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Hochschild Cohomology Theories in White Noise Analysis*

Rémi LÉANDRE

Institut de Mathématiques de Bourgogne, Université de Bourgogne, 21000, Dijon, France
E-mail: *Remi.leandre@u-bourgogne.fr*

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Abstract. We show that the continuous Hochschild cohomology and the differential Hochschild cohomology of the Hida test algebra endowed with the normalized Wick product are the same.

Key words: white noise analysis; Hochschild cohomology

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1 Introduction

Hochschild cohomology is a basic tool in the deformation theory of algebras. Gerstenhaber has remarked that in his seminal work (we refer to [15] and references therein for that). Deformation quantization [2, 3] in quantum field theory leads to some important problems [10, 11, 14, 40]. Motivated by that, Dito [12] has defined the Moyal product on a Hilbert space. It is easier to work with models of stochastic analysis although they are similar to models of quantum field. In order to illustrate the difference between these two theories, we refer to:

- The paper on Dirichlet forms in infinite dimensions of Albeverio–Hoegh–Krohn [1] which used measures on the space of distributions, the traditional space of quantum field theory.
- The seminal paper of Malliavin [28] which used the traditional Brownian motion and the space of continuous functions as a topological space. This allows Malliavin to introduce stochastic differential equations in infinite-dimensional analysis, and to interpret some traditional tools of quantum field theory in stochastic analysis.

This remark lead Dito and Léandre [13] to construct of the Moyal product on the Malliavin test algebra on the Wiener space.

It is very classical in theoretical physics [9] that the vacuum expectation of some operator algebras on some Hilbert space is formally represented by formal path integrals on the fields. In the case of infinite-dimensional Gaussian measure, this isomorphism is mathematically well established and is called the Wiener–Itô–Segal isomorphism between the Bosonic Fock space and the L^2 of a Gaussian measure. The operator algebra is the algebra of annihilation and creation operators with the classical commutation relations. In the case of the classical Brownian motion B_t on $\mathbb{R}$, B_t is identified with $A_t + A_t^*$ where A_t is the annihilation operator associated to $1_{[0,t]}$ and A_t^* the associated creation operator.

Let us give some details on this identification [31]. Let H be the Hilbert space of L^2 maps from $\mathbb{R}^+$ into $\mathbb{R}$. We consider the symmetric tensor product $\hat{H}^{\otimes n}$ of H. It can be realized as the set of maps h^n from $(\mathbb{R}^+)^n$ into $\mathbb{R}$ such that $\int_{(\mathbb{R}^+)^n} |h^n(s_1,\dots,s_n)|^2 ds_1 \cdots ds_n = \|h^n\|^2 < \infty$. Moreover these maps $h^n(s_1,\dots,s_n)$ are symmetric in $(s_1,\dots,s_n)$. The symmetric Fock space W_0

*This paper is a contribution to the Special Issue on Deformation Quantization. The full collection is available at http://www.emis.de/journals/SIGMA/Deformation_Quantization.html coincides with the set of formal series $\sum h^n$ such that $\sum n!\|h^n\|^2 < \infty$. The annihilation operators A_t and the creation operators are densely defined on W_0, mutually adjoint and therefore closable. To h^n we associate the multiple Wiener chaos $I^n(h^n)$

$$I^n(h^n) = \int_{(\mathbb{R}^+)^n} h^n(s_1, \dots, s_n)\delta B_{s_1} \cdots \delta B_{s_n},$$

where $s \to B_s$ is the classical Brownian motion on $\mathbb{R}$. $E_{dP}[|I^n(h^n)^2|]$for the law of the Brownian motion dP is $n!\|h^n\|^2$. $I^n(h^n)$ and $I^m(h^m)$ are mutually orthogonal on $L^2(dP)$. If F belongs to $L^2(dP)$, F can be written in a unique way $F = \sum I^n(h^n)$ where $\sum h^n$ belongs to the symmetric Fock space W_0. This identification, called chaotic decomposition of Wiener functionals, realizes an isometry between $L^2(dP)$ and the symmetric Fock space. B_t can be assimilated to the densely defined closable operator on $L^2(dP)$

$$F \to B_t F.$$

This operator is nothing else but the operator $A_t + A_t^*$ on the symmetric Fock space.

White noise analysis [18] is concerned with the time derivative of B_t (the white noise) as a distribution (an element of $W_{-\infty}$) acting on some weighted Fock space $W_{-\infty}$ (we refer to [4, 19, 35] for textbooks on white noise analysis). Let us recall namely that the Brownian motion is only continuous! The theory of Hida distribution leads to new insight in stochastic analysis.

One of the main points of interest in the white noise analysis is that we can compute the elements of $L(W_{\infty-}, W_{-\infty})$ [20, 21, 35] in terms of kernels. We refer to [21, 27, 29] for a well established theory of kernels on the Fock space. This theory was motivated by the heuristic constructions of quantum field theory [5, 16, 17]. Elements of $L(W_{\infty-}, W_{-\infty})$ can be computed in a sum of multiple integrals of the elementary creation and annihilation operators.

This theorem plays the same role as the theorem of Pinczon [37, 38]: the operators acting on $\mathbb{C}(x_1, \dots, x_d)$, the complex polynomial algebra on $\mathbb{R}^d$, are series of differential operators with polynomial components. This theorem of Pinczon allowed Nadaud [32, 33] to show that the continuous Hochschild cohomology on $C^\infty(\mathbb{R}^d)$ is equal to the differential Hochschild cohomology of the same algebra (we refer to papers of Connes [8] and Pflaum [36] for other proofs).

In the framework of white noise analysis we have an analogous theorem to the theorem of Pinczon [37]. Therefore we can repeat in this framework the proof of Nadaud. We show that the continuous Hochschild cohomology [22] of the Hida Fock space (we consider *series* of kernels) is equal to the differential Hochschild cohomology (we consider *finite* sums of kernels).

In the first part of this work, we recall the theorem of Obata which computes the operators on the Hida Fock space: Obata considers standard creation operators and standard annihilation operators. We extend this theorem in the second part to continuous multilinear operators on the Hida algebra, endowed with the normalized Wick product. This Hida test algebra was used by Léandre [23, 24] in order to define some star products in white noise analysis.

We refer to the review paper of Léandre for deformation quantization in infinite-dimensional analysis [25].

2 A brief review on Obata's theorem

We consider the Hilbert space $H = L^2(\mathbb{R}, dt)$. We consider the operator $\Delta = 1 + t^2 - d^2/dt^2$. It has eigenvalues $\mu_j = (2j + 2)$ associated to the normalized eigenvectors e_j, $j \geq 0$. We consider the Hilbert space H_k of series $f = \sum \lambda_j e_j$ such that

$$\|f\|_k^2 = \sum |\lambda_j|^2 \mu_j^{2k} < \infty.$$

It is the Sobolev space associated to Δ^k. Since $\mu_j > 1$, $H_k \subseteq H_{k'}$ if $k > k'$, and the system of Sobolev norms $\|\cdot\|_k$ increases when k increases. Therefore we can define the test functional space $H_{\infty-}$ of functions f such that all norms $\|f\|_k < \infty$. A functional F with values in this space is continuous if and only if it is continuous for all Sobolev norms $\|\cdot\|_k$, $k > 0$,

$$H_{\infty-} = \cap_{k \geq 0} H_k.$$

The topological dual of $H_{\infty-}$ is the space of Schwartz distributions:

$$H_{-\infty} = \cup_{k<0} H_k.$$

σ is called a distribution if the following condition holds: let f be in H_∞. For some $k > 0$, there exists C_k such that for all $f \in H_{\infty-}$, $|\langle \sigma, f\rangle| \leq C_k \|f\|_k$ Therefore we get a Gel'fand triple

$$H_{\infty-} \subseteq H \subseteq H_{-\infty}.$$

We complexify all these spaces (We take the same notation). It is important to complexify these spaces to apply Potthoff–Streit theorem [39].

Let $A = ((i_1, r_1), \ldots, (i_n, r_n))$ where $i_1 < i_2 < \cdots < i_n$ and $r_i > 0$. We put

$$|A| = \sum r_i \tag{1}$$

and

$$e_A = \hat{\otimes} e_{i_j}^{r_j},$$

where we consider a normalized symmetric tensor product. We introduce the Hida weight

$$\|A\| = \prod_{(i_j, r_j) \in A} (2i_j + 2)^{r_j}.$$

We consider the weighted Fock space W_k of series

$$\phi = \sum_A \lambda_A e_A$$

such that

$$\|\phi\|_k = \sum_A |\lambda_A|^2 \|A\|^{2k} |A|! < \infty$$

(λ_A is complex). These systems of norms increase when k increases.

We consider

$$W_{\infty-} = \cap_{k>0} W_k$$

endowed with the projective topology and its topological dual (called the space of Hida distributions)

$$W_{-\infty} = \cup_{k<0} W_k$$

endowed with the inductive topology. We get a Gel'fand triple

$$W_{\infty-} \subseteq W_0 \subseteq W_{-\infty},$$

W_0 is the classical Fock space of quantum physics.

We consider $\xi \in H_{\infty-}$ and the classical coherent vector

$$\phi_\xi = \sum_{n \in \mathbb{N}} \frac{\xi^{\otimes n}}{n!},$$

ϕ_ξ belongs to the Hida test functional space $W_{\infty-}$.

Definition 1. Let Ξ belong to $L(W_{\infty-}, W_{-\infty})$. Its symbol is the function $\hat{\Xi}$ from $H_{\infty-} \times H_{\infty-}$ into $\mathbb{C}$ defined by

$$\hat{\Xi}(\xi, \eta) = \langle \Xi\phi_\xi, \phi_\eta \rangle_0.$$

If Ξ belongs to $L(W_{\infty-}, W_{-\infty})$, its symbol satisfies clearly the following properties:

(P$_1$) For any $\xi_1, \xi_2, \eta_1, \eta_2 \in H_{\infty-}$, the function

$$(z, w) \to \hat{\Xi}(z\xi_1 + \xi_2, w\eta_1 + \eta_2)$$

is an entire holomorphic function on $\mathbb{C} \times \mathbb{C}$.

(P$_2$) There exists a constant K and a constant $k > 0$ such that

$$|\hat{\Xi}(\xi, \eta)|^2 \leq K \exp\left[\|\xi\|_k^2\right] \exp\left[\|\eta\|_k^2\right].$$

The converse of this theorem also holds. It is a result of Obata [34] which generalizes the theorem of Potthoff–Streit characterizing distribution in white noise analysis [39]. If a function $\hat{\Xi}$ from $H_{\infty-} \times H_{\infty-}$ into $\mathbb{C}$ satisfies (P$_1$) and (P$_2$), it is the symbol of an element of $L(W_{\infty-}, W_{-\infty})$. Its continuity norms can be estimated in a universal way linearly in the data of (P$_2$).

This characterization theorem allows Obata to show that an element Ξ of $L(W_{\infty-}, W_{-\infty})$ can be decomposed into a sum

$$\Xi = \sum_{l,m} \Xi_{l,m}(k_{l,m}),$$

where $\Xi_{l,m}(k_{l,m})$ is defined by the following considerations.

Let A be as given above (1). Let a_i^* be the standard creation operator

$$a_i^* e_A = c(r_i) e_{A^i},$$

where

$$A^i = ((i_1, r_1), \ldots, (i_l, r_l), (i, r_i + 1), (i_{l+1}, r_{l+1}), \ldots, (i_n, r_n))$$

if $i_l < i < i_{l+1}$. If i does not appear in A, we put $r_i = 1$. We consider the standard annihilation operator e_i defined

$$a_i e_A = 0$$

if i does not appear in A and equals to $c'(r_i) e_{A_i}$ where

$$A_i = ((i_1, r_i), \ldots, (i_l, r_l), (i, r_i - 1), \ldots, (i_n, r_n)).$$

The constants $c(r_i)$ and $c'(r_i)$ are computed in [31]. Their choice is motivated by the use of Hermite polynomial on the associated Gaussian space by the Wiener–Itô–Segal isomorphism: the role of Hermite polynomial in infinite dimensions is played by the theory of chaos decomposition through the theory of multiple Wiener integrals. The annihilation operator a_i corresponds to the stochastic derivative in the direction of e_i on the corresponding Wiener space. a_i^* is its adjoint obtained by integrating by parts on the Wiener space. We consider the operator $\Xi_{I,J}$

$$\phi \to a_{i_1}^* \cdots a_{i_l}^* a_{j_1} \cdots a_{j_m} \phi = a_I^* a_J \phi.$$

It belongs to $L(W_{\infty-}, W_{-\infty})$ and its symbol is [34]

$$\exp[\langle \xi, \eta \rangle_0] \prod_{j_k \in J} \langle e_{j_k}, \xi \rangle_0 \prod_{i_k \in I} \langle e_{i_k}, \eta \rangle_0.$$

Therefore we can consider

$$\Xi_{l,m}(k_{l,m}) = \sum_{|I|=l, |J|=m} \lambda_{I,J} a_I^* a_J,$$

where

$$\sum |\lambda_{I,J}|^2 \|I\|^{-k} \|J\|^{-k} < \infty$$

for some $k > 0$. $\Xi_{l,m} = \sum \lambda_{I,J} e_I \otimes e_J$ defines an element of $H_{-\infty}^{\otimes(l+m)}$. $\Xi_{I,J}$ can be extended by linearty to

$$\sum \lambda_{I,J} \Xi_{I,J} = \Xi_{l,m}(k_{l,m}).$$

$\Xi_{l,m}(k_{l,m})$ belongs to $L(W_{\infty-}, W_{-\infty})$ if $k_{l,m}$ belongs to $H_{-\infty}^{\otimes(l+m)}$. This last space is $\cup_{k>0} H_{-k}^{\otimes(l+m)}$ endowed with the inductive topology. $\Xi_{l,m}(k_{l,m})$ belongs to $L(W_{\infty-}, W_{\infty-})$ if $k_{l,m}$ belongs to $H_{\infty-}^{\otimes l} \otimes H_{-\infty}^{\otimes m}$. This means that there exists k such that for all k'

$$\sum |\lambda_{I,J}|^2 \|I\|^{k'} \|J\|^{-k} < \infty$$

for some $k > 0$. The symbol of $\Xi_{l,m}(k_{l,m})$ satisfies

$$\hat{\Xi}_{l,m}(k_{l,m})(\eta, \xi) = \langle k_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle \exp[\langle \xi, \eta \rangle_0].$$

Following the heuristic notation of quantum field theory [5, 6, 16, 17, 30], the operator $\Xi_{l,m}(k_{l,m})$ can be written as

$$\Xi_{l,m}(k_{l,m}) = \int_{\mathbb{R}^{l+m}} k(s_1, \ldots, s_l, t_1, \ldots, t_m) a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m.$$

The “elementary” creation operators a_s^* and the “elementary” annihilation operators a_t satisfy the canonical commutation relations

$$[a_s^*, a_t^*] = [a_s, a_t] = 0, \qquad [a_s^*, a_t] = \delta(s - t),$$

where $\delta(\cdot)$ is the Dirac function in 0.

3 Fock expansion of continuous multilinear operators

We are motivated in this work by the Hochschild cohomology in white noise analysis. For that, we require that $W_{\infty-}$ is an algebra. In order to be self-consistent we will take the model of [24] or [26].

We will take the normalized Wick product

$$: e_A . e_B := e_{A \cup B},$$

where $A \cup B$ is obtained by concatenating the indices and adding the length of these when the same appears twice.

$W_{\infty-}$ is not the same space as before. We consider another Hida Fock space. $W_{k,C}$ is the space of $\phi = \sum_A \lambda_A e_A$ such that

$$\|\phi\|_{k,C}^2 = \sum_A |\lambda_A|^2 C^{2|A|} \|A\|^{2k} |A|! < \infty.$$

$W_{k,C}$ can be identified with the Bosonic Fock space associated to the Hilbert Sobolev space associated to the operator $C\Delta^k$. $W_{\infty-}$ is the intersection of $W_{k,C}$, $k > 0$, $C > 0$. This space is endowed with the projective topology.

By a small improvement of [24] and [26], we get:

Theorem 1. *$W_{\infty-}$ is a topological algebra for the normalized Wick product.*

Proof. The only new ingredient in the proof of [24, 26] is that

$$|A \cup B|! \le 2^{|A|+|B|} |A|! |B|!.$$

Let us give some details. Let us consider

$$\phi^1 = \sum_A \lambda_A^1 e_A, \qquad \phi^2 = \sum_A \lambda_A^2 e_A.$$

We have

$$: \phi^1 \cdot \phi^2 := \sum_A \mu_A e_A, \qquad \text{where} \qquad \mu_A = \sum_{B \cup D = A} \lambda_B^1 \lambda_D^2.$$

There are at most $2^{|A|}$ terms in the previous sum. By Jensen inequality

$$|\mu_A|^2 \le C_1^{|A|} \sum_{B \cup D = A} |\lambda_B^1|^2 |\lambda_D^2|^2.$$

Therefore

$$\| : \phi^1 \cdot \phi^2 : \|_{k,C}^2 \le \sum_A (C_1 C)^{2|A|} C^{2|A|} \|A\|^{2k} |A|! \sum_{B \cup D = A} |\lambda_B^1|^2 |\lambda_D^2|^2.$$

But

$$\|A\|^{2k} \le \|B\|^{2k} \|D\|^{2k} \qquad \text{and} \qquad |A|! \le 2^{|B|+|D|} |B|! |D|!$$

if the concatenation of B and D equals A. Therefore, for some C_1

$$\| : \phi^1 \cdot \phi^2 : \|_{k,C}^2 \le \sum_A \sum_{B \cup D = A} (C_1 C)^{|B|+|D|} \|B\|^{2k} \|D\|^{2k} |B|! |D|! \le \|\phi^1\|_{k,C_1 C}^2 \|\phi^2\|_{k,C_1 C}^2.$$

This shows the result. ∎

Let $L(W_{\infty-}^n, W_{\infty-})$ be the space of n-multilinear continuous applications from $W_{\infty-}$ into $W_{\infty-}$.

Definition 2. The symbol $\hat{\Xi}$ of an element Ξ of $L(W_{\infty-}^n, W_{\infty-})$ is the map from $H_{\infty-}^n \times H_{\infty-}$ into $\mathbb{C}$ defined by

$$\xi^1 \times \xi^2 \times \cdots \times \xi^n \times \eta \to \langle \Xi(\phi_{\xi^1}, \ldots, \phi_{\xi^n}), \phi_\eta \rangle_0 = \hat{\Xi}(\xi^1, \ldots, \xi^n, \eta).$$

Ξ belongs to $L(W^n_{\infty-}, W_{\infty-})$ if for any (k, C), there exists (k_1, C_1, K_1) such that

$$\|\Xi(\phi^1, \dots, \phi^n)\|_{k,C} \le K_2 \prod \|\phi^i\|_{k_2,C_2}.$$

If Ξ belongs to $L(W^n_{\infty-}, W_{\infty-})$, its symbol satisfies clearly the following properties:

(O_1) For any elements $\xi^1_1, \dots, \xi^n_1$, $\xi^1_2, \dots, \xi^n_2$, η_1, η_2 of $H_{\infty-}$, the map

$$(z_1, \dots, z_n, w) \to \hat{\Xi}(z_1\xi^1_1 + \xi^1_2, \dots, z_n\xi^n_1 + \xi^n_2, w\eta_1 + \eta_2)$$

is an entire holomorphic map from $\mathbb{C}^n \times \mathbb{C}$ into $\mathbb{C}$.

(O_2) For all $k > 0$, $K > 0$, there exists numbers C, $k_1 > k$, $K_1 > 0$ such that

$$|\hat{\Xi}(\xi^1, \dots, \xi^n, \eta)|^2 \le C \exp\left[K_1 \sum_{i=1}^n \|\xi^i\|^2_{k_1} + K\|\eta\|^2_{-k}\right].$$

We prove the converse of this result. It is a small improvement of the theorem of Ji and Obata [20].

Theorem 2. *If a function $\hat{\Xi}$ from $\mathbb{C}^n \times \mathbb{C}$ into $\mathbb{C}$ satisfies to* (O_1) *and* (O_2)*, it is the symbol of an element Ξ of $L(W^n_{\infty-}, W_{\infty-})$. The different modulus of continuity can be estimated in terms of the data in* (O_2)*.*

Proof. It is an adaptation of the proof of a result of Obata [34], the result which was generalizing Potthoff–Streit theorem. We omit all the details. This classical Potthoff–Streit theorem is the following. Let Φ in $W_{-\infty}$ be the topological dual of $W_{\infty-}$. We define its S-transform as follows

$$S(\xi) = \langle \Phi, \phi_\xi \rangle_0$$

for $\xi \in H_{-\infty}$. The S-transform of Φ satisfies the following properties:

i) the function $z \to S(z\xi_1 + \xi_2)$ is entire holomorphic;

ii) for some $K_1 > 0$, $K_2 > 0$ and some $k \in \mathbb{R}$

$$|S(\xi)|^2 \le K_1 \exp[K_2\|\xi\|^2_k].$$

Potthoff–Streit theorem states the opposite [20, Lemma 3.2]: if a function S from $H_{\infty-}$ into $\mathbb{C}$ satisfies i) and ii), it is the S-transform of a distribution Φ. Moreover, there exists C depending only of K_2 such that

$$\|\Phi\|^2_{-k-r,C} \le K_1$$

for all $r > 0$.

From this theorem, we deduce that there exists a distribution $\Phi_{\xi^1,\dots,\xi^{n-1},\eta}$ such that

$$\hat{\Xi}(\xi^1, \dots, \xi^n, \eta) = \langle \Phi_{\xi^1,\dots,\xi^{n-1},\eta}, \phi_\xi \rangle_0.$$

Moreover there exists C independent of η, $\xi^1, \dots, \xi^{n-1}$ such that

$$\|\Phi_{\xi^1,\dots,\xi^{n-1},\eta}\|^2_{-k_1-r,C} \le K_2 \exp\left[K_1 \sum_{i=1}^{n-1} \|\xi^i\|^2_{k_1} + K\|\eta\|^2_{-k}\right].$$

If ϕ belongs to $W_{\infty-}$ we put

$$G_\phi(\xi^1, \dots, \xi^{n-1}, \eta) = \langle \Phi_{\xi^1, \dots, \xi^{n-1}, \eta}, \phi \rangle_0.$$

We have for some K_2, C_2, K_1, k_1, k_2 depending only on the previous datas that

$$\|G_\phi(\xi^1, \dots, \xi^{n-1}, \eta)\|^2 \leq K_2 \|\phi\|^2_{k_2, C_2} \exp\left[K_1 \sum_{i=1}^{n-1} \|\xi^i\|^2_{k_1} + K\|\eta\|^2_{-k}\right].$$

The two properties (O_1) and (O_2) are satisfied at the step $n-1$. By induction, we deduce that

$$G_\phi(\xi^1, \dots, \xi^{n-1}, \eta) = \hat{\Xi}_\phi(\xi^1, \dots, \xi^{n-1}, \eta).$$

Moreover, we get that

$$G_\phi(\xi^1, \dots, \xi^{n-1}, \eta) = \langle \Xi_\phi(\phi_{\xi^1}, \dots, \phi_{\xi^{n-1}}), \phi_\eta \rangle_0,$$

where Ξ_ϕ is an element of $L(W^{n-1}_{\infty-}, W_{\infty-})$ depending linearly and continuously from $\phi \in W_{\infty-}$. We put

$$\Xi(\phi^1, \dots, \phi^{n-1}, \phi) = \Xi_\phi(\phi^1, \dots, \phi^{n-1}).$$

It remains to prove the result for $n = 1$. It is a small improvement of the proof of the result of Ji and Obata [20]. Let us give some details.

By using Potthoff–Streit theorem, we deduce that there is a distribution Φ_η such that

$$\hat{\Xi}(\xi, \eta) = \langle \Phi_\eta, \phi_\xi \rangle_0.$$

Moreover there exists C independent of η such that

$$\|\Phi_\eta\|^2_{-k_1 - r, C} \leq K_2 \exp[K\|\eta\|^2_{-k}]$$

If ϕ belongs to $W_{\infty-}$, we set

$$G_\phi(\eta) = \langle \Phi_\eta, \phi \rangle_0.$$

We have

$$|G_\phi(\eta)|^2 \leq K_3 \|\phi|^2_{k_2, C_2} \exp[K\|\eta\|^2_{-k}]$$

for some $k_2 > 0$. We apply Potthoff–Streit theorem (see [20, Lemma 3.3]). There exists an element $\Xi(\phi)$ of $W_{k-r,C}$ where $k > 0$ depending continuously of ϕ such that

$$G_\phi(\eta) = \langle \Xi(\phi), \phi_\eta \rangle.$$

We have clearly

$$\langle \Xi(\phi_\xi), \phi_\eta \rangle = \hat{\Xi}(\xi, \eta).$$

This shows the result. ■

The following statements follow closely [7, Appendix].

Let Ξ be an element of $L(W^n_{\infty-}, W_{\infty-})$. Let $\hat{\Xi}$ be its symbol. We put:

$$\Psi(\xi^1, \dots, \xi^n, \eta) = \exp\left[-\sum_{i=1}^n \langle \xi_i, \eta \rangle_0\right] \hat{\Xi}(\xi^1, \dots, \xi^n, \eta).$$

Clearly Ψ satisfies to (O_1) and (O_2). We put

$$\psi(z_1^1,\dots,z_{m_1}^1,z_1^2,\dots,z_{m_2}^2,\dots,z_1^n,\dots,z_{m_n}^n,w_1,\dots,w_l)$$
$$=\Psi(z_1^1\xi_1^1+\cdots+z_{m_1}^1\xi_{m_1}^1,\dots,z_1^n\xi_1^n+\cdots+z_{m_n}^n\xi_{m_n}^n,w_1\eta_1+\cdots+w_l\eta_l).$$

We put $M=(m_1,\dots,m_n)$ and

$$K_{l,M}(\xi_1^1,\dots,\xi_{m_1}^1,\dots,\xi_1^n,\dots,\xi_{m_n}^n,\eta_1,\dots,\eta_l)$$
$$=\frac{1}{l!m_1!\cdots m_n!}\frac{\partial^{l+\sum m_i}}{\partial z_1^1\cdots\partial z_{m_1}^1\cdots\partial z_1^n\cdots\partial z_{m_n}^n\partial w_1\cdots\partial w_l}\psi(0,0,\dots,0).$$

$K_{L,M}$ is an $l+\sum m_i$ multilinear map.

Since ψ is holomorphic, we have a Cauchy type representation of the considered expression $K_{l,M}(\xi_1^1,\dots,\xi_{m_1}^1,\dots,\xi_1^n,\dots,\xi_{m_n}^n,\eta_1,\dots,\eta_l)$

$$K_{l,M}(\xi_1^1,\dots,\xi_{m_1}^1,\dots,\xi_1^n,\dots,\xi_{m_n}^n,\eta_1,\dots,\eta_l)=\frac{1}{l!m_1!\cdots m_n!}\prod_{j=1}^{n}\prod_{i=1}^{m_j}\frac{1}{2\pi}\int_{|z_i^j|=r_i^j}\frac{|dz_i^j|}{(z_i^j)^2}$$
$$\times\prod_{k=1}^{l}\int_{|w_k|=s_k}\frac{|dw_k|}{w_k^2}\psi(z_1^1,\dots,z_{m_1}^1,\dots,z_1^n,\dots,z_{m_n}^n,w_1,\dots,w_l).$$

We deduce from $(O)_2$ a bound of $K_{l,M}$ of the type (D.5) in [7]

$$|K_{l,M}(\xi_1^1,\dots,\xi_{m_1}^1,\dots,\xi_1^n,\dots,\xi_{m_n}^n,\eta_1,\dots,\eta_l)|\le\frac{C}{l!\prod m_i!}\frac{1}{r_1^1\cdots r_{m_1}^1\cdots r_1^n\cdots r_{m_n}^n s_1\cdots s_l}$$
$$\times\exp\left[K_1\left\{\sum_{i,j}r_i^j\|\xi_i^j\|_{k_1}\right\}^2\right]\exp\left[K\left\{\sum_j s_j\|\eta_j\|_{-k}\right\}^2\right]$$

for some $k_1>k$ and K, $k>0$ and some $K_1>0$.

According to [7, (D.6)], we choose

$$r_i^j=\frac{R}{Cm_j\|\xi_i^j\|_{k_1}},\qquad s_j=\frac{S}{Cl\|\eta_j\|_{-k}}$$

and we deduce a bound of $K_{l,M}$ in

$$\frac{C}{l!\prod m_i!}\prod\left(\frac{Cm_i}{R}\right)^{m_i}\left(\frac{Cl}{S}\right)^{l}\prod\|\xi_i^j\|_{k_1}\prod\|\eta_i\|_{-k}\exp\left[KR^2\right]\exp\left[KS^2\right].\tag{2}$$

Clearly, $K_{l,M}$ is a multilinear application in ξ_i^j, w_k. By (2), $K_{l,M}$ is continuous. Therefore $K_{l,M}$ can be identified with an element of $H_{\infty-}^{\otimes l}\otimes H_{-\infty}^{\otimes\sum m_i}$. We consider

$$\hat{\Xi}_{l,M}(\xi^1,\dots,\xi^n,\eta)=K_{l,M}(\xi^1,\dots,\xi^1,\xi^2,\dots,\xi^2,\dots,\xi^n,\dots,\xi^n,\eta,\dots,\eta),$$

where ξ_i is taken m_i times and η l times.

By holomorphy,

$$\hat{\Xi}=\sum\hat{\Xi}_{l,M}\exp\left[\sum_{i=1}^{n}\langle\xi^i,\eta\rangle_0\right].$$

and the series converges in the sense of (O_1) and (O_2). Only the second statement presents some difficulties. We remark for that by [7, page 557] ($\alpha(n) = 1$, $G_\alpha(s) = \exp[s]$)

$$\inf_{s>0} \exp[s]s^{-n} \leq Cn!n^{-2n}.$$

We deduce a bound analog to the bound (D.7) in [7]

$$|\hat{\Xi}_{l,M}|^2 \leq \frac{1}{(l^l \prod m_i^{m_i})} C^l \prod C^{m_i} \exp\left[D \sum \|\xi^i\|^2_{k_1} + D_1 \|\eta\|^2_{-k}\right].$$

$x^n \exp[-D_1 x^2]$ has a bound in $\exp[-C_1 n] C^n n^{n/2}$. If D_1 is large, C can be chosen very small and C_1 very large. We deduce the following bound

$$|\hat{\Xi}_{l,M}|^2 \leq C^l C^{\sum m_i} \exp\left[-C_1 l - C_1 \sum m_i\right] \exp\left[D_2 \sum \|\xi^i\|^2_{k_1} + D_2 \|\eta\|^2_{-k}\right].$$

We remark if D_2 is very large that C can be chosen very small and C_1 can be chosen very large. We remark if C_1 is large

$$\sum_{l,M} \exp\left[-C_1 l - C_1 \sum m_i\right] < \infty$$

in order to see that the series $\Xi_{l,M}$ converges in $L(W^n_{\infty-}, W_{\infty-})$.

Definition 3. The series $\sum_{l,M} \Xi_{l,M} = \Xi$ is called the Fock expansion of the element Ξ belonging to $L(W^n_{\infty-}, W_{\infty-})$.

4 Isomorphism of Hochschild cohomology theories

In this part, we prove the main theorem of this work.

Lemma 1. *If ξ belongs to $H_{\infty-}$,*

$$\xi^{\otimes n} =: \xi :^n .$$

Proof. We put $\xi = \sum \lambda_i e_i$ such that

$$\xi^{\otimes n} = \sum_{i_1,\ldots,i_n} \lambda_{i_1} \cdots \lambda_{i_n} e_{i_1} \otimes_0 \cdots \otimes_0 e_{i_n},$$

where $\otimes_0$ denotes the traditional tensor product. By regrouping various element, we deduce that

$$\begin{aligned} \xi^{\otimes n} &= \sum_{\substack{i<1<\cdots<i_r;\ n_1\neq 0,\ldots,n_r\neq 0;\\ n_1+\cdots+n_r=n}} \lambda_{i_1}^{n_1} \cdots \lambda_{i_r}^{n_r} \frac{n!}{n_1! \cdots n_r!} e_{i_1}^{\otimes n_1} \hat{\otimes} \cdots \hat{\otimes} e_{i_r}^{n_r} \\ &= \sum_{\substack{i<1<\cdots<i_r;\ n_1\neq 0,\ldots,n_r\neq 0;\\ n_1+\cdots+n_r=n}} \lambda_{i_1}^{n_1} \cdots \lambda_{i_r}^{n_r} \frac{n!}{n_1! \cdots n_r!} : e_{i_1} :^{\otimes n_1} \cdots : e_{i_r} :^{n_r} =: \xi :^n . \end{aligned}$$

This shows the result. ■

Corollary 1. *If $\xi_1 \in H_{\infty-}$ and if $\xi_2 \in H_{\infty-}$*

$$\phi_{\xi_1+\xi_2} =: \phi_{\xi_1} \phi_{\xi_2} : .$$

Definition 4. Let Ξ belong to $L(W^r_{\infty-}, W_{\infty-})$. Its Hochschild coboundary δ^r is defined as follows:

$$\delta^r\Xi(\phi^1, \dots, \phi^{r+1}) =: \phi^1\Xi(\phi^2, \dots, \phi^r) + \sum_{i=1}^{r}(-1)^i\Xi(\phi^1, \dots, :\phi^i\phi^{i+1}:, \dots, \phi^{r+1}) + (-1)^{r+1} :\Xi(\phi^1, \dots, \phi^r)\phi^{r+1}: .$$

Classically $\delta^{r+1}\delta^r = 0$.

Definition 5. We say that an element Ξ of $L(W^r_{\infty-}, W_{\infty-})$ is a homogeneous polydifferential operator of order (l, m) if its symbol $\hat{\Xi}(\xi^1, \dots, \xi^r, \eta)$ is equal to

$$\Psi(\xi^1, \dots, \xi^r, \eta) \exp\left[\sum \langle \xi^i, \eta\rangle\right], \tag{3}$$

where Ψ is a homogeneous polynomial in the ξ^i of degree m and in η of degree l.

Proposition 1. *If Ξ is an r-polydifferential operator of degree (l, m), $\delta^r\Xi$ is an $(r+1)$-polydifferential operator of degree (l, m).*

Proof. Since $:\phi_{\xi^1}\phi_{\xi^2} := \phi_{\xi^1+\xi^2}$, the only problem is to show that

$$(\phi^1, \dots, \phi^{r+1}) \to: \phi^1\Xi(\phi^2, \dots, \phi^{r+1}):$$

is still a polydifferential operator of degree (l, m).

Let $\eta \in H_{\infty-}$ be such that $\|\eta\|_0 = 1$. Let us compute

$$\langle \Xi(\phi_{\xi^2}, \dots, \phi_{\xi^{r+1}}), \phi_{\lambda\eta})\rangle_0 = \Psi(\xi^2, \dots, \xi^{r+1}, \lambda\eta) \exp\left[\lambda \sum_{i=2}^{r+1}\langle \xi^i, \eta\rangle\right].$$

If we compute the component of $\Xi(\phi_{\xi^2}, \dots, \phi_{\xi^{r+1}})$ along $\frac{\eta^{\otimes n}}{n!}$, it is the element of degree n in the expansion in λ of the (3). Since Ψ is homogeneous of degree l in η, the term of degree n in the expansion in λ of (3) is

$$\frac{\sum_{i=2}^{r+1}\langle \xi_i, \eta\rangle^{n-l}}{n-l!} C_l(\xi^2, \dots, \xi^{r+1}, \eta).$$

Because the component of ϕ_{ξ^1} along $\frac{\eta^{\otimes n}}{n!}$ is $\frac{\langle \xi^1, \eta\rangle^n}{n!}$, this shows that the component of

$$:\phi_{\xi^1}\Xi(\phi_{\xi^2}, \dots, \phi_{\xi^{r+1}}, \lambda\eta):$$

along $\frac{\eta^{\otimes n}}{n!}$ is

$$C_l(\xi^2, \dots, \xi^{r+1}, \eta) \sum_{n_1+n_2=n} \frac{\langle \sum_{i=2}^{r+1} \xi^i, \eta\rangle^{n_1-l}}{(n_1-l)!} \frac{\langle \xi^1, \eta\rangle^{n_2}}{n_2!} = C_l(\xi^2, \dots, \xi^{r+1}, \eta) \frac{\langle \sum_{i=1}^{r+1} \xi_i, \eta\rangle^{n-l}}{(n-l)!}.$$

The result follows directly. ∎

Definition 6. The continuous Hochschild cohomology $H^r_{\text{cont}}(W_{\infty-}, W_{\infty-})$ of the Hida test algebra is the space $\operatorname{Ker}\delta^r/\operatorname{Im}\delta^{r-1}$, where the Hochschild coboundary acts on $L(W^r_{\infty-}, W_{\infty-})$.

We consider cochains which are *finite* sums of polydifferential operators of degree $(l, m) \in (\mathbb{N} \times \mathbb{N})$. We call the space of polydifferential operators $L_{\mathrm{dif}}(W^r_{\infty-}, W_{\infty-})$. By the previous proposition, δ^r applies $L_{\mathrm{dif}}(W^r_{\infty-}, W_{\infty-})$ into $L_{\mathrm{dif}}(W^{r+1}_{\infty-}, W_{\infty-})$.

Definition 7. The differential Hochschild cohomology $H^r_{\mathrm{dif}}(W_{\infty-}, W_{\infty-})$ of the Hida test algebra is the space $\operatorname{Ker}\delta^r/\operatorname{Im}\delta^{r-1}$ where δ^r acts on $L_{\mathrm{dif}}(W^r_{\infty-}, W_{\infty-})$.

We get the main theorem of this work:

Theorem 3. *The differential Hochschild cohomology groups of the Hida test algebra are equal to the continuous Hochschild cohomology groups of the Hida test algebra.*

Proof. This comes from the Fock expansion of the previous part and from the following fact: if $\delta\Xi$ is a polydifferential operator for a continuous cochain Ξ, there exists a polydifferential operator Ξ_1 such that $\delta\Xi = \delta\Xi_1$ by Proposition 1. ■

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