Symmetry, Integrability and Geometry: Methods and Applications SIGMA 4 (2008), 064, 26 pages

Hochschild Homology and Cohomology of Klein Surfaces*

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Received April 09, 2008, in final form September 04, 2008; Published online September 17, 2008
Original article is available at http://www.emis.de/journals/SIGMA/2008/064/

Abstract. Within the framework of deformation quantization, a first step towards the study of star-products is the calculation of Hochschild cohomology. The aim of this article is precisely to determine the Hochschild homology and cohomology in two cases of algebraic varieties. On the one hand, we consider singular curves of the plane; here we recover, in a different way, a result proved by Fronsdal and make it more precise. On the other hand, we are interested in Klein surfaces. The use of a complex suggested by Kontsevich and the help of Groebner bases allow us to solve the problem.

Key words: Hochschild cohomology; Hochschild homology; Klein surfaces; Groebner bases; quantization; star-products

2000 Mathematics Subject Classification: 53D55; 13D03; 30F50; 13P10

1 Introduction

1.1 Deformation quantization

Given a mechanical system $(M, \mathcal{F}(M))$, where M is a Poisson manifold and $\mathcal{F}(M)$ the algebra of regular functions on M, it is important to be able to quantize it, in order to obtain more precise results than through classical mechanics. An available method is deformation quantization, which consists of constructing a star-product on the algebra of formal power series $\mathcal{F}(M)[[\hbar]]$. The first approach for this construction is the computation of Hochschild cohomology of $\mathcal{F}(M)$.

We consider such a mechanical system given by a Poisson manifold M, endowed with a Poisson bracket $\{\cdot, \cdot\}$. In classical mechanics, we study the (commutative) algebra $\mathcal{F}(M)$ of regular functions (i.e., for example, $\mathcal{C}^\infty$, holomorphic or polynomial) on M, that is to say the observables of the classical system. But quantum mechanics, where the physical system is described by a (non commutative) algebra of operators on a Hilbert space, gives more correct results than its classical analogue. Hence the importance to get a quantum description of the classical system $(M, \mathcal{F}(M))$, such an operation is called a quantization.

One option is geometric quantization, which allows us to construct in an explicit way a Hilbert space and an algebra of operators on this space (see the book [10] on the Virasoro group and algebra for a nice introduction to geometric quantization). This very interesting method presents the drawback of being seldom applicable.

That is why other methods, such as asymptotic quantization and deformation quantization, have been introduced. The latter, described in 1978 by F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer in [5], is a good alternative: instead of constructing an

*This paper is a contribution to the Special Issue on Deformation Quantization. The full collection is available at http://www.emis.de/journals/SIGMA/Deformation_Quantization.html algebra of operators on a Hilbert space, we define a formal deformation of $\mathcal{F}(M)$. This is given by the algebra of formal power series $\mathcal{F}(M)[[\hbar]]$, endowed with some associative, but not always commutative, star-product,

$$f * g = \sum_{j=0}^{\infty} m_j(f,\, g)\hbar^j, \tag{1}$$

where the maps m_j are bilinear and where $m_0(f,\, g) = fg$. Then quantization is given by the map $f \mapsto \widehat{f}$, where the operator $\widehat{f}$ satisfies $\widehat{f}(g) = f * g$.

In which cases does a Poisson manifold admit such a quantization? The answer was given by Kontsevich in [11]: in fact he constructed a star-product on every Poisson manifold. Besides, he proved that if M is a smooth manifold, then the equivalence classes of formal deformations of the zero Poisson bracket are in bijection with equivalence classes of star-products. Moreover, as a consequence of the Hochschild–Kostant–Rosenberg theorem, every Abelian star-product is equivalent to a trivial one.

In the case where M is a singular algebraic variety, say

$$M = \{\mathbf{z} \in \mathbb{C}^n \,/\, f(\mathbf{z}) = 0\},$$

with $n = 2$ or 3, where f belongs to $\mathbb{C}[\mathbf{z}]$ – and this is the case which we shall study – we shall consider the algebra of functions on M, i.e. the quotient algebra $\mathbb{C}[\mathbf{z}] \,/\, \langle f \rangle$. So the above mentioned result is not always valid. However, the deformations of the algebra $\mathcal{F}(M)$, defined by the formula (1), are always classified by the Hochschild cohomology of $\mathcal{F}(M)$, and we are led to the study of the Hochschild cohomology of $\mathbb{C}[\mathbf{z}] \,/\, \langle f \rangle$.

1.2 Cohomologies and quotients of polynomial algebras

We shall now consider $R := \mathbb{C}[z_1, \ldots, z_n] = \mathbb{C}[\mathbf{z}]$, the algebra of polynomials in n variables with complex coefficients. We also fix m elements $f_1, \ldots, f_m$ of R, and we define the quotient algebra $A := R \,/\, \langle f_1, \ldots, f_m \rangle = \mathbb{C}[z_1, \ldots, z_n] \,/\, \langle f_1, \ldots, f_m \rangle$.

Recent articles were devoted to the study of particular cases, for Hochschild as well as for Poisson homology and cohomology:

C. Roger and P. Vanhaecke, in [16], calculate the Poisson cohomology of the affine plane $\mathbb{C}^2$, endowed with the Poisson bracket $f_1\, \partial_{z_1} \wedge \partial_{z_2}$, where f_1 is a homogeneous polynomial. They express it in terms of the number of irreducible components of the singular locus $\{\mathbf{z} \in \mathbb{C}^2 \,/\, f_1(\mathbf{z}) = 0\}$ (in this case, we have a symplectic structure outside the singular locus), the algebra of regular functions on this curve being the quotient algebra $\mathbb{C}[z_1,\, z_2] \,/\, \langle f \rangle$.

M. Van den Bergh and A. Pichereau, in [18, 13] and [14], are interested in the case where $n = 3$ and $m = 1$, and where f_1 is a weighted homogeneous polynomial with an isolated singularity at the origin. They compute the Poisson homology and cohomology, which in particular may be expressed in terms of the Milnor number of the space $\mathbb{C}[z_1,\, z_2,\, z_3] \,/\, \langle \partial_{z_1} f_1,\, \partial_{z_2} f_1,\, \partial_{z_3} f_1 \rangle$ (the definition of this number is given in [3]).

Once more in the case where $n = 3$ and $m = 1$, in [2], J. Alev and T. Lambre compare the Poisson homology in degree 0 of Klein surfaces with the Hochschild homology in degree 0 of $A_1(\mathbb{C})^G$, where $A_1(\mathbb{C})$ is the Weyl algebra and G the group associated to the Klein surface. We shall give more details about those surfaces in Section 4.1.

In [1], J. Alev, M.A. Farinati, Th. Lambre and A.L. Solotar establish a fundamental result: they compute all the Hochschild homology and cohomology spaces of $A_n(\mathbb{C})^G$, where $A_n(\mathbb{C})$

is the Weyl algebra, for every finite subgroup G of $\mathbf{Sp}_{2n}\mathbb{C}$. It is an interesting and classical question to compare the Hochschild homology and cohomology of $A_n(\mathbb{C})^G$ with the Poisson homology and cohomology of the ring of invariants $\mathbb{C}[\mathbf{x},\, \mathbf{y}]^G$, which is a quotient algebra of the form $\mathbb{C}[\mathbf{z}] \,/\, \langle f_1, \dots, f_m \rangle$.

C. Fronsdal studies in [8] Hochschild homology and cohomology in two particular cases: the case where $n = 1$ and $m = 1$, and the case where $n = 2$ and $m = 1$. Besides, the appendix of this article gives another way to calculate the Hochschild cohomology in the more general case of complete intersections.

In this paper, we propose to calculate the Hochschild homology and cohomology in two particularly important cases.

- The case of singular curves of the plane, with polynomials f_1 which are weighted homogeneous polynomials with a singularity of modality zero: these polynomials correspond to the normal forms of weighted homogeneous functions of two variables and of modality zero, given in the classification of weighted homogeneous functions of [3] (this case already held C. Fronsdal's attention).
- The case of Klein surfaces $\mathcal{X}_\Gamma$ which are the quotients $\mathbb{C}^2 \,/\, \Gamma$, where Γ is a finite subgroup of $\mathbf{SL}_2\mathbb{C}$ (this case corresponds to $n = 3$ and $m = 1$). The latter have been the subject of many works; their link with the finite subgroups of $\mathbf{SL}_2\mathbb{C}$, with the Platonic polyhedra, and with McKay correspondence explains this large interest. Moreover, the preprojective algebras, to which [6] is devoted, constitute a family of deformations of the Klein surfaces, parametrized by the group which is associated to them: this fact justifies once again the calculation of their cohomology.

The main result of the article is given by two propositions:

Proposition 1. *Given a singular curve of the plane, defined by a polynomial $f \in \mathbb{C}[\mathbf{z}]$, of type A_k, D_k or E_k. For $j \in \mathbb{N}$, let HH^j (resp. HH_j) be the Hochschild cohomology (resp. homology) space in degree j of $A := \mathbb{C}[\mathbf{z}] \,/\, \langle f \rangle$, and let ∇f be the gradient of f. Then $HH^0 \simeq HH_0 \simeq A$, $HH^1 \simeq A \oplus \mathbb{C}^k$ and $HH_1 \simeq A^2 \,/\, (A\nabla f)$, and for all $j \geq 2$, $HH^j \simeq HH_j \simeq \mathbb{C}^k$.*

Proposition 2. *Let Γ be a finite subgroup of $\mathbf{SL}_2\mathbb{C}$ and $f \in \mathbb{C}[\mathbf{z}]$ such that $\mathbb{C}[x,\, y]^\Gamma \simeq \mathbb{C}[\mathbf{z}] \,/\, \langle f \rangle$. For $j \in \mathbb{N}$, let HH^j (resp. HH_j) be the Hochschild cohomology (resp. homology) space in degree j of $A := \mathbb{C}[\mathbf{z}] \,/\, \langle f \rangle$, and let ∇f be the gradient of f. Then $HH^0 \simeq HH_0 \simeq A$, $HH^1 \simeq (\nabla f \wedge A^3) \oplus \mathbb{C}^\mu$ and $HH_1 \simeq \nabla f \wedge A^3$, $HH^2 \simeq A \oplus \mathbb{C}^\mu$ and $HH_2 \simeq A^3 \,/\, (\nabla f \wedge A^3)$, and for all $j \geq 3$, $HH^j \simeq HH_j \simeq \mathbb{C}^\mu$, where μ is the Milnor number of $\mathcal{X}_\Gamma$.*

For explicit computations, we shall make use of, and develop a method suggested by M. Kontsevich in the appendix of [8].

We will first study the case of singular curves of the plane in Section 3: we will use this method to recover the result that C. Fronsdal proved by direct calculations. Then we will refine it by determining the dimensions of the cohomology and homology spaces by means of multivariate division and Groebner bases.

Next, in Section 4, we will consider the case of Klein surfaces $\mathcal{X}_\Gamma$. For $j \in \mathbb{N}$, we denote by HH^j the Hochschild cohomology space in degree j of $\mathcal{X}_\Gamma$. We will first prove that HH^0 identifies with the space of polynomial functions on the singular surface $\mathcal{X}_\Gamma$. We will then prove that HH^1 and HH^2 are infinite-dimensional. We will also determine, for j greater or equal to 3, the dimension of HH^j, by showing that it is equal to the Milnor number of the surface $\mathcal{X}_\Gamma$. Finally, we will compute the Hochschild homology spaces.

In Section 1.3 we begin by recalling important classical results about deformations.

1.3 Hochschild homology and cohomology and deformations of algebras

Consider an associative $\mathbb{C}$-algebra, denoted by A. The Hochschild cohomological complex of A is

$$C^0(A) \xrightarrow{d^{(0)}} C^1(A) \xrightarrow{d^{(1)}} C^2(A) \xrightarrow{d^{(2)}} C^3(A) \xrightarrow{d^{(3)}} C^4(A) \xrightarrow{d^{(4)}} \cdots,$$

where the space $C^p(A)$ of p-cochains is defined by $C^p(A) = 0$ for $p \in -\mathbb{N}^*$, $C^0(A) = A$, and for $p \in \mathbb{N}^*$, $C^p(A)$ is the space of $\mathbb{C}$-linear maps from $A^{\otimes p}$ to A. The differential $d = \bigoplus_{i=0}^{\infty} d^{(p)}$ is given by

$$\begin{aligned}\forall\, f \in C^p(A), \quad d^{(p)} f(a_0, \ldots, a_p) &= a_0 f(a_1, \ldots, a_p) \\ &- \sum_{i=0}^{p-1} (-1)^i f(a_0, \ldots, a_i a_{i+1}, \ldots, a_p) + (-1)^p f(a_0, \ldots, a_{p-1}) a_p.\end{aligned}$$

We may write it in terms of the Gerstenhaber bracket[1] $[\cdot, \cdot]_G$ and of the product μ of A, as follows

$$d^{(p)} f = (-1)^{p+1} [\mu, f]_G.$$

Then we define the Hochschild cohomology of A as the cohomology of the Hochschild cohomological complex associated to A, i.e. $HH^0(A) := \operatorname{Ker} d^{(0)}$ and for $p \in \mathbb{N}^*$, $HH^p(A) := \operatorname{Ker} d^{(p)} / \operatorname{Im} d^{(p-1)}$.

We denote by $\mathbb{C}[[\hbar]]$ (resp. $A[[\hbar]]$) the algebra of formal power series in the parameter $\hbar$, with coefficients in $\mathbb{C}$ (resp. A). A deformation of the map μ is a map m from $A[[\hbar]] \times A[[\hbar]]$ to $A[[\hbar]]$ which is $\mathbb{C}[[\hbar]]$-bilinear and such that

$$\begin{aligned}&\forall\, (s, t) \in A[[\hbar]]^2, \qquad m(s, t) = st \mod \hbar A[[\hbar]], \\ &\forall\, (s, t, u) \in A[[\hbar]]^3, \qquad m(s, m(t, u)) = m(m(s, t), u).\end{aligned}$$

This means that there exists a sequence of bilinear maps m_j from $A \times A$ to A of which the first term m_0 is the product of A and such that

$$\begin{aligned}&\forall\, (a, b) \in A^2, \qquad m(a, b) = \sum_{j=0}^{\infty} m_j(a, b) \hbar^j, \\ &\forall\, n \in \mathbb{N}, \qquad \sum_{i+j=n} m_i(a, m_j(b, c)) = \sum_{i+j=n} m_i(m_j(a, b), c), \\ &\text{that is to say} \quad \sum_{i+j=n} [m_i, m_j]_G = 0.\end{aligned}$$

We say that $(A[[\hbar]], m)$ is a deformation of the algebra (A, μ). We say that the deformation is of order p if the previous formulae are satisfied (only) for $n \leq p$.

The Hochschild cohomology plays an important role in the study of deformations of the algebra A, by helping us to classify them. In fact, if $\pi \in C^2(A)$, we may construct a first order deformation m of A such that $m_1 = \pi$ if and only if $\pi \in \operatorname{Ker} d^{(2)}$. Moreover, two first order

[1] Recall that for $F \in C^p(A)$ and $H \in C^q(A)$, the Gerstenhaber product is the element $F \bullet H \in C^{p+q-1}(A)$ defined by $F \bullet H(a_1, \ldots, a_{p+q-1}) = \sum_{i=0}^{p-1} (-1)^{i(q+1)} F(a_1, \ldots, a_i, H(a_{i+1}, \ldots, a_{i+q}), a_{i+q+1}, \ldots, a_{p+q-1})$, and the Gerstenhaber bracket is $[F, H]_G := F \bullet H - (-1)^{(p-1)(q-1)} H \bullet F$. See for example [9], and [4, page 38].

deformations are equivalent[2] if and only if their difference is an element of $\operatorname{Im} d^{(1)}$. So the set of equivalence classes of first order deformations is in bijection with $HH^2(A)$.

If $m = \sum_{j=0}^{p} m_j \hbar^j$, $m_j \in C^2(A)$ is a deformation of order p, then we may extend m to a deformation of order $p+1$ if and only if there exists m_{p+1} such that

$$\forall\, (a,b,c) \in A^3, \qquad \sum_{i=1}^{p} \left(m_i(a, m_{p+1-i}(b,c)) - m_i(m_{p+1-i}(a,b),c)\right) = -d^{(2)} m_{p+1}(a,b,c),$$

$$\text{i.e.} \quad \sum_{i=1}^{p} [m_i,\, m_{p+1-i}]_G = 2\, d^{(2)} m_{p+1}.$$

According to the graded Jacobi identity for $[\cdot,\cdot]_G$, the last sum belongs to $\operatorname{Ker} d^{(3)}$. So $HH^3(A)$ contains the obstructions to extend a deformation of order p to a deformation of order $p+1$.

The Hochschild homological complex of A is

$$\cdots \xrightarrow{d_5} C_4(A) \xrightarrow{d_4} C_3(A) \xrightarrow{d_3} C_2(A) \xrightarrow{d_2} C_1(A) \xrightarrow{d_1} C_0(A),$$

where the space of p-chains is given by $C_p(A) = 0$ for $p \in -\mathbb{N}^*$, $C_0(A) = A$, and for $p \in \mathbb{N}^*$, $C_p(A) = A \otimes A^{\otimes p}$. The differential $d = \bigoplus_{i=0}^{\infty} d_p$ is given by

$$\begin{aligned} d_p\,(a_0 \otimes a_1 \otimes \cdots \otimes a_p) &= a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_p \\ &+ \sum_{i=1}^{p-1} (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_p + (-1)^p a_p a_0 \otimes a_1 \otimes \cdots \otimes a_{p-1}. \end{aligned}$$

We define the Hochschild homology of A as the homology of the Hochschild homological complex associated to A, i.e. $HH_0(A) := A \,/\, \operatorname{Im}\, d_1$ and for $p \in \mathbb{N}^*$, $HH_p(A) := \operatorname{Ker} d_p \,/\, \operatorname{Im} d_{p+1}$.

2 Presentation of the Koszul complex

We recall in this section some results about the Koszul complex used below (see the appendix of [8]).

2.1 Kontsevich theorem and notations

As in Section 1.2, we consider $R = \mathbb{C}[\mathbf{z}]$ and $(f_1, \ldots, f_m) \in R^m$, and we denote by A the quotient $R \,/\, \langle f_1, \ldots, f_m \rangle$. We assume that we have a *complete intersection*, i.e. the dimension of the set of solutions of the system $\{f_1(\mathbf{z}) = \cdots = f_m(\mathbf{z}) = 0\}$ is $n - m$.

We consider the differential graded algebra

$$\widetilde{T} = A[\eta_1, \ldots, \eta_n; b_1, \ldots,\, b_m] = \frac{\mathbb{C}[z_1, \ldots, z_n]}{\langle f_1, \ldots, f_m \rangle}[\eta_1, \ldots, \eta_n; b_1, \ldots, b_m],$$

[2]Two deformations $m = \sum_{j=0}^{p} m_j\, \hbar^j$, $m_j \in C^2(A)$ and $m' = \sum_{j=0}^{p} m'_j\, \hbar^j$, $m'_j \in C^2(A)$ are called equivalent if there exists a sequence of linear maps φ_j from A to A of which the first term φ_0 is the identity of A and such that

$$\forall\, a \in A, \qquad \varphi(a) = \sum_{j=0}^{\infty} \varphi_j(a) \hbar^j,$$

$$\forall\, n \in \mathbb{N}, \qquad \sum_{i+j=n} \varphi_i(m_j(a,b)) = \sum_{i+j+k=n} m'_i(\varphi_j(a), \varphi_k(b)).$$

where $\eta_i := \frac{\partial}{\partial z_i}$ is an odd variable (i.e. the η_i's anticommute), and b_j an even variable (i.e. the b_j's commute).

$\widetilde{T}$ is endowed with the differential

$$d_{\widetilde{T}} = \sum_{j=1}^{n}\sum_{i=1}^{m} \frac{\partial f_i}{\partial z_j} b_i \frac{\partial}{\partial \eta_j},$$

and the Hodge grading, defined by $\deg(z_i) = 0$, $\deg(\eta_i) = 1$, $\deg(b_j) = 2$.

We may now state the main theorem which will allow us to calculate the Hochschild cohomology:

Theorem 1 (Kontsevich). *Under the previous assumptions, the Hochschild cohomology of A is isomorphic to the cohomology of the complex $(\widetilde{T}, d_{\widetilde{T}})$ associated with the differential graded algebra $\widetilde{T}$.*

Remark 1. Theorem 1 may be seen as a generalization of the Hochschild–Kostant–Rosenberg theorem to the case of non-smooth spaces.

There is no element of negative degree. So the complex is as follows

$$\widetilde{T}(0) \xrightarrow{\widetilde{0}} \widetilde{T}(1) \xrightarrow{d_{\widetilde{T}}^{(1)}} \widetilde{T}(2) \xrightarrow{d_{\widetilde{T}}^{(2)}} \widetilde{T}(3) \xrightarrow{d_{\widetilde{T}}^{(3)}} \widetilde{T}(4) \xrightarrow{d_{\widetilde{T}}^{(4)}} \cdots .$$

For each degree p, we choose a basis $\mathcal{B}_p$ of $\widetilde{T}(p)$. For example for $p = 0, \ldots, 3$, we may take

$$\begin{aligned}
\widetilde{T}(0) &= A,\\
\widetilde{T}(1) &= A\eta_1 \oplus \cdots \oplus A\eta_n,\\
\widetilde{T}(2) &= Ab_1 \oplus \cdots \oplus Ab_m \oplus \bigoplus_{i<j} A\,\eta_i\eta_j,\\
\widetilde{T}(3) &= \bigoplus_{\substack{i=1,\ldots,m\\ j=1,\ldots,n}} A\,b_i\eta_j \oplus \bigoplus_{i<j<k} A\,\eta_i\eta_j\eta_k.
\end{aligned}$$

Below we shall make use of the explicit matrices $\mathrm{Mat}_{\mathcal{B}_p,\mathcal{B}_{p+1}}(d_{\widetilde{T}}^{(p)})$.

Set $H^0 := A$, $H^1 := \operatorname{Ker} d_{\widetilde{T}}^{(1)}$ and for $j \geq 2$, $H^p := \operatorname{Ker} d_{\widetilde{T}}^{(p)} / \operatorname{Im} d_{\widetilde{T}}^{(p-1)}$. According to Theorem 1, we have, for $p \in \mathbb{N}$, $HH^p(A) \simeq H^p$.

There is an analogous of Theorem 1 for the Hochschild homology. We consider the complex

$$\widetilde{\Omega} = A[\xi_1, \ldots, \xi_n; a_1, \ldots, a_m],$$

where ξ_i is an odd variable and a_j an even variable. $\widetilde{\Omega}$ is endowed with the differential

$$d_{\widetilde{\Omega}} = \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{\partial f_j}{\partial z_i} \xi_i \frac{\partial}{\partial a_j},$$

and the Hodge grading, defined by $\deg(z_i) = 0$, $\deg(\xi_i) = -1$, $\deg(a_j) = -2$.

Theorem 2 (Kontsevich). *Under the previous assumptions, the Hochschild homology of A is isomorphic to the cohomology of the complex $(\widetilde{\Omega}, d_{\widetilde{\Omega}})$*

$$\cdots \xrightarrow{d_{\widetilde{\Omega}}^{(-5)}} \widetilde{\Omega}(-4) \xrightarrow{d_{\widetilde{\Omega}}^{(-4)}} \widetilde{\Omega}(-3) \xrightarrow{d_{\widetilde{\Omega}}^{(-3)}} \widetilde{\Omega}(-2) \xrightarrow{d_{\widetilde{\Omega}}^{(-2)}} \widetilde{\Omega}(-1) \xrightarrow{d_{\widetilde{\Omega}}^{(-1)}} \widetilde{\Omega}(0) \ .$$

For each degree p, we will choose a basis $\mathcal{V}_p$ of $\widetilde{\Omega}(p)$ and we will make use of the explicit matrices $\mathrm{Mat}_{\mathcal{V}_p,\mathcal{V}_{p+1}}(d_{\widetilde{\Omega}}^{(p)})$. Set $L^0 := A\,/\operatorname{Im} d_{\widetilde{\Omega}}^{(-1)}$, and for $p \geq 1$, $L^{-p} := \operatorname{Ker} d_{\widetilde{\Omega}}^{(-p)}/\operatorname{Im} d_{\widetilde{\Omega}}^{(-p-1)}$. According to Theorem 2, we have, for $p \in \mathbb{N}$, $HH_p(A) \simeq L^{-p}$.

For each ideal J of $\mathbb{C}[\mathbf{z}]$, we denote by J_A the image of J by the canonical projection

$$\mathbb{C}[\mathbf{z}] \to A = \mathbb{C}[\mathbf{z}]/\langle f_1, \dots, f_m\rangle.$$

Similarly if $(g_1, \dots, g_r) \in A^r$ we denote by $\langle g_1, \dots, g_r\rangle_A$ the ideal of A generated by $(g_1, \dots, g_r)$. Besides, if $g \in \mathbb{C}[\mathbf{z}]$, and if J is an ideal of $\mathbb{C}[\mathbf{z}]$, we set

$$\operatorname{Ann}_J(g) := \{h \in \mathbb{C}[\mathbf{z}] \,/\, hg = 0 \mod J\}.$$

In particular, g does not divide 0 in $\mathbb{C}[\mathbf{z}]\,/\,J$ if and only if $\operatorname{Ann}_J(g) = J$. Finally, we denote by ∇g the gradient of a polynomial $g \in \mathbb{C}[\mathbf{z}]$.

From now on, we consider the case $m = 1$ and set $f := f_1$. Moreover, we use the notation ∂_j for the partial derivative with respect to z_j.

2.2 Particular case where $n = 1$ and $m = 1$

In the case where $n = 1$ and $m = 1$, according to what we have seen, we have for $p \in \mathbb{N}$,

$$\widetilde{T}(2p) = Ab_1^p, \qquad \widetilde{T}(2p+1) = Ab_1^p\eta_1, \qquad \widetilde{\Omega}(-2p) = Aa_1^p, \qquad \widetilde{\Omega}(-2p-1) = Aa_1^p\xi_1.$$

We deduce

$$H^0 = L^0 = A, \qquad H^1 = \{g\eta_1 \,/\, g \in A \text{and } g\partial_1 f = 0\},$$

and for $p \in \mathbb{N}^*$,

$$H^{2p} = \frac{Ab_1^p}{\{g(\partial_1 f)b_1^p \,/\, g \in A\}}, \qquad \text{and} \qquad H^{2p+1} = \{gb_1^p\eta_1 \,/\, g \in A \text{ and } g\partial_1 f = 0\}.$$

Similarly, for $p \in \mathbb{N}^*$,

$$L^{-2p} = \{ga_1^p \,/\, g \in A \text{ and } g\partial_1 f = 0\},$$

and for $p \in \mathbb{N}$,

$$L^{-2p-1} = \frac{Aa_1^p\xi_1}{\{g(\partial_1 f)a_1^p\xi_1 \,/\, g \in A\}}.$$

Now if $f = z_1^k$, then

$$H^0 = L^0 = A = \mathbb{C}[z_1]\,/\,\langle z_1^k\rangle \simeq \mathbb{C}^k,$$

$$H^1 = \{g\eta_1 \,/\, g \in A \text{ and } kgz_1^{k-1} = 0\} \simeq \mathbb{C}^{k-1}, \qquad L^{-1} = \frac{A\xi_1}{\{g(kz_1^{k-1})\xi_1 \,/\, g \in A\}} \simeq \mathbb{C}^{k-1},$$

and for $p \in \mathbb{N}^*$,

$$H^{2p} \simeq L^{-2p-1} \simeq \frac{Ab_1^p}{\{g(kz_1^{k-1})b_1^p \,/\, g \in A\}} \simeq \mathbb{C}^{k-1},$$

and for $p \in \mathbb{N}^*$,

$$H^{2p+1} \simeq L^{-2p} \simeq \{gb_1^p\eta_1 \,/\, g \in A \text{ and } kgz_1^{k-1} = 0\} \simeq \mathbb{C}^{k-1}.$$

See [12] for a similar calculation.

3 Case $n = 2$, $m = 1$. Singular curves of the plane

3.1 Singular curves of the plane

In this section, we recall a result about the weighted homogeneous functions, given in [3, page 181].

Theorem 3 (Classification of weighted homogeneous functions, [3]). *The weighted homogeneous functions of two variables and of modality zero reduce, up to equivalence, to the following list of normal forms*

Type	A_k	D_k	E_6	E_7	E_8
Normal form	$z_1^{k+1} + z_2^2$	$z_1^2 z_2 + z_2^{k-1}$	$z_1^3 + z_2^4$	$z_1^3 + z_1 z_2^3$	$z_1^3 + z_2^5$

The singularities of types A_k, D_k, E_6, E_7, E_8 are called simple singularities. In the two following sections, we will study the Hochschild cohomology of $\mathbb{C}[\mathbf{z}] \,/\, \langle f \rangle$, where f is one of the normal forms of the preceding table.

3.2 Description of the cohomology spaces

With the help of Theorem 1 we calculate the Hochschild cohomology of $A := \mathbb{C}[z_1, z_2] \,/\, \langle f \rangle$, where $f \in \mathbb{C}[z_1, z_2]$. We begin by making cochains and differentials explicit, by using the notations of Section 2.1.

The various spaces of the complex are given by

$$
\begin{aligned}
\widetilde{T}(0) &= A, & \widetilde{T}(5) &= Ab_1^2\eta_1 \oplus Ab_1^2\eta_2,\\
\widetilde{T}(1) &= A\eta_1 \oplus A\eta_2, & \widetilde{T}(6) &= Ab_1^3 \oplus Ab_1^2\eta_1\eta_2,\\
\widetilde{T}(2) &= Ab_1 \oplus A\eta_1\eta_2, & \widetilde{T}(7) &= Ab_1^3\eta_1 \oplus Ab_1^3\eta_2,\\
\widetilde{T}(3) &= Ab_1\eta_1 \oplus Ab_1\eta_2, & \widetilde{T}(8) &= Ab_1^4 \oplus Ab_1^3\eta_1\eta_2,\\
\widetilde{T}(4) &= Ab_1^2 \oplus Ab_1\eta_1\eta_2, & \widetilde{T}(9) &= Ab_1^4\eta_1 \oplus Ab_1^4\eta_2,
\end{aligned}
$$

i.e., for an arbitrary $p \in \mathbb{N}^*$,

$$\widetilde{T}(2p) = Ab_1^p \oplus Ab_1^{p-1}\eta_1\eta_2,$$

and for an arbitrary $p \in \mathbb{N}$,

$$\widetilde{T}(2p+1) = Ab_1^p\eta_1 \oplus Ab_1^p\eta_2.$$

As in [8], we denote by $\frac{\partial}{\partial \eta_k}$ the partial derivative with respect to the variable η_k, for $k \in \{1, 2\}$. So, for $\{k, l\} = \{1, 2\}$, we have

$$\frac{\partial}{\partial \eta_k}(\eta_k \wedge \eta_l) = 1 \wedge \eta_l = -\eta_l \wedge 1,$$

hence

$$d_{\widetilde{T}}^{(2)}(\eta_k\eta_l) = -\frac{\partial f}{\partial z_k} b_1\eta_l + \frac{\partial f}{\partial z_l} b_1\eta_k.$$

The matrices of $d_{\widetilde{T}}$ are therefore given by

$$\mathrm{Mat}_{\mathcal{B}_{2p},\mathcal{B}_{2p+1}}(d_{\widetilde{T}}^{(2p)}) = \begin{pmatrix} 0 & \partial_2 f \\ 0 & -\partial_1 f \end{pmatrix},$$

$$\mathrm{Mat}_{\mathcal{B}_{2p+1},\mathcal{B}_{2p+2}}\big(d_{\widetilde{T}}^{(2p+1)}\big) = \begin{pmatrix} \partial_1 f & \partial_2 f \\ 0 & 0 \end{pmatrix}.$$

We deduce a simpler expression for the cohomology spaces

$$\begin{aligned} H^0 &= A, \\ H^1 &= \{g_1\eta_1 + g_2\eta_2 \,/\, (g_1, g_2) \in A^2 \text{ and } g_1\partial_1 f + g_2\partial_2 f = 0\} \\ &\simeq \left\{\mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in A^2 \,/\, \mathbf{g}\cdot\nabla f = 0\right\}. \end{aligned}$$

For $p \in \mathbb{N}^*$,

$$\begin{aligned} H^{2p} &= \frac{\{g_1 b_1^p + g_2 b_1^{p-1}\eta_1\eta_2 \,/\, (g_1, g_2) \in A^2 \text{ and } g_2\partial_1 f = g_2\partial_2 f = 0\}}{\{(g_1\partial_1 f + g_2\partial_2 f)b_1^p \,/\, (g_1, g_2) \in A^2\}} \\ &\simeq \frac{\left\{\mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in A^2 \,/\, g_2\partial_1 f = g_2\partial_2 f = 0\right\}}{\left\{\begin{pmatrix} \mathbf{g}\cdot\nabla f \\ 0 \end{pmatrix} \,/\, \mathbf{g} \in A^2\right\}} \\ &\simeq \frac{A}{\langle \partial_1 f, \partial_2 f\rangle_A} \oplus \{g \in A \,/\, g\,\partial_1 f = g\partial_2 f = 0\}, \\ H^{2p+1} &= \frac{\{g_1 b_1^p\eta_1 + g_2 b_1^p\eta_2 \,/\, (g_1, g_2) \in A^2 \text{ and } g_1\partial_1 f + g_2\partial_2 f = 0\}}{\{g_2(\partial_2 f b_1^p\eta_1 - \partial_1 f b_1^p\eta_2) \,/\, g_2 \in A\}} \\ &\simeq \frac{\left\{\mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in A^2 \,/\, \mathbf{g}\cdot\nabla f = 0\right\}}{\left\{g_2\begin{pmatrix} \partial_2 f \\ -\partial_1 f \end{pmatrix} \,/\, g_2 \in A\right\}}. \end{aligned}$$

Remark 2. We recover a result of [8] (here, we use the notations of [8]). According to Theorem 3.8 of [8], we have $\mathrm{Hoch}_{2p} = \mathrm{Hoch}_{2p,p} \oplus \mathrm{Hoch}_{2p,p+1}$ and $\mathrm{Hoch}_{2p+1} = \mathrm{Hoch}_{2p+1,p+1}$, so $\mathrm{Hoch}_{2p,k} = 0$ if $k \notin \{p, p+1\}$, and $Hoch_{2p+1,k} = 0$ if $k \neq p+1$. By using Section 4.1 of [8], we deduce $H^{2p,k} = 0$ if $k \notin \{p, p+1\}$, and $H^{2p+1,k} = 0$ if $k \neq p+1$. Hence $H^{2p} = H^{2p,p} \oplus H^{2p,p+1}$ and $H^{2p+1} = H^{2p+1,p+1}$. So Theorem 4.9 of [8] gives the cohomology spaces which we have just obtained.

It remains to determine these spaces more explicitly. This will be done in the two following sections.

3.3 Explicit calculations in the particular case where f has separate variables

In this section, we consider the polynomial $f = a_1 z_1^k + a_2 z_2^l$, with $k \geq 2$, $l \geq 2$, and $(a_1, a_2) \in (\mathbb{C}^*)^2$. The partial derivatives of f are $\partial_1 f = k a_1 z_1^{k-1}$ and $\partial_2 f = l a_2 z_2^{l-1}$.

We already have

$$H^0 = \mathbb{C}[z_1, z_2] \,/\, \langle a_1 z_1^k + a_2 z_2^l\rangle.$$

Besides, as f is weighted homogeneous, Euler's formula gives $\frac{1}{k} z_1\partial_1 f + \frac{1}{l} z_2\partial_2 f = f$. So we have the inclusion $\langle f\rangle \subset \langle \partial_1 f, \partial_2 f\rangle$, hence

$$\frac{A}{\langle \partial_1 f, \partial_2 f\rangle_A} \simeq \frac{\mathbb{C}[z_1, z_2]}{\langle \partial_1 f, \partial_2 f\rangle} \simeq \mathrm{Vect}\big(z_1^i z_2^j \,/\, i \in [\![0, k-2]\!],\ j \in [\![0, l-2]\!]\big).$$

But $\partial_1 f$ and f are relatively prime, just as $\partial_2 f$ and f are, hence if $g \in A$ satisfies $g\partial_1 f = 0 \mod \langle f\rangle$, then $g \in \langle f\rangle$, i.e. g is zero in A. So,

$$H^{2p} \simeq \mathrm{Vect}\big(z_1^i z_2^j \,/\, i \in [\![0, k-2]\!],\ j \in [\![0, l-2]\!]\big) \simeq \mathbb{C}^{(k-1)(l-1)}.$$

We now determine the set

$$\left\{\mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in A^2 \,/\, \mathbf{g}\cdot\nabla f = 0\right\}.$$

First we have

$$\langle f, \partial_1 f\rangle = \langle a_1 z_1^k + a_2 z_2^l, z_1^{k-1}\rangle = \langle z_2^l, z_1^{k-1}\rangle.$$

So the only monomials which are not in this ideal are the elements $z_1^i z_2^j$ with $i \in [\![0, k-2]\!]$ and $j \in [\![0, l-1]\!]$. Every polynomial $P \in \mathbb{C}[\mathbf{z}]$ may be written in the form

$$P = \alpha f + \beta\partial_1 f + \sum_{\substack{i=0,\dots,k-2\\ j=0,\dots,l-1}} a_{ij} z_1^i z_2^j,$$

with $\alpha, \beta \in \mathbb{C}[\mathbf{z}]$ and $a_{ij} \in \mathbb{C}$. Therefore, the polynomials $P \in \mathbb{C}[\mathbf{z}]$ such that $P\partial_2 f \in \langle f, \partial_1 f\rangle$ are the elements

$$P = \alpha f + \beta\partial_1 f + \sum_{\substack{i=0,\dots,k-2\\ j=1,\dots,l-1}} a_{ij} z_1^i z_2^j.$$

So we have calculated $\mathrm{Ann}_{\langle f,\partial_1 f\rangle}(\partial_2 f)$. Let $\mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in A^2$ satisfy the equation

$$\mathbf{g}\cdot\nabla f = 0 \mod \langle f\rangle. \tag{2}$$

Then we have

$$g_2\partial_2 f = 0 \mod \langle f, \partial_1 f\rangle,$$

i.e. $g_2 \in \mathrm{Ann}_{\langle f,\partial_1 f\rangle}(\partial_2 f)$, i.e. again

$$g_2 = \alpha f + \beta\partial_1 f + \sum_{\substack{i=0,\dots,k-2\\ j=1,\dots,l-1}} a_{ij} z_1^i z_2^j, \qquad \text{with } (\alpha,\, \beta) \in \mathbb{C}[\mathbf{z}]^2.$$

It follows that

$$g_1\partial_1 f + \alpha f\partial_2 f + \beta\partial_1 f\partial_2 f + \sum_{\substack{i=0,\dots,k-2\\ j=1,\dots,l-1}} a_{ij} z_1^i z_2^j \partial_2 f \in \langle f\rangle.$$

From the equality $z_2\partial_2 f = lf - \frac{l}{k} z_1\partial_1 f$, one deduces

$$\partial_1 f \left(g_1 + \beta\partial_2 f - \frac{l}{k} \sum_{\substack{i=0,\dots,k-2\\ j=1,\dots,l-1}} a_{ij} z_1^{i+1} z_2^{j-1} \right) \in \langle f\rangle,$$

i.e.

$$g_1 = -\beta\partial_2 f + \frac{l}{k}\sum_{\substack{i=0,\dots,k-2\\ j=1,\dots,l-1}} a_{ij} z_1^{i+1} z_2^{j-1} + \delta f, \qquad \text{with} \quad \delta \in \mathbb{C}[\mathbf{z}].$$

Then we verify that the elements g_1 and g_2 obtained in this way are indeed solutions of equation (2).

Finally, we have

$$\begin{aligned}&\left\{\mathbf{g} \in A^2 \,/\, \mathbf{g}\cdot\nabla f = 0\right\}\\ &\quad = \left\{ -\beta\begin{pmatrix} \partial_2 f \\ -\partial_1 f\end{pmatrix} + \sum_{\substack{i=0,\dots,k-2\\ j=1,\dots,l-1}} a_{ij} z_1^i z_2^{j-1}\begin{pmatrix} \frac{l}{k} z_1 \\ z_2\end{pmatrix} \Big/ \beta \in A \text{ and } a_{ij} \in \mathbb{C}\right\}.\end{aligned}$$

We immediately deduce the cohomology spaces of odd degree:

$$\begin{aligned}\forall\, p \geq 1, \quad & H^{2p+1} \simeq \mathbb{C}^{(k-1)(l-1)},\\ & H^1 \simeq \mathbb{C}^{(k-1)(l-1)} \oplus \mathbb{C}[z_1,\, z_2]/\langle a_1 z_1^k + a_2 z_2^l\rangle,\end{aligned}$$

where the direct sum results from the following argument: if we have

$$-\beta\begin{pmatrix} \partial_2 f \\ -\partial_1 f\end{pmatrix} = \sum_{\substack{i=0,\dots,k-2\\ j=1,\dots,l-1}} a_{ij} z_1^i z_2^{j-1}\begin{pmatrix} \frac{l}{k} z_1 \\ z_2\end{pmatrix} \quad \operatorname{mod} \langle f\rangle,$$

then

$$v := -\beta l a_2 z_2^{l-1} - \sum_{\substack{i=0,\dots,k-2\\ j=1,\dots,l-1}} \frac{l}{k} a_{ij} z_1^{i+1} z_2^{j-1} \in \langle f\rangle.$$

And by a Euclidian division in $(\mathbb{C}[z_2])\,[z_1]$, we may write $\beta = fq + r$, where the z_1-degree of r is smaller or equal to $k-1$. So the z_1-degree of v is also smaller or equal to $k-1$, thus $v \in \langle f\rangle$ implies $\beta = 0$ and $a_{ij} = 0$.

Remark 3. We obtain in particular the cohomology for the cases where $f = z_1^{k+1} + z_2^2$ ($k \in \mathbb{N}^*$), $f = z_1^3 + z_2^4$ and $f = z_1^3 + z_2^5$. These cases correspond respectively to the weighted homogeneous functions of types A_k, E_6 and E_8 given in Theorem 3.

The table below summarizes the results we have just obtained for the three particular cases

	H^0	H^1	$H^p,\ p \geq 2$
A_k	$\mathbb{C}[\mathbf{z}]\,/\,\langle z_1^{k+1} + z_2^2\rangle$	$\mathbb{C}[\mathbf{z}]\,/\,\langle z_1^{k+1} + z_2^2\rangle \oplus \mathbb{C}^k$	$\mathbb{C}^k$
E_6	$\mathbb{C}[\mathbf{z}]\,/\,\langle z_1^3 + z_2^4\rangle$	$\mathbb{C}[\mathbf{z}]\,/\,\langle z_1^3 + z_2^4\rangle \oplus \mathbb{C}^6$	$\mathbb{C}^6$
E_8	$\mathbb{C}[\mathbf{z}]\,/\,\langle z_1^3 + z_2^5\rangle$	$\mathbb{C}[\mathbf{z}]\,/\,\langle z_1^3 + z_2^5\rangle \oplus \mathbb{C}^8$	$\mathbb{C}^8$

The cases where $f = z_1^2 z_2 + z_2^{k-1}$ and $f = z_1^3 + z_1 z_2^3$, i.e. respectively D_k and E_7, will be studied in the next section.

3.4 Explicit calculations for D_k and E_7

To study these particular cases, we use the following result about Groebner bases (Theorem 4). First, recall the definition of a Groebner basis. For $g \in \mathbb{C}[\mathbf{z}]$, we denote by $\mathrm{lt}(g)$ its leading term (for the lexicographic order). Given a non-trivial ideal J of $\mathbb{C}[\mathbf{z}]$, a *Groebner basis* of J is a finite subset G_J of $J \setminus \{0\}$ such that for all $f \in J \setminus \{0\}$, there exists $g \in G_J$ such that $\mathrm{lt}(g)$ divides $\mathrm{lt}(f)$. See [15] for more details.

Definition 1. Let J be a non-trivial ideal of $\mathbb{C}[\mathbf{z}]$ and let $G_J := [g_1, \dots, g_r]$ be a Groebner basis of J. We call *set of the G_J-standard terms*, the set of all monomials of $\mathbb{C}[\mathbf{z}]$ that are not divisible by any of $\mathrm{lt}(g_1), \dots, \mathrm{lt}(g_r)$.

Theorem 4 (Macaulay). *The set of the G_J-standard terms forms a basis of the quotient vector space $\mathbb{C}[\mathbf{z}] \,/\, J$.*

3.4.1 Case of $f = z_1^2 z_2 + z_2^{k-1}$, i.e. D_k

Here we have

$$f = z_1^2 z_2 + z_2^{k-1}, \qquad \partial_1 f = 2 z_1 z_2 \qquad \text{and} \qquad \partial_2 f = z_1^2 + (k-1) z_2^{k-2}.$$

A Groebner basis of the ideal $\langle f, \partial_2 f \rangle$ is

$$B := [b_1, b_2] = [z_1^2 + (k-1) z_2^{k-2}, z_2^{k-1}].$$

So the set of the standard terms is

$$\{z_1^i z_2^j \,/\, i \in \{0,1\} \text{ and } j \in [\![0,\ k-2]\!]\}.$$

We may now solve the equation $p \partial_1 f = 0$ in $\mathbb{C}[\mathbf{z}] \,/\, \langle f, \partial_2 f \rangle$. In fact, by writing

$$p := \sum_{\substack{i=0,1 \\ j=0,\dots,k-2}} a_{ij} z_1^i z_2^j,$$

the equation becomes

$$q := \sum_{\substack{i=0,1 \\ j=0,\dots,k-2}} a_{ij} z_1^{i+1} z_2^{j+1} \in \langle f, \partial_2 f \rangle.$$

We look for the normal form of the element q modulo the ideal $\langle f, \partial_2 f \rangle$.

The multivariate division of q by B is $q = q_1 b_1 + q_2 b_2 + r$ with $r = \sum_{j=0}^{k-3} a_{0,j} z_1 z_2^{j+1}$. Thus the solution in $\mathbb{C}[\mathbf{z}] \,/\, \langle f,\, \partial_2 f \rangle$ is

$$p = a_{0,k-2} z_2^{k-2} + \sum_{j=0}^{k-2} a_{1,j} z_1 z_2^j.$$

But the equation

$$\mathbf{g} \cdot \nabla f = 0 \mod \langle f \rangle$$

yields

$$g_1 \partial_1 f = 0 \mod \langle f, \partial_2 f \rangle,$$

i.e.

$$g_1 = \alpha f + \beta \partial_2 f + a z_2^{k-2} + \sum_{j=0}^{k-2} b_j z_1 z_2^j, \qquad \text{with } (\alpha, \beta) \in \mathbb{C}[\mathbf{z}]^2 \text{ and } a, b_j \in \mathbb{C}.$$

Hence

$$g_2 \partial_2 f + \beta \partial_1 f \, \partial_2 f + a z_2^{k-2} \partial_1 f + \sum_{j=0}^{k-2} b_j z_1 z_2^j \partial_1 f \in \langle f \rangle.$$

And with the equalities,

$$z_2^{k-1} = \frac{1}{2-k}(f - z_2 \partial_2 f) = -\frac{1}{2-k} z_2 \partial_2 f \mod \langle f \rangle,$$

and

$$\frac{k-2}{2} z_1 \partial_1 f + z_2 \partial_2 f = (k-1) f \quad \text{(Euler)},$$

we obtain

$$\partial_2 f \left(g_2 + \beta \partial_1 f - \frac{2a}{2-k} z_1 z_2 + \sum_{j=0}^{k-2} b_j \frac{2}{2-k} z_2^{j+1} \right) \in \langle f \rangle.$$

i.e.,

$$g_2 = -\beta \partial_1 f + \frac{2a}{2-k} z_1 z_2 - \sum_{j=0}^{k-2} b_j \frac{2}{2-k} z_2^{j+1} + \delta f, \qquad \text{with } \delta \in \mathbb{C}[\mathbf{z}].$$

So

$$\begin{aligned} &\{\mathbf{g} \in A^2 \,/\, \mathbf{g} \cdot \nabla f = 0\} \\ &= \left\{ \beta \begin{pmatrix} \partial_2 f \\ -\partial_1 f \end{pmatrix} + a \begin{pmatrix} z_2^{k-2} \\ \frac{2}{2-k} z_1 z_2 \end{pmatrix} + \sum_{j=0}^{k-2} b_j z_2^j \begin{pmatrix} z_1 \\ -\frac{2}{2-k} z_2 \end{pmatrix} \Big/ \beta \in A,\ a,\, b_j \in \mathbb{C} \right\}. \end{aligned}$$

On the other hand, a Groebner basis of $\langle \partial_1 f, \partial_2 f \rangle$ is $[z_1^2 + (k-1) z_2^{k-2}, z_1 z_2, z_2^{k-1}]$, thus

$\mathbb{C}[\mathbf{z}] \,/\, \langle \partial_1 f, \partial_2 f \rangle \simeq \mathrm{Vect}(z_1, 1, z_2, \ldots, z_2^{k-2})$.

Let us summarize (by using, for the direct sum, the same argument as in Section 3.2):

$H^0 = \mathbb{C}[\mathbf{z}] \,/\, \langle z_1^2 z_2 + z_2^{k-1} \rangle$,
$H^1 \simeq \mathbb{C}[\mathbf{z}] \,/\, \langle z_1^2 z_2 + z_2^{k-1} \rangle \oplus \mathbb{C}^k$,
$H^{2p} \simeq \mathbb{C}^k$,
$H^{2p+1} \simeq \mathbb{C}^k$.

3.4.2 Case of $f = z_1^3 + z_1 z_2^3$, i.e. E_7

Here we have $\partial_1 f = 3z_1^2 + z_2^3$ and $\partial_2 f = 3 z_1 z_2^2$. A Groebner basis of the ideal $\langle f, \partial_1 f \rangle$ is $[3z_1^2 + z_2^3, z_1 z_2^3, z_2^6]$, and a Groebner basis of $\langle \partial_1 f, \partial_2 f \rangle$ is $[3z_1^2 + z_2^3, z_1 z_2^2, z_2^5]$. By an analogous proof, we obtain

$H^0 = \mathbb{C}[\mathbf{z}] \,/\, \langle z_1^3 + z_1 z_2^3 \rangle$,
$H^1 \simeq \mathbb{C}[\mathbf{z}] \,/\, \langle z_1^3 + z_1 z_2^3 \rangle \oplus \mathbb{C}^7$,
$H^{2p} \simeq \mathbb{C}^7$,
$H^{2p+1} \simeq \mathbb{C}^7$.

3.5 Homology

The study is the same as the one of the Hochschild cohomology: to get the Hochschild homology is equivalent to compute the cohomology of the complex $(\widetilde{\Omega},\ d_{\widetilde{\Omega}})$ described in Section 2.1. We have $\widetilde{\Omega}(0) = A$, $\widetilde{\Omega}(-2p) = Aa_1^p \oplus Aa_1^{p-1}\xi_1\xi_2$ for $p \in \mathbb{N}^*$, and $\widetilde{\Omega}(-2p-1) = Aa_1^p\xi_1 \oplus Aa_1^p\xi_2$ for $p \in \mathbb{N}$. This defines the bases $\mathcal{V}_p$. The differential is $d_{\widetilde{\Omega}} = (\xi_1\partial_1 f + \xi_2\partial_2 f)\frac{\partial}{\partial a_1}$.

So we obtain, for $p \in \mathbb{N}^*$, the matrices

$$\mathrm{Mat}_{\mathcal{V}_{-2p},\mathcal{V}_{-2p+1}}\left(d_{\widetilde{\Omega}}^{(-2p)}\right) = \begin{pmatrix} p\partial_1 f & 0 \\ p\partial_2 f & 0 \end{pmatrix},$$

and

$$\mathrm{Mat}_{\mathcal{V}_{-2p-1},\mathcal{V}_{-2p}}\left(d_{\widetilde{\Omega}}^{(-2p-1)}\right) = \begin{pmatrix} 0 & 0 \\ -p\partial_2 f & p\partial_1 f \end{pmatrix}.$$

The cohomology spaces read as

$$L^0 = A, \qquad L^{-1} = \frac{A^2}{\{g\nabla f \,/\, g \in A\}}.$$

For $p \in \mathbb{N}^*$,

$$\begin{aligned} L^{-2p} &\simeq \frac{\left\{\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in A^2 \Big/ pg_1\partial_1 f = pg_1\partial_2 f = 0\right\}}{\left\{\begin{pmatrix} 0 \\ -(p+1)g_1\partial_2 f + (p+1)g_2\partial_1 f \end{pmatrix} \Big/ \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in A^2\right\}} \\ &\simeq \{g \in A \,/\, g\partial_1 f = g\partial_2 f = 0\} \oplus \frac{A}{\langle \nabla f\rangle_A}. \end{aligned}$$

For $p \in \mathbb{N}^*$,

$$L^{-2p-1} \simeq \frac{\left\{\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in A^2 \Big/ -pg_1\partial_2 f + pg_2\partial_1 f = 0\right\}}{\left\{\begin{pmatrix} (p+1)g_1\partial_1 f \\ (p+1)g_1\partial_2 f \end{pmatrix} \Big/ \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in A^2\right\}} \simeq \frac{\{\mathbf{g} \in A^2 \,/\, \det(\nabla f,\, \mathbf{g}) = 0\}}{\{g\nabla f \ /\ g \in A\}}.$$

From now on, we assume that f has separate variables, or f is of type D_k or E_7. Then we have $\{g \in A \,/\, g\partial_1 f = g\partial_2 f = 0\} = \{0\}$, and according to Euler's formula, for $p \in \mathbb{N}$, $L^{-2p} \simeq \frac{A}{\langle \nabla f\rangle_A} \simeq \frac{\mathbb{C}[\mathbf{z}]}{\langle \nabla f\rangle}$. For the computation of $\{\mathbf{g} \in A^2 \,/\, \det(\nabla f, \mathbf{g}) = 0\}$ and $\frac{A^2}{\{g\nabla f \,/\, g \in A\}}$, we proceed with Groebner bases as in Section 3.3. For example, we do it for $f = z_1^2 z_2 + z_2^{k-1}$ (i.e. type D_k).

Let $\mathbf{g} \in A^2$ be such that $\det(\nabla f, \mathbf{g}) = 0$. Then $g_2\partial_1 f = 0 \mod \langle f, \partial_2 f\rangle$, i.e., according to Section 3.4.1,

$$g_2 = \alpha f + \beta\partial_2 f + a z_2^{k-2} + \sum_{j=0}^{k-2} b_j z_1 z_2^j,$$

with $(\alpha, \beta) \in \mathbb{C}[\mathbf{z}]^2$ and $a, b_j \in \mathbb{C}$. Hence

$$-g_1\partial_2 f + \alpha f\partial_1 f + \beta\partial_2 f\partial_1 f + a z_2^{k-2}\partial_1 f + \sum_{j=0}^{k-2} b_j z_1 z_2^j \partial_1 f \in \langle f\rangle.$$

With the equalities,

$$z_2^{k-1} = \frac{1}{2-k}(f - z_2\partial_2 f) = -\frac{1}{2-k} z_2\partial_2 f \mod \langle f \rangle,$$

and

$$\frac{k-2}{2} z_1\partial_1 f + z_2\partial_2 f = (k-1)f \quad \text{(Euler)},$$

we obtain

$$\partial_2 f \left(-g_1 + \beta\partial_1 f - \frac{2a}{2-k} z_1 z_2 + \sum_{j=0}^{k-2} b_j \frac{2}{2-k} z_2^{j+1} \right) \in \langle f \rangle.$$

i.e.,

$$g_1 = \beta\partial_1 f - \frac{2a}{2-k} z_1 z_2 + \sum_{j=0}^{k-2} b_j \frac{2}{2-k} z_2^{j+1} + \delta f, \qquad \text{with } \delta \in \mathbb{C}[\mathbf{z}].$$

So

$$\begin{aligned} &\left\{ \mathbf{g} \in A^2 \,/\, \det(\nabla f, \mathbf{g}) = 0 \right\} \\ &\quad = \left\{ \beta\nabla f + a \begin{pmatrix} -\frac{2}{2-k} z_1 z_2 \\ z_2^{k-2} \end{pmatrix} + \sum_{j=0}^{k-2} b_j z_2^j \begin{pmatrix} \frac{2}{2-k} z_2 \\ z_1 \end{pmatrix} \Big/ \beta \in A,\ a, b_j \in \mathbb{C} \right\}. \end{aligned}$$

We have $\{g\nabla f \,/\, g \in A\} \subset \{\mathbf{g} \in A^2 \,/\, \det(\nabla f, \mathbf{g}) = 0\}$, thus

$$\dim \left(A^2 \,/\, \{g\nabla f \,/\, g \in A\} \right) \geq \dim \left(A^2 \,/\, \{\mathbf{g} \in A^2 \,/\, \det(\nabla f, \mathbf{g}) = 0\} \right).$$

Since $A^2 \,/\, \{\mathbf{g} \in A^2 \,/\, \det(\nabla f, \mathbf{g}) = 0\} \simeq \{\det(\nabla f, \mathbf{g}) \,/\, \mathbf{g} \in A^2\}$, and since the map

$$g \in A \mapsto \det\left(\nabla f, \begin{pmatrix} g \\ 0 \end{pmatrix} \right) \in \{\det(\nabla f, \mathbf{g}) \,/\, \mathbf{g} \in A^2\}$$

is injective, we deduce that $A^2 \,/\, \{g\nabla f \,/\, g \in A\}$ is infinite-dimensional.

We collect in the following table the results for the Hochschild homology in the various cases

Type	$HH_0 = A$	HH_1	$HH_p,\ p \geq 2$
A_k	$\mathbb{C}[\mathbf{z}] \,/\, \langle z_1^{k+1} + z_2^2 \rangle$	$A^2 \,/\, A\nabla f$	$\mathbb{C}^k$
D_k	$\mathbb{C}[\mathbf{z}] \,/\, \langle z_1^2 z_2 + z_2^{k-1} \rangle$	$A^2 \,/\, A\nabla f$	$\mathbb{C}^k$
E_6	$\mathbb{C}[\mathbf{z}] \,/\, \langle z_1^3 + z_2^4 \rangle$	$A^2 \,/\, A\nabla f$	$\mathbb{C}^6$
E_7	$\mathbb{C}[\mathbf{z}] \,/\, \langle z_1^3 + z_1 z_2^3 \rangle$	$A^2 \,/\, A\nabla f$	$\mathbb{C}^7$
E_8	$\mathbb{C}[\mathbf{z}] \,/\, \langle z_1^3 + z_2^5 \rangle$	$A^2 \,/\, A\nabla f$	$\mathbb{C}^8$

4 Case $n = 3$, $m = 1$. Klein surfaces

4.1 Klein surfaces

Given a finite group G acting on $\mathbb{C}^n$, we associate to it, according to Erlangen program of Klein, the quotient space $\mathbb{C}^n/G$, i.e. the space whose points are the orbits under the action of G; it is an algebraic variety, and the polynomial functions on this variety are the polynomial functions on $\mathbb{C}^n$ which are G-invariant. In the case of $\mathbf{SL}_2\mathbb{C}$, invariant theory allows us to associate a polynomial to any finite subgroup, as explained in Proposition 4. Thus, to every finite subgroup of $\mathbf{SL}_2\mathbb{C}$ is associated the zero set of this polynomial; it is an algebraic variety, called a Klein surface.

In this section we recall some results about these surfaces. See the references [17] and [7] for more details.

Proposition 3. *Every finite subgroup of* $\mathbf{SL}_2\mathbb{C}$ *is conjugate to one of the following groups:*

- A_k *(cyclic),* $k \geq 1$, $|A_k| = k$;
- D_k *(dihedral),* $k \geq 1$, $|D_k| = 4k$;
- E_6 *(tetrahedral),* $|E_6| = 24$;
- E_7 *(octahedral),* $|E_7| = 48$;
- E_8 *(icosahedral),* $|E_8| = 120$.

Proposition 4. *Let G be one of the groups of the preceding list. The ring of invariants is the following*

$$\mathbb{C}[x,y]^G = \mathbb{C}[e_1,e_2,e_3] = \mathbb{C}[e_1,e_2] \oplus e_3\mathbb{C}[e_1,e_2] \simeq \mathbb{C}[z_1,z_2,z_3] \,/\, \langle f \rangle,$$

where the invariants e_j are homogeneous polynomials, with e_1 and e_2 algebraically independent, and where f is a weighted homogeneous polynomial with an isolated singularity at the origin. These polynomials are given in the following table.

We call Klein surface the algebraic hyper-surface defined by $\{\mathbf{z} \in \mathbb{C}^3 \,/\, f(\mathbf{z}) = 0\}$.

G	e_1, e_2, e_3	f	$\mathbb{C}[z_1,z_2,z_3]\,/\,\langle \partial_1 f, \partial_2 f, \partial_3 f\rangle$
A_k	$e_1 = x^k$ $e_2 = y^k$ $e_3 = xy$	$-k(z_1z_2 - z_3^k)$	$\mathrm{Vect}(1, z_3, \ldots, z_3^{k-2})$ $\dim = k-1$
D_k	$e_1 = x^{2k+1}y + (-1)^{k+1}xy^{2k+1}$ $e_2 = x^{2k} + (-1)^k y^{2k}$ $e_3 = x^2y^2$	$\lambda_k((-1)^k z_1^2 + (-1)^{k+1} z_2^2 z_3 + 4z_3^{k+1})$ with $\lambda_k = 2k(-1)^{k+1}$	$\mathrm{Vect}(1, z_2, z_3, \ldots, z_3^{k-1})$ $\dim = k+1$
E_6	$e_1 = 33y^8x^4 - y^{12} + 33y^4x^8 - x^{12}$ $e_2 = 14y^4x^4 + x^8 + y^8$ $e_3 = x^5y - xy^5$	$4(z_1^2 - z_2^3 + 108z_3^4)$	$\mathrm{Vect}(1, z_2, z_3, z_2z_3, z_2z_3^2, z_3^2)$ $\dim = 6$
E_7	$e_1 = -34x^5y^{13} - yx^{17} + 34y^5x^{13} + xy^{17}$ $e_2 = -3y^{10}x^2 + 6y^6x^6 - 3y^2x^{10}$ $e_3 = 14y^4x^4 + x^8 + y^8$	$8(3z_1^2 - 12z_2^3 + z_2z_3^3)$	$\mathrm{Vect}(1, z_2, z_2^2, z_3, z_2z_3, z_2^2z_3, z_3^2)$ $\dim = 7$
E_8	$e_1 = x^{30} + 522x^{25}y^5 - 10\,005x^{20}y^{10}$ $-10\,005x^{10}y^{20} - 522x^5y^{25} + y^{30}$ $e_2 = x^{20} - 228x^{15}y^5 + 494x^{10}y^{10}$ $+ 228x^5y^{15} + y^{20}$ $e_3 = x^{11}y + 11x^6y^6 - xy^{11}$	$10(-z_1^2 + z_2^3 + 1\,728z_3^5)$	$\mathrm{Vect}(z_2^i z_3^j)_{\substack{i=0,1,\\ j=0,\ldots,3}}$ $\dim = 8$

Before carrying on with our study, we make a digression in order to draw a parallel between the Poisson and the Hochschild cohomologies of Klein surfaces, by recalling the result of A. Pichereau.

Theorem 5 (Pichereau). *Consider the Poisson bracket defined on $\mathbb{C}[z_1, z_1, z_3]$ by*

$$\{\cdot,\cdot\}_f = \partial_3 f \partial_1 \wedge \partial_2 + \partial_1 f \partial_2 \wedge \partial_3 + \partial_2 f \partial_3 \wedge \partial_1 = i(df)(\partial_1 \wedge \partial_2 \wedge \partial_3),$$

where i is the contraction of a multiderivation by a differential form. Denote by HP_f^* (resp. HP_*^f) the Poisson cohomology (resp. homology) for this bracket. Under the previous assumptions, the Poisson cohomology HP_f^* and the Poisson homology HP_*^f of $(\mathbb{C}[z_1, z_1, z_3] \,/\, \langle f\rangle, \{\cdot,\cdot\}_f)$ are given by

$$
\begin{aligned}
&HP_f^0 = \mathbb{C}, \qquad HP_f^1 \simeq HP_f^2 = \{0\},\\
&HP_0^f \simeq HP_2^f \simeq \mathbb{C}[z_1, z_2, z_3] \,/\, \langle \partial_1 f, \partial_2 f, \partial_3 f\rangle,\\
&\dim(HP_1^f) = \dim(HP_0^f) - 1,\\
&HP_j^f = HP_f^j = \{0\} \quad \text{if} \;\; j \geq 3.
\end{aligned}
$$

The algebra $\mathbb{C}[x, y]$ is a Poisson algebra for the standard symplectic bracket $\{\cdot,\cdot\}_{\text{std}}$. As G is a subgroup of the symplectic group $\mathbf{Sp}_2\mathbb{C}$ (since $\mathbf{Sp}_2\mathbb{C} = \mathbf{SL}_2\mathbb{C}$), the invariant algebra $\mathbb{C}[x, y]^G$ is a Poisson subalgebra of $\mathbb{C}[x, y]$. The following proposition allows us to deduce, from Theorem 5, the Poisson cohomology and homology of $\mathbb{C}[x, y]^G$ for the standard symplectic bracket.

Proposition 5. *With the choice made in the preceding table for the polynomial f, the isomorphism of associative algebras*

$$\pi : \;(\mathbb{C}[x,\, y]^G, \{\cdot,\cdot\}_{\text{std}}) \to (\mathbb{C}[z_1, z_1, z_3]/\langle f\rangle, \{\cdot,\cdot\}_f), \qquad e_j \mapsto \overline{z_j}$$

is a Poisson isomorphism.

In the sequel, we will calculate the Hochschild cohomology of $\mathbb{C}[z_1, z_1, z_3]/\langle f\rangle$, and we will immediately deduce the Hochschild cohomology of $\mathbb{C}[x, y]^G$, with the help of the isomorphism π. Note that the fact that π preserves the Poisson structures has no incidence on the computation of the Hochschild cohomology. Therefore, so as to simplify the calculations, we may replace the polynomial f by a simpler one, given in the following table

G	A_k	D_k	E_6	E_7	E_8
f	$z_1^2 + z_2^2 + z_3^k$	$z_1^2 + z_2^2 z_3 + z_3^k$	$z_1^2 + z_2^3 + z_3^4$	$z_1^2 + z_2^3 + z_2 z_3^3$	$z_1^2 + z_2^3 + z_3^5$

Indeed, the linear maps defined by

$$
\begin{aligned}
&\mathbb{C}[\mathbf{z}] \to \mathbb{C}[\mathbf{z}],\\
&(z_1, z_2, z_3) \mapsto (\alpha_1 z_1, \alpha_2 z_2, \alpha_3 z_3),\\
&(z_1, z_2, z_3) \mapsto (\alpha_1(z_1 + z_2), \alpha_2(z_1 + z_2), \alpha_3 z_3)
\end{aligned}
$$

are isomorphisms of associative algebras.

4.2 Description of the cohomology spaces

We consider now the case $A := \mathbb{C}[z_1, z_2, z_3], / \langle f\rangle$ and we want to calculate the Hochschild cohomology of A. We use the notations of Section 2.1, but we change the ordering of the basis: we shall take $(\eta_1\eta_2, \eta_2\eta_3, \eta_3\eta_1)$ instead of $(\eta_1\eta_2, \eta_1\eta_3, \eta_2\eta_3)$. The different spaces of the complex are now given by

$$
\begin{aligned}
&\widetilde{T}(0) = A,\\
&\widetilde{T}(1) = A\eta_1 \oplus A\eta_2 \oplus A\eta_3,\\
&\widetilde{T}(2) = Ab_1 \oplus A\eta_1\eta_2 \oplus A\eta_2\eta_3 \oplus A\eta_3\eta_1,\\
&\widetilde{T}(3) = Ab_1\eta_1 \oplus Ab_1\eta_2 \oplus Ab_1\eta_3 \oplus A\eta_1\eta_2\eta_3,
\end{aligned}
$$

$$\widetilde{T}(4) = Ab_1^2 \oplus Ab_1\eta_1\eta_2 \oplus Ab_1\eta_2\eta_3 \oplus Ab_1\eta_3\eta_1,$$
$$\widetilde{T}(5) = Ab_1^2\eta_1 \oplus Ab_1^2\eta_2 \oplus Ab_1^2\eta_3 \oplus Ab_1\eta_1\eta_2\eta_3,$$

i.e., for an arbitrary $p \in \mathbb{N}^*$,

$$\widetilde{T}(2p) = Ab_1^p \oplus Ab_1^{p-1}\eta_1\eta_2 \oplus Ab_1^{p-1}\eta_2\eta_3 \oplus Ab_1^{p-1}\eta_3\eta_1,$$

and

$$\widetilde{T}(2p+1) = Ab_1^p\eta_1 \oplus Ab_1^p\eta_2 \oplus Ab_1^p\eta_3 \oplus Ab_1^{p-1}\eta_1\eta_2\eta_3.$$

We have

$$\frac{\partial}{\partial \eta_1}(\eta_1 \wedge \eta_2 \wedge \eta_3) = 1 \wedge \eta_2 \wedge \eta_3 = \eta_2 \wedge \eta_3 \wedge 1,$$

thus

$$d_{\widetilde{T}}^{(3)}(\eta_1\eta_2\eta_3) = \frac{\partial f}{\partial z_1} b_1\eta_2\eta_3 + \frac{\partial f}{\partial z_2} b_1\eta_3\eta_1 + \frac{\partial f}{\partial z_3} b_1\eta_1\eta_2.$$

The matrices of $d_{\widetilde{T}}$ are therefore given by

$$\mathrm{Mat}_{\mathcal{B}_1,\mathcal{B}_2}(d_{\widetilde{T}}^{(1)}) = \begin{pmatrix} \partial_1 f & \partial_2 f & \partial_3 f \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\forall\, p \in \mathbb{N}^*, \quad \mathrm{Mat}_{\mathcal{B}_{2p},\mathcal{B}_{2p+1}}(d_{\widetilde{T}}^{(2p)}) = \begin{pmatrix} 0 & \partial_2 f & 0 & -\partial_3 f \\ 0 & -\partial_1 f & \partial_3 f & 0 \\ 0 & 0 & -\partial_2 f & \partial_1 f \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\forall\, p \in \mathbb{N}^*, \quad \mathrm{Mat}_{\mathcal{B}_{2p+1},\mathcal{B}_{2p+2}}(d_{\widetilde{T}}^{(2p+1)}) = \begin{pmatrix} \partial_1 f & \partial_2 f & \partial_3 f & 0 \\ 0 & 0 & 0 & \partial_3 f \\ 0 & 0 & 0 & \partial_1 f \\ 0 & 0 & 0 & \partial_2 f \end{pmatrix}.$$

We deduce

$$H^0 = A,$$
$$H^1 = \{g_1\eta_1 + g_2\eta_2 + g_3\eta_3 \,/\, (g_1, g_2, g_3) \in A^3 \text{ and } g_1\partial_1 f + g_2\partial_2 f + g_3\partial_3 f = 0\}$$
$$\simeq \left\{ \mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \in A^3 \,/\, \mathbf{g} \cdot \nabla f = 0 \right\},$$

$$H^2 = \frac{\left\{ g_0 b_1 + g_3\eta_1\eta_2 + g_1\eta_2\eta_3 + g_2\eta_3\eta_1 \Big/ \begin{array}{l} (g_0, g_1, g_2, g_3) \in A^4 \text{ and} \\ g_3\partial_2 f - g_2\partial_3 f = g_1\partial_3 f - g_3\partial_1 f \\ = g_2\partial_1 f - g_1\partial_2 f = 0 \end{array} \right\}}{\{(g_1\partial_1 f + g_2\partial_2 f + g_3\partial_3 f) b_1 \,, / \,(g_1, g_2, g_3) \in A^3\}}$$
$$\simeq \left\{ \mathbf{g} = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix} \in A^4 \Big/ \nabla f \wedge \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = 0 \right\} \Big/ \left\{ \begin{pmatrix} \mathbf{g} \cdot \nabla f \\ \mathbf{0}_{3,1} \end{pmatrix} / \, \mathbf{g} \in A^3 \right\}$$

$$\simeq \frac{A}{\langle \partial_1 f, \partial_2 f, \partial_3 f\rangle_A} \oplus \{\mathbf{g} \in A^3 \,/\, \nabla f \wedge \mathbf{g} = 0\}.$$

For $p \geq 2$,

$$H^{2p} = \frac{\left\{ g_0 b_1^p + g_3 b_1^{p-1}\eta_1\eta_2 + g_1 b_1^{p-1}\eta_2\eta_3 + g_2 b_1^{p-1}\eta_3\eta_1 \Big/ \begin{array}{l} (g_0, g_1, g_2, g_3) \in A^4 \text{ and} \\ g_3\partial_2 f - g_2 \partial_3 f \\ = g_1 \partial_3 f - g_3 \partial_1 f \\ = g_2 \partial_1 f - g_1 \partial_2 f = 0 \end{array} \right\}}{\left\{ \begin{array}{l} (g_1\partial_1 f + g_2 \partial_2 f + g_3 \partial_3 f) b_1^p \\ + g_0(\partial_3 f b_1^{p-1}\eta_1\eta_2 + \partial_1 f b_1^{p-1}\eta_2\eta_3 + \partial_2 f b_1^{p-1}\eta_3\eta_1) \end{array} \Big/ (g_0, g_1, g_2, g_3) \in A^3 \right\}}$$

$$\simeq \left\{ \mathbf{g} = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix} \in A^4 \Big/ \nabla f \wedge \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = 0 \right\} \Big/ \left\{ \begin{pmatrix} \mathbf{g} \cdot \nabla f \\ g_0\, \partial_1 f \\ g_0\, \partial_2 f \\ g_0\, \partial_3 f \end{pmatrix} / \, \mathbf{g} \in A^3 \text{ and } g_0 \in A \right\}$$

$$\simeq \frac{A}{\langle \partial_1 f, \partial_2 f, \partial_3 f\rangle_A} \oplus \frac{\{\mathbf{g} \in A^3 \,/\, \nabla f \wedge \mathbf{g} = 0\}}{\{g \nabla f \,/\, g \in A\}}.$$

For $p \in \mathbb{N}^*$,

$$H^{2p+1} = \frac{\left\{ g_1 b_1^p \eta_1 + g_2 b_1^p \eta_2 + g_3 b_1^p \eta_3 + g_0 b_1^{p-1}\eta_1\eta_2\eta_3 \Big/ \begin{array}{l} (g_0, g_1, g_2 g_3) \in A^4 \text{ and} \\ g_1 \partial_1 f + g_2 \partial_2 f + g_3 \partial_3 f = 0, \\ g_0 \partial_3 f = g_0 \partial_1 f = g_0 \partial_2 f = 0 \end{array} \right\}}{\left\{ \begin{array}{l} (g_3\partial_2 f - g_2 \partial_3 f) b_1^p \eta_1 + (g_1 \partial_3 f - g_3 \partial_1 f) b_1^p \eta_2 \\ + (g_2 \partial_1 f - g_1 \partial_2 f) b_1^p \eta_3 \end{array} \Big/ (g_1, g_2, g_3) \in A^3 \right\}}$$

$$\simeq \left\{ \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_0 \end{pmatrix} \in A^4 \Big/ \begin{array}{l} \nabla f \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = 0 \\ g_0\, \partial_3 f = g_0\, \partial_1 f = g_0\, \partial_2 f = 0 \end{array} \right\} \Big/ \left\{ \begin{pmatrix} \nabla f \wedge \mathbf{g} \\ 0 \end{pmatrix} \Big/ \mathbf{g} \in A^3 \right\}$$

$$\simeq \frac{\{\mathbf{g} \in A^3 \,/\, \nabla f \cdot \mathbf{g} = 0\}}{\{\nabla f \wedge \mathbf{g} \,/\, \mathbf{g} \in A^3\}} \oplus \{g \in A \,/\, g\partial_3 f = g \partial_1 f = g \partial_2 f = 0\}.$$

The following section will allow us to make those various spaces more explicit.

4.3 Explicit calculations in the particular case where f has separate variables

In this section, we consider the polynomial $f = a_1 z_1^i + a_2 z_2^j + a_3 z_3^k$, with $2 \leq i \leq j \leq k$ and $a_j \in \mathbb{C}^*$. Its partial derivatives are $\partial_1 f = i a_1 z_1^{i-1}$, $\partial_2 f = j a_2 z_2^{j-1}$ and $\partial_3 f = k a_3 z_3^{k-1}$.

We already have

$$H^0 = \mathbb{C}[z_1, z_2, z_3] \,/\, \langle a_1 z_1^i + a_2 z_2^j + a_3 z_3^k \rangle.$$

Moreover, as f is weighted homogeneous, Euler's formula gives

$$\frac{1}{i} z_1 \partial_1 f + \frac{1}{j} z_2 \partial_2 f + \frac{1}{k} z_3 \partial_3 f = f.$$

So we have the inclusion $\langle f \rangle \subset \langle \partial_1 f, \partial_2 f, \partial_3 f\rangle$, thus

$$\begin{aligned} \frac{A}{\langle \partial_1 f, \partial_2 f, \partial_3 f\rangle_A} &\simeq \frac{\mathbb{C}[z_1, z_2, z_3]}{\langle \partial_1 f, \partial_2 f, \partial_3 f\rangle} \\ &\simeq \operatorname{Vect}\left(z_1^p z_2^q z_3^r \,/\, p \in [\![0, i-2]\!],\ q \in [\![0, j-2]\!],\ r \in [\![0, k-2]\!] \right). \end{aligned}$$

Finally, as $\partial_1 f$ and f are relatively prime, if $g \in A$ verifies $g\partial_1 f = 0 \mod \langle f \rangle$, then $g \in \langle f \rangle$, i.e. g is zero in A.

Now we determine the set

$$\left\{ \mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \in A^3 \,/\, \mathbf{g} \cdot \nabla f = 0 \right\}.$$

First we have

$$\langle f, \partial_1 f, \partial_2 f \rangle = \langle a_1 z_1^i + a_2 z_2^j + a_3 z_3^k, z_1^{i-1}, z_2^{j-1} \rangle = \langle z_1^{i-1}, z_2^{j-1}, z_3^k \rangle.$$

Thus the only monomials which are not in this ideal are the elements $z_1^p z_2^q z_3^r$ with $p \in [\![0, i-2]\!]$, $q \in [\![0, j-2]\!]$, and $r \in [\![0, k-1]\!]$.

So every polynomial $P \in \mathbb{C}[\mathbf{z}]$ may be written in the form

$$P = \alpha f + \beta \partial_1 f + \gamma \partial_2 f + \sum_{\substack{p=0,\dots,i-2 \\ q=0,\dots,j-2 \\ r=0,\dots,k-1}} a_{pqr} z_1^p z_2^q z_3^r.$$

The polynomials $P \in \mathbb{C}[\mathbf{z}]$ such that $P\partial_3 f \in \langle f, \partial_1 f, \partial_2 f \rangle$ are therefore the following ones

$$P = \alpha f + \beta \partial_1 f + \gamma \partial_2 f + \sum_{\substack{p=0,\dots,i-2 \\ q=0,\dots,j-2 \\ r=1,\dots,k-1}} a_{pqr} z_1^p z_2^q z_3^r.$$

So we have calculated $\mathrm{Ann}_{\langle f, \partial_1 f, \partial_2 f \rangle}(\partial_3 f)$. The equation

$$\mathbf{g} \cdot \nabla f = 0 \mod \langle f \rangle$$

leads to $g_3 \in \mathrm{Ann}_{\langle f, \partial_1 f, \partial_2 f \rangle}(\partial_3 f)$, i.e.

$$g_3 = \alpha f + \beta \partial_1 f + \gamma \partial_2 f + \sum_{\substack{p=0,\dots,i-2 \\ q=0,\dots,j-2 \\ r=1\dots k-1}} a_{pqr} z_1^p z_2^q z_3^r,$$

with $(\alpha, \beta, \gamma) \in \mathbb{C}[\mathbf{z}]^3$. Hence

$$g_2 \partial_2 f + \gamma \partial_2 f \partial_3 f + \sum_{\substack{p=0,\dots,i-2 \\ q=0,\dots,j-2 \\ r=1,\dots,k-1}} a_{pqr} z_1^p z_2^q z_3^r \partial_3 f \in \langle f, \partial_1 f \rangle.$$

Thus, according to Euler's formula,

$$\partial_2 f \left(g_2 + \gamma \partial_3 f - \frac{k}{j} \sum_{\substack{p=0,\dots,i-2 \\ q=0\dots j-2 \\ r=1,\dots,k-1}} a_{pqr} z_1^p z_2^{q+1} z_3^{r-1} \right) \in \langle f,\, \partial_1 f \rangle.$$

Since $\mathrm{Ann}_{\langle f, \partial_1 f \rangle}(\partial_2 f) = \langle f,\, \partial_1 f \rangle$, this equation is equivalent to

$$g_2 = -\gamma \partial_3 f + \frac{k}{j} \sum_{\substack{p=0,\dots,i-2 \\ q=0,\dots,j-2 \\ r=1,\dots,k-1}} a_{pqr} z_1^p z_2^{q+1} z_3^{r-1} + \delta f + \varepsilon \partial_1 f,$$

with $\delta, \varepsilon \in \mathbb{C}[\mathbf{z}]$. It follows that

$$\begin{aligned}&g_1\partial_1 f + \beta\partial_1 f\partial_3 f + \varepsilon\partial_1 f\partial_2 f\\ &\qquad + \sum_{\substack{p=0,\dots,i-2\\ q=0,\dots,j-2\\ r=1,\dots,k-1}} a_{pqr} z_1^p z_2^q z_3^r \partial_3 f + \frac{k}{j} \sum_{\substack{p=0,\dots,i-2\\ q=0,\dots,j-2\\ r=1,\dots,k-1}} a_{pqr} z_1^p z_2^{q+1} z_3^{r-1} \partial_2 f \in \langle f\rangle.\end{aligned}$$

And, according to Euler's formula,

$$\partial_1 f \left(g_1 + \beta\partial_3 f + \varepsilon\partial_2 f - \frac{k}{i} \sum_{\substack{p=0,\dots,i-2\\ q=0,\dots,j-2\\ r=1,\dots,k-1}} a_{pqr} z_1^{p+1} z_2^q z_3^{r-1} \right) \in \langle f\rangle,$$

i.e.

$$g_1 = -\beta\partial_3 f - \varepsilon\partial_2 f + \frac{k}{i} \sum_{\substack{p=0,\dots,i-2\\ q=0,\dots,j-2\\ r=1,\dots,k-1}} a_{pqr} z_1^{p+1} z_2^q z_3^{r-1} + \eta f,$$

with $\eta \in \mathbb{C}[\mathbf{z}]$. Finally

$$\begin{aligned}&\left\{\mathbf{g} \in A^3 \,/\, \mathbf{g}\cdot\nabla f = 0\right\}\\ &\quad = \left\{ \nabla f \wedge \begin{pmatrix} -\gamma \\ \beta \\ -\varepsilon \end{pmatrix} + \sum_{\substack{p=0,\dots,i-2\\ q=0,\dots,j-2\\ r=1,\dots,k-1}} a_{pqr} z_1^p z_2^q z_3^{r-1} \begin{pmatrix} \frac{k}{i} z_1 \\ \frac{k}{j} z_2 \\ z_3 \end{pmatrix} \Big/ (\beta,\gamma,\varepsilon) \in A^3 \text{ and } a_{pqr} \in \mathbb{C} \right\}.\end{aligned}$$

We deduce immediately the cohomology spaces of odd degrees

$$\begin{aligned}\forall\, p \geq 1,\quad & H^{2p+1} \simeq \mathbb{C}^{(i-1)(j-1)(k-1)},\\ & H^1 \simeq \nabla f \wedge (\mathbb{C}[\mathbf{z}] \,/\, \langle f\rangle)^3 \oplus \mathbb{C}^{(i-1)(j-1)(k-1)}.\end{aligned}$$

It remains to determine the set

$$\left\{ \mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \in A^3 \,/\, \nabla f \wedge \mathbf{g} = 0 \right\}.$$

Let $\mathbf{g} \in A^3$ be such that $\nabla f \wedge \mathbf{g} = 0$. This means that, modulo $\langle f\rangle$, $\mathbf{g}$ verifies the system

$$\partial_2 f g_3 - \partial_3 f g_2 = 0, \qquad \partial_3 f g_1 - \partial_1 f g_3 = 0, \qquad \partial_1 f g_2 - \partial_2 f g_1 = 0.$$

The first equation gives, modulo $\langle f, \partial_2 f\rangle$, $\partial_3 f g_2 = 0$. Now $\mathrm{Ann}_{\langle f,\partial_2 f\rangle}(\partial_3 f) = \langle f, \partial_2 f\rangle$, therefore $g_2 = \alpha f + \beta\partial_2 f$. Hence

$$\partial_2 f (g_3 - \beta\partial_3 f) = 0 \mod \langle f\rangle,$$

i.e. $g_3 = \gamma f + \beta\partial_3 f$. Finally, we obtain

$$\partial_3 f (g_1 - \beta\partial_1 f) = 0 \mod \langle f\rangle,$$

i.e. $g_1 = \delta f + \beta\partial_1 f$. So, $\{\mathbf{g} \in A^3 \,/\, \nabla f \wedge \mathbf{g} = 0\} = \{\beta\nabla f \,/\, \beta \in A\}$.

We deduce the cohomology spaces of even degrees (for the direct sum, we use the same argument as in Section 3.2)

$$\begin{aligned}
\forall\, p \geq 2, \quad H^{2p} &\simeq A \,/\, \langle \partial_1 f, \partial_2 f, \partial_3 f \rangle \simeq \mathbb{C}[\mathbf{z}] \,/\, \langle z_1^{i-1}, z_2^{j-1}, z_3^{k-1} \rangle \\
&\simeq \mathrm{Vect}\,(z_1^p z_2^q z_3^r \,/\, p \in [\![0, i-2]\!],\ q \in [\![0, j-2]\!], r \in [\![0, k-2]\!]) \\
&\simeq \mathbb{C}^{(i-1)(j-1)(k-1)}, \\
H^2 &\simeq \{\beta \nabla f \,/\, \beta \in A\} \oplus \mathbb{C}^{(i-1)(j-1)(k-1)} \\
&\simeq \mathbb{C}[\mathbf{z}] \,/\, \langle a_1 z_1^i + a_2 z_2^j + a_3 z_3^k \rangle \oplus \mathbb{C}^{(i-1)(j-1)(k-1)}.
\end{aligned}$$

Remark 4. We also have

$$\nabla f \wedge (\mathbb{C}[\mathbf{z}] \,/\, \langle f \rangle)^3 \simeq (\mathbb{C}[\mathbf{z}] \,/\, \langle f \rangle)^3 \,/\, \{\mathbf{g} \,/\, \nabla f \wedge \mathbf{g} = 0\} = (\mathbb{C}[\mathbf{z}] \,/\, \langle f \rangle)^3 \,/\, (\mathbb{C}[\mathbf{z}] \,/\, \langle f \rangle) \nabla f.$$

Moreover the map

$$(\mathbb{C}[\mathbf{z}] \,/\, \langle f \rangle)^2 \to \nabla f \wedge (\mathbb{C}[\mathbf{z}] \,/\, \langle f \rangle)^3, \qquad \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \mapsto \nabla f \wedge \begin{pmatrix} g_1 \\ g_2 \\ 0 \end{pmatrix}$$

is injective, thus $\nabla f \wedge (\mathbb{C}[\mathbf{z}] \,/\, \langle f \rangle)^3$ is infinite-dimensional.

Remark 5. In particular, we obtain the cohomology for the cases where $f = z_1^2 + z_2^2 + z_3^k$, $f = z_1^2 + z_2^3 + z_3^4$ and $f = z_1^2 + z_2^3 + z_3^5$. These cases correspond respectively to the types A_k, E_6 and E_8 of the Klein surfaces.

The following table sums up the results of those three special cases:

	H^0	H^1	H^2	$H^p,\ p \geq 3$
A_k	$\mathbb{C}[\mathbf{z}] \,/\, \langle z_1^2 + z_2^2 + z_3^k \rangle$	$\nabla f \wedge (\mathbb{C}[\mathbf{z}] \,/\, \langle f \rangle)^3 \oplus \mathbb{C}^{k-1}$	$\mathbb{C}[\mathbf{z}] \,/\, \langle z_1^2 + z_2^2 + z_3^k \rangle \oplus \mathbb{C}^{k-1}$	$\mathbb{C}^{k-1}$
E_6	$\mathbb{C}[\mathbf{z}] \,/\, \langle z_1^2 + z_2^3 + z_3^4 \rangle$	$\nabla f \wedge (\mathbb{C}[\mathbf{z}] \,/\, \langle f \rangle)^3 \oplus \mathbb{C}^6$	$\mathbb{C}[\mathbf{z}] \,/\, \langle z_1^2 + z_2^3 + z_3^4 \rangle \oplus \mathbb{C}^6$	$\mathbb{C}^6$
E_8	$\mathbb{C}[\mathbf{z}] \,/\, \langle z_1^2 + z_2^3 + z_3^5 \rangle$	$\nabla f \wedge (\mathbb{C}[\mathbf{z}] \,/\, \langle f \rangle)^3 \oplus \mathbb{C}^8$	$\mathbb{C}[\mathbf{z}] \,/\, \langle z_1^2 + z_2^3 + z_3^5 \rangle \oplus \mathbb{C}^8$	$\mathbb{C}^8$

The cases where $f = z_1^2 + z_2^2 z_3 + z_3^k$ and $f = z_1^2 + z_2^3 + z_2 z_3^3$, i.e. respectively D_k and E_7 are studied in the following section.

4.4 Explicit calculations for D_k and E_7

4.4.1 Case of $f = z_1^2 + z_2^2 z_3 + z_3^k$, i.e. D_k

In this section, we consider the polynomial $f = z_1^2 + z_2^2 z_3 + z_3^k$, with $k \geq 3$. Its partial derivatives are $\partial_1 f = 2z_1$, $\partial_2 f = 2z_2 z_3$ and $\partial_3 f = z_2^2 + k z_3^{k-1}$.

We already have

$$H^0 = \mathbb{C}[\mathbf{z}] \,/\, \langle z_1^2 + z_2^2 z_3 + z_3^k \rangle.$$

Besides, since f is weighted homogeneous, Euler's formula gives

$$\frac{k}{2} z_1 \partial_1 f + \frac{k-1}{2} z_2 \partial_2 f + z_3 \partial_3 f = kf. \tag{3}$$

Thus, we have the inclusion $\langle f \rangle \subset \langle \partial_1 f, \partial_2 f, \partial_3 f \rangle$. Moreover, a Groebner basis of $\langle \partial_1 f, \partial_2 f, \partial_3 f \rangle$ is $[z_3^k, z_2 z_3, z_2^2 + k z_3^{k-1}, z_1]$, therefore

$$\frac{A}{\langle \partial_1 f, \partial_2 f, \partial_3 f \rangle_A} \simeq \frac{\mathbb{C}[z_1, z_2, z_3]}{\langle \partial_1 f, \partial_2 f, \partial_3 f \rangle} \simeq \mathrm{Vect}\big(z_2, 1, z_3, \ldots, z_3^{k-1}\big).$$

Finally, as $\partial_1 f$ and f are relatively prime, if $g \in A$ verifies $g\partial_1 f = 0 \mod \langle f \rangle$, then $g \in \langle f \rangle$, i.e. g is zero in A, thus $\{g \in A \,/\, g\partial_3 f = g\partial_1 f = g\,\partial_2 f = 0\} = 0$.

Now we determine the set

$$\left\{ \mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \in A^3 \,/\, \mathbf{g} \cdot \nabla f = 0 \right\}.$$

A Groebner basis of $\langle f, \partial_1 f, \partial_3 f \rangle$ is $[z_1, z_3^k, z_2^2 + k z_3^{k-1}]$, thus a basis of $\mathbb{C}[\mathbf{z}] \,/\, \langle f, \partial_1 f, \partial_3 f \rangle$ is $\{z_2^i z_3^j \,/\, i \in \{0,1\},\ j \in [\![0, k-1]\!]\}$. We have already solved the equation $p\partial_2 f = 0$ in this space; the solutions of this equation in $\mathbb{C}[\mathbf{z}] \,/\, \langle f, \partial_1 f, \partial_3 f \rangle$ are of the form

$$p = a_{0,k-1} z_3^{k-1} + \sum_{j=0}^{k-1} a_{1,j} z_2 z_3^j,$$

where $a_{0,k-1}, a_{1,j} \in \mathbb{C}$.

Let $\mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \in A^3$ satisfy the equation

$$\mathbf{g} \cdot \nabla f = 0 \mod \langle f \rangle.$$

Then we have

$$g_2 \partial_2 f = 0 \mod \langle f, \partial_1 f, \partial_3 f \rangle,$$

hence

$$g_2 = \alpha f + \beta \partial_1 f + \gamma \partial_3 f + a z_3^{k-1} + \sum_{j=0}^{k-1} b_j z_2 z_3^j,$$

with $(\alpha, \beta, \gamma) \in \mathbb{C}[\mathbf{z}]^3$. And

$$g_3 \partial_3 f + \gamma \partial_3 f \partial_2 f + a z_3^{k-1} \partial_2 f + \sum_{j=0}^{k-1} b_j z_2 z_3^j \partial_2 f \in \langle f, \partial_1 f \rangle. \tag{4}$$

Now according to Euler's formula (3) and the equality

$$z_3^k z_2 = \frac{1}{1-k} \left(z_2 f - z_2 z_3 \partial_3 f - \frac{1}{2} z_2 z_1 \partial_1 f \right) = -\frac{1}{1-k} z_2 z_3 \partial_3 f \mod \langle f, \partial_1 f \rangle,$$

Equation (4) becomes

$$\partial_3 f \left(g_3 + \gamma \partial_2 f - \frac{2a}{1-k} z_2 z_3 - \sum_{j=0}^{k-1} b_j \frac{2}{k-1} z_3^{j+1} \right) \in \langle f, \partial_1 f \rangle.$$

As $\mathrm{Ann}_{\langle f, \partial_1 f \rangle}(\partial_3 f) = \langle f, \partial_1 f \rangle$, this equation is equivalent to

$$g_3 = -\gamma \partial_2 f + \frac{2a}{1-k} z_2 z_3 + \sum_{j=0}^{k-1} b_j \frac{2}{k-1} z_3^{j+1} + \delta f + \varepsilon \partial_1 f,$$

with $\delta, \varepsilon \in \mathbb{C}[\mathbf{z}]$. We find

$$g_1 = -\beta\partial_2 f - \varepsilon\partial_3 f + \sum_{j=0}^{k-1} b_j \frac{k}{k-1} z_1 z_3^j + \frac{a}{1-k} z_2 z_1 + \eta f,$$

with $\eta \in \mathbb{C}[\mathbf{z}]$. Finally, we have

$$\begin{aligned} &\left\{\mathbf{g} \in A^3 \,/\, \mathbf{g}\cdot\nabla f = 0\right\} \\ &= \left\{ \nabla f \wedge \begin{pmatrix} \gamma \\ \varepsilon \\ -\beta \end{pmatrix} + \sum_{j=0}^{k-1} b_j \begin{pmatrix} \frac{k}{k-1} z_1 z_3^j \\ z_2 z_3^j \\ -\frac{2}{1-k} z_3^{j+1} \end{pmatrix} + a \begin{pmatrix} \frac{1}{1-k} z_2 z_1 \\ z_3^{k-1} \\ \frac{2}{1-k} z_2 z_3 \end{pmatrix} \middle/ \begin{array}{c} (\beta, \gamma, \varepsilon) \in A^3 \\ \text{and } a, b_j \in \mathbb{C} \end{array} \right\}, \end{aligned}$$

as well as cohomology spaces of odd degrees (for the direct sum, we use the same argument as in Section 3.2)

$$\begin{aligned} \forall\, p \geq 1, \quad & H^{2p+1} \simeq \mathbb{C}^{k+1}, \\ & H^1 \simeq \nabla f \wedge (\mathbb{C}[\mathbf{z}] \,/\, \langle f \rangle)^3 \oplus \mathbb{C}^{k+1}. \end{aligned}$$

To show $\{\mathbf{g} \in A^3 \,/\, \nabla f \wedge \mathbf{g} = 0\} = \left\{ f\mathbf{g} + \beta\nabla f \,/\, \mathbf{g} \in A^3,\ \beta \in A \right\}$, we proceed as in the case of separate variables. We deduce the cohomology spaces of even degrees

$$\begin{aligned} \forall\, p \geq 2, \quad & H^{2p} \simeq A \,/\, \langle \partial_1 f, \partial_2 f, \partial_3 f \rangle \simeq \operatorname{Vect}\left(z_2, 1, z_3, \ldots, z_3^{k-1}\right) \simeq \mathbb{C}^{k+1}, \\ & H^2 \simeq \{\beta\nabla f \,/\, \beta \in A\} \oplus \mathbb{C}^{k+1} \simeq \mathbb{C}[\mathbf{z}] \,/\, \langle z_1^2 + z_2^2 z_3 + z_3^k \rangle \oplus \mathbb{C}^{k+1}. \end{aligned}$$

4.4.2 Case of $f = z_1^2 + z_2^3 + z_2 z_3^3$, i.e. E_7

Here we have $\partial_1 f = 2z_1, \partial_2 f = 3z_2^2 + z_3^3$ and $\partial_3 f = 3z_2 z_3^2$. The proof is similar to that of the previous cases. A Groebner basis of $\langle \partial_1 f, \partial_2 f, \partial_3 f \rangle$ is $[z_3^5, z_2 z_3^2, 3z_2^2 + z_3^3, z_1]$. Similarly, a Groebner basis of $\langle f, \partial_1 f, \partial_2 f \rangle$ is $[z_3^6, z_2 z_3^3, 3z_2^2 + z_3^3, z_1]$. We obtain the following results

$$\begin{aligned} \forall\, p \geq 1, \quad & H^{2p+1} \simeq \mathbb{C}^7, \\ & H^1 \simeq \nabla f \wedge (\mathbb{C}[\mathbf{z}] \,/\, \langle f \rangle)^3 \oplus \mathbb{C}^7, \\ \forall\, p \geq 2, \quad & H^0 = \mathbb{C}[\mathbf{z}] \,/\, \langle z_1^2 + z_2^3 + z_2 z_3^3 \rangle, \\ & H^{2p} \simeq A \,/\, \langle \partial_1 f, \partial_2 f, \partial_3 f \rangle \simeq \operatorname{Vect}\left(z_2, z_2^2, 1, z_3, z_3^2, z_3^3, z_3^4\right) \simeq \mathbb{C}^7, \\ & H^2 \simeq \{\beta\nabla f \,/\, \beta \in A\} \oplus \mathbb{C}^7 \simeq \mathbb{C}[\mathbf{z}] \,/\, \langle z_1^2 + z_2^3 + z_2 z_3^3 \rangle \oplus \mathbb{C}^7. \end{aligned}$$

Remark 6. In all the previously studied cases, there exists a triple (i, j, k) such that $\{i, j, k\} = \{1, 2, 3\}$, and such that the map

$$\begin{aligned} & \mathbb{C}[\mathbf{z}] \,/\, \langle \partial_1 f, \partial_2 f, \partial_3 f \rangle \to \{g \in \mathbb{C}[\mathbf{z}] \,/\, \langle f, \partial_j f, \partial_k f \rangle \,/\, g\partial_i f = 0\}, \\ & P \mod \langle \partial_1 f, \partial_2 f, \partial_3 f \rangle \mapsto z_i P \mod \langle f, \partial_j f, \partial_k f \rangle \end{aligned}$$

is an isomorphism of vector spaces.

4.5 Homology

The study is the same as the one of the Hochschild cohomology, and we proceed as in Section 3.5. Here, we have

$$\widetilde{\Omega}(0) = A, \qquad \widetilde{\Omega}(-1) = A\xi_1 \oplus A\xi_2 \oplus A\xi_3,$$

$$\forall\, p \in \mathbb{N}^*, \quad \widetilde{\Omega}(-2p) = Aa_1^p \oplus Aa_1^{p-1}\xi_1\xi_2 \oplus Aa_1^{p-1}\xi_2\xi_3 \oplus Aa_1^{p-1}\xi_3\xi_1,$$
$$\forall\, p \in \mathbb{N}^*, \quad \widetilde{\Omega}(-2p-1) = Aa_1^p\xi_1 \oplus Aa_1^p\xi_2 \oplus Aa_1^p\xi_3 \oplus Aa_1^{p-1}\xi_1\xi_2\xi_3.$$

This defines the bases $\mathcal{V}_p$. The differential is $d_{\widetilde{\Omega}} = (\xi_1\partial_1 f + \xi_2\partial_2 f + \xi_3\partial_3 f)\frac{\partial}{\partial a_1}$.

By setting $Df := (\partial_3 f\ \partial_1 f\ \partial_2 f)$, we deduce the matrices

$$\mathrm{Mat}_{\mathcal{V}_{-2},\mathcal{V}_{-1}}\left(d_{\widetilde{\Omega}}^{(-2)}\right) = \begin{pmatrix} \nabla f & \mathbf{0}_{3,3} \end{pmatrix},$$
$$\forall\, p \geq 2, \quad \mathrm{Mat}_{\mathcal{V}_{-2p},\mathcal{V}_{-2p+1}}\left(d_{\widetilde{\Omega}}^{(-2p)}\right) = \begin{pmatrix} \nabla f & \mathbf{0}_{3,3} \\ 0 & (p-1)Df \end{pmatrix},$$
$$\forall\, p \geq 1, \quad \mathrm{Mat}_{\mathcal{V}_{-2p-1},\mathcal{V}_{-2p}}\left(d_{\widetilde{\Omega}}^{(-2p-1)}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -p\partial_2 f & p\partial_1 f & 0 & 0 \\ 0 & -p\partial_3 f & p\partial_2 f & 0 \\ p\partial_3 f & 0 & -p\partial_1 f & 0 \end{pmatrix}.$$

The cohomology spaces read as

$$L^0 = A, \qquad L^{-1} = \frac{A^3}{\{g\nabla f \,/\, g \in A\}},$$
$$L^{-2} = \{g \in A \,/\, g\partial_1 f = g\partial_2 f = g\partial_3 f = 0\} \oplus \frac{A^3}{\{\nabla f \wedge \mathbf{g} \,/\, \mathbf{g} \in A^3\}}.$$

For $p \geq 2$,

$$L^{-2p} \simeq \{g \in A \,/\, g\partial_1 f = g\partial_2 f = g\partial_3 f = 0\} \oplus \frac{\{\mathbf{g} \in A^3 \,/\, \mathbf{g}\cdot\nabla f = 0\}}{\{\nabla f \wedge \mathbf{g} \,/\, \mathbf{g} \in A^3\}}.$$

For $p \in \mathbb{N}^*$,

$$L^{-2p-1} \simeq \frac{\{\mathbf{g} \in A^3 \,/\, \nabla f \wedge \mathbf{g} = 0\}}{\{g\nabla f \,/\, g \in A\}} \oplus \frac{A}{\langle \nabla f\rangle_A}.$$

From now on, we assume that either f has separate variables, or f is of type D_k or E_7. Then we have $\{g \in A \,/\, g\partial_1 f = g\partial_2 f = g\partial_3 f = 0\} = \{0\}$, and according to Euler's formula,

$$\frac{A}{\langle \nabla f\rangle_A} \simeq \frac{\mathbb{C}[\mathbf{z}]}{\langle \nabla f\rangle}.$$

Most of the spaces have already been computed in Sections 4.3 and 4.4. In particular, we have $A^3 \,/\, A\nabla f \simeq \nabla f \wedge A^3$. Moreover, $\{\nabla f \wedge \mathbf{g} \,/\, \mathbf{g} \in A^3\} \subset \{\mathbf{g} \in A^3 \,/\, \mathbf{g}\cdot\nabla f = 0\}$, thus

$$\dim\left(A^3 \,/\, \{\nabla f \wedge \mathbf{g} \,/\, \mathbf{g} \in A^3\}\right) \geq \dim\left(A^3 \,/\, \{\mathbf{g} \in A^3 \,/\, \mathbf{g}\cdot\nabla f = 0\}\right).$$

And $A^3 \,/\, \{\mathbf{g} \in A^3 \,/\, \mathbf{g}\cdot\nabla f = 0\} \simeq \{\mathbf{g}\cdot\nabla f \,/\, \mathbf{g} \in A^3\}$. Since the map

$$g \in A \mapsto \begin{pmatrix} g \\ 0 \\ 0 \end{pmatrix} \cdot \nabla f \in \{\mathbf{g}\cdot\nabla f \,/\, \mathbf{g} \in A^3\}$$

is injective, $A^3 \,/\, \{\nabla f \wedge \mathbf{g} \,/\, \mathbf{g} \in A^3\}$ is infinite-dimensional.

In the following table we collect the results for the Hochschild homology in the various cases

Type	$HH_0 = A$	HH_1	HH_2	$HH_p,\ p \geq 3$
A_k	$\mathbb{C}[\mathbf{z}] \,/\, \langle z_1^2 + z_2^2 + z_3^k\rangle$	$\nabla f \wedge A^3$	$A^3 \,/\, (\nabla f \wedge A^3)$	$\mathbb{C}^{k-1}$
D_k	$\mathbb{C}[\mathbf{z}] \,/\, \langle z_1^2 + z_2^2 z_3 + z_3^k\rangle$	$\nabla f \wedge A^3$	$A^3 \,/\, (\nabla f \wedge A^3)$	$\mathbb{C}^{k+1}$
E_6	$\mathbb{C}[\mathbf{z}] \,/\, \langle z_1^2 + z_2^3 + z_3^4\rangle$	$\nabla f \wedge A^3$	$A^3 \,/\, (\nabla f \wedge A^3)$	$\mathbb{C}^6$
E_7	$\mathbb{C}[\mathbf{z}] \,/\, \langle z_1^2 + z_2^3 + z_2 z_3^3\rangle$	$\nabla f \wedge A^3$	$A^3 \,/\, (\nabla f \wedge A^3)$	$\mathbb{C}^7$
E_8	$\mathbb{C}[\mathbf{z}] \,/\, \langle z_1^2 + z_2^3 + z_3^5\rangle$	$\nabla f \wedge A^3$	$A^3 \,/\, (\nabla f \wedge A^3)$	$\mathbb{C}^8$

Acknowledgements

I would like to thank my thesis advisors Gadi Perets and Claude Roger for their efficient and likeable help, for their great availability, and for the time that they devoted to me all along this study. I also thank Daniel Sternheimer who paid attention to my work and the referees for their relevant remarks and their judicious advice. And I am grateful to Serge Parmentier for the rereading of my English text.

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