisfied by this transform which will be used in the following sections (see [5, 9, 10]).

Notation. We denote by $\mathbb{H}(\mathbb{C}^d)$ the space of entire functions on $\mathbb{C}^d$ which are rapidly decreasing and of exponential type. We equip this space with the classical topology.

The Dunkl transform of a function f in $\mathcal{S}(\mathbb{R}^d)$ is given by

$$\forall\, y \in \mathbb{R}^d, \qquad \mathcal{F}_D(f)(y) = \int_{\mathbb{R}^d} f(x)K(x, -iy)\omega_k(x)dx. \tag{3.1}$$

This transform satisfies the relation

$$\mathcal{F}_D(f) = \mathcal{F} \circ {}^tV_k(f), \qquad f \in \mathcal{S}(\mathbb{R}^d), \tag{3.2}$$

where $\mathcal{F}$ is the classical Fourier transform on $\mathbb{R}^d$ given by

$$\forall\, y \in \mathbb{R}^d, \qquad \mathcal{F}(f)(y) = \int_{\mathbb{R}^d} f(x)e^{-i\langle x, y\rangle} dx, \qquad f \in \mathcal{S}(\mathbb{R}^d).$$

The following theorems are proved in [9, 10].

Theorem 3.1. *The transform $\mathcal{F}_D$ is a topological isomorphism*

i) *from $\mathcal{D}(\mathbb{R}^d)$ onto $\mathbb{H}(\mathbb{C}^d)$,*

ii) *from $\mathcal{S}(\mathbb{R}^d)$ onto itself.*

The inverse transform is given by

$$\forall\, x \in \mathbb{R}^d, \qquad \mathcal{F}_D^{-1}(h)(x) = \frac{c_k^2}{2^{2\gamma+d}} \int_{\mathbb{R}^d} h(y)K(x, iy)\omega_k(y)dy. \tag{3.3}$$

Remark 3.1. Another proof of Theorem 3.1 is given in [17].

When the multiplicity function satisfies $k(\alpha) \in \mathbb{N}$ for all $\alpha \in R_+$, M.F.E. de Jeu has proved in [10] the following geometrical form of Paley–Wiener theorem for functions.

Theorem 3.2. *Let E be a W-invariant compact convex set of $\mathbb{R}^d$ and f an entire function on $\mathbb{C}^d$. Then f is the Dunkl transform of a function in $\mathcal{D}(\mathbb{R}^d)$ with support in E, if and only if for all $q \in \mathbb{N}$ there exists a positive constant C_q such that*

$$\forall\, z \in \mathbb{C}^d, \qquad |f(z)| \leq C_q(1 + ||z||)^{-q} e^{I_E(\operatorname{Im} z)},$$

where I_E is the gauge associated to the polar of E, given by

$$\forall\, y \in \mathbb{R}^d, \qquad I_E(y) = \sup_{x \in E} \langle x, y\rangle. \tag{3.4}$$

4 The Dunkl convolution product and the Dunkl transform of distributions

4.1 The Dunkl translation operators and the Dunkl convolution product of functions

The definitions and properties of the Dunkl translation operators and the Dunkl convolution product of functions presented in this subsection are given in the seventh section of [17, pages 33–37].

The Dunkl translation operators τ_x, $x \in \mathbb{R}^d$, are defined on $\mathcal{E}(\mathbb{R}^d)$ by

$$\forall\, y \in \mathbb{R}^d, \qquad \tau_x f(y) = (V_k)_x (V_k)_y [V_k^{-1}(f)(x + y)]. \tag{4.1}$$

For f in $\mathcal{S}(\mathbb{R}^d)$ the function $\tau_x f$ can also be written in the form

$$\forall\, y \in \mathbb{R}^d, \qquad \tau_x f(y) = (V_k)_x ({}^tV_k^{-1})_y [{}^tV_k(f)(x + y)]. \tag{4.2}$$

Using properties of the operators V_k and ${}^tV_k$ we deduce that for f in $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}(\mathbb{R}^d)$) and $x \in \mathbb{R}^d$, the function $y \to \tau_x f(y)$ belongs to $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}(\mathbb{R}^d)$) and we have

$$\forall\, t \in \mathbb{R}^d, \qquad \mathcal{F}_D(\tau_x f)(t) = K(ix, t)\mathcal{F}_D(f)(t). \tag{4.3}$$

The Dunkl convolution product of f and g in $\mathcal{D}(\mathbb{R}^d)$ is the function $f *_D g$ defined by

$$\forall\, x \in \mathbb{R}^d, \qquad f *_D g(x) = \int_{\mathbb{R}^d} \tau_x f(-y) g(y) \omega_k(y) dy.$$

For f, g in $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}(\mathbb{R}^d)$) the function $f *_D g$ belongs to $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}(\mathbb{R}^d)$) and we have

$$\forall\, t \in \mathbb{R}^d, \qquad \mathcal{F}_D(f *_D g)(t) = \mathcal{F}_D(f)(t)\mathcal{F}_D(g)(t).$$

4.2 The Dunkl convolution product of tempered distributions

Definition 4.1. Let S be in $\mathcal{S}'(\mathbb{R}^d)$ and φ in $\mathcal{S}(\mathbb{R}^d)$. The Dunkl convolution product of S and φ is the function $S *_D \varphi$ defined by

$$\forall\, x \in \mathbb{R}^d, \qquad S *_D \varphi(x) = \langle S_y, \tau_x \varphi(-y)\rangle.$$

Proposition 4.1. *For S in $\mathcal{S}'(\mathbb{R}^d)$ and φ in $\mathcal{S}(\mathbb{R}^d)$ the function $S *_D \varphi$ belongs to $\mathcal{E}(\mathbb{R}^d)$ and we have*

$$T^\mu(S *_D \varphi) = S *_D (T^\mu(\varphi)),$$

where

$$T^\mu = T_1^{\mu_1} \circ T_2^{\mu_2} \circ \cdots \circ T_d^{\mu_d} \qquad \text{with} \quad \mu = (\mu_1, \mu_2, \dots, \mu_d) \in \mathbb{N}^d.$$

Proof. We remark first that the topology of $\mathcal{S}(\mathbb{R}^d)$ is also generated by the seminorms

$$Q_{k,l}(\psi) = \sup_{\substack{|\mu|\le k\\ x\in\mathbb{R}^d}} \left(1 + ||x||^2\right)^l |T^\mu\psi(x)|, \qquad k, l \in \mathbb{N}.$$

i) Let $x_0 \in \mathbb{R}^d$. We prove first that $S *_D \varphi$ is continuous at x_0. We have

$$\forall\, x \in \mathbb{R}^d, \qquad S *_D \varphi(x) - S *_D \varphi(x_0) = \langle S_y, (\tau_x\varphi - \tau_{x_0}\varphi)(-y)\rangle.$$

We must prove that $(\tau_x\varphi - \tau_{x_0}\varphi)$ converges to zero in $\mathcal{S}(\mathbb{R}^d)$ when x tends to x_0.

Let $k, \ell \in \mathbb{N}$ and $\mu \in \mathbb{N}^d$ such that $|\mu| \le k$. From (4.3), Theorem 3.1 and the relations (2.1), (2.2) we have

$$\begin{aligned}
&\left(1 + \|y\|^2\right)^\ell T^\mu(\tau_x\varphi - \tau_{x_0}\varphi)(-y)\\
&\quad = \frac{i^{|\mu|}c_k^2}{2^{2\gamma+d}} \int_{\mathbb{R}^d} (1 + \|\lambda\|^2)^p K(i\lambda, -y)(I - \Delta_k)^\ell \Big[\lambda^\mu(K(-ix, \lambda)\\
&\quad - K(-ix_0, \lambda))\mathcal{F}_D(\varphi)(\lambda)\Big] \frac{\omega_k(\lambda)}{(1 + \|\lambda\|^2)^p} d\lambda,
\end{aligned}$$

with $\lambda^\mu = \lambda_1^{\mu_1}\lambda_2^{\mu_2}\cdots\lambda_d^{\mu_d}$, $\Delta_k = \sum_{j=1}^d T_j^2$ the Dunkl Laplacian and $p \in \mathbb{N}$ such that $p > \gamma + \frac{d}{2} + 1$.

Using (2.4) and (2.5) we deduce that

$$Q_{k,\ell}(\tau_x\varphi - \tau_{x_0}\varphi) = \sup_{\substack{|\mu|\le k\\ y\in\mathbb{R}^d}} (1 + \|y\|)^\ell |T^\mu(\tau_x\varphi - \tau_{x_0}\varphi)(-y)| \to 0 \qquad \text{as} \quad x \to x_0.$$

Then the function $S *_D \varphi$ is continuous at x_0, and thus it is continuous on $\mathbb{R}^d$.

Now we will prove that $S *_D \varphi$ admits a partial derivative on $\mathbb{R}^d$ with respect to the variable x_j. Let $h \in \mathbb{R}\backslash\{0\}$. We consider the function f_h defined on $\mathbb{R}^d$ by

$$f_h(y) = \frac{1}{h}\left(\tau_{(x_1,\dots,x_j+h,\dots,x_d)}\varphi(-y) - \tau_{(x_1,\dots,x_j,\dots,x_d)}\varphi(-y)\right) - \frac{\partial}{\partial x_j}\tau_x\varphi(-y).$$

Using the formula

$$\forall\, y \in \mathbb{R}^d, f_h(y) = \frac{1}{h}\int_{x_j}^{x_j+h}\left(\int_{x_j}^{u_j} \frac{\partial^2}{\partial t_j^2}\tau_{(x_1,\dots,t_j,\dots,x_d)}\varphi(-y)dt_j\right) du_j,$$

we obtain for all $k, \ell \in \mathbb{N}$ and $\mu \in \mathbb{N}^d$ such that $|\mu| \le k$:

$$\begin{aligned}
\forall\, y \in \mathbb{R}^d, \qquad &(1 + \|y\|^2)^\ell T^\mu f_h(y)\\
&= \frac{1}{h}\int_{x_j}^{x_j+h}\left(\int_{x_j}^{u_j} \left(1 + \|y\|^2\right)^\ell T^\mu \frac{\partial^2}{\partial t_j^2}\tau_{(x_1,\dots,t_j,x_d)}\varphi(-y)dt_j\right) du_j.
\end{aligned} \tag{4.4}$$

By applying the preceding method to the function

$$\left(1+\|y\|^2\right)^{\ell} T^{\mu} \frac{\partial^2}{\partial t_j^2} \tau_{(x_1,\dots,t_j,\dots,x_d)} \varphi(-y),$$

we deduce from the relation (4.4) that

$$Q_{k,\ell}(f_h) = \sup_{\substack{|\mu|\leq k \\ y\in\mathbb{R}^d}} \left(1+\|y\|^2\right)^{\ell} |T^{\mu} f_h(y)| \to 0 \qquad \text{as} \quad h \to 0.$$

Thus the function $S *_D \varphi(x)$ admits a partial derivative at x_0 with respect to x_j and we have

$$\frac{\partial}{\partial x_j} S *_D \varphi(x_0) = \langle S_y, \frac{\partial}{\partial x_j} \tau_{x_0} \varphi(-y)\rangle.$$

These results is true on $\mathbb{R}^d$. Moreover the partial derivatives are continuous on $\mathbb{R}^d$. By proceeding in a similar way for partial derivatives of all order with respect to all variables, we deduce that $S *_D \varphi$ belongs to $\mathcal{E}(\mathbb{R}^d)$.

ii) From the i) we have

$$\forall\, x \in \mathbb{R}^d, \qquad \frac{\partial}{\partial x_j} S *_D \varphi(x) = \langle S_y, \frac{\partial}{\partial x_j} \tau_x \varphi(-y)\rangle.$$

On the other hand using the definition of the Dunkl operator T_j and the relation

$$T_j(\tau_x \varphi(-y)) = \tau_x(T_j \varphi)(-y),$$

we obtain

$$\forall\, x \in \mathbb{R}^d, \qquad T_j(S *_D \varphi)(x) = \langle S_y, \tau_x(T_j\varphi)(-y)\rangle = S *_D (T_j\varphi)(x).$$

By iteration we get

$$\forall\, x \in \mathbb{R}^d, \qquad T^{\mu}(S *_D \varphi)(x) = S *_D (T^{\mu}\varphi)(x).$$ ■

4.3 The Dunkl transform of distributions

Definition 4.2.

i) The Dunkl transform of a distribution S in $\mathcal{S}'(\mathbb{R}^d)$ is defined by

$$\langle \mathcal{F}_D(S), \psi\rangle = \langle S, \mathcal{F}_D(\psi)\rangle, \psi \in \mathcal{S}(\mathbb{R}^d).$$

ii) We define the Dunkl transform of a distribution S in $\mathcal{E}'(\mathbb{R}^d)$ by

$$\forall\, y \in \mathbb{R}^d, \qquad \mathcal{F}_D(S)(y) = \langle S_x, K(-iy, x)\rangle. \tag{4.5}$$

Remark 4.1. When the distribution S in $\mathcal{E}'(\mathbb{R}^d)$ is given by the function $g\omega_k$ with g in $\mathcal{D}(\mathbb{R}^d)$, and denoted by $T_{g\omega_k}$, the relation (4.5) coincides with (3.1).

Notation. We denote by $\mathcal{H}(\mathbb{C}^d)$ the space of entire functions on $\mathbb{C}^d$ which are slowly increasing and of exponential type. We equip this space with the classical topology.

The following theorem is given in [17, page 27].

Theorem 4.1. *The transform $\mathcal{F}_D$ is a topological isomorphism*

i) *from $\mathcal{S}'(\mathbb{R}^d)$ onto itself;*

ii) *from $\mathcal{E}'(\mathbb{R}^d)$ onto $\mathcal{H}(\mathbb{C}^d)$.*

Theorem 4.2. *Let S be in $\mathcal{S}'(\mathbb{R}^d)$ and φ in $\mathcal{S}(\mathbb{R}^d)$. Then, the distribution on $\mathbb{R}^d$ given by $(S *_D \varphi)\omega_k$ belongs to $\mathcal{S}'(\mathbb{R}^d)$ and we have*

$$\mathcal{F}_D(T_{(S*_D\varphi)\omega_k}) = \mathcal{F}_D(\varphi)\mathcal{F}_D(S). \tag{4.6}$$

Proof. i) As S belongs to $\mathcal{S}'(\mathbb{R}^d)$ then there exists a positive constant C_0 and $k_0, \ell_0 \in \mathbb{N}$ such that

$$|S *_D \varphi(x)| = |\langle S_y, \tau_x\varphi(-y)\rangle| \leq C_0 Q_{k_0,\ell_0}(\tau_x\varphi). \tag{4.7}$$

But by using the inequality

$$\forall\, x, y \in \mathbb{R}^d, \qquad 1 + \|x+y\|^2 \leq 2\big(1+\|x\|^2\big)\big(1+\|y\|^2\big),$$

the relations (4.2), (1.3) and the properties of the operator ${}^tV_k$ (see Theorem 3.2 of [17]), we deduce that there exists a positive constant C_1 and $k, \ell \in \mathbb{N}$ such that

$$Q_{\ell_0,\ell_0}(\tau_x\varphi) \leq C_1\big(1+\|x\|^2\big)^{\ell_0} Q_{k,\ell}(\varphi).$$

Thus from (4.7) we obtain

$$|S *_D \varphi(x)| \leq C\big(1+\|x\|^2\big)^{\ell_0} Q_{k,\ell}(\varphi), \tag{4.8}$$

where C is a positive constant. This inequality shows that the distribution on $\mathbb{R}^d$ associated with the function $(S *_D \varphi)\omega_k$ belongs to $\mathcal{S}'(\mathbb{R}^d)$.

ii) Let ψ be in $\mathcal{S}(\mathbb{R}^d)$. We shall prove first that

$$\langle T_{(S*_D\varphi)\omega_k}, \psi\rangle = \langle \check{S}, \varphi *_D \check{\psi}\rangle, \tag{4.9}$$

where $\check{S}$ is the distribution in $\mathcal{S}'(\mathbb{R}^d)$ given by

$$\langle \check{S}, \phi\rangle = \langle S, \check{\phi}\rangle,$$

with

$$\forall\, x \in \mathbb{R}^d, \qquad \check{\phi}(x) = \phi(-x).$$

We consider the two sequences $\{\varphi_n\}_{n\in\mathbb{N}}$ and $\{\psi_m\}_{m\in\mathbb{N}}$ in $\mathcal{D}(\mathbb{R}^d)$ which converge respectively to φ and ψ in $\mathcal{S}(\mathbb{R}^d)$. We have

$$\begin{aligned}\langle T_{(S*_D\varphi_n)\omega_k}, \psi_m\rangle &= \int_{\mathbb{R}^d} \langle S_y, \tau_x\varphi_n(-y)\rangle \psi_m(x)\omega_k(x)dx,\\ &= \langle S_y, \int_{\mathbb{R}^d} \psi_m(x)\tau_x\varphi_n(-y)\omega_k(x)dx\rangle = \langle S_y, \int_{\mathbb{R}^d} \check{\psi}_m(x)\tau_{-x}\varphi_n(-y)\omega_k(x)dx\rangle.\end{aligned}$$

Thus

$$\langle T_{(S*_D\varphi_n)\omega_k}, \psi_m\rangle = \langle \check{S}, \varphi_n *_D \check{\psi}_m\rangle. \tag{4.10}$$

But

$$\langle T_{(S*_D\varphi_n)\omega_k},\psi_m\rangle-\langle T_{(S*_D\varphi)\omega_k},\psi_m\rangle=\int_{\mathbb{R}^d}\check{S}*_D(\varphi_n-\varphi)(x)\check{\psi}_m(x)\omega_k(x)dx.$$

Thus from (4.8) there exist a positive constant M and $k,\ell\in\mathbb{N}$ such that

$$|T_{(S*_D\varphi_n)\omega_k},\psi_m\rangle-\langle T_{(S*_D\varphi)\omega_k},\psi_m\rangle|\leq MQ_{k,\ell}(\varphi_n-\varphi).$$

Thus

$$\langle T_{(S*_D\varphi_n)\omega_k},\psi_m\rangle\underset{n\to+\infty}{\longrightarrow}\langle T_{(S*_D\varphi)\omega_k},\psi_m\rangle. \tag{4.11}$$

On the other hand we have

$$\langle T_{(S*_D\varphi)\omega_k},\psi_m\rangle\underset{m\to+\infty}{\longrightarrow}\langle T_{(S*_D\varphi)\omega_k},\psi\rangle, \tag{4.12}$$

and

$$\varphi_n*_D\check{\psi}_m\underset{\substack{n\to+\infty\\ m\to+\infty}}{\longrightarrow}\varphi*_D\check{\psi}, \tag{4.13}$$

the limit is in $\mathcal{S}(\mathbb{R}^d)$.

We deduce (4.9) from (4.10), (4.11), (4.12) and (4.13).

We prove now the relation (4.6). Using (4.9) we obtain for all ψ in $\mathcal{S}(\mathbb{R}^d)$

$$\langle\mathcal{F}_D(T_{(S*_D\varphi)\omega_k}),\psi\rangle=\langle T_{(S*_D\varphi)\omega_k},\mathcal{F}_D(\psi)\rangle,=\langle\check{S},\varphi*_D(\mathcal{F}_D(\psi))\check{}\rangle.$$

But

$$\varphi*_D(\mathcal{F}_D(\psi))\check{}=(\mathcal{F}_D[\mathcal{F}_D(\varphi)\psi])\check{}.$$

Thus

$$\langle\check{S},\varphi*_D(\mathcal{F}_D(\psi))\check{}\rangle=\langle S,\mathcal{F}_D[\mathcal{F}_D(\varphi)\psi]\rangle,=\langle\mathcal{F}_D(\varphi)\mathcal{F}_D(S),\psi\rangle.$$

Then

$$\langle\mathcal{F}_D(T_{(S*_D\varphi)\omega_k}),\psi\rangle=\langle\mathcal{F}_D(\varphi)\mathcal{F}_D(S),\psi\rangle.$$

This completes the proof of (4.6). ■

We consider the positive function φ in $\mathcal{D}(\mathbb{R}^d)$ which is radial for $d\geq 2$ and even for $d=1$, with support in the closed ball of center 0 and radius 1, satisfying

$$\int_{\mathbb{R}^d}\varphi(x)\omega_k(x)dx=1,$$

and ϕ the function on $[0,+\infty[$ given by

$$\varphi(x)=\phi(\|x\|)=\phi(r)\qquad\text{with}\quad r=\|x\|.$$

For $\varepsilon\in]0,1]$, we denote by φ_ε the function on $\mathbb{R}^d$ defined by

$$\forall\,x\in\mathbb{R}^d,\qquad\varphi_\varepsilon(x)=\frac{1}{\varepsilon^{2\gamma+d}}\phi(\frac{\|x\|}{\varepsilon}). \tag{4.14}$$

This function satisfies the following properties:

i) Its support is contained in the closed ball B_ε of center 0, and radius ε.

ii) From [13, pages 585–586] we have

$$\forall\, y \in \mathbb{R}^d, \qquad \mathcal{F}_D(\varphi_\varepsilon)(y) = \frac{2^{\gamma+\frac{d}{2}}}{c_k} \mathcal{F}_B^{\gamma+\frac{d}{2}-1}(\phi)(\varepsilon\|y\|), \tag{4.15}$$

where $\mathcal{F}_B^{\gamma+\frac{d}{2}-1}(f)(\lambda)$ is the Fourier–Bessel transform given by

$$\forall\, \lambda \in \mathbb{R}, \qquad \mathcal{F}_B^{\gamma+\frac{d}{2}-1}(f)(\lambda) = \int_0^\infty f(r) j_{\gamma+\frac{d}{2}-1}(\lambda r) \frac{r^{2\gamma+d-1}}{2^{\gamma+\frac{d}{2}}\Gamma\left(\gamma+\frac{d}{2}\right)} dr, \tag{4.16}$$

with $j_{\gamma+\frac{d}{2}-1}(\lambda r)$ the normalized Bessel function.

iii) There exists a positive constant M such that

$$\forall\, y \in \mathbb{R}^d, \qquad |\mathcal{F}_D(\varphi_\varepsilon)(y) - 1| \leq \varepsilon M \|y\|^2. \tag{4.17}$$

Theorem 4.3. *Let S be in $\mathcal{S}'(\mathbb{R}^d)$. We have*

$$\lim_{\varepsilon \to 0} (S *_D \phi_\varepsilon)\omega_k = S, \tag{4.18}$$

where the limit is in $\mathcal{S}'(\mathbb{R}^d)$.

Proof. We deduce (4.18) from (4.6), (4.15), (4.17) and Theorem 4.1. ■

Definition 4.3. Let S_1 be in $\mathcal{S}'(\mathbb{R}^d)$ and S_2 in $\mathcal{E}'(\mathbb{R}^d)$. The Dunkl convolution product of S_1 and S_2 is the distribution $S_1 *_D S_2$ on $\mathbb{R}^d$ defined by

$$\langle S_1 *_D S_2, \psi\rangle = \langle S_{1,x}, \langle S_{2,y}, \tau_x\psi(y)\rangle\rangle, \qquad \psi \in \mathcal{D}(\mathbb{R}^d). \tag{4.19}$$

Remark 4.2. The relation (4.19) can also be written in the form

$$\langle S_1 *_D S_2, \psi\rangle = \langle S_1, \check{S}_2 *_D \psi\rangle. \tag{4.20}$$

Theorem 4.4. *Let S_1 be in $\mathcal{S}'(\mathbb{R}^d)$ and S_2 in $\mathcal{E}'(\mathbb{R}^d)$. Then the distribution $S_1 *_D S_2$ belongs to $\mathcal{S}'(\mathbb{R}^d)$ and we have*

$$\mathcal{F}_D(S_1 *_D S_2) = \mathcal{F}_D(S_2) \cdot \mathcal{F}_D(S_1).$$

Proof. We deduce the result from (4.20), the relation

$$T_{(\check{S}_2 *_D \mathcal{F}_D(\psi))\omega_k} = \check{S}_2 *_D T_{\mathcal{F}_D(\psi)\omega_k},$$

and Theorem 4.2. ■

4.4 Another proof of the geometrical form of the Paley–Wiener–Schwartz theorem for the Dunkl transform

In this subsection we suppose that the multiplicity function satisfies $k(\alpha) \in \mathbb{N}\backslash\{0\}$ for all $\alpha \in R_+$.

The main result is to give another proof of the geometrical form of Paley–Wiener–Schwartz theorem for the transform $\mathcal{F}_D$, given in [17, pages 23–33].

Theorem 4.5. *Let E be a W-invariant compact convex set of $\mathbb{R}^d$ and f an entire function on $\mathbb{C}^d$. Then f is the Dunkl transform of a distribution in $\mathcal{E}'(\mathbb{R}^d)$ with support in E if and only if there exist a positive constant C and $N \in \mathbb{N}$ such that*

$$\forall\, z \in \mathbb{C}^d, \qquad |f(z)| \leq C(1 + \|z\|^2)^N e^{I_E(\operatorname{Im} z)}, \tag{4.21}$$

where I_E is the function given by (3.4)*.*

Proof. *Necessity condition.* We consider a distribution S in $\mathcal{E}'(\mathbb{R}^d)$ with support in E.

Let $\mathcal{X}$ be in $\mathcal{D}(\mathbb{R}^d)$ equal to 1 in a neighborhood of E, and θ in $\mathcal{E}(\mathbb{R})$ such that

$$\theta(t) = \begin{cases} 1, & \text{if } t \leq 1, \\ 0, & \text{if } t > 2. \end{cases}$$

We put $\eta = \operatorname{Im} z$, $z \in \mathbb{C}^d$ and we take $\varepsilon > 0$. We denote by ψ_z the function defined on $\mathbb{R}^d$ by

$$\psi_z(x) = \chi(x) K(-ix, z) |W|^{-1} \sum_{w \in W} \theta(\|z\|^\varepsilon (\langle wx, \eta\rangle - I_E(\eta))).$$

This function belongs to $\mathcal{D}(\mathbb{R}^d)$ and as E is W-invariant, then it is equal to $K(-ix, z)$ in a neighborhood of E. Thus

$$\forall\, z \in \mathbb{C}^d, \qquad \mathcal{F}_D(S)(z) = \langle S_x, \psi_z(x)\rangle.$$

As S is with compact support, then it is of finite order N. Then there exists a positive constant C_0 such that

$$\forall\, z \in \mathbb{C}^d, \qquad |\mathcal{F}_D(S)(z)| \leq C_0 \sum_{|p| \leq N} \sup_{x \in \mathbb{R}^d} |D^p \psi_z(x)|. \tag{4.22}$$

Using the Leibniz rule, we obtain

$$\begin{aligned} \forall\, x \in \mathbb{R}^d, \qquad D^p \psi_z(x) = &\sum_{q+r+s=p} \frac{p!}{q!r!s!} D^q \mathcal{X}(x) D^r K(-ix, z) \\ &\times D^s |W|^{-1} \sum_{w \in W} \theta(\|z\|^\varepsilon (\langle wx, \eta\rangle - I_E(\eta))). \end{aligned} \tag{4.23}$$

We have

$$\forall\, x \in \mathbb{R}^d, \qquad |D^q \chi(x)| \leq \text{const}, \tag{4.24}$$

and if M is the estimate of $\sup_{t \in \mathbb{R}} |\theta^{(k)}(t)|$, $k \leq N$, we obtain

$$\forall\, x \in \mathbb{R}^d, \qquad \left| D^s \left(\sum_{w \in W} \theta(\|z\|^\varepsilon (\langle wx, \eta\rangle - I_E(\eta))) \right) \right| \leq M (\|z\|^\varepsilon \|\eta\|)^{|s|}. \tag{4.25}$$

On the other hand from (2.3) we have

$$\forall\, x \in \mathbb{R}^d, \qquad |D^r K(-ix, z)| \leq \|z\|^r e^{\max_{w \in W} \langle wx, \eta\rangle}. \tag{4.26}$$

Using inequalities (4.24), (4.25), (4.26) and (4.23) we deduce that there exists a positive constant C_1 such that

$$\forall\, x \in \mathbb{R}^d, \qquad |D^p \psi_z(x)| \leq C_1 (1 + \|z\|^2)^{N(1+\varepsilon)} e^{\max_{w \in W} \langle wx, \eta\rangle}.$$

From this relation and (4.22) we obtain

$$\forall\ z \in \mathbb{C}^d, \qquad |\mathcal{F}_D(S)(z)| \leq C_2(1+\|z\|^2)^{N(1+\varepsilon)} \sup_{x\in E} e^{\max_{w\in W}\langle wx,\eta\rangle}, \tag{4.27}$$

where C_2 is a positive constant, and the supremum is calculated when $\|z\| \geq 1$, for

$$\langle wx, \eta\rangle \leq I_E(\eta) + \frac{2}{\|z\|^{\varepsilon}},$$

because if not we have $\theta = 0$. This inequality implies

$$\sup_{x\in E} e^{\max_{w\in W}\langle wx,\eta\rangle} \leq e^2 \cdot e^{I_E(\eta)}. \tag{4.28}$$

From (4.27), (4.28) we deduce that there exists a positive constant C_3 independent from ε such that

$$\forall\ z \in \mathbb{C}^d, \qquad \|z\| \geq 1, \qquad |\mathcal{F}_D(S)(z)| \leq C_3(1+\|z\|^2)^{N(1+\varepsilon)} e^{I_E(\eta)}.$$

If we make $\varepsilon \to 0$ in this relation we obtain (4.21) for $\|z\| \geq 1$. But this inequality is also true (with another constant) for $\|z\| \leq 1$, because in the set $\{z \in \mathbb{C}^d, \|z\| \leq 1\}$ the function $\mathcal{F}_D(S)(z)e^{-I_E(\eta)}$ is bounded.

Sufficient condition. Let f be an entire function on $\mathbb{C}^d$ satisfying the condition (4.21). It is clear that the distribution given by the restriction of $f\omega_k$ to $\mathbb{R}^d$ belongs to $\mathcal{S}'(\mathbb{R}^d)$. Thus from Theorem 4.1i there exists a distribution S in $\mathcal{S}'(\mathbb{R}^d)$ such that

$$T_{f\omega_k} = \mathcal{F}_D(S). \tag{4.29}$$

We shall show that the support of S is contained in E. Let φ_ε be the function given by the relation (4.14). We consider the distribution

$$T_{f_\varepsilon\omega_k} = \mathcal{F}_D(T_{(S*_D\varphi_\varepsilon)\omega_k}). \tag{4.30}$$

From Theorem 4.2 and (4.29), (4.30) we deduce that

$$f_\varepsilon = \mathcal{F}_D(\varphi_\varepsilon)f.$$

The properties of the function f and (4.15), (4.16) and (4.17) show that the function f_ε can be extended to an entire function on $\mathbb{C}^d$ which satisfies: for all $q \in \mathbb{N}$ there exists a positive constant C_q such that

$$\forall\ z \in \mathbb{C}^d, \qquad |f_\varepsilon(z)| \leq C_q(1+\|z\|)^{-q} e^{I_{E+B_\varepsilon}(\operatorname{Im} z)}. \tag{4.31}$$

Then from (4.31), Theorem 3.2 and (4.30), the function $(S*\varphi_\varepsilon)\omega_k$ belongs to $\mathcal{D}(\mathbb{R}^d)$ with support in $E + B_\varepsilon$. But from Theorem 4.3, the family $(S*\varphi_\varepsilon)\omega_k$ converges to S in $\mathcal{S}'(\mathbb{R}^d)$ when ε tends to zero. Thus for all $\varepsilon > 0$, the support of S is in $E + B_\varepsilon$, then it is contained in E. ■

Remark 4.3. In the following we give an ameliorated version of the proof of Proposition 6.3 of [17, page 30].

Let E be a W-invariant compact convex set of $\mathbb{R}^d$ and $x \in E$. The function $f(x,\cdot)$ defined on $\mathbb{C}^d$ by

$$f(x,z) = e^{-i\left(\sum\limits_{j=1}^{d} x_j z_j\right)},$$

is entire on $\mathbb{C}^d$ and satisfies

$$\forall\, z \in \mathbb{C}^d, \qquad |f(x,z)| \leq e^{I_E(\operatorname{Im} z)}.$$

Thus from Theorem 4.5 there exists a distribution $\tilde{\eta}_x$ in $\mathcal{E}'(\mathbb{R}^d)$ with support in E such that

$$\forall\, y \in \mathbb{R}^d, \qquad f(x,y) = e^{-i\langle x,y\rangle} = \langle \tilde{\eta}_x, K(-iy,\cdot)\rangle.$$

Applying now the remainder of the proof given in [17, page 32], we deduce that the support of the representing distribution η_x of the inverse Dunkl intertwining operator V_k^{-1} is contained in E.

5 Inversion formulas for the Dunkl intertwining operator and its dual

5.1 The pseudo-differential operators P

Definition 5.1. We define the pseudo-differential operator P on $\mathcal{S}(\mathbb{R}^d)$ by

$$\forall\, x \in \mathbb{R}^d, \qquad P(f)(x) = \frac{\pi^d c_k^2}{2^{2\gamma}} \mathcal{F}^{-1}[\omega_k \mathcal{F}(f)](x). \tag{5.1}$$

Proposition 5.1. *The distribution T_{ω_k} given by the function ω_k, is in $\mathcal{S}'(\mathbb{R}^d)$ and for all f in $\mathcal{S}(\mathbb{R}^d)$ we have*

$$\forall\, x \in \mathbb{R}^d, \qquad P(f)(x) = \frac{\pi^d c_k^2}{2^{2\gamma}} \mathcal{F}(T_{\omega_k}) * \breve{f}(-x).$$

where $*$ is the classical convolution production of a distribution and a function on $\mathbb{R}^d$.

Proof. It is clear that the distribution T_{ω_k} given by the function ω_k belongs to $\mathcal{S}'(\mathbb{R}^d)$. On the other hand from the relation (5.1) we have

$$\forall\, x \in \mathbb{R}^d, \qquad P(f)(x) = \frac{\pi^d c_k^2}{2^{2\gamma}} \int_{\mathbb{R}^d} \mathcal{F}(f(\xi + x))(y)\omega_k(y)dy.$$

Thus

$$\forall\, x \in \mathbb{R}^d, \qquad P(f)(x) = \frac{\pi^d c_k^2}{2^{2\gamma}} \langle \mathcal{F}(T_{\omega_k})_y, f(x+y)\rangle. \tag{5.2}$$

With the definition of the classical convolution product of a distribution and a function on $\mathbb{R}^d$, the relation (5.2) can also be written in the form

$$\forall\, x \in \mathbb{R}^d, \qquad P(f)(x) = \frac{\pi^d c_k^2}{2^{2\gamma}} \mathcal{F}(T_{\omega_k}) * \breve{f}(-x). \qquad \blacksquare$$

Proposition 5.2. *For all f in $\mathcal{S}(\mathbb{R}^d)$ the function $P(f)$ is of class C^∞ on $\mathbb{R}^d$ and we have*

$$\forall\, x \in \mathbb{R}^d, \qquad \frac{\partial}{\partial x_j} P(f)(x) = P\left(\frac{\partial}{\partial \xi_j} f\right)(x), \qquad j = 1, 2, \ldots, d. \tag{5.3}$$

Proof. By derivation under the integral sign, and by using the relation

$$\forall\, y \in \mathbb{R}^d, \qquad iy_j \mathcal{F}(f)(y) = \mathcal{F}\left(\frac{\partial}{\partial \xi_j} f\right)(y),$$

we obtain (5.3). $\blacksquare$

5.2 Inversion formulas for the Dunkl intertwining operator and its dual on the space $\mathcal{S}(\mathbb{R}^d)$

Theorem 5.1. *For all f in $\mathcal{S}(\mathbb{R}^d)$ we have*

$$\forall\, x \in \mathbb{R}^d, \qquad {}^tV_k^{-1}(f)(x) = V_k(P(f))(x). \tag{5.4}$$

Proof. From [15, Theorem 4.1] for all f in $\mathcal{S}(\mathbb{R}^d)$, the function ${}^tV_k^{-1}(f)$ belongs to $\mathcal{S}(\mathbb{R}^d)$. Then from Theorem 3.1 we have

$$\forall\, x \in \mathbb{R}^d, \qquad {}^tV_k^{-1}(f)(x) = \frac{c_k^2}{2^{2\gamma+d}} \int_{\mathbb{R}^d} K(iy,x)\mathcal{F}_D({}^tV_k^{-1}(f))(y)\omega_k(y)dy. \tag{5.5}$$

But from the relations (3.2), (1.7), (1.3), we have

$$\forall\, y \in \mathbb{R}^d, \qquad \mathcal{F}_D({}^tV_k^{-1}(f))(y) = \mathcal{F}(f)(y),$$

and

$$\forall\, y \in \mathbb{R}^d, \qquad K(iy,x) = \mathcal{F}(\breve{\mu}_x)(y),$$

where $\breve{\mu}_x$ is the probability measure given for a continuous function f on $\mathbb{R}^d$ by

$$\int_{\mathbb{R}^d} f(t)d\breve{\mu}_x(t) = \int_{\mathbb{R}^d} f(-t)d\mu_x(t).$$

Thus (5.5) can also be written in the form

$$\forall\, x \in \mathbb{R}^d, \qquad {}^tV_k^{-1}(f)(x) = \frac{c_k^2}{2^{2\gamma+d}} \int_{\mathbb{R}} \mathcal{F}(\breve{\mu}_x)(y)\omega_k(y)\mathcal{F}(f)(y)dy.$$

Then by using (5.1), the properties of the Fourier transform $\mathcal{F}$ and Fubini's theorem we obtain

$$\forall\, x \in \mathbb{R}^d, \qquad {}^tV_k^{-1}(f)(x) = \frac{c_k^2}{2^{2\gamma+d}} \int_{\mathbb{R}^d} \mathcal{F}[\omega_k\mathcal{F}(f)](y)d\breve{\mu}_x(y) = \int_{\mathbb{R}^d} P(f)(y)d\mu_x(y).$$

Thus

$$\forall\, x \in \mathbb{R}^d, \qquad {}^tV_k^{-1}(f)(x) = V_k(P(f))(x).$$

■

Theorem 5.2. *For all f in $\mathcal{S}(\mathbb{R}^d)$ we have*

$$\forall\, x \in \mathbb{R}^d, \qquad V_k^{-1}(f)(x) = P\,{}^tV_k(f)(x). \tag{5.6}$$

Proof. We deduce the relation (5.6) by replacing f by ${}^tV_k(f)$ in (5.4) and by using the fact that the operator V_k is an isomorphism from $\mathcal{E}(\mathbb{R}^d)$ onto itself. ■

5.3 Inversion formulas for the dual Dunkl intertwining operator on the space $\mathcal{E}'(\mathbb{R}^d)$

The dual Dunkl intertwining operator ${}^tV_k$ on $\mathcal{E}'(\mathbb{R}^d)$ is defined by

$$\langle {}^tV_k(S), f\rangle = \langle S, V_k(f)\rangle, \qquad f \in \mathcal{E}(\mathbb{R}^d).$$

The operator ${}^tV_k$ is a topological isomorphism from $\mathcal{E}'(\mathbb{R}^d)$ onto itself. The inverse operator is given by

$$\langle {}^tV_k^{-1}(S), f\rangle = \langle S, V_k^{-1}(f)\rangle, \qquad f \in \mathcal{E}(\mathbb{R}^d), \tag{5.7}$$

see [17, pages 26–27].

Theorem 5.3. *For all S in $\mathcal{E}'(\mathbb{R}^d)$ the operator ${}^tV_k^{-1}$ satisfies also the relation*

$$\langle {}^tV_k^{-1}(S), f\rangle = \langle S, P\,{}^tV_k(f)\rangle, \qquad f \in \mathcal{S}(\mathbb{R}^d). \tag{5.8}$$

Proof. We deduce (5.8) from (5.6) and (5.7). ■

6 Other expressions of the inversion formulas for the Dunkl intertwining operator and its dual when the multiplicity function is integer

In this section we suppose that the multiplicity function satisfies $k(\alpha) \in \mathbb{N}\backslash\{0\}$ for all $\alpha \in R_+$. The following two Propositions give some other properties of the operator P defined by (5.1).

Proposition 6.1. *Let E be a compact convex set of $\mathbb{R}^d$. Then for all f in $\mathcal{D}(\mathbb{R}^d)$ we have*

$$\operatorname{supp} f \subset E \Rightarrow \operatorname{supp} P(f) \subset E.$$

Proof. From the relation (5.1) we have

$$\forall\, x \in \mathbb{R}^d, \qquad P(f)(x) = \frac{\pi^d c_k^2}{2^{2\gamma}} \int_{\mathbb{R}^d} \mathcal{F}f(y) e^{i\langle x,y\rangle} \omega_k(y) dy. \tag{6.1}$$

We consider the function F defined by

$$\forall\, z \in \mathbb{C}^d, \qquad F(z) = \left(\prod_{\alpha \in R_+} (\langle \alpha, z\rangle)^{2k(\alpha)} \right) \mathcal{F}(f)(z).$$

This function is entire on $\mathbb{C}^d$ and by using Theorem 2.6 of [1] we deduce that for all $q \in \mathbb{N}$, there exists a positive constant C_q such that

$$\forall\, z \in \mathbb{C}^d, \qquad |F(z)| \le C_q (1 + \|z\|^2)^{-q} e^{I_E(\operatorname{Im} z)}, \tag{6.2}$$

where I_E is the function given by (3.4).

The relation (6.1) can also be written in the form

$$\forall\, x \in \mathbb{R}^d, \qquad P(f)(x) = \frac{\pi^d c_k^2}{2^{2\gamma}} \int_{\mathbb{R}^d} F(y) e^{i\langle x,y\rangle} dy. \tag{6.3}$$

Thus (6.3), (6.2) and Theorem 2.6 of [1], imply that $\operatorname{supp} Pf \subset E$. ■

Proposition 6.2. *For all f in $\mathcal{S}(\mathbb{R}^d)$ we have*

$$P(f) = \frac{\pi^d c_k^2}{2^{2\gamma}} \left[\prod_{\alpha \in R_+} (-1)^{k(\alpha)} \left(\alpha_1 \frac{\partial}{\partial \xi_1} + \cdots + \alpha_d \frac{\partial}{\partial \xi_d} \right)^{2k(\alpha)} \right] (f). \tag{6.4}$$

Proof. For all f in $\mathcal{S}(\mathbb{R}^d)$, we have

$$\forall\, y \in \mathbb{R}^d, \quad \omega_k(y) \mathcal{F}(f)(y) = \prod_{\alpha \in R_+} (\langle \alpha, y\rangle)^{2k(\alpha)} \mathcal{F}(f)(y). \tag{6.5}$$

But

$$\forall\, y \in \mathbb{R}^d, \qquad \langle \alpha, y\rangle \mathcal{F}(f)(y) = \mathcal{F}\left[-i \left(\alpha_1 \frac{\partial}{\partial \xi_1} + \cdots + \alpha_d \frac{\partial}{\partial \xi_d} \right) f \right](y). \tag{6.6}$$

From (6.5), (6.6) we obtain

$$\forall\, y \in \mathbb{R}^d, \qquad \omega_k(y) \mathcal{F}(f)(y) = \mathcal{F}\left[\prod_{\alpha \in R_+} (-1)^{k(\alpha)} \left(\alpha_1 \frac{\partial}{\partial \xi_1} + \cdots + \alpha_d \frac{\partial}{\partial \xi_d} \right)^{2k(\alpha)} f \right](y).$$

This relation, Definition 5.1 and the inversion formula for the Fourier transform $\mathcal{F}$ imply (6.4). ■

Remark 6.1. In this case the operator P is not a pseudo-differential operator but it is a partial differential operator.

6.1 The differential-difference operator Q

Definition 6.1. We define the differential-difference operator Q on $\mathcal{S}(\mathbb{R}^d)$ by

$$\forall\, x \in \mathbb{R}^d, \qquad Q(f)(x) = {}^tV_k^{-1} \circ P \circ {}^tV_k(f)(x).$$

Proposition 6.3.

i) *The operator Q is linear and continuous from $\mathcal{S}(\mathbb{R}^d)$ into itself.*

ii) *For all f in $\mathcal{S}(\mathbb{R}^d)$ we have*

$$\forall\, x \in \mathbb{R}^d, \qquad T_jQ(f)(x) = Q(T_jf)(x), \qquad j = 1, \dots, d,$$

where T_j, $j = 1, 2, \dots, d$, are the Dunkl operators.

Proof. We deduce the result from the properties of the operator ${}^tV_k$ (see Theorem 3.2 of [17]), and Proposition 5.2. ■

Proposition 6.4. *For all f in $\mathcal{S}(\mathbb{R}^d)$ we have*

$$\forall\, x \in \mathbb{R}^d, \qquad Q(f)(x) = \frac{\pi^d c_k^2}{2^{2\gamma}} \mathcal{F}_D^{-1}(\omega_k \mathcal{F}_D(f))(x). \tag{6.7}$$

Proof. Using the relations (3.2), (5.1) and the properties of the operator ${}^tV_k$ (see Theorem 3.2 of [17]), we deduce from Definition 6.1 that

$$\forall\, x \in \mathbb{R}^d, \qquad Q(f)(x) = \mathcal{F}_D^{-1}\{\mathcal{F} \circ P({}^tV_k(f))\}(x) = \frac{\pi^d c_k^2}{2^{2\gamma}} \mathcal{F}_D^{-1}\{\mathcal{F} \circ \mathcal{F}^{-1}[\omega_k \mathcal{F}_D(f)]\}(x).$$

As the function $\omega_k \mathcal{F}_D(f)$ belongs to $\mathcal{S}(\mathbb{R}^d)$, then by applying the fact that the classical Fourier transform $\mathcal{F}$ is bijective from $\mathcal{S}(\mathbb{R}^d)$ onto itself, we obtain

$$\forall\, x \in \mathbb{R}^d, \qquad Q(f)(x) = \frac{\pi^d c_k^2}{2^{2\gamma}} \mathcal{F}_D^{-1}(\omega_k \mathcal{F}_D(f))(x).$$ ■

Proposition 6.5. *The distribution $T_{\omega_k^2}$ given by the function ω_k^2 is in $\mathcal{S}'(\mathbb{R}^d)$ and for all f in $\mathcal{S}(\mathbb{R}^d)$ we have*

$$\forall\, x \in \mathbb{R}^d, \qquad Q(f)(x) = \frac{\pi^d c_k^4}{2^{4\gamma+d}} \mathcal{F}_D(T_{\omega_k^2}) *_D \check{f}(-x),$$

where $*_D$ is the Dunkl convolution product of a distribution and a function on $\mathbb{R}^d$.

Proof. It is clear that the distribution $T_{\omega_k^2}$ given by the function ω_k^2 belongs to $\mathcal{S}'(\mathbb{R}^d)$. On the other hand from the relations (6.7), (3.3) and (4.3) we obtain

$$\begin{aligned} \forall\, x \in \mathbb{R}^d, \qquad Q(f)(x) &= \frac{\pi^d c_k^4}{2^{4\gamma+d}} \int_{\mathbb{R}^d} \mathcal{F}_D(\tau_x(f))(y) \omega_k^2(y) dy \\ &= \frac{\pi^d c_k^4}{2^{4\gamma+d}} \langle \mathcal{F}(T_{\omega_k^2})_y, \tau_x(f)(y) \rangle. \end{aligned}$$

Thus Definition 4.1 implies

$$\forall\, x \in \mathbb{R}^d, \qquad Q(f)(x) = \frac{\pi^d c_k^4}{2^{4\gamma+d}} \mathcal{F}_D(T_{\omega_k^2}) *_D \check{f}(-x).$$ ■

Proposition 6.6. *For all f in $\mathcal{S}(\mathbb{R}^d)$ we have*

$$Q(f) = \frac{\pi^d c_k^2}{2^{2\gamma}} \left[\prod_{\alpha \in R_+} (-1)^{k(\alpha)} (\alpha_1 T_1 + \cdots + \alpha_d T_d)^{2k(\alpha)} \right] (f). \tag{6.8}$$

Proof. For all f in $\mathcal{S}(\mathbb{R}^d)$, we have

$$\forall\, y \in \mathbb{R}^d, \qquad \omega_k(y) \mathcal{F}_D(f)(y) = \prod_{\alpha \in R_+} (\langle \alpha, y \rangle)^{2k(\alpha)} \mathcal{F}_D(f)(y). \tag{6.9}$$

But using (2.1), (2.2) we deduce that

$$\forall\, y \in \mathbb{R}^d, \qquad \langle \alpha, y \rangle \mathcal{F}_D(f)(y) = \mathcal{F}_D\big[- i(\alpha_1 T_1 + \cdots + \alpha_d T_d) f \big](y). \tag{6.10}$$

From (6.9), (6.10) we obtain

$$\forall\, y \in \mathbb{R}^d, \qquad \omega_k(y) \mathcal{F}_D(f)(y) = \mathcal{F}_D \left[\prod_{\alpha \in R_+} (-1)^{k(\alpha)} (\alpha_1 T_1 + \cdots + \alpha_d T_d)^{2k(\alpha)} f \right] (y).$$

This relation, Propositions 6.3, 6.4 and Theorem 3.1 imply (6.8). ■

6.2 Other expressions of the inversion formulas for the Dunkl intertwining operator and its dual on spaces of functions and distributions

In this subsection we give other expressions of the inversion formulas for the operators V_k and ${}^tV_k$ and we deduce the expressions of the representing distributions of the operators V_k^{-1} and ${}^tV_k^{-1}$.

Theorem 6.1. *For all f in $\mathcal{S}(\mathbb{R}^d)$ we have*

$$\forall\, x \in \mathbb{R}^d, \qquad V_k^{-1}(f)(x) = {}^tV_k(Q(f))(x). \tag{6.11}$$

Proof. We obtain this result by using of Proposition 6.3, Theorem 5.2 and Definition 6.1. ■

Proposition 6.7. *Let E be a W-invariant compact convex set of $\mathbb{R}^d$. Then for all f in $\mathcal{D}(\mathbb{R}^d)$ we have*

$$\operatorname{supp} f \subset E \Longleftrightarrow \operatorname{supp} {}^tV_k(f) \subset E. \tag{6.12}$$

Proof. For all f in $\mathcal{D}(\mathbb{R}^d)$, we obtain from (3.2) the relations

$$ {}^tV_k(f) = \mathcal{F}^{-1} \circ \mathcal{F}_D(f), $$
$$ {}^tV_k^{-1}(f) = \mathcal{F}_D^{-1} \circ \mathcal{F}(f). $$

We deduce (6.12) from these relations, Theorem 3.2 and Theorem 2.6 of [1]. ■

Proposition 6.8. *Let E be a W-invariant compact convex set of $\mathbb{R}^d$. Then for all f in $\mathcal{D}(\mathbb{R}^d)$ we have*

$$\operatorname{supp} f \subset E \Rightarrow \operatorname{supp} Q(f) \subset E. \tag{6.13}$$

Proof. We obtain (6.13) from Definition 6.1, Propositions 6.1 and 6.7. ■

Theorem 6.2. *For all S in $\mathcal{E}'(\mathbb{R}^d)$ the operator ${}^tV_k^{-1}$ satisfies also the relation*

$$\langle {}^tV_k^{-1}(S), f\rangle = \langle S, {}^tV_k(Q(f))\rangle, \qquad f \in \mathcal{S}(\mathbb{R}^d). \tag{6.14}$$

Proof. We deduce (6.14) from (5.7) and (6.11). ∎

Corollary 6.1. *Let E be a W-invariant compact convex set of $\mathbb{R}^d$. For all S in $\mathcal{E}'(\mathbb{R}^d)$ with* $\operatorname{supp} S \subset E$, *we have*

$$\operatorname{supp} {}^tV_k^{-1}(S) \subset E.$$

Definition 6.2. We define the transposed operators ${}^tP$ and ${}^tQ$ of the operators P and Q on $\mathcal{S}'(\mathbb{R}^d)$ by

$$\langle {}^tP(S), f\rangle = \langle S, P(f)\rangle, \qquad f \in \mathcal{S}(\mathbb{R}^d),$$
$$\langle {}^tQ(S), f\rangle = \langle S, Q(f)\rangle, \qquad f \in \mathcal{S}(\mathbb{R}^d).$$

Proposition 6.9. *For all S in $\mathcal{S}'(\mathbb{R}^d)$ we have*

$${}^tP(S) = \frac{\pi^d c_k^2}{2^{2\gamma}}\left[\prod_{\alpha\in R_+}\left(\alpha\frac{\partial}{\partial\xi_1} + \cdots + \alpha_d\frac{\partial}{\partial\xi_d}\right)^{2k(\alpha)}\right] S,$$
$${}^tQ(S) = \frac{\pi^d c_k^2}{2^{2\gamma}}\left[\prod_{\alpha\in R_+}(\alpha T_1 + \cdots + \alpha_d T_d)^{2k(\alpha)}\right] S,$$

where T_j, $j = 1, 2, \ldots, d$, are the Dunkl operators defined on $\mathcal{S}'(\mathbb{R}^d)$ by

$$\langle T_j S, f\rangle = -\langle S, T_j f\rangle, \qquad f \in \mathcal{S}(\mathbb{R}^d).$$

Proposition 6.10. *For all S in $\mathcal{S}'(\mathbb{R}^d)$ we have*

$$\mathcal{F}^{-1}({}^tP(S)) = \frac{\pi^d c_k^2}{2^{2\gamma}}\mathcal{F}^{-1}(S)\omega_k,$$
$$\mathcal{F}_D^{-1}({}^tQ(S)) = \frac{\pi^d c_k^2}{2^{2\gamma}}\mathcal{F}_D^{-1}(S)\omega_k.$$

Proof. We deduce these relations from (5.1), (6.7) and the definitions of the classical Fourier transform and the Dunkl transform of tempered distributions on $\mathbb{R}^d$. ∎

Theorem 6.3. *The representing distributions η_x and Z_x of the inverse of the Dunkl intertwining operator and its dual, are given by*

$$\forall\, x \in \mathbb{R}^d, \qquad \eta_x = {}^tQ(\nu_x) \tag{6.15}$$

and

$$\forall\, x \in \mathbb{R}^d, \qquad Z_x = {}^tP(\mu_x), \tag{6.16}$$

where μ_x and ν_x are the representing measures of the Dunkl intertwining operator V_k and its dual ${}^tV_k$.

Proof. From (1.5), for all f in $\mathcal{S}(\mathbb{R}^d)$ we have

$$\forall\, x \in \mathbb{R}^d, \qquad {}^tV_k(Q(f))(x) = \langle \nu_x, Q(f)\rangle = \langle {}^tQ(\nu_x), f\rangle. \tag{6.17}$$

On the other hand from (1.4)

$$\forall\, x \in \mathbb{R}^d, \qquad V_k^{-1}(f)(x) = \langle \eta_x, f\rangle.$$

We obtain (6.15) from this relation, (6.17) and (6.11).

Using (1.3), for all f in $\mathcal{S}(\mathbb{R}^d)$ we can also write the relation (5.4) in the form

$$\forall\, x \in \mathbb{R}^d, \qquad {}^tV_k^{-1}(f)(x) = \langle \mu_x, P(f)\rangle = \langle {}^tP(\mu_x), f\rangle. \tag{6.18}$$

But from (1.6) we have

$$\forall\, x \in \mathbb{R}^d, \qquad {}^tV^{-1}(f)(x) = \langle Z_x, f\rangle.$$

We deduce (6.16) from this relation and (6.18). ■

Corollary 6.2. *We have*

$$\forall\, x \in \mathbb{R}^d, \qquad \eta_x = \frac{\pi^d c_k^2}{2^{2\gamma}} \left[\prod_{\alpha\in R_+} (\alpha_1 T_1 + \cdots + \alpha_d T_d)^{2k(\alpha)} \right] (\nu_x)$$

and

$$\forall\, x \in \mathbb{R}^d, \qquad Z_x = \frac{\pi^d c_k^2}{2^{2\gamma}} \left[\prod_{\alpha\in R_+} \left(\alpha_1 \frac{\partial}{\partial \xi_1} + \cdots + \alpha_d \frac{\partial}{\partial \xi_d} \right)^{2k(\alpha)} \right] (\mu_x).$$

Proof. We deduce these relations from Theorem 6.3 and Proposition 6.9. ■

7 Applications

7.1 Other proof of the sufficiency condition of Theorem 4.4

Let f be an entire function on $\mathbb{C}^d$ satisfying the condition (4.21). Then from Theorem 2.6 of [1], the distribution $\mathcal{F}^{-1}(f)$ belongs to $\mathcal{E}'(\mathbb{R}^d)$ and we have

$$\operatorname{supp} \mathcal{F}^{-1}(f) \subset E.$$

From the relation

$$\mathcal{F}_D^{-1}(f) = {}^tV_k^{-1} \circ \mathcal{F}^{-1}(f)$$

given in [17, page 27] and Corollary 6.1, we deduce that the distribution $\mathcal{F}_D^{-1}(f)$ is in $\mathcal{E}'(\mathbb{R}^d)$ and its support is contained in E.

7.2 Other expressions of the Dunkl translation operators

We consider the Dunkl translation operators τ_x, $x \in \mathbb{R}^d$, given by the relations (4.1), (4.2).

Theorem 7.1.

i) *When the multiplicity function $k(\alpha)$ satisfies $k(\alpha) > 0$ for all $\alpha \in R_+$, we have*

$$\forall\, y \in \mathbb{R}^d, \qquad \tau_x(f)(y) = \mu_x * \mu_y(P^t V_k(f)), \qquad f \in \mathcal{S}(\mathbb{R}^d), \tag{7.1}$$

where $*$ is the classical convolution product of measures on $\mathbb{R}^d$.

ii) *When the multiplicity function satisfies $k(\alpha) \in \mathbb{N}\backslash\{0\}$ for all $\alpha \in R_+$, we have*

$$\forall\, y \in \mathbb{R}^d, \qquad \tau_x(f)(y) = \mu_x * \mu_y({}^t V_k(Q(f))), \qquad f \in \mathcal{S}(\mathbb{R}^d). \tag{7.2}$$

Proof. i) From the relations (4.1) and (1.3), for f in $\mathcal{S}(\mathbb{R}^d)$ we have

$$\forall\, x, y \in \mathbb{R}^d, \qquad \tau_x(f)(y) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V_k^{-1}(f)(\xi+\eta)d\mu_x(\xi)d\mu_y(\eta).$$

By using the definition of the classical convolution product of two measures with compact support on $\mathbb{R}^d$, we obtain

$$\forall\, x, y \in \mathbb{R}^d, \qquad \tau_x(f)(y) = \mu_x * \mu_y(V_k^{-1}(f)).$$

Thus Theorem 5.2 implies the relation (7.1).

ii) The same proof as for i) and Theorem 6.1 give the relation (7.2). ∎

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