Symmetry, Integrability and Geometry: Methods and Applications SIGMA 4 (2008), 065, 19 pages

$\mathfrak{sl}(2)$-Trivial Deformations of $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$-Modules of Symbols*

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Received January 14, 2008, in final form September 05, 2008; Published online September 18, 2008
Original article is available at `http://www.emis.de/journals/SIGMA/2008/065/`

Abstract. We consider the action of $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$ by Lie derivative on the spaces of symbols of differential operators. We study the deformations of this action that become trivial once restricted to $\mathfrak{sl}(2)$. Necessary and sufficient conditions for integrability of infinitesimal deformations are given.

Key words: tensor densities, cohomology, deformations

2000 Mathematics Subject Classification: 17B56; 17B66; 53D55

1 Introduction

Let $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$ be the Lie algebra of polynomial vector fields on $\mathbb{R}$. Consider the 1-parameter action of $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$ on the space $\mathbb{R}[x]$ of polynomial functions on $\mathbb{R}$ defined by

$$L^{\lambda}_{X\frac{d}{dx}}(f) = Xf' + \lambda X'f,$$

where $X, f \in \mathbb{R}[x]$ and $X' := \frac{dX}{dx}$. Denote by $\mathcal{F}_\lambda$ the $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$-module structure on $\mathbb{R}[x]$ defined by this action for a fixed λ. Geometrically, $\mathcal{F}_\lambda$ is the space of polynomial weighted densities of weight λ on $\mathbb{R}$

$$\mathcal{F}_\lambda = \left\{ f dx^\lambda \mid f \in \mathbb{R}[x] \right\}.$$

The space $\mathcal{F}_\lambda$ coincides with the space of vector fields, functions and differential 1-forms for $\lambda = -1, 0$ and 1, respectively.

Denote by $\mathcal{D}_{\nu,\mu} := \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_\nu, \mathcal{F}_\mu)$ the $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$-module of linear differential operators with the $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$-action given by the formula

$$L^{\nu,\mu}_X(A) = L^\mu_X \circ A - A \circ L^\nu_X. \tag{1}$$

Each module $\mathcal{D}_{\nu,\mu}$ has a natural filtration by the order of differential operators; the graded module $\mathcal{S}_{\nu,\mu} := \mathrm{gr}\mathcal{D}_{\nu,\mu}$ is called the *space of symbols*. The quotient-module $\mathcal{D}^k_{\nu,\mu}/\mathcal{D}^{k-1}_{\nu,\mu}$ is isomorphic to the module of tensor densities $\mathcal{F}_{\mu-\nu-k}$, the isomorphism is provided by the principal symbol σ defined by

$$A = \sum_{i=0}^{k} a_i(x)\partial_x^i \mapsto \sigma(A) = a_k(x)(dx)^{\mu-\nu-k}$$

*This paper is a contribution to the Special Issue on Deformation Quantization. The full collection is available at http://www.emis.de/journals/SIGMA/Deformation_Quantization.html (see, e.g., [10]). As $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$-module, the space $\mathcal{S}_{\nu,\mu}$ depends only on the difference $\delta = \mu - \nu$, so that $\mathcal{S}_{\nu,\mu}$ can be written as $\mathcal{S}_\delta$, and we have

$$\mathcal{S}_\delta = \bigoplus_{k=0}^{\infty} \mathcal{F}_{\delta-k}$$

as $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$-modules. The space of symbols of order $\leq n$ is

$$\mathcal{S}_\delta^n := \bigoplus_{j=0}^{n} \mathcal{F}_{\delta-j}.$$

The space $\mathcal{D}_{\nu,\mu}$ cannot be isomorphic as a $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$-module to the corresponding space of symbols, but is its deformation in the sense of Richardson–Neijenhuis [12]; however, they are isomorphic as $\mathfrak{sl}(2)$-modules (see [9]). In the last two decades, deformations of various types of structures have assumed an ever increasing role in mathematics and physics. For each such deformation problem a goal is to determine if all related deformation obstructions vanish and many beautiful techniques had been developed to determine when this is so. Deformations of Lie algebras with base and versal deformations were already considered by Fialowski [5]. It was further developed, with introduction of a complete local algebra base (local means a commutative algebra which has a unique maximal ideal) by Fialowski [6]. Also, in [6], the notion of miniversal (or formal versal) deformation was introduced in general, and it was proved that under some cohomology restrictions, a versal deformation exists. Later Fialowski and Fuchs, using this framework, gave a construction for the versal deformation [7].

We use the framework of Fialowski [6] (see also [1] and [2]) and consider (multi-parameter) deformations over complete local algebras. We construct the miniversal deformation of this action and define the complete local algebra related to this deformation.

According to Nijenhuis–Richardson [12], deformation theory of modules is closely related to the computation of cohomology. More precisely, given a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module V, the infinitesimal deformations of the $\mathfrak{g}$-module structure on V, i.e., deformations that are linear in the parameter of deformation, are related to $\mathrm{H}^1(\mathfrak{g}; \mathrm{End}(V))$. The obstructions to extension of any infinitesimal deformation to a formal one are related to $\mathrm{H}^2(\mathfrak{g}; \mathrm{End}(V))$. More generally, if $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, then the $\mathfrak{h}$-relative cohomology space $\mathrm{H}^1(\mathfrak{g}, \mathfrak{h}; \mathrm{End}(V))$ measures the infinitesimal deformations that become trivial once the action is restricted to $\mathfrak{h}$ ($\mathfrak{h}$-*trivial deformations*), while the obstructions to extension of any $\mathfrak{h}$-trivial infinitesimal deformation to a formal one are related to $\mathrm{H}^2(\mathfrak{g}, \mathfrak{h}; \mathrm{End}(V))$ (see, e.g., [3]).

Denote $\mathcal{D} := \mathcal{D}(n, \delta)$ the $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$-module of differential operators on $\mathcal{S}_\delta^n$. The infinitesimal deformations of the $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$-module $\mathcal{S}_\delta^n$ are classified by the first differential cohomology space,

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}); \mathcal{D}) = \oplus_{\lambda,k} \mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}); \mathcal{D}_{\lambda,\lambda+k}),$$

while the obstructions for integrability of infinitesimal deformations belong to the second differential cohomology space,

$$\mathrm{H}^2_{\mathrm{diff}}(\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}); \mathcal{D}) = \oplus_{\lambda,k} \mathrm{H}^2_{\mathrm{diff}}(\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}); \mathcal{D}_{\lambda,\lambda+k}),$$

where, hereafter, $\delta - \lambda$ and k are integers satisfying $\delta - n \leq \lambda, \lambda + k \leq \delta$.

Here we study the $\mathfrak{sl}(2)$-trivial deformations, thus we consider the differential $\mathfrak{sl}(2)$-relative cohomology spaces. The first space

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}), \mathfrak{sl}(2); \mathcal{D}) = \oplus_{\lambda,k} \mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}), \mathfrak{sl}(2); \mathcal{D}_{\lambda,\lambda+k})$$

was calculated by Bouarroudj and Ovsienko [4]. For the second space

$$\mathrm{H}^2_{\mathrm{diff}}(\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}), \mathfrak{sl}(2); \mathcal{D}) = \oplus_{\lambda,k}\mathrm{H}^2_{\mathrm{diff}}(\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}), \mathfrak{sl}(2); \mathcal{D}_{\lambda,\lambda+k})$$

we give, in this paper, explicit expressions of some $\mathfrak{sl}(2)$-relative nontrivial 2-cocycles.

This paper is organized as follows. In Section 2 we study some properties of the $\mathfrak{sl}(2)$-invariant differential operators. These properties are related to the $\mathfrak{sl}(2)$-relative cohomology. In Section 3 we study the first and the second $\mathfrak{sl}(2)$-relative cohomology spaces which are closely related to the deformation theory. Especially we explain some $\mathfrak{sl}(2)$-relative 2-cocycles which naturally appear as obstructions to integrate any $\mathfrak{sl}(2)$-trivial infinitesimal deformation to a formal one. In Section 4 we give an outline of the general deformation theory: definitions, equivalence, integrability conditions and miniversal deformations. In Section 5 we give the first main result of this paper: Theorem 2. That is, we explain all second-order integrability conditions for any infinitesimal $\mathfrak{sl}(2)$-trivial deformation of the $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$-module $\mathcal{S}^n_\delta$. In Section 6 we complete the list of integrability conditions by computing those of third-order. We prove that these conditions are necessary and sufficient to integrate any infinitesimal $\mathfrak{sl}(2)$-trivial deformation to a formal one. Moreover, we prove that any $\mathfrak{sl}(2)$-trivial deformation is, in fact, equivalent to a polynomial one of degree ≤ 2: Theorem 3. Finally, in Section 7, we complete our study by given a few examples of deformations.

2 Invariant differential operators

In this paper we study the $\mathfrak{sl}(2)$-trivial deformations of the space of symbols of differential operators which is a $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$-module, so we begin by studying some properties of $\mathfrak{sl}(2)$-invariant bilinear differential operators.

Let us consider the space of bilinear differential operators $c : \mathcal{F}_\lambda \times \mathcal{F}_\mu \to \mathcal{F}_\tau$. The Lie algebra, $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$, acts on this space by the Lie derivative:

$$L_X(c)(f dx^\lambda, g dx^\mu) = L^\tau_X\big(c(f dx^\lambda, g dx^\mu)\big) - c\big(L^\lambda_X(f dx^\lambda), g dx^\mu\big) - c\big(f dx^\lambda, L^\mu_X(g dx^\mu)\big).$$

A bilinear differential operator $c : \mathcal{F}_\tau \times \mathcal{F}_\lambda \to \mathcal{F}_\mu$ is called $\mathfrak{sl}(2)$-invariant if, for all $X \in \mathfrak{sl}(2)$, we have $L_X(c) = 0$, or equivalently

$$L^\mu_X\big(c(f dx^\tau, g dx^\lambda)\big) = c\big(L^\tau_X(f dx^\tau), g dx^\lambda\big) + c\big(f dx^\tau, L^\lambda_X(g dx^\lambda)\big). \tag{2}$$

That is, the set of such $\mathfrak{sl}(2)$-invariant bilinear differential operators is the subspace on which the subalgebra $\mathfrak{sl}(2)$ acts trivially.

Now, let us consider a linear map $c : \mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}) \to \mathcal{D}_{\lambda,\mu}$, then we can see c as a bilinear differential operator $c : \mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}) \times \mathcal{F}_\lambda \to \mathcal{F}_\mu$ (or $c : \mathcal{F}_{-1} \times \mathcal{F}_\lambda \to \mathcal{F}_\mu$ since $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$ is isomorphic to $\mathcal{F}_{-1}$) defined by $c(X, f dx^\lambda) = c(X)(f dx^\lambda)$. So, the $\mathfrak{sl}(2)$-invariance property (2) of c reads, for all $X \in \mathfrak{sl}(2)$ and $Y \in \mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$,

$$L^\mu_X \circ c(Y) = c([X, Y]) + c(Y) \circ L^\lambda_X,$$

or equivalently

$$L^{\lambda,\mu}_X(c(Y)) = c([X, Y]).$$

The $\mathfrak{sl}(2)$-invariant bilinear differential operators were calculated by Gordan. We recall here the results and we need to add some precision concerning the space of the $\mathfrak{sl}(2)$-invariant differential operators from $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$ to $\mathcal{D}_{\lambda,\mu}$ vanishing on $\mathfrak{sl}(2)$.

Proposition 1 ([11]). *There exist $\mathfrak{sl}(2)$-invariant bilinear differential operators, called transvectants,*

$$J_k^{\tau,\lambda}: \quad \mathcal{F}_\tau \times \mathcal{F}_\lambda \to \mathcal{F}_{\tau+\lambda+k}, \qquad (\varphi dx^\tau, \phi dx^\lambda) \mapsto J_k^{\tau,\lambda}(\varphi,\phi) dx^{\tau+\lambda+k}$$

given by

$$J_k^{\tau,\lambda}(\varphi,\phi) = \sum_{i+j=k} c_{i,j} \varphi^{(i)} \phi^{(j)},$$

where $k \in \mathbb{N}$ and the coefficients $c_{i,j}$ are characterized as follows:

i) *If neither τ nor λ belong to the set $\{0, -\frac{1}{2}, -1, \dots, -\frac{k-1}{2}\}$ then*

$$c_{i,j} = (-1)^j \binom{2\tau+k}{j} \binom{2\lambda+k}{i},$$

where $\binom{x}{i}$ is the standard binomial coefficient $\binom{x}{i} = \frac{x(x-1)\cdots(x-i+1)}{i!}$.

ii) *If τ or $\lambda \in \{0, -\frac{1}{2}, -1, \dots, -\frac{k-1}{2}\}$, the coefficients $c_{i,j}$ satisfy the recurrence relation*

$$(i+1)(i+2\tau)c_{i+1,j} + (j+1)(j+2\lambda)c_{i,j+1} = 0. \tag{3}$$

Moreover, the space of solutions of the system (3) is two-dimensional if $2\lambda = -s$ and $2\tau = -t$ with $t > k-s-2$, and one-dimensional otherwise.

iii) *For $k \geq 3$, the space of $\mathfrak{sl}(2)$-invariant linear differential operator from $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$ to $\mathcal{D}_{\lambda,\lambda+k-1}$ vanishing on $\mathfrak{sl}(2)$ is one-dimensional.*

Proof. We need to prove only part iii), for the other statements see, for instance, [11] or [3]. First, we recall that $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$ is isomorphic to $\mathcal{F}_{-1}$ as $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$-module. So, according to the formulae (3), if $k \geq 3$, the space of $\mathfrak{sl}(2)$-invariant bilinear differential operator from $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}) \times \mathcal{F}_\lambda$ to $\mathcal{F}_{\lambda-1+k}$ is 2-dimensional if and only if $2\lambda \in \{1-k, 2-k, 3-k\}$. Let us consider the transvectant $J_k^{-1,\lambda}$ defined by, for $\varphi\frac{d}{dx} \in \mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$ and $\phi dx^\lambda \in \mathcal{F}_\lambda$,

$$J_k^{-1,\lambda}(\varphi,\phi) = \sum_{i+j=k, i\geq 3} c_{i,j} \varphi^{(i)} \phi^{(j)}, \tag{4}$$

where the coefficients $c_{i,j}$ satisfy (3). If $2\lambda \in \{1-k, 2-k, 3-k\}$ the space of $\mathfrak{sl}(2)$-invariant linear differential operator from $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$ to $\mathcal{D}_{\lambda,\lambda+k-1}$ is spanned by $J_k^{-1,\lambda}$ and $I_k^{-1,\lambda}$ where

$$I_k^{-1,\lambda}(\varphi,\phi) = \begin{cases} \varphi\phi^{(k)} & \text{if} \quad 2\lambda = 1-k, \\ \varphi\phi^{(k)} + \frac{k}{2}\varphi'\phi^{(k-1)} & \text{if} \quad 2\lambda = 2-k, \\ \varphi\phi^{(k)} + k\varphi'\phi^{(k-1)} + \frac{k(k-1)}{2}\varphi''\phi^{(k-2)} & \text{if} \quad 2\lambda = 3-k. \end{cases}$$

If $2\lambda \notin \{1-k, 2-k, 3-k\}$ the corresponding space is one-dimensional and it is spanned by $J_k^{-1,\lambda}$. We see obviously that only the operators $J_k^{-1,\lambda}$ vanish on $\mathfrak{sl}(2)$. Part iii) of Proposition 1 is proved. ■

3 Cohomology spaces

Let $\mathfrak{g}$ be a Lie algebra acting on a space V and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. The space of $\mathfrak{h}$-relative n-cochains of $\mathfrak{g}$ with values in V is the $\mathfrak{g}$-module

$$C^n(\mathfrak{g}, \mathfrak{h}; V) := \mathrm{Hom}_{\mathfrak{h}}(\wedge^n(\mathfrak{g}/\mathfrak{h}); V).$$

The *coboundary operator* $\partial^n : C^n(\mathfrak{g},\mathfrak{h};V) \to C^{n+1}(\mathfrak{g},\mathfrak{h};V)$ is a $\mathfrak{g}$-map satisfying $\partial^n \circ \partial^{n-1} = 0$. The kernel of ∂^n, denoted $Z^n(\mathfrak{g},\mathfrak{h};V)$, is the space of $\mathfrak{h}$-relative *n-cocycles*, among them, the elements in the range of ∂^{n-1} are called $\mathfrak{h}$-relative *n-coboundaries.* We denote $B^n(\mathfrak{g},\mathfrak{h};V)$ the space of n-coboundaries.

By definition, the n^{th} $\mathfrak{h}$-relative cohomolgy space is the quotient space

$$H^n(\mathfrak{g},\mathfrak{h};V) = Z^n(\mathfrak{g},\mathfrak{h};V)/B^n(\mathfrak{g},\mathfrak{h};V).$$

We will only need the formula of ∂^n (which will be simply denoted ∂) in degrees 0, 1 and 2: for $v \in C^0(\mathfrak{g},\mathfrak{h};V) = V^{\mathfrak{h}}$, $\partial v(X) := Xv$, for $b \in C^1(\mathfrak{g},\mathfrak{h};V)$,

$$\partial b(X,Y) := Xb(Y) - Yb(X) - b([X,Y])$$

and for $\Omega \in C^2(\mathfrak{g};\mathfrak{h},V)$,

$$\partial\Omega(X,Y,Z) := X\Omega(Y,Z) - \Omega([X,Y],Z) + \circlearrowleft (X,Y,Z), \tag{5}$$

where $\circlearrowleft (X,Y,Z)$ denotes the summands obtained from the two written ones by the cyclic permutation of the symbols X, Y, Z.

In this paper, we are interested to the differential $\mathfrak{sl}(2)$-relative cohomology spaces

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}),\mathfrak{sl}(2);\mathcal{D}_{\lambda,\lambda+k}) \qquad \text{and} \qquad \mathrm{H}^2_{\mathrm{diff}}(\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}),\mathfrak{sl}(2);\mathcal{D}_{\lambda,\lambda+k}).$$

Proposition 2.

i) *Any* 1*-cocycle* $c : \mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}) \to \mathcal{D}_{\lambda,\lambda+k}$ *vanishing on* $\mathfrak{sl}(2)$ *coincides (up to a scalar factor) with the transvectant* $J^{-1,\lambda}_{k+1}$ *defined here and below by the formulae* (4)*.*

ii) *Any* 2*-cocycle vanishing on* $\mathfrak{sl}(2)$ *is* $\mathfrak{sl}(2)$*-invariant.*

iii) *Let* $\Omega \in Z^2(\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}),\mathfrak{sl}(2);\mathcal{D}_{\lambda,\lambda+k})$*. If* Ω *is a* $\mathfrak{sl}(2)$*-relative* 2*-coboundary then (up to a scalar factor) we have* $\Omega = \partial J^{-1,\lambda}_{k+1}$*.*

Proof. i) The 1-cocycle relation reads:

$$c([X,Y]) = L^{\mu}_X \circ c(Y) - c(Y) \circ L^{\lambda}_X - L^{\mu}_Y \circ c(X) + c(X) \circ L^{\lambda}_X.$$

Consider $X \in \mathfrak{sl}(2)$. Since $c(X) = 0$, one easily sees that

$$L^{\mu}_X c(Y) = c([X,Y]) + c(Y) \circ L^{\lambda}_X. \tag{6}$$

The equation (6) expresses the $\mathfrak{sl}(2)$-invariance property of the bilinear map c. Thus, according to Proposition 1, the map c coincides with the transvectant $J^{-1,\lambda}_{k+1}$.

ii) Let $\Omega \in Z^2(\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R});\mathcal{D}_{\lambda,\lambda+k})$. Since $\Omega(X,Y) = 0$ for all $X \in \mathfrak{sl}(2)$, we deduce from the 2-cocycle relation (5) that, for all $X \in \mathfrak{sl}(2)$ and Y, $Z \in \mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$, we have

$$X\Omega(Y,Z) - \Omega([X,Y],Z) - \Omega(Y,[X,Z]) = 0.$$

This last relation is nothing but the $\mathfrak{sl}(2)$-invariance property of the bilinear map Ω.

iii) Let $\Omega = \partial b$. For all X, $Y \in \mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$ we have

$$\partial b(X,Y) := L^{\lambda,\lambda+k}_X b(Y) - L^{\lambda,\lambda+k}_Y b(X) - b([X,Y]).$$

Since $\partial b(X,Y) = b(X) = 0$ for all $X \in \mathfrak{sl}(2)$ we deduce that b is $\mathfrak{sl}(2)$-invariant:

$$L^{\lambda,\lambda+k}_X b(Y) = b([X,Y]).$$

According to Proposition 1, the space of $\mathfrak{sl}(2)$-invariant linear differential operator from $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$ to $\mathcal{D}_{\lambda,\lambda+k}$ vanishing on $\mathfrak{sl}(2)$ is one dimensional and it is spanned by $J^{-1,\lambda}_{k+1}$. Thus, up to a scalar factor, $b = J^{-1,\lambda}_{k+1}$. Proposition 2 is proved. ■

The $\mathfrak{sl}(2)$-trivial deformations are closely related to the $\mathfrak{sl}(2)$-relative cohomology spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}), \mathfrak{sl}(2); \mathcal{D}_{\lambda,\lambda+k})$ and $\mathrm{H}^2_{\mathrm{diff}}(\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}), \mathfrak{sl}(2); \mathcal{D}_{\lambda,\lambda+k})$. Therefore, we will describe briefly these two spaces.

3.1 The first cohomology space

Note that, by Proposition 2, we can describe the space $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}), \mathfrak{sl}(2); \mathcal{D}_{\lambda,\lambda+k})$. This space is, in fact, one-dimensional if and only if the corresponding transvectant $J^{-1,\lambda}_{k+1}$ is a non-trivial $\mathfrak{sl}(2)$-relative 1-cocycle, otherwise it is trivial. However, this space was computed by Bouarroudj and Ovsienko, the result is as follows:

Theorem 1 ([4]). $\dim \mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}), \mathfrak{sl}(2); \mathcal{D}_{\lambda,\mu}) = 1$ *if*

$$\begin{array}{lll} \mu-\lambda=2 & \textit{and} & \lambda\neq-\frac{1}{2},\\ \mu-\lambda=3 & \textit{and} & \lambda\neq-1,\\ \mu-\lambda=4 & \textit{and} & \lambda\neq-\frac{3}{2},\\ \mu-\lambda=5 & \textit{and} & \lambda=0,-4,\\ \mu-\lambda=6 & \textit{and} & \lambda=-\frac{5\pm\sqrt{19}}{2}. \end{array}$$

Otherwise, $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}), \mathfrak{sl}(2); \mathcal{D}_{\lambda,\mu}) = 0$.

These spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}), \mathfrak{sl}(2); \mathcal{D}_{\lambda,\lambda+k})$ are generated by the cohomology classes of the $\mathfrak{sl}(2)$-relative 1-cocycles, $C_{\lambda,\lambda+k} : \mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}) \to \mathcal{D}_{\lambda,\lambda+k}$ that are collected in the following table.

Table 1.

$C_{\lambda,\lambda+2}(X,f) = X^{(3)}f, \qquad \lambda\neq-\frac{1}{2}$
$C_{\lambda,\lambda+3}(X,f) = X^{(3)}f' - \frac{\lambda}{2}X^{(4)}f, \qquad \lambda\neq-1$
$C_{\lambda,\lambda+4}(X,f) = X^{(3)}f'' - \frac{2\lambda+1}{2}X^{(4)}f' + \frac{\lambda(2\lambda+1)}{10}X^{(5)}f, \qquad \lambda\neq-\frac{3}{2}$
$C_{0,5}(X,f) = -3X^{(5)}f' + 15X^{(4)}f'' - 10X^{(3)}f^{(3)}$
$C_{-4,1}(X,f) = 28X^{(6)}f + 63X^{(5)}f' + 45X^{(4)}f'' + 10X^{(3)}f^{(3)}$
$C_{a_i,a_i+6}(X,f) = \alpha_i X^{(7)}f - 14\beta_i X^{(6)}f' - 126\gamma_i X^{(5)}f'' - 210\tau_i X^{(4)}f^{(3)} + 210X^{(3)}f^4$

where

$$\begin{array}{lllll} a_1=-\frac{5+\sqrt{19}}{2}, & \alpha_1=-\frac{22+5\sqrt{19}}{4}, & \beta_1=\frac{31+7\sqrt{19}}{2}, & \gamma_1=\frac{25+7\sqrt{19}}{2}, & \tau_1=-2+\sqrt{19},\\ a_2=-\frac{5-\sqrt{19}}{2}, & \alpha_2=-\frac{22-5\sqrt{19}}{4}, & \beta_2=\frac{31-7\sqrt{19}}{2}, & \gamma_2=\frac{25-7\sqrt{19}}{2}, & \tau_2=-2-\sqrt{19}. \end{array}$$

For $X\frac{d}{dx} \in \mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$ and $f dx^\lambda \in \mathcal{F}_\lambda$, we write

$$C_{\lambda,\lambda+k}(X\frac{d}{dx})(f dx^\lambda) = C_{\lambda,\lambda+k}(X,f)dx^{\lambda+k}.$$

The maps $C_{\lambda,\lambda+j}(X)$ are naturally extended to $\mathcal{S}^n_\delta = \bigoplus_{j=0}^n \mathcal{F}_{\delta-j}$.

3.2 The second cohomology space

Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$ and V a $\mathfrak{g}$-module, the *cup-product* is defined, for arbitrary linear maps $c_1, c_2 : \mathfrak{g} \to \mathrm{End}(V)$, by

$$[\![c_1,c_2]\!] : \ \mathfrak{g}\otimes\mathfrak{g} \to \mathrm{End}(V), \qquad [\![c_1,c_2]\!](x,y) = [c_1(x),c_2(y)] + [c_2(x),c_1(y)]. \tag{7}$$

Therefore, it is easy to check that for any two $\mathfrak{h}$-relative 1-cocycles c_1 and $c_2 \in Z^1(\mathfrak{g}, \mathfrak{h}; \mathrm{End}(V))$, the bilinear map $[\![c_1, c_2]\!]$ is a $\mathfrak{h}$-relative 2-cocycle. Moreover, if one of the cocycles c_1 or c_2 is a $\mathfrak{h}$-relative 1-coboundary, then $[\![c_1, c_2]\!]$ is a $\mathfrak{h}$-relative 2-coboundary. Therefore, we naturally deduce that the operation (7) defines a bilinear map

$$\mathrm{H}^1(\mathfrak{g}, \mathfrak{h}; \mathrm{End}(V)) \otimes \mathrm{H}^1(\mathfrak{g}, \mathfrak{h}; \mathrm{End}(V)) \to \mathrm{H}^2(\mathfrak{g}, \mathfrak{h}; \mathrm{End}(V)).$$

Thus, by computing the cup-products of the 1-cocycles $C_{\lambda,\lambda+k}$ generating the spaces

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}), \mathfrak{sl}(2); \mathcal{D}_{\lambda,\lambda+k}),$$

we can exhibit explicit expressions of some $\mathfrak{sl}(2)$-relative 2-cocycles

$$\Omega_{\lambda,\lambda+k} : \mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}) \to \mathcal{D}_{\lambda,\lambda+k}.$$

For $X\frac{d}{dx}, Y\frac{d}{dx} \in \mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$ and $f dx^\lambda \in \mathcal{F}_\lambda$, we write

$$\Omega_{\lambda,\lambda+k}\left(X\frac{d}{dx}, Y\frac{d}{dx}\right)(f dx^\lambda) = \Omega_{\lambda,\lambda+k}(X, Y, f) dx^{\lambda+k}.$$

Proposition 3.

i) *The map $\Omega_{\lambda,\lambda+5}$ is defined by*

$$(\lambda+4)\Omega_{\lambda,\lambda+5} = 2[\![C_{\lambda+2,\lambda+5}, C_{\lambda,\lambda+2}]\!]$$

is a nontrivial $\mathfrak{sl}(2)$-relative 2-cocycles if and only if $\lambda \in \{0, -2, -4\}$. Moreover

$$-2[\![C_{\lambda+3,\lambda+5}, C_{\lambda,\lambda+3}]\!] = \lambda\Omega_{\lambda,\lambda+5}.$$

ii) *The map $\Omega_{\lambda,\lambda+6}$ is defined by*

$$(2\lambda+9)\Omega_{\lambda,\lambda+6} = -2[\![C_{\lambda+2,\lambda+6}, C_{\lambda,\lambda+2}]\!]$$

is a nontrivial $\mathfrak{sl}(2)$-relative 2-cocycles if and only if $\lambda \in \left\{-\frac{5}{2}, -\frac{22-5\sqrt{19}}{4}, -\frac{22+5\sqrt{19}}{4}\right\}$. Moreover,

$$5(2\lambda+1)\Omega_{\lambda,\lambda+6} = -2(2\lambda+1)[\![C_{\lambda+3,\lambda+6}, C_{\lambda,\lambda+3}]\!] = 10[\![C_{\lambda+4,\lambda+6}, C_{\lambda,\lambda+4}]\!].$$

Proof. By a straightforward computation we get

$$\Omega_{\lambda,\lambda+5}(X, Y, f) = \left(X^{(4)}Y^{(3)} - X^{(3)}Y^{(4)}\right)f,$$
$$\Omega_{\lambda,\lambda+6}(X, Y, f) = \left(X^{(3)}Y^{(4)} - X^{(4)}Y^{(3)}\right)f' - \tfrac{\lambda}{5}\left(X^{(3)}Y^{(5)} - X^{(5)}Y^{(3)}\right)f.$$

Moreover, we show also by a direct computation that

$$3\partial J_6^{-1,\lambda} = -\lambda(\lambda^2+6\lambda+8)\Omega_{\lambda,\lambda+5}, \qquad \text{and} \qquad 3\partial J_7^{-1,\lambda} = (4\lambda^3+30\lambda^2+56\lambda+15)\Omega_{\lambda,\lambda+6}$$

where

$$J_6^{-1,\lambda}(X)(f) = 3X^{(3)}f^{(3)} - \tfrac{9}{2}(\lambda+1)X^{(4)}f'' + \tfrac{9}{10}(\lambda+1)(2\lambda+1)X^{(5)}f' - \lambda\tfrac{2\lambda^2+3\lambda+1}{10}X^{(6)}f$$

and

$$J_7^{-1,\lambda}(X)(f) = X^{(3)}f^{(4)} - (2\lambda+3)X^{(4)}f^{(3)} + \tfrac{6\lambda^2+15\lambda+9}{5}X^{(5)}f''$$
$$- \tfrac{4\lambda^3+12\lambda^2+11\lambda+3}{15}X^{(6)}f' + \tfrac{\lambda(4\lambda^3+12\lambda^2+11\lambda+3)}{210}X^{(7)}f.$$

Thus, we conclude by using Proposition 2. ∎

In the following proposition we prove that the space $\mathrm{H}^2_{\mathrm{diff}}(\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}),\mathfrak{sl}(2);\mathcal{D}_{\lambda,\lambda+7})$ is, at least, 2-dimensional. Thus, Theorem 1 of [3] is not entirely correct.

Proposition 4. *For $\lambda\notin\{0,-6\}$, the cup products $[\![C_{\lambda+3,\lambda+7},C_{\lambda,\lambda+3}]\!]$ and $[\![C_{\lambda+4,\lambda+7},C_{\lambda,\lambda+4}]\!]$ are nontrivial $\mathfrak{sl}(2)$-relative 2-cocycles and they are non-cohomologous. Thus,*

$$\dim\mathrm{H}^2_{\mathrm{diff}}(\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}),\mathfrak{sl}(2),\mathcal{D}_{\lambda,\lambda+7})\geq 2.$$

Proof. The transvectant $J_8^{-1,\lambda}$ is given by, for $X\frac{d}{dx}\in\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$ and $f dx^\lambda\in\mathcal{F}_\lambda$,

$$\begin{aligned}J_8^{-1,\lambda}(X)(f)=&\,X^{(3)}f^{(5)}-\tfrac{5}{2}(\lambda+2)X^{(4)}f^{(4)}+(\lambda+2)(2\lambda+3)X^{(5)}f^{(3)}\\&-\tfrac{1}{3}(\lambda+1)(\lambda+2)(2\lambda+3)X^{(6)}f''+\tfrac{1}{42}(\lambda+1)(\lambda+2)(2\lambda+3)(2\lambda+1)X^{(7)}f'\\&-\tfrac{1}{840}\lambda(\lambda+1)(\lambda+2)(2\lambda+3)(2\lambda+1)X^{(8)}f.\end{aligned}$$

Therefore, by a direct computation, we show that

$$\begin{aligned}\partial J_8^{-1,\lambda}(X,Y)(f)=&\,\tfrac{1}{30}\lambda\Big((\lambda+1)(\lambda+2)(2\lambda+3)(2\lambda+11)+30\Big)X^{(3)}Y^{(6)}f\\&-\lambda(\lambda+2)\Big(-\tfrac{1}{60}(\lambda+1)(2\lambda+3)(2\lambda+1)+2\lambda+\tfrac{11}{2}\Big)X^{(4)}Y^{(5)}f\\&-\Big((\lambda+2)(2\lambda+3)\big(\tfrac{1}{3}(\lambda+1)(2\lambda+1)+3\lambda+1\big)-5\lambda-1\Big)X^{(3)}Y^{(5)}f'\\&+5\Big((\lambda+2)\big[\tfrac{1}{3}(\lambda+1)(2\lambda+3)+3\lambda+2\big]+2\lambda+1\Big)X^{(3)}Y^{(4)}f''-(X\leftrightarrow Y).\end{aligned}$$

Let us define $\Omega_{\lambda,\lambda+7}$ and $\widetilde{\Omega}_{\lambda,\lambda+7}$ by

$$\Omega_{\lambda,\lambda+7}=[\![C_{\lambda+3,\lambda+7},C_{\lambda,\lambda+3}]\!]\qquad\textit{and}\qquad\widetilde{\Omega}_{\lambda,\lambda+7}=[\![C_{\lambda+4,\lambda+7},C_{\lambda,\lambda+4}]\!].$$

Thus,

$$\begin{aligned}\Omega_{\lambda,\lambda+7}(X,Y,f)=&\Big(\tfrac{-\lambda(2\lambda+7)(\lambda+8)}{20}X^{(5)}Y^{(4)}-\tfrac{\lambda}{2}X^{(3)}Y^{(6)}\Big)f+\tfrac{2\lambda^2+23\lambda+11}{10}X^{(5)}Y^{(3)}f'\\&+\tfrac{\lambda+11}{2}X^{(3)}Y^{(4)}f''-(X\leftrightarrow Y),\\ \widetilde{\Omega}_{\lambda,\lambda+7}(X,Y,f)=&\Big(\tfrac{\lambda(2\lambda+1)}{10}X^{(3)}Y^{(6)}-\tfrac{\lambda(\lambda+4)(2\lambda+1)}{20}X^{(4)}Y^{(5)}\Big)f+\tfrac{(\lambda-5)(2\lambda+1)}{10}X^{(3)}Y^{(5)}f'\\&+\tfrac{5-\lambda}{2}X^{(3)}Y^{(4)}f''-(X\leftrightarrow Y).\end{aligned}$$

Now, it is easy to show that the three maps $\partial J_8^{-1,\lambda}$, $\Omega_{\lambda,\lambda+7}$ and $\widetilde{\Omega}_{\lambda,\lambda+7}$ are linearly independent if and only if $\lambda\notin\{-6,0\}$. So, according to Proposition 2, the maps $\Omega_{\lambda,\lambda+7}$ and $\widetilde{\Omega}_{\lambda,\lambda+7}$ are nontrivial $\mathfrak{sl}(2)$-relative 2-cocycles and they are non-cohomologous. ■

Proposition 5. *The cup product $[\![C_{\lambda,\lambda+4},C_{\lambda+4,\lambda+8}]\!]$ is a nontrivial $\mathfrak{sl}(2)$-relative 2-cocycle.*

Proof. Let $\Omega_{\lambda,\lambda+8}=[\![C_{\lambda,\lambda+4},C_{\lambda+4,\lambda+8}]\!]$. By a straightforward computation we show that

$$\begin{aligned}\Omega_{\lambda,\lambda+8}(X,Y,f)=&-\Big(\tfrac{\lambda(2\lambda+1)(2\lambda+9)}{20}X^{(4)}Y^{(6)}-\tfrac{\lambda(2\lambda+1)}{10}X^{(3)}Y^{(7)}\Big)f\\&-\Big(\tfrac{9(2\lambda+1)(2\lambda+9)}{20}X^{(5)}Y^{(4)}-\tfrac{(2\lambda+1)(2\lambda-5)}{10}X^{(3)}Y^{(6)}\Big)f'\\&+\tfrac{18(1+\lambda)}{5}X^{(5)}Y^{(3)}f''-6X^{(4)}Y^{(3)}f^{(3)}-(X\leftrightarrow Y).\end{aligned}$$

As before, we show that this 2-cocycle $\Omega_{\lambda,\lambda+8}$ is nontrivial: $\Omega_{\lambda,\lambda+8}\neq\partial J_9^{-1,\lambda}$. ■

Now, we collect in the following proposition some $\mathfrak{sl}(2)$-relative nontrivial 2-cocycles $\Omega_{\lambda,\lambda+k}$ for $k=9,10$ and for singular values of λ.

Proposition 6. *The following cup-products are* $\mathfrak{sl}(2)$*-relative nontrivial* 2*-cocycles*

$$\begin{array}{ll} \Omega_{0,9}=[\![C_{0,5},C_{5,9}]\!], & \Omega_{-4,5}=[\![C_{-4,0},C_{0,5}]\!], \\ \Omega_{-8,1}=[\![C_{-8,-4},C_{-4,1}]\!], & \Omega_{a_i,a_i+9}=[\![C_{a_i,a_i+6},C_{a_i+6,a_i+9}]\!], \\ \Omega_{a_i-3,a_i+6}=[\![C_{a_i-3,a_i},C_{a_i,a_i+6}]\!], & \Omega_{a_i,a_i+10}=[\![C_{a_i,a_i+6},C_{a_i+6,a_i+10}]\!], \\ \Omega_{a_i-4,a_i+6}=[\![C_{a_i-4,a_i},C_{a_i,a_i+6}]\!]. & \end{array}$$

4 The general framework

In this section we define deformations of Lie algebra homomorphisms and introduce the notion of miniversal deformations over complete local algebras. Deformation theory of Lie algebra homomorphisms was first considered with only one-parameter deformation [7, 12, 15]. Recently, deformations of Lie (super)algebras with multi-parameters were intensively studied (see, e.g., [1, 2, 13, 14]). Here we give an outline of this theory.

4.1 Infinitesimal deformations

Let $\rho_0:\mathfrak{g}\to\mathrm{End}(V)$ be an action of a Lie algebra $\mathfrak{g}$ on a vector space V and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. When studying $\mathfrak{h}$-trivial deformations of the $\mathfrak{g}$-action ρ_0, one usually starts with infinitesimal deformations

$$\rho=\rho_0+tC,$$

where $C:\mathfrak{g}\to\mathrm{End}(V)$ is a linear map vanishing on $\mathfrak{h}$ and t is a formal parameter. The homomorphism condition

$$[\rho(x),\rho(y)]=\rho([x,y]),$$

where $x,y\in\mathfrak{g}$, is satisfied in order 1 in t if and only if C is a $\mathfrak{h}$-relative 1-cocycle. That is, the map C satisfies

$$[\rho_0(x),C(y)]-[\rho_0(y),C(x)]-C([x,y])=0.$$

Moreover, two $\mathfrak{h}$-trivial infinitesimal deformations $\rho=\rho_0+t\,C_1$, and $\rho=\rho_0+t\,C_2$, are equivalents if and only if C_1-C_2 is $\mathfrak{h}$-relative coboundary:

$$(C_1-C_2)(x)=[\rho_0(x),A]:=\partial A(x),$$

where $A\in\mathrm{End}(V)^{\mathfrak{h}}$ and ∂ stands for differential of cochains on $\mathfrak{g}$ with values in $\mathrm{End}(V)$. So, the space $\mathrm{H}^1(\mathfrak{g},\mathfrak{h};\mathrm{End}(V))$ determines and classifies the $\mathfrak{h}$-trivial infinitesimal deformations up to equivalence. (see, e.g., [8, 12]). If $\mathrm{H}^1(\mathfrak{g},\mathfrak{h};\mathrm{End}(V))$ is multi-dimensional, it is natural to consider multi-parameter $\mathfrak{h}$-trivial deformations. More precisely, if $\dim\mathrm{H}^1(\mathfrak{g},\mathfrak{h};\mathrm{End}(V))=m$, then choose $\mathfrak{h}$-relative 1-cocycles $C_1,\dots,C_m$ representing a basis of $\mathrm{H}^1(\mathfrak{g},\mathfrak{h};\mathrm{End}(V))$ and consider the $\mathfrak{h}$-trivial infinitesimal deformation

$$\rho=\rho_0+\sum_{i=1}^m t_iC_i,$$

with independent parameters $t_1,\dots,t_m$.

In our study, we are interested in the infinitesimal $\mathfrak{sl}(2)$-trivial deformation of the $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$-action on $\mathcal{S}^n_\delta = \bigoplus_{j=0}^n \mathcal{F}_{\delta-j}$, the space of symbols of differential operators, where $n \in \mathbb{N}$ and $\delta \in \mathbb{R}$. Thus, we consider the $\mathfrak{sl}(2)$-relative cohomology space $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}), \mathfrak{sl}(2); \mathcal{D})$. Any infinitesimal $\mathfrak{sl}(2)$-trivial deformation is then of the form

$$\mathcal{L}_X = L_X + \mathcal{L}^{(1)}_X, \tag{8}$$

where L_X is the Lie derivative of $\mathcal{S}^n_\delta$ along the vector field $X\frac{d}{dx}$ defined by (1), and

$$\mathcal{L}^{(1)}_X = \sum_\lambda \sum_{j=2}^{6} t_{\lambda,\lambda+j} C_{\lambda,\lambda+j}(X) \tag{9}$$

and where $t_{\lambda,\lambda+j}$ are independent parameters, $\delta-\lambda \in \mathbb{N}$, $\delta-n \leq \lambda, \lambda+j \leq \delta$ and the $\mathfrak{sl}(2)$-relative 1-cocycles $C_{\lambda,\lambda+j}$ are defined in Table 1.

Note that for $(j,\lambda) = (2,-\frac{1}{2}), (3,-1), (4,-\frac{3}{2})$, or $j = 5$ and $\lambda \notin \{0,-4\}$ or $j = 6$ and $\lambda \neq -\frac{5\pm\sqrt{19}}{2}$ we have $C_{\lambda,\lambda+j} = 0$, then there are no corresponding parameters $t_{\lambda,\lambda+j}$.

4.2 Integrability conditions

Consider the problem of integrability of infinitesimal deformations. Starting with the infinitesimal deformation (8), we look for a formal series

$$\mathcal{L}_X = L_X + \mathcal{L}^{(1)}_X + \mathcal{L}^{(2)}_X + \mathcal{L}^{(3)}_X + \cdots, \tag{10}$$

where $\mathcal{L}^{(k)}_X$ is an homogenous polynomial of degree k in the parameters $(t_{\lambda,\lambda+j})$ and with coefficients in $\mathcal{D}$ such that $\mathcal{L}^{(k)}_X = 0$ if $X\frac{d}{dx} \in \mathfrak{sl}(2)$. This formal series (10) must satisfy the homomorphism condition in any order in the parameters $(t_{\lambda,\lambda+j})$

$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}. \tag{11}$$

The homomorphism condition (11) gives the following (Maurer–Cartan) equations

$$\partial\mathcal{L}^{(k)} = -\frac{1}{2} \sum_{i+j=k} [\![\mathcal{L}^{(i)}, \mathcal{L}^{(j)}]\!]. \tag{12}$$

However, quite often the above problem has no solution. Note here that the right side of (12) must be a coboundary of a 1-cochain vanishing on $\mathfrak{sl}(2)$, so, the obstructions for integrability of infinitesimal deformations belong to the second $\mathfrak{sl}(2)$-relative cohomology space $\mathrm{H}^2_{\mathrm{diff}}(\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R}), \mathfrak{sl}(2); \mathcal{D})$.

Following [7] and [2], we will impose extra algebraic relations on the parameters $(t_{\lambda,\lambda+j})$. Let $\mathcal{R}$ be an ideal in $\mathbb{C}[[t_{\lambda,\lambda+j}]]$ generated by some set of relations, the quotient

$$\mathcal{A} = \mathbb{C}[[t_{\lambda,\lambda+j}]]/\mathcal{R} \tag{13}$$

is a complete local algebra with unity, and one can speak about deformations with base $\mathcal{A}$, see [7] for details.

Given an infinitesimal deformation (8), one can always consider it as a deformation with base (13), where $\mathcal{R}$ is the ideal generated by all the quadratic monomials. Our aim is to find $\mathcal{A}$ which is big as possible, or, equivalently, we look for relations on the parameters $(t_{\lambda,\lambda+j})$ which are necessary and sufficient for integrability (cf. [1, 2]).

4.3 Equivalence and the miniversal deformation

The notion of equivalence of deformations over complete local algebras has been considered in [6].

Definition 1. Two deformations, ρ and ρ' with the same base $\mathcal{A}$ are called equivalent if there exists an inner automorphism Ψ of the associative algebra $\mathrm{End}(V)\otimes\mathcal{A}$ such that

$$\Psi\circ\rho=\rho' \text{ and } \Psi(\mathbb{I})=\mathbb{I},$$

where $\mathbb{I}$ is the unity of the algebra $\mathrm{End(V)}\otimes\mathcal{A}$.

The following notion of miniversal deformation is fundamental. It assigns to a $\mathfrak{g}$-module V a canonical commutative associative algebra $\mathcal{A}$ and a canonical deformation with base $\mathcal{A}$.

Definition 2. A deformation ρ with base $\mathcal{A}$ is called miniversal, if

(i) for any other deformation, ρ' with base (local) $\mathcal{A}'$, there exists a homomorphism $\psi:\mathcal{A}'\to\mathcal{A}$ satisfying $\psi(1)=1$, such that

$$\rho=(\mathbb{I}d\otimes\psi)\circ\rho'.$$

(ii) in the notations of (i), if $\mathcal{A}$ is infinitesimal then ψ is unique.

If ρ satisfies only the condition (i), then it is called versal.

The miniversal deformation corresponds to the smallest ideal $\mathcal{R}$. We refer to [7] for a construction of miniversal deformations of Lie algebras and to [2] for miniversal deformations of $\mathfrak{g}$-modules.

5 Second-order integrability conditions

In this section we obtain the integrability conditions for the infinitesimal deformation (8). We will give all necessary conditions, but, any space of symbols of differential operators, $\mathcal{S}^n_\delta$, is concerned only by relations between monomials $t_{\lambda,\lambda+j}t_{\lambda+j,\lambda+k}$, where $\delta-n\le\lambda,\lambda+k\le\delta$ and $2\le j\le k\le 10$. Assume that the infinitesimal deformation (8) can be integrated to a formal deformation

$$\mathcal{L}_X=L_X+\mathcal{L}^{(1)}_X+\mathcal{L}^{(2)}_X+\mathcal{L}^{(3)}_X+\cdots,$$

where $\mathcal{L}^{(1)}_X$ is given by (9) and $\mathcal{L}^{(2)}_X$ is a quadratic polynomial in t whose coefficients are elements of $\mathcal{D}$ vanishing on $\mathfrak{sl}(2)$. We compute the conditions for the second-order terms $\mathcal{L}^{(2)}$. The homomorphism condition

$$[\mathcal{L}_X,\mathcal{L}_Y]=\mathcal{L}_{[X,Y]},$$

gives for the second-order terms the following (Maurer–Cartan) equation

$$\partial\mathcal{L}^{(2)}=-\tfrac{1}{2}[\![\mathcal{L}^{(1)},\mathcal{L}^{(1)}]\!]. \tag{14}$$

The right hand side of (14) is a cup-product of $\mathfrak{sl}(2)$-relative 1-cocycles, so it is automatically a $\mathfrak{sl}(2)$-relative 2-cocycle. More precisely, the equation (14) can be expressed as follows

$$\partial\mathcal{L}^{(2)}=-\tfrac{1}{2}[\![\sum_\lambda\sum_{j=2}^{6}t_{\lambda,\lambda+j}\,C_{\lambda,\lambda+j},\sum_\lambda\sum_{j=2}^{6}t_{\lambda,\lambda+j}\,C_{\lambda,\lambda+j}]\!], \tag{15}$$

therefore, let us consider the $\mathfrak{sl}(2)$-relative 2-cocycles $B_{\lambda,\lambda+k} \in Z^2_{\rm diff}(\mathrm{Vect}_{\rm Pol}(\mathbb{R}), \mathfrak{sl}(2), \mathcal{D}_{\lambda,\lambda+k})$, for $k = 4, \dots, 10$, defined by

$$B_{\lambda,\lambda+k} = -\sum_{j=2}^{k} t_{\lambda+j,\lambda+k} t_{\lambda,\lambda+j} \, [\![C_{\lambda+j,\lambda+k}, C_{\lambda,\lambda+j}]\!].$$

It is easy to see that $B_{\lambda,\lambda+4} = 0$. The second order integrability conditions follow from the fact that any map 2-cocycles $B_{\lambda,\lambda+k}$, for $k = 5, \dots, 10$, must be a $\mathfrak{sl}(2)$-relative 2-coboundary. More precisely, $B_{\lambda,\lambda+k}$ must coincide, up to a scalar factor, with $\partial J^{-1,\lambda}_{k+1}$. We split these conditions in two family which we explain in the two following propositions. Let us first consider the following functions in t where t is the family of parameters $(t_{\lambda,\lambda+j})$

$$\begin{aligned}
&\omega_{\lambda,\lambda+5}(t) = \tfrac{\lambda+4}{2} t_{\lambda,\lambda+2} t_{\lambda+2,\lambda+5} - \tfrac{\lambda}{2} t_{\lambda,\lambda+3} t_{\lambda+3,\lambda+5},\\
&\omega_{\lambda,\lambda+6}(t) = -\tfrac{2\lambda+9}{2} t_{\lambda,\lambda+2} t_{\lambda+2,\lambda+6} - \tfrac{5}{2} t_{\lambda,\lambda+3} t_{\lambda+3,\lambda+6} + \tfrac{2\lambda+1}{2} t_{\lambda,\lambda+4} t_{\lambda+4,\lambda+6},\\
&\omega_{0,7}(t) = -\tfrac{1}{7}\left(\tfrac{11}{10} t_{0,3} t_{3,7} + \tfrac{1}{2} t_{0,4} t_{4,7} + 3 t_{0,5} t_{5,7}\right),\\
&\omega_{-6,1}(t) = \tfrac{1}{14}\left(t_{-6,-3} t_{-3,1} - 6 t_{-6,-4} t_{-4,1} + \tfrac{11}{5} t_{-6,-2} t_{-2,1}\right),\\
&\omega_{0,8}(t) = \tfrac{2}{11} t_{0,5} t_{5,8},\\
&\omega_{-7,1}(t) = \tfrac{2}{15} t_{-7,-4} t_{-4,1}.
\end{aligned}$$

These functions $\omega_{\lambda,\lambda+k}(t)$, $k = 5, 6, 7$, will appear as coefficients for some maps from $\mathcal{F}_\lambda$ to $\mathcal{F}_{\lambda+k}$ and they will be used in the expressions of integrability conditions. More precisely, we will show that the second term $\mathcal{L}^{(2)}$ is of the form $\mathcal{L}^{(2)} = \sum_{\lambda,k} \omega_{\lambda,\lambda+k}(t) J^{-1,\lambda}_{k+1}$.

Proposition 7. *For $k = 5, 6, 7$, we have the following second-order integrability conditions of the infinitesimal deformation* (8)

$$\begin{aligned}
&\omega_{\lambda,\lambda+5}(t) = 0 && \text{if } \lambda \in \{0, -2, -4\},\\
&\omega_{\lambda,\lambda+6}(t) = 0 && \text{if } \lambda \in \left\{-\tfrac{5\pm\sqrt{19}}{2}, -\tfrac{5}{2}\right\},\\
&t_{\lambda,\lambda+3} t_{\lambda+3,\lambda+7} = 0 && \text{if } \lambda \notin \{0, -2, -4, -6\},\\
&t_{\lambda,\lambda+4} t_{\lambda+4,\lambda+7} = 0 && \text{if } \lambda \notin \{0, -6\},\\
&10 t_{-2,0} t_{0,5} - t_{-2,1} t_{1,5} = 0,\\
&10 t_{-4,1} t_{1,3} + t_{-4,-1} t_{-1,3} = 0.
\end{aligned} \tag{16}$$

Proof. 1) For $k = 5$, we have

$$B_{\lambda,\lambda+5} = -t_{\lambda,\lambda+2} t_{\lambda+2,\lambda+5} [\![C_{\lambda+2,\lambda+5}, C_{\lambda,\lambda+2}]\!] - t_{\lambda,\lambda+3} t_{\lambda+3,\lambda+5} [\![C_{\lambda+3,\lambda+5}, C_{\lambda,\lambda+5}]\!],$$

hence, according to Proposition 3, we have

$$B_{\lambda,\lambda+5} = \omega_{\lambda,\lambda+5}(t) \Omega_{\lambda,\lambda+5}.$$

Thus, by Proposition 3, the $\mathfrak{sl}(2)$-relative 2-cocycle $\Omega_{\lambda,\lambda+5}$ is nontrivial if and only if $\lambda \in \{0, -2, -4\}$. Hence, for $\lambda \in \{0, -2, -4\}$, the condition $\omega_{\lambda,\lambda+5}(t) = 0$ holds.

2) For $k = 6$, as before, we have

$$B_{\lambda,\lambda+6} = \omega_{\lambda,\lambda+6}(t) \Omega_{\lambda,\lambda+6}.$$

Thus, if $\lambda \in \left\{-\frac{5\pm\sqrt{19}}{2}, -\frac{5}{2}\right\}$ the condition $\omega_{\lambda,\lambda+6}(t) = 0$ must be satisfied.

3) Let $k = 7$. Note that, hereafter, some singular values of the parameter λ appear because the $\mathfrak{sl}(2)$-relative 2-cocycles $C_{\lambda,\lambda+5}$ exist only for $\lambda = 0, -4$ and $C_{\lambda,\lambda+6}$ exist only for $\lambda = -\frac{5\pm\sqrt{19}}{2}$.

i) For $\lambda \notin \{0, -2, -4, -6\}$, we have

$$B_{\lambda,\lambda+7} = -t_{\lambda,\lambda+3}t_{\lambda+3,\lambda+7}\Omega_{\lambda,\lambda+7} - t_{\lambda,\lambda+4}t_{\lambda+4,\lambda+7}\widetilde{\Omega}_{\lambda,\lambda+7}.$$

Therefore, the following conditions follow from Proposition 4

$$t_{\lambda,\lambda+3}t_{\lambda+3,\lambda+7} = t_{\lambda,\lambda+4}t_{\lambda+4,\lambda+7} = 0.$$

ii) By a direct computation, we show that

$$B_{0,7} = \omega_{0,7}(t)\partial J_8^{-1,0} \qquad \text{and} \qquad B_{-6,1} = \omega_{-6,1}(t)\partial J_8^{-1,-6}.$$

Hence, there are no conditions on $B_{0,7}$ and $B_{-6,1}$.

iii) For the other two singular values of λ we have

$$B_{-2,5} = -t_{-2,0}t_{0,5}[\![C_{0,5}, C_{-2,0}]\!] - t_{-2,1}t_{1,5}\Omega_{-2,5} - t_{-2,2}t_{2,5}\widetilde{\Omega}_{-2,5},$$
$$B_{-4,3} = -t_{-4,1}t_{1,3}[\![C_{1,3}, C_{-4,1}]\!] - t_{-4,-1}t_{-1,3}\Omega_{-4,3} - t_{-4,0}t_{0,3}\widetilde{\Omega}_{-4,3}.$$

But, we show that $[\![C_{0,5}, C_{-2,0}]\!] = -10\Omega_{-2,5}$ and $[\![C_{1,3}, C_{-4,1}]\!] = 10\Omega_{-2,5}$. So, the following integrability conditions follow again from Proposition 4

$$10t_{-2,0}t_{0,5} - t_{-2,1}t_{1,5} = t_{-2,2}t_{2,5} = 10t_{-4,1}t_{1,3} + t_{-4,-1}t_{-1,3} = t_{-4,0}t_{0,3} = 0.$$

Therefore, the previous equation: $t_{\lambda,\lambda+4}t_{\lambda+4,\lambda+7} = 0$ holds also for $\lambda = -2, -4$. ■

Proposition 8. *For $k = 8, 9, 10$, we have the following second-order integrability conditions of the infinitesimal deformation* (8)*, where in the first line $\lambda \notin \{0, -3, -4, -7\}$,*

$$\begin{aligned}
&t_{\lambda,\lambda+4}t_{\lambda+4,\lambda+8} = 11t_{0,4}t_{4,8} + 10t_{0,5}t_{5,8} = 0,\\
&t_{-3,1}t_{1,5} - 10t_{-3,0}t_{0,5} = t_{-4,0}t_{0,4} + 10t_{-4,1}t_{1,4} = 0,\\
&11t_{-7,-3}t_{-3,1} - 10t_{-7,-4}t_{-4,1} = t_{a_i,a_i+6}t_{a_i+6,a_i+8} = 0,\\
&t_{a_i-2,a_i}t_{a_i,a_i+6} = t_{0,5}t_{5,9} = 0,\\
&t_{-4,0}t_{0,5} - t_{-4,1}t_{1,5} = t_{-8,-4}t_{-4,1} = 0,\\
&t_{a_i,a_i+6}t_{a_i+6,a_i+9} = t_{a_i-3,a_i}t_{a_i,a_i+6} = 0,\\
&t_{a_i,a_i+6}t_{a_i+6,a_i+10} = t_{a_i-4,a_i}t_{a_i,a_i+6} = 0.
\end{aligned} \tag{17}$$

Proof. 1) For $k = 8$, we first recall that the cup-product $\Omega_{\lambda,\lambda+8} = [\![C_{\lambda,\lambda+4}, C_{\lambda+4,\lambda+8}]\!]$ is a $\mathfrak{sl}(2)$-relative nontrivial 2-cocycle. Moreover, for $\lambda \notin \{0, -3, -4, -7, a_1, a_2, a_1 - 2, a_2 - 2\}$, we have

$$B_{\lambda,\lambda+8} = -t_{\lambda,\lambda+4}t_{\lambda+4,\lambda+8}\Omega_{\lambda,\lambda+8}.$$

For the singular values, we easily check that

$$[\![C_{5,8}, C_{0,5}]\!] = \tfrac{10}{11}\Omega_{0,8} + \tfrac{2}{11}\partial J_9^{-1,0}, \qquad [\![C_{0,5}, C_{-3,0}]\!] = 10\Omega_{-3,5},$$
$$[\![C_{-4,1}, C_{-7,-4}]\!] = \tfrac{10}{11}\Omega_{-7,1} + \frac{2}{15}\partial J_9^{-1,-7}, \qquad [\![C_{1,4}, C_{-4,1}]\!] = -10\Omega_{-4,4}$$

and we show that $[\![C_{a_i+6,a_i+8}, C_{a_i,a_i+6}]\!]$, and $[\![C_{a_i,a_i+6}, C_{a_i-2,a_i}]\!]$ are also nontrivial 2-cocycles.

Thus, we deduce all integrability conditions corresponding to the case $k = 8$.

2) For $k = 9$, the integrability conditions follow from the fact that any corresponding cup-product of 1-cocycle is nontrivial. Moreover, we have only singular cases and we also show that $[\![C_{1,5}, C_{-4,1}]\!] = -[\![C_{0,5}, C_{-4,0}]\!]$.

3) For $k = 10$ and $\lambda \neq a_i, a_i - 4$ we have $B_{\lambda,\lambda+10} = 0$. For $\lambda = a_i, a_i - 4$ we have

$$B_{a_i,a_i+10} = -t_{a_i,a_i+6}t_{a_i+6,a_i+10}[\![C_{a_i,a_i+6}, C_{a_i+6,a_i+10}]\!] = -t_{a_i,a_i+6}t_{a_i+6,a_i+10}\Omega_{a_i,a_i+10},$$
$$B_{a_i-4,a_i+6} = -t_{a_i-4,a_i}t_{a_i,a_i+6}[\![C_{a_i-4,a_i}, C_{a_i,a_i+6}]\!] = -t_{a_i-4,a_i}t_{a_i,a_i+6}\Omega_{a_i-4,a_i+6}.$$

Like in the previous case we prove that the 2-cocycles Ω_{a_i,a_i+10} and Ω_{a_i-4,a_i+6} are nontrivial and then we deduce the corresponding integrability conditions. ■

Our main result in this section is the following

Theorem 2. *The conditions* (16) *and* (17) *are necessary and sufficient for second-order integrability of the* $\mathfrak{sl}(2)$*-trivial infinitesimal deformation* (8)*.*

Proof. Of course, these conditions are necessary as, it was shown in Proposition 7 and Proposition 8. Now, under these conditions, the second term $\mathcal{L}^{(2)}$ of the the $\mathfrak{sl}(2)$-trivial infinitesimal deformation (8) is a solution of the Maurer–Cartan equation (15). This solution is defined up to a 1-coboundary and it has been shown in [7, 2] that different choices of solutions of the Maurer–Cartan equation correspond to equivalent deformations. Thus, we can always choose

$$\mathcal{L}^{(2)} = \frac{1}{2}\sum_{\lambda\neq 0,-2,-4} \omega_{\lambda,\lambda+5}(t)J_6^{-1,\lambda} + \frac{1}{2}\sum_{\lambda\neq a_i,-\frac{5}{2}} \omega_{\lambda,\lambda+6}(t)J_7^{-1,\lambda}$$
$$+ \frac{1}{2}\sum_{\lambda=0,-6} \omega_{\lambda,\lambda+7}(t)J_8^{-1,\lambda} + \frac{1}{2}\sum_{\lambda=0,-7} \omega_{\lambda,\lambda+8}(t)J_9^{-1,\lambda}.$$

Of course, any $t_{\lambda,\lambda+k}$ appear in the expressions of $\mathcal{L}^{(1)}$ or $\mathcal{L}^{(2)}$ if and only if $\delta - \lambda$ and k are integers satisfying $\delta - n \leq \lambda, \lambda + k \leq \delta$. Theorem 2 is proved. ■

6 Third-order integrability conditions

6.1 Computing the third-order Maurer–Cartan equation

Now we reconsider the formal deformation (10) which is a formal power series in the parameters $t_{\lambda,\lambda+j}$ with coefficients in $\mathcal{D}$. We suppose that the second-order integrability conditions are satisfied. So, the third-order terms of (10) are solutions of the (Maurer–Cartan) equation

$$\partial\mathcal{L}^{(3)} = -\frac{1}{2}\sum_{i+j=3}[\![\mathcal{L}^{(i)}, \mathcal{L}^{(j)}]\!]. \tag{18}$$

As in the previous section we can write

$$\partial\mathcal{L}^{(3)} = -\frac{1}{2}\sum_{k,\lambda} E_{\lambda,\lambda+k}, \tag{19}$$

where $E_{\lambda,\lambda+k}$ are maps from $\mathrm{Vect_{Pol}}(\mathbb{R}) \times \mathrm{Vect_{Pol}}(\mathbb{R})$ to $\mathcal{D}_{\lambda,\lambda+k}$. The third-order term $\mathcal{L}^{(3)}$ of the $\mathfrak{sl}(2)$-trivial formal deformation (10) is a solution of (19). So, the 2-cochains $E_{\lambda,k}$ must satisfy $E_{\lambda,k} = \partial J_{k+1}^{-1,\lambda}$ and then the third-order integrability conditions are deduced from this fact.

It is easy to see that $E_{\lambda,\lambda+k} = 0$ for $k \leq 6$ or $k \geq 13$, so we compute successively the $E_{\lambda,\lambda+k}$ for $k = 7, \ldots, 12$ and we resolve $E_{\lambda,\lambda+k} = \partial J_{k+1}^{-1,\lambda}$ to get the corresponding third-order integrability conditions.

Here, we mention that the maps $E_{\lambda,\lambda+k}$ are 2-cochains, but they are not necessarily 2-cocycles because they are not cup-products of 1-cocycles like the maps $B_{\lambda,\lambda+k}$. Indeed, $\mathcal{L}^{(2)}$ is not necessarily 1-cocycle.

6.2 Third-order integrability conditions

Proposition 9. *For $k = 7, 8$, we have the following third-order integrability conditions of the infinitesimal deformation* (8)*, for all* λ

$$\begin{aligned} &t_{\lambda,\lambda+2}\omega_{\lambda+2,\lambda+7}(t) = \omega_{\lambda,\lambda+5}(t)t_{\lambda+5,\lambda+7} = 0,\\ &t_{\lambda,\lambda+2}\,\omega_{\lambda+2,\lambda+8}(t) = \omega_{\lambda,\lambda+6}(t)t_{\lambda+6,\lambda+8} = 0,\\ &t_{\lambda,\lambda+3}\omega_{\lambda+3,\lambda+8}(t) = \omega_{\lambda,\lambda+5}(t)t_{\lambda+5,\lambda+8} = 0. \end{aligned} \tag{20}$$

Proof. For $k = 7$ and $\lambda \notin \{0, -2, -4, -6\}$ we have

$$\begin{aligned} &E_{\lambda,\lambda+7} = t_{\lambda,\lambda+2}\omega_{\lambda+2,\lambda+7}(t)[\![J_6^{-1,\lambda+2},\, C_{\lambda,\lambda+2}]\!] + t_{\lambda+5,\lambda+7}\omega_{\lambda,\lambda+5}(t)[\![C_{\lambda+5,\lambda+7},\, J_6^{-1,\lambda}]\!],\\ &E_{0,7} = t_{0,2}\omega_{2,7}(t)[\![J_6^{-1,2},\, C_{0,2}]\!], \qquad E_{-6,1} = t_{-1,1}\omega_{-6,-1}(t)[\![C_{-1,1},\, J_6^{-1,-6}]\!] \end{aligned}$$

and

$$E_{-2,5} = E_{-4,3} = 0.$$

By a direct computation, we show that the three maps $[\![J_6^{-1,\lambda+2},\, C_{\lambda,\lambda+2}]\!]$, $[\![C_{\lambda+5,\lambda+7},\, J_6^{-1,\lambda}]\!]$ and $\partial J_8^{-1,\lambda}$ are linearly independent, for all λ. Thus,

$$\begin{aligned} &t_{\lambda,\lambda+2}\omega_{\lambda+2,\lambda+7}(t) = 0, \qquad \text{for} \quad \lambda \neq -2, -4, -6,\\ &t_{\lambda+5,\lambda+7}\omega_{\lambda,\lambda+5}(t) = 0, \qquad \text{for} \quad \lambda \neq 0, -2, -4. \end{aligned}$$

But, under the second-order integrability conditions: $\omega_{\lambda,\lambda+5}(t) = 0$ for $\lambda \in \{0, -2, -4\}$, the conditions

$$t_{\lambda,\lambda+2}\omega_{\lambda+2,\lambda+7}(t) = \omega_{\lambda,\lambda+5}(t) = 0$$

hold for all λ.

Now, for $k = 8$ and $\lambda \notin \{a_1, a_2, -\frac{5}{2}, a_1 - 2, a_2 - 2, -\frac{9}{2}, 0, -2, -4, -3, -5, -7\}$ we have

$$\begin{aligned} E_{\lambda,\lambda+8} &= t_{\lambda,\lambda+2}\omega_{\lambda+2,\lambda+8}(t)[\![J_7^{-1,\lambda+2}, C_{\lambda,\lambda+2}]\!] + t_{\lambda+6,\lambda+8}\omega_{\lambda,\lambda+6}(t)[\![C_{\lambda+6,\lambda+8}, J_7^{-1,\lambda}]\!]\\ &\quad + t_{\lambda,\lambda+3}\omega_{\lambda+3,\lambda+8}(t)[\![J_6^{-1,\lambda+3}, C_{\lambda,\lambda+3}]\!] + t_{\lambda+5,\lambda+8}\omega_{\lambda,\lambda+5}(t)[\![C_{\lambda+5,\lambda+8}, J_6^{-1,\lambda}]\!]. \end{aligned}$$

As before, we show that

$$t_{\lambda,\lambda+2}\omega_{\lambda+2,\lambda+8}(t) = t_{\lambda,\lambda+3}\omega_{\lambda+3,\lambda+8}(t) = t_{\lambda+5,\lambda+8}\omega_{\lambda,\lambda+5}(t) = t_{\lambda+6,\lambda+8}\omega_{\lambda,\lambda+6}(t) = 0.$$

We get the same results for $\lambda \in \{a_1, a_2, -\frac{5}{2}, a_1 - 2, a_2 - 2, -\frac{9}{2}, 0, -2, -4, -3, -5, -7\}$ by considering the second-order integrability conditions. ∎

Proposition 10. *For $k = 9$, we have the following third-order integrability conditions of the infinitesimal deformation* (8)

$$\begin{aligned} &t_{\lambda,\lambda+3}\omega_{\lambda+3,\lambda+9}(t) = \omega_{\lambda,\lambda+6}(t)t_{\lambda+6,\lambda+9} = 0 \qquad \textit{for all} \quad \lambda,\\ &t_{\lambda,\lambda+4}\omega_{\lambda+4,\lambda+9}(t) = \omega_{\lambda,\lambda+5}(t)t_{\lambda+5,\lambda+9} = 0, \qquad \textit{for all} \quad \lambda,\\ &t_{\lambda-2,\lambda}\omega_{\lambda,\lambda+7}(t) = \omega_{\lambda,\lambda+7}(t)t_{\lambda+7,\lambda+9} = 0, \qquad \textit{for} \quad \lambda = 0, -6. \end{aligned} \tag{21}$$

Proof. For $k = 9$ and $\lambda \notin \{0, -2, -4, -6, -8, a_i, -\frac{5}{2}, a_i - 3, -\frac{11}{2}\}$ we have

$$\begin{aligned} E_{\lambda,\lambda+9} &= t_{\lambda,\lambda+3}\omega_{\lambda+3,\lambda+9}(t)[\![J_7^{-1,\lambda+3}, C_{\lambda,\lambda+3}]\!] + t_{\lambda,\lambda+4}\omega_{\lambda+4,\lambda+9}(t)[\![J_6^{-1,\lambda+4}, C_{\lambda,\lambda+4}]\!]\\ &\quad + t_{\lambda+5,\lambda+9}\omega_{\lambda,\lambda+5}(t)[\![C_{\lambda+5,\lambda+9}, J_6^{-1,\lambda}]\!] + t_{\lambda+6,\lambda+9}\omega_{\lambda,\lambda+6}(t)[\![C_{\lambda+6,\lambda+9}, J_7^{-1,\lambda}]\!]. \end{aligned}$$

The equation $E_{\lambda,\lambda+9} = \partial J_{10}^{-1,\lambda}$ gives

$$t_{\lambda,\lambda+3}\omega_{\lambda+3,\lambda+9}(t) = t_{\lambda,\lambda+4}\omega_{\lambda+4,\lambda+9}(t) = t_{\lambda+5,\lambda+9}\omega_{\lambda,\lambda+5}(t) = t_{\lambda+6,\lambda+9}\omega_{\lambda,\lambda+6}(t) = 0.$$

By considering the second-order integrability conditions, we get the same results for each $\lambda \in \{0, -2, -4, -6, -8, a_i, -\frac{5}{2}, a_i - 3, -\frac{11}{2}\}$. ∎

Proposition 11. *For $k = 10$, we have the following third-order integrability conditions of the infinitesimal deformation* (8)

$$\begin{aligned} &t_{\lambda-2,\lambda}\omega_{\lambda,\lambda+8}(t) = \omega_{\lambda,\lambda+8}(t)t_{\lambda+8,\lambda+10} = 0 && \textit{for} \quad \lambda = 0, -7,\\ &t_{\lambda-3,\lambda}\omega_{\lambda,\lambda+7}(t) = \omega_{\lambda,\lambda+7}(t)t_{\lambda+7,\lambda+10} = 0 && \textit{for} \quad \lambda = 0, -6,\\ &t_{\lambda,\lambda+4}\omega_{\lambda+4,\lambda+10}(t) = \omega_{\lambda,\lambda+6}(t)t_{\lambda+6,\lambda+10} = 0 && \textit{for all} \quad \lambda,\\ &t_{\lambda,\lambda+5}\omega_{\lambda+5,\lambda+10}(t) = \omega_{\lambda-5,\lambda}(t)\, t_{\lambda,\lambda+5} = 0 && \textit{for} \quad \lambda = 0, -4. \end{aligned} \tag{22}$$

Proof. For $k = 10$ and $\lambda \notin \{-9, -7, -\frac{13}{2}, -6, -5, -4, -3, -2, -\frac{5}{2}, 0, a_1, a_1 - 4, a_2, a_2 - 4\}$ we have

$$E_{\lambda,\lambda+10} = t_{\lambda,\lambda+4}\omega_{\lambda+4,\lambda+10}(t)[\![J_7^{-1,\lambda+4}, C_{\lambda,\lambda+4}]\!] + t_{\lambda+6,\lambda+10}\omega_{\lambda,\lambda+6}(t)[\![C_{\lambda+6,\lambda+10}, J_7^{-1,\lambda}]\!].$$

The equation $E_{\lambda,\lambda+10} = \partial J_{11}^{-1,\lambda}$ gives the conditions

$$t_{\lambda,\lambda+4}\omega_{\lambda+4,\lambda+10}(t) = t_{\lambda+6,\lambda+10}\omega_{\lambda,\lambda+6}(t) = 0.$$

We check that, for $\lambda \in \{-9, -7, -\frac{13}{2}, -6, -5, -4, -3, -2, -\frac{5}{2}, 0, a_1, a_1 - 4, a_2, a_2 - 4\}$, these latter conditions must be also satisfied. The others conditions follow from the singular values of λ. ∎

Proposition 12. *For $k = 11$, we have the following second-order integrability conditions of the infinitesimal deformation* (8)

$$\begin{aligned} &t_{\lambda,\lambda+6}\,\omega_{\lambda+6,\lambda+11}(t) = \omega_{\lambda-5,\lambda}(t)t_{\lambda,\lambda+6} = 0 && \textit{for} \quad \lambda = -\tfrac{5\pm\sqrt{19}}{2},\\ &t_{\lambda,\lambda+5}\,\omega_{\lambda+5,\lambda+11}(t) = \omega_{\lambda-6,\lambda}(t)t_{\lambda,\lambda+5} = 0 && \textit{for} \quad \lambda = 0, -4,\\ &t_{\lambda-4,\lambda}\omega_{\lambda,\lambda+7}(t) = \omega_{\lambda,\lambda+7}(t)t_{\lambda+7,\lambda+11} = 0 && \textit{for} \quad \lambda = 0, -6,\\ &t_{\lambda-3,\lambda}\omega_{\lambda,\lambda+8}(t) = \omega_{\lambda,\lambda+8}(t)t_{\lambda+8,\lambda+11} = 0 && \textit{for} \quad \lambda = 0, -7. \end{aligned} \tag{23}$$

Proposition 13. *For $k = 12$, we have the following third-order integrability conditions of the infinitesimal deformation* (8)

$$\begin{aligned} &t_{\lambda,\lambda+6}\omega_{\lambda+6,\lambda+12}(t) = \omega_{\lambda-6,\lambda}(t)t_{\lambda,\lambda+6} = 0 && \textit{for} \quad \lambda = -\tfrac{5\pm\sqrt{19}}{2},\\ &t_{\lambda-4,\lambda}\omega_{\lambda,\lambda+8}(t) = \omega_{\lambda,\lambda+8}(t)t_{\lambda+8,\lambda+12} = 0 && \textit{for} \quad \lambda = 0, -7. \end{aligned} \tag{24}$$

Proof. For $k = 11$ and $k = 12$, the 2-cochains $E_{\lambda,\lambda+k}$ are defined only for some particular values of λ. We compute these 2-cochains and then we check the corresponding integrability conditions. ∎

The following theorem is our main result.

Theorem 3. *The second-order integrability conditions* (16) *and* (17) *together with the third-order conditions* (20)–(24) *are necessary and sufficient for the complete integrability of the infinitesimal deformation* (8)*. Moreover, any formal $\mathfrak{sl}(2)$-trivial deformation of the Lie derivative L_X on the space of symbols $\mathcal{S}_\delta^n$ is equivalent to a polynomial one of degree equal or less than* 2.

Proof. Clearly, all these conditions are necessary. So, let us prove that they are also sufficient. As in the proof of Theorem 2, the solution $\mathcal{L}^{(3)}$ of the Maurer–Cartan equation (18) is defined up to a 1-coboundary, thus, we can always reduce $\mathcal{L}^{(3)}$ to zero by equivalence. Moreover, by recurrence, the highest-order terms $\mathcal{L}^{(m)}$ satisfy the equation $\partial\mathcal{L}^{(m)} = 0$ and can also be reduced to the identically zero map. This completes the proof of Theorem 3. ■

Remark 1. The majority of integrability conditions concern some parameters $t_{\lambda,\lambda+k}$ with singular values of λ. All these singular values of λ are negatives. So, let us consider the space $\mathcal{S}^n_\delta$ with generic δ, for example, $\delta - n > 0$. In this case, the second-order integrability conditions are reduced to the following equations

$$t_{\lambda,\lambda+3}t_{\lambda+3,\lambda+7} = t_{\lambda,\lambda+4}t_{\lambda+4,\lambda+7} = t_{\lambda,\lambda+4}t_{\lambda+4,\lambda+8} = 0,$$

and the third-order integrability conditions are reduced to

$$\begin{aligned}
&t_{\lambda,\lambda+2}\omega_{\lambda+2,\lambda+7}(t) = \omega_{\lambda,\lambda+5}(t)t_{\lambda+5,\lambda+7} = t_{\lambda,\lambda+2}\omega_{\lambda+2,\lambda+8}(t) = \omega_{\lambda,\lambda+6}t_{\lambda+6,\lambda+8}(t) = 0,\\
&t_{\lambda,\lambda+3}\omega_{\lambda+3,\lambda+8}(t) = \omega_{\lambda,\lambda+5}(t)t_{\lambda+5,\lambda+8} = t_{\lambda,\lambda+3}\omega_{\lambda+3,\lambda+9}(t) = t_{\lambda+6,\lambda+9}\omega_{\lambda,\lambda+6}(t) = 0,\\
&\omega_{\lambda,\lambda+5}(t)t_{\lambda+5,\lambda+9} = \omega_{\lambda+4,\lambda+9}(t)t_{\lambda,\lambda+4} = t_{\lambda,\lambda+4}\omega_{\lambda+4,\lambda+10}(t) = \omega_{\lambda,\lambda+6}(t)t_{\lambda+6,\lambda+10} = 0.
\end{aligned}$$

7 Examples

Example 1. Let us consider the space of symbols $\mathcal{S}^4_{\lambda+4}$.

Proposition 14. *Any formal $\mathfrak{sl}(2)$-trivial deformation of the* $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$*-action on the space $\mathcal{S}^4_{\lambda+4}$ is equivalent to his infinitesimal part, without any conditions on the parameters (independent parameters). That is, the miniversal deformation is here with base $\mathbb{C}[[t]]$ where t designates the family of all parameters.*

Proof. The infinitesimal $\mathfrak{sl}(2)$-trivial deformation, in this case, is given by

$$\mathcal{L} = L + \mathcal{L}^{(1)},$$

where L_X is the Lie derivative of $\mathcal{S}^4_{\lambda+4}$ along the vector field $X\frac{d}{dx}$ defined by (1), and

$$\begin{aligned}
\mathcal{L}^{(1)} &= t_{\lambda,\lambda+2}C_{\lambda,\lambda+2} + t_{\lambda,\lambda+3}C_{\lambda,\lambda+3} + t_{\lambda,\lambda+4}C_{\lambda,\lambda+4}\\
&\quad + t_{\lambda+1,\lambda+3}C_{\lambda+1,\lambda+3}(X) + t_{\lambda+1,\lambda+4}C_{\lambda+1,\lambda+4} + t_{\lambda+2,\lambda+4}C_{\lambda+2,\lambda+4}.
\end{aligned}$$

There are no conditions to integrate this infinitesimal deformation to a formal one. The solution $\mathcal{L}^{(2)}$ of (12) is defined up to a 1-coboundary and different choices of solutions of the Maurer–Cartan equation correspond to equivalent deformations. Thus, we can always reduce $\mathcal{L}^{(2)}$ to zero by equivalence. Then, by recurrence, the highest-order terms $\mathcal{L}^{(m)}$ satisfy the equation $\partial\mathcal{L}^{(m)} = 0$ and $\mathcal{L}^{(m)}$ can also be reduced to the identically zero map. ■

Remark 2. We have the same results for $\mathcal{S}^k_{\lambda+k}$ if $k \le 4$. Indeed, for $k \le 4$ there are no integrability conditions.

Example 2. Let us consider the $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$-module $\mathcal{S}^6_7$.

Proposition 15. *Any formal $\mathfrak{sl}(2)$-trivial deformation of the* $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$*-action on the space $\mathcal{S}^6_7$ is equivalent to*

$$\mathcal{L}_X = L + \mathcal{L}^{(1)} + \mathcal{L}^{(2)}, \tag{25}$$

where

$$\mathcal{L}^{(1)} = t_{1,3}C_{1,3} + t_{1,4}C_{1,4} + t_{1,5}C_{1,5} + t_{2,4}C_{2,4} + t_{2,5}C_{2,5} + t_{2,6}C_{2,6}$$
$$+ t_{3,5}C_{3,5} + t_{3,6}C_{3,6} + t_{3,7}C_{3,7} + t_{4,6}C_{4,6} + t_{4,7}C_{4,7}$$

and

$$\mathcal{L}^{(2)} = \tfrac{1}{2}\omega_{1,6}(t)J_6^{-1,1} + \tfrac{1}{2}\omega_{2,7}(t)J_6^{-1,2} + \tfrac{1}{2}\omega_{1,7}(t)J_7^{-1,1}.$$

The formal deformation (25) *is defined without any condition on the parameters (independent parameters). That is, the miniversal deformation is here with base* $\mathbb{C}[[t]]$ *where* t *designates the family of all parameters.*

For $k \leq 6$, generically there are no integrability conditions which is the case of the previous example (see Remark 1). Now, we study a generic example with $k = 7$.

Example 3. Let us consider the $\mathcal{S}^7_{\lambda+7}$ for generic λ, (for example $\lambda > 0$).

Proposition 16. *The* $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$*-module* $\mathcal{S}^7_{\lambda+7}$ *admits four* $\mathfrak{sl}(2)$*-trivial deformations with* 11 *independent parameters. It admits a miniversal* $\mathfrak{sl}(2)$*-trivial deformations with* 15 *parameters. These deformations are polynomial of degree* 2.

Proof. Any formal $\mathfrak{sl}(2)$-trivial deformation of the $\mathrm{Vect}_{\mathrm{Pol}}(\mathbb{R})$-action on the space $\mathcal{S}^7_{\lambda+7}$ is equivalent to

$$\mathcal{L} = L + \mathcal{L}^{(1)} + \mathcal{L}^{(2)},$$

where

$$\mathcal{L}^{(1)} = t_{\lambda,\lambda+2}C_{\lambda,\lambda+2} + t_{\lambda,\lambda+3}C_{\lambda,\lambda+3} + t_{\lambda,\lambda+4}C_{\lambda,\lambda+4} + t_{\lambda+1,\lambda+3}C_{\lambda+1,\lambda+3}$$
$$+ t_{\lambda+1,\lambda+4}C_{\lambda+1,\lambda+4} + t_{\lambda+1,\lambda+5}C_{\lambda+1,\lambda+5} + t_{\lambda+2,\lambda+4}C_{\lambda+2,\lambda+4} + t_{\lambda+2,\lambda+5}C_{\lambda+2,\lambda+5}$$
$$+ t_{\lambda+2,\lambda+6}C_{\lambda+2,\lambda+6} + t_{\lambda+3,\lambda+5}C_{\lambda+3,\lambda+5} + t_{\lambda+3,\lambda+6}C_{\lambda+3,\lambda+6} + t_{\lambda+3,\lambda+7}C_{\lambda+3,\lambda+7}$$
$$+ t_{\lambda+4,\lambda+6}C_{\lambda+4,\lambda+6} + t_{\lambda+4,\lambda+7}C_{\lambda+4,\lambda+7} + t_{\lambda+5,\lambda+7}C_{\lambda+5,\lambda+7}$$

and

$$\mathcal{L}^{(2)} = \tfrac{1}{2}\omega_{\lambda,\lambda+5}(t)J_6^{-1,\lambda} + \tfrac{1}{2}\omega_{\lambda+1,\lambda+6}(t)J_6^{-1,\lambda+1} + \tfrac{1}{2}\omega_{\lambda+2,\lambda+7}(t)J_6^{-1,\lambda+2}$$
$$+ \tfrac{1}{2}\omega_{\lambda,\lambda+6}(t)J_7^{-1,\lambda} + \tfrac{1}{2}\omega_{\lambda+1,\lambda+7}(t)J_7^{-1,\lambda+1}.$$

There are only 4 integrability conditions

$$t_{\lambda,\lambda+3}t_{\lambda+3,\lambda+7} = t_{\lambda,\lambda+4}t_{\lambda+4,\lambda+7} = t_{\lambda,\lambda+2}\omega_{\lambda+2,\lambda+7}(t) = \omega_{\lambda,\lambda+5}(t)t_{\lambda+5,\lambda+7} = 0.$$

The formal $\mathfrak{sl}(2)$-trivial deformations with the greatest number of independent parameters are those corresponding to $t_{\lambda,\lambda+3}t_{\lambda+3,\lambda+7} = t_{\lambda,\lambda+4}t_{\lambda+4,\lambda+7} = t_{\lambda,\lambda+2} = t_{\lambda+5,\lambda+7} = 0$. So, we must kill at least four parameters and there are four choices. Thus, there are four deformations with 11 independent parameters. Of course, there are many formal deformations with less then 11 independent parameters. The deformation $\mathcal{L} = L + \mathcal{L}^{(1)} + \mathcal{L}^{(2)}$, is the miniversal $\mathfrak{sl}(2)$-trivial deformation of $\mathcal{S}^7_{\lambda+7}$ with base $\mathcal{A} = \mathbb{C}[t]/\mathcal{R}$, where t is the family of all parameters given in the expression of $\mathcal{L}^{(1)}$ and $\mathcal{R}$ is the ideal generated by the polynomials $t_{\lambda,\lambda+3}t_{\lambda+3,\lambda+7}$, $t_{\lambda,\lambda+4}t_{\lambda+4,\lambda+7}$, $t_{\lambda,\lambda+2}\omega_{\lambda+2,\lambda+7}(t)$ and $\omega_{\lambda,\lambda+5}(t)t_{\lambda+5,\lambda+7}$. ■

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