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Zhedanov's Algebra $AW(3)$ and the Double Affine Hecke Algebra in the Rank One Case. II. The Spherical Subalgebra*

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Abstract. This paper builds on the previous paper by the author, where a relationship between Zhedanov's algebra $AW(3)$ and the double affine Hecke algebra (DAHA) corresponding to the Askey–Wilson polynomials was established. It is shown here that the spherical subalgebra of this DAHA is isomorphic to $AW(3)$ with an additional relation that the Casimir operator equals an explicit constant. A similar result with q-shifted parameters holds for the antispherical subalgebra. Some theorems on centralizers and centers for the algebras under consideration will finally be proved as corollaries of the characterization of the spherical and antispherical subalgebra.

Key words: Zhedanov's algebra $AW(3)$; double affine Hecke algebra in rank one; Askey–Wilson polynomials; spherical subalgebra

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1 Introduction

Zhedanov [15] introduced in 1991 an algebra $AW(3)$ with three generators K_0, K_1, K_2 and three relations in the form of q-commutators, which describes deeper symmetries of the Askey–Wilson polynomials. In the *basic representation* (or *polynomial representation*) of $AW(3)$ on the space of symmetric Laurent polynomials in z, K_0 acts as the second order q-difference operator $D_{\rm sym}$ for which the Askey–Wilson polynomials are eigenfunctions and K_1 acts as multiplication by $z+z^{-1}$. The Casimir operator Q for $AW(3)$ becomes a scalar Q_0 in this representation. Let $AW(3,Q_0)$ be $AW(3)$ with the additional relation $Q=Q_0$. Then the basic representation $AW(3,Q_0)$ is faithful, see [7]. There is a parameter changing anti-algebra isomorphism of $AW(3)$ which interchanges K_0 and K_1, and hence interchanges $D_{\rm sym}$ and $z+z^{-1}$ in the basic representation. In the basic representation this *duality isomorphism* can be realized by an integral transform having the Askey–Wilson polynomial $P_n[z]$ as kernel which maps symmetric Laurent polynomials to infinite sequences $\{c_n\}_{n=0,1,\dots}$.

In 1992 Cherednik [2] introduced double affine Hecke algebras associated with root systems (DAHA's). This was the first of an important series of papers by the same author, where a representation of the DAHA was given in terms of q-difference-reflection operators (q-analogues of Dunkl operators), joint eigenfunctions of such operators were identified as non-symmetric Macdonald polynomials, and Macdonald's conjectures for ordinary (symmetric) Macdonald polynomials associated with root systems could be proved. The idea of nonsymmetric polynomials

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was very fruitful, on the one hand as an important extension of traditional harmonic analysis involving orthogonal systems of special functions, on the other hand because the nonsymmetric point of view was helpful for understanding the symmetric case better. For instance, a duality anti-algebra isomorphism and shift operators (as studied earlier by Opdam [11] in the symmetric framework) occur naturally in the non-symmetric context.

Related to Askey–Wilson polynomials the DAHA of type $(C_1^\vee, C_1)$ (four parameters) was studied by Sahi [13, 14], Noumi & Stokman [9], and Macdonald [8, Ch. 6]. See also the author's previous paper [7]. The same phenomena as described in the previous paragraph occur here, but in a very explicit form. See for instance Remarks 2.7 and 4.5 (referring to [9]) about the duality anti-algebra isomorphism and the shift operators, respectively.

In [7] I also discussed how the algebra $AW(3, Q_0)$ is related to the double affine Hecke algebra (DAHA) of type $(C_1^\vee, C_1)$. In the basic (or polynomial) representation of this DAHA (denoted by $\tilde{\mathfrak{H}}$) on the space of Laurent polynomials in z, the nonsymmetric Askey–Wilson polynomials occur as eigenfunctions of a suitable element Y of $\tilde{\mathfrak{H}}$ (see [13, 9], [8, Ch. 6]). It turns out (see [7]) that a central extension $\widetilde{AW}(3, Q_0)$ of $AW(3, Q_0)$ can be embedded as a subalgebra of $\tilde{\mathfrak{H}}$. As pointed out in the present paper (see Remark 2.7), the duality anti-algebra isomorphisms for $\widetilde{AW}(3, Q_0)$ and $\tilde{\mathfrak{H}}$ are compatible with this embedding.

It would be interesting, also for possible generalisations to higher rank, to have a more conceptual way of decribing the relationship between Zhedanov's algebra and the double affine Hecke algebra. The present paper establishes this by showing that the algebra $AW(3, Q_0)$ is isomorphic to the spherical subalgebra of $\tilde{\mathfrak{H}}$. The definition of a *spherical subalgebra* of a DAHA goes back to Etingof & Ginzburg [3], where a similar object was defined in the context of Cherednik algebras, see also [1]. The definition of spherical subalgebra for the DAHA of type $(C_1^\vee, C_1)$ was given by Oblomkov [10]. In general, the spherical subalgebra of a DAHA $\tilde{\mathfrak{H}}$ is the algebra $P_{\rm sym}\tilde{\mathfrak{H}}P_{\rm sym}$, where $P_{\rm sym}$ is the symmetrizer idempotent in $\tilde{\mathfrak{H}}$.

The proof of the isomorphism of $AW(3, Q_0)$ with $P_{\rm sym}\tilde{\mathfrak{H}}P_{\rm sym}$ in Section 3 is somewhat technical. It heavily uses the explicit relations given in [7] for the algebras $AW(3, Q_0)$, $\widetilde{AW}(3, Q_0)$ and $\tilde{\mathfrak{H}}$. In Section 4 a similar isomorphism is proved between $AW(3, Q_0)$ with two of its parameters q-shifted and $P_{\rm sym}^-\tilde{\mathfrak{H}}P_{\rm sym}^-$, where $P_{\rm sym}^-$ is the antisymmetrizing idempotent in $\tilde{\mathfrak{H}}$. In the final Section 5 it is shown as a corollary of these two isomorphisms that $\widetilde{AW}(3, Q_0)$ is isomorphic with the centralizer of T_1 in $\tilde{\mathfrak{H}}$. There it is also shown that the center of $AW(3, Q_0)$ is trivial, with a proof in the same spirit as the proof of the faithfulness of the basic representation of $AW(3, Q_0)$ given in [7]. Combination of the various results finally gives as a corollary that the center of $\tilde{\mathfrak{H}}$ is trivial.

Conventions

Throughout assume that q and a, b, c, d are complex constants such that

$$q \neq 0, \quad q^m \neq 1 \ (m = 1, 2, \ldots), \quad a, b, c, d \neq 0, \quad abcd \neq q^{-m} \ (m = 0, 1, 2, \ldots). \tag{1.1}$$

Let e_1, e_2, e_3, e_4 be the elementary symmetric polynomials in a, b, c, d:

$$\begin{aligned} &e_1 := a + b + c + d, \qquad e_2 := ab + ac + bc + ad + bd + cd,\\ &e_3 := abc + abd + acd + bcd, \qquad e_4 := abcd. \end{aligned} \tag{1.2}$$

For Laurent polynomials f in z the z-dependence will be written as $f[z]$. Symmetric Laurent polynomials $f[z] = \sum_{k=-n}^{n} c_k z^k$ (where $c_k = c_{-k}$) are related to ordinary polynomials $f(x)$ in $x = \frac{1}{2}(z + z^{-1})$ by $f(\frac{1}{2}(z + z^{-1})) = f[z]$.

2 Summary of earlier results

This section summarizes some results from [7], while Remarks 2.1 and 2.7 about the duality anti-algebra isomorphism are new here.

2.1 Askey–Wilson polynomials

The *Askey–Wilson polynomials* are given by

$$p_n\big(\tfrac{1}{2}(z+z^{-1});a,b,c,d\mid q\big):=\frac{(ab,ac,ad;q)_n}{a^n}\,{}_4\phi_3\left(\begin{matrix}q^{-n},q^{n-1}abcd,az,az^{-1}\\ ab,ac,ad\end{matrix};q,q\right)$$

(see [4] for the definition of q-shifted factorials $(a;q)_n$ and of q-hypergeometric series ${}_r\phi_s$). These polynomials are symmetric in a, b, c, d. We will work with the renormalized version which is *monic* as a Laurent polynomial in z (i.e., the coefficient of z^n equals 1):

$$P_n[z]=P_n[z;a,b,c,d\mid q]:=\frac{1}{(abcdq^{n-1};q)_n}\,p_n\big(\tfrac{1}{2}(z+z^{-1});a,b,c,d\mid q\big). \tag{2.1}$$

The polynomials $P_n[z]$ are eigenfunctions of the operator $D_{\rm sym}$ acting on the space $\mathcal{A}_{\rm sym}$ of symmetric Laurent polynomials $f[z]=f[z^{-1}]$:

$$\begin{aligned}(D_{\rm sym}f)[z]:={}&\frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}\big(f[qz]-f[z]\big)\\&+\frac{(a-z)(b-z)(c-z)(d-z)}{(1-z^2)(q-z^2)}\big(f[q^{-1}z]-f[z]\big)+(1+q^{-1}abcd)f[z].\end{aligned} \tag{2.2}$$

The eigenvalue equation is

$$D_{\rm sym}P_n=\lambda_nP_n,\qquad \lambda_n:=q^{-n}+abcdq^{n-1}. \tag{2.3}$$

Under condition (1.1) all eigenvalues in (2.3) are distinct.

The three-term recurrence relation for the monic Askey–Wilson polynomials is

$$(z+z^{-1})P_n[z]=P_{n+1}[z]+\beta_nP_n[z]+\gamma_nP_{n-1}[z]\qquad (n=0,1,2,\ldots),$$

where β_n and γ_n are suitable constants and $P_{-1}[z]:=0$ (see [6, (3.1.5)]).

2.2 Zhedanov's algebra

Zhedanov's algebra $AW(3)$ (see [15, 5]) can in the q-case be described as an algebra with two generators K_0, K_1 and with two relations

$$(q+q^{-1})K_1K_0K_1-K_1^2K_0-K_0K_1^2=B\,K_1+C_0\,K_0+D_0, \tag{2.4}$$

$$(q+q^{-1})K_0K_1K_0-K_0^2K_1-K_1K_0^2=B\,K_0+C_1\,K_1+D_1. \tag{2.5}$$

Here the *structure constants* B, C_0, C_1, D_0, D_1 are fixed complex constants.

There is a *Casimir operator* Q commuting with K_0, K_1:

$$\begin{aligned}Q={}&K_1K_0K_1K_0-(q^2+1+q^{-2})K_0K_1K_0K_1+(q+q^{-1})K_0^2K_1^2\\&+(q+q^{-1})(C_0K_0^2+C_1K_1^2)+B\big((q+1+q^{-1})K_0K_1+K_1K_0\big)\\&+(q+1+q^{-1})(D_0K_0+D_1K_1).\end{aligned} \tag{2.6}$$

Let the structure constants be expressed in terms of a, b, c, d by means of e_1, e_2, e_3, e_4 (see (1.2)) as follows:

$$\begin{aligned}
&B := (1-q^{-1})^2(e_3+qe_1),\\
&C_0 := (q-q^{-1})^2,\\
&C_1 := q^{-1}(q-q^{-1})^2 e_4,\\
&D_0 := -q^{-3}(1-q)^2(1+q)(e_4+qe_2+q^2),\\
&D_1 := -q^{-3}(1-q)^2(1+q)(e_1e_4+qe_3).
\end{aligned} \tag{2.7}$$

Then there is a representation (the *basic representation* or *polynomial representation*) of the algebra $AW(3)$ with structure constants (2.7) on the space $\mathcal{A}_{\rm sym}$ of symmetric Laurent polynomials as follows:

$$(K_0 f)[z] := (D_{\rm sym} f)[z], \qquad (K_1 f)[z] := (Z+Z^{-1})f[z] := (z+z^{-1})f[z], \tag{2.8}$$

where $D_{\rm sym}$ is the operator (2.2) having the *Askey–Wilson polynomials* as eigenfunctions. The Casimir operator Q becomes constant in this representation:

$$(Qf)(z) = Q_0\, f(z), \tag{2.9}$$

where

$$\begin{aligned}
Q_0 := q^{-4}(1-q)^2\Big(&q^4(e_4-e_2)+q^3(e_1^2-e_1e_3-2e_2)\\
&-q^2(e_2e_4+2e_4+e_2)+q(e_3^2-2e_2e_4-e_1e_3)+e_4(1-e_2)\Big).
\end{aligned} \tag{2.10}$$

(Note the slight error in this formula in [7, (2.8)], version `v3`. It is corrected in `v4`.)

Remark 2.1. Write $AW(3) = AW(3; K_0, K_1; a, b, c, d; q)$ in order to emphasize the dependence of $AW(3)$ on the generators and the parameters (by (2.4), (2.5), (2.7)). There are several symmetries of this algebra. First of all it is invariant under permutations of a, b, c, d. The following one will be compatible with the duality of the double affine Hecke algebra to be discussed below:

There is an anti-algebra isomorphism

$$\begin{aligned}
AW(3; K_0, K_1; a, b, c, d; q) \to AW\Bigg(&3; aK_1, (q^{-1}abcd)^{-\frac12}K_0;\\
&\frac{1}{(q^{-1}abcd)^{\frac12}}, \frac{ab}{(q^{-1}abcd)^{\frac12}}, \frac{ac}{(q^{-1}abcd)^{\frac12}}, \frac{ad}{(q^{-1}abcd)^{\frac12}}; q\Bigg).
\end{aligned} \tag{2.11}$$

See [15, § 2]. Under this mapping Q is sent to $qa(bcd)^{-1}Q$ and Q_0 to $qa(bcd)^{-1}Q_0$.

Note that there is also a trivial anti-algebra isomorphism

$$AW(3; K_0, K_1; a, b, c, d; q) \to AW(3; K_0, K_1; a, b, c, d; q).$$

This sends Q to Q and Q_0 to Q_0.

There is also an algebra isomorphism

$$AW(3; K_0, K_1; a, b, c, d; q) \to AW\left(3; \frac{q}{abcd}K_0, K_1; a^{-1}, b^{-1}, c^{-1}, d^{-1}; q^{-1}\right).$$

This sends Q to $q^2(abcd)^{-2}Q$ and Q_0 to $q^2(abcd)^{-2}Q_0$.

Let $AW(3,Q_0)$ be the algebra generated by K_0, K_1 with relations (2.4), (2.5) and $Q=Q_0$, assuming the structure constants (2.7). Then the basic representation of $AW(3)$ is also a representation of $AW(3,Q_0)$. We have the following theorem (see [7, Theorem 2.2]):

Theorem 2.2. *The elements*

$$K_0^n(K_1K_0)^lK_1^m \qquad (m,n=0,1,2,\dots,\ l=0,1)$$

form a basis of $AW(3,Q_0)$ *and the representation* (2.8) *of* $AW(3,Q_0)$ *is faithful.*

Note that the anti-algebra isomorphism (2.11) induces an anti-algebra isomorphisms for $AW(3,Q_0)$.

2.3 The double affine Hecke algebra of type $(C_1^\vee, C_1)$

One of the ways to describe the *double affine Hecke algebra* of type $(C_1^\vee, C_1)$, denoted by $\tilde{\mathfrak{H}}$, is as follows (see [7, Proposition 5.2]):

Definition 2.3. $\tilde{\mathfrak{H}}$ is the algebra generated by T_1, Y, Y^{-1}, Z, Z^{-1} with relations $YY^{-1}=1=Y^{-1}Y$, $ZZ^{-1}=1=Z^{-1}Z$ and

$$\begin{aligned}
&T_1^2=-(ab+1)T_1-ab,\\
&T_1Z=Z^{-1}T_1+(ab+1)Z^{-1}-(a+b),\\
&T_1Z^{-1}=ZT_1-(ab+1)Z^{-1}+(a+b),\\
&T_1Y=q^{-1}abcdY^{-1}T_1-(ab+1)Y+ab(1+q^{-1}cd),\\
&T_1Y^{-1}=q(abcd)^{-1}YT_1+q(abcd)^{-1}(1+ab)Y-q(cd)^{-1}(1+q^{-1}cd),\\
&YZ=qZY+(1+ab)cd\,Z^{-1}Y^{-1}T_1-(a+b)cd\,Y^{-1}T_1-(1+q^{-1}cd)Z^{-1}T_1\\
&\qquad -(1-q)(1+ab)(1+q^{-1}cd)Z^{-1}+(c+d)T_1+(1-q)(a+b)(1+q^{-1}cd),\\
&YZ^{-1}=q^{-1}Z^{-1}Y-q^{-2}(1+ab)cd\,Z^{-1}Y^{-1}T_1+q^{-2}(a+b)cd\,Y^{-1}T_1\\
&\qquad +q^{-1}(1+q^{-1}cd)Z^{-1}T_1-q^{-1}(c+d)T_1,\\
&Y^{-1}Z=q^{-1}ZY^{-1}-q(ab)^{-1}(1+ab)Z^{-1}Y^{-1}T_1+(ab)^{-1}(a+b)Y^{-1}T_1\\
&\qquad +q(abcd)^{-1}(1+q^{-1}cd)Z^{-1}T_1+q(abcd)^{-1}(1-q)(1+ab)(1+q^{-1}cd)Z^{-1}\\
&\qquad -(abcd)^{-1}(c+d)T_1-(abcd)^{-1}(1-q)(1+ab)(c+d),\\
&Y^{-1}Z^{-1}=qZ^{-1}Y^{-1}+q(ab)^{-1}(1+ab)Z^{-1}Y^{-1}T_1-(ab)^{-1}(a+b)Y^{-1}T_1\\
&\qquad -q^2(abcd)^{-1}(1+q^{-1}cd)Z^{-1}T_1+q(abcd)^{-1}(c+d)T_1.
\end{aligned}\tag{2.12}$$

By adding the relations for TZ and TZ^{-1} and by combining the relations for TY and TY^{-1} we see that

$$T_1(Z+Z^{-1})=(Z+Z^{-1})T_1, \qquad T_1(Y+q^{-1}abcdY^{-1})=(Y+q^{-1}abcdY^{-1})T_1.$$

For the following theorem see Sahi [12] in the general rank case. In [7, Theorem 5.3] it is proved for the rank one case only.

Theorem 2.4. *A basis of* $\tilde{\mathfrak{H}}$ *is provided by the elements* $Z^mY^nT_1^i$, *where* $m,n\in\mathbb{Z}$, $i=0,1$.

The *basic* or *polynomial* representation of $\tilde{\mathfrak{H}}$ is a representation of $\tilde{\mathfrak{H}}$ on the space $\mathcal{A}$ of Laurent polynomials $f[z]$ in z such that $(Zf)[z]:=z\,f[z]$ and T_1 and Y act as q-difference-reflection operators given by [7, (3.11), (3.13)]. This representation is faithful. If $ab\neq 1$ then T_1 acting on $\mathcal{A}$ has eigenspaces $\mathcal{A}_{\rm sym}$ (for eigenvalue $-ab$) and $\mathcal{A}_{\rm sym}^-$ (for eigenvalue -1), and $\mathcal{A}$ is the direct sum of $\mathcal{A}_{\rm sym}$ and $\mathcal{A}_{\rm sym}^-$.

In the basic representation of $\tilde{\mathfrak{H}}$ the operator $Y+q^{-1}abcdY^{-1}$ has eigenvalues λ_n (see (2.3)), of multiplicity 2 for $n=1,2,\ldots$ and of multiplicity 1 for $n=0$. If $n\ge 1$ and $ab\ne 1$ then the eigenspace of λ_n splits as a one-dimensional part in $\mathcal{A}_{\rm sym}$ spanned by the Askey–Wilson polynomial $P_n[z]$ given by (2.1) and a one-dimensional part in $\mathcal{A}_{\rm sym}^-$ spanned by

$$Q_n[z]:=a^{-1}b^{-1}z^{-1}(1-az)(1-bz)\,P_{n-1}[z;qa,qb,c,d\mid q]. \tag{2.13}$$

In [7, § 6] the following algebra $\widetilde{AW}(3,Q_0)$ was defined, which is a central extension of $AW(3,Q_0)$:

Definition 2.5. The algebra $\widetilde{AW}(3,Q_0)$ is generated by K_0, K_1, T_1 such that $(T_1+ab)(T_1+1)=0$, T_1 commutes with K_0, K_1, and with further relations

$$\begin{aligned}&(q+q^{-1})K_1K_0K_1-K_1^2K_0-K_0K_1^2=B\,K_1+C_0\,K_0+D_0\\&\qquad+E\,K_1(T_1+ab)+F_0(T_1+ab),\end{aligned} \tag{2.14}$$

$$\begin{aligned}&(q+q^{-1})K_0K_1K_0-K_0^2K_1-K_1K_0^2=B\,K_0+C_1\,K_1+D_1\\&\qquad+E\,K_0(T_1+ab)+F_1(T_1+ab),\end{aligned} \tag{2.15}$$

$$\begin{aligned}&Q_0=(K_1K_0)^2-(q^2+1+q^{-2})K_0(K_1K_0)K_1+(q+q^{-1})K_0^2K_1^2\\&\qquad+(q+q^{-1})(C_0K_0^2+C_1K_1^2)+\big(B+E(T_1+ab)\big)\big((q+1+q^{-1})K_0K_1+K_1K_0\big)\\&\qquad+(q+1+q^{-1})\big(D_0+F_0(T_1+ab)\big)K_0+(q+1+q^{-1})\big(D_1+F_1(T_1+ab)\big)K_1\\&\qquad+G(T_1+ab).\end{aligned} \tag{2.16}$$

Here the structure constants are given by (2.7) together with

$$\begin{aligned}&E:=-q^{-2}(1-q)^3(c+d),\\&F_0:=q^{-3}(1-q)^3(1+q)(cd+q),\\&F_1:=q^{-3}(1-q)^3(1+q)(a+b)cd,\\&G:=-q^{-4}(1-q)^3\Big((a+b)(c+d)\big(cd(q^2+1)+q\big)-q(ab+1)\big((c^2+d^2)(q+1)-cd\big)\\&\qquad+(cd+e_4)(q^2+1)+(e_2+e_4-ab)q^3\Big),\end{aligned}$$

and Q_0 is given by (2.10).

Then it was proved in [7, Theorem 6.2, Corollary 6.3]:

Theorem 2.6. *$\widetilde{AW}(3,Q_0)$ has a basis consisting of*

$$K_0^n(K_1K_0)^iK_1^mT_1^j\qquad(m,n=0,1,2,\ldots,\ i,j=0,1).$$

There is a unique algebra isomorphism from $\widetilde{AW}(3,Q_0)$ into $\tilde{\mathfrak{H}}$ such that $K_0\mapsto Y+q^{-1}abcdY^{-1}$, $K_1\mapsto Z+Z^{-1}$, $T_1\mapsto T_1$. The elements in the image commute with T_1.

Remark 2.7. Write $\tilde{\mathfrak{H}}=\tilde{\mathfrak{H}}(Y,Z,T_1;a,b,c,d;q)$ in order to emphasize the dependence of $\tilde{\mathfrak{H}}$ on the generators and the parameters. Then there is an anti-algebra isomorphism

$$\begin{aligned}&\tilde{\mathfrak{H}}(Y,Z,T_1;a,b,c,d;q)\to\tilde{\mathfrak{H}}\Bigg(aZ^{-1},(q^{-1}abcd)^{-\frac12}Y^{-1},T_1;\\&\qquad\qquad\frac{1}{(q^{-1}abcd)^{\frac12}},\frac{ab}{(q^{-1}abcd)^{\frac12}},\frac{ac}{(q^{-1}abcd)^{\frac12}},\frac{ad}{(q^{-1}abcd)^{\frac12}};q\Bigg),\end{aligned}$$

see [9, Proposition 8.5(i)]. Also write $\widetilde{AW}(3,Q_0)=\widetilde{AW}(3,Q_0;K_0,K_1,T_1;a,b,c,d;q)$. Then, by a slight adaptation of (2.11), there is an anti-algebra isomorphism

$$\widetilde{AW}(3,Q_0;K_0,K_1,T_1;a,b,c,d;q)\to\widetilde{AW}\Bigg(3;qa(bcd)^{-1}Q_0;aK_1,(q^{-1}abcd)^{-\frac{1}{2}}K_0;\\ \frac{1}{(q^{-1}abcd)^{\frac{1}{2}}},\frac{ab}{(q^{-1}abcd)^{\frac{1}{2}}},\frac{ac}{(q^{-1}abcd)^{\frac{1}{2}}},\frac{ad}{(q^{-1}abcd)^{\frac{1}{2}}};q\Bigg).$$

The two anti-algebra isomorphisms are compatible under the algebra embedding of $\widetilde{AW}(3,Q_0)$ into $\tilde{\mathfrak{H}}$ given in Theorem 2.6.

3 The spherical subalgebra

From now on assume $ab\neq 1$. In $\tilde{\mathfrak{H}}$ put

$$P_{\rm sym}:=(1-ab)^{-1}(T_1+1).$$

Then

$$P_{\rm sym}^2=P_{\rm sym}.$$

In the basic representation of $\tilde{\mathfrak{H}}$ we have for $f\in\mathcal{A}$:

$$P_{\rm sym}f=\begin{cases} f & \text{if } T_1f=-abf,\\ 0 & \text{if } T_1f=-f.\end{cases}$$

$P_{\rm sym}$ projects $\mathcal{A}$ onto $\mathcal{A}_{\rm sym}$. Define the linear map $S\colon\tilde{\mathfrak{H}}\to\tilde{\mathfrak{H}}$ by

$$S(U):=P_{\rm sym}UP_{\rm sym}\qquad(U\in\tilde{\mathfrak{H}}).$$

Then

$$S(U)\,S(V)=S(UP_{\rm sym}V)\qquad(U,V\in\tilde{\mathfrak{H}}).$$

Hence the image $S(\tilde{\mathfrak{H}})$ is a subalgebra of $\tilde{\mathfrak{H}}$. We call it the *spherical subalgebra* of $\tilde{\mathfrak{H}}$.

For $U\in\tilde{\mathfrak{H}}$ we have in the basic representation:

$$S(U)\,f=P_{\rm sym}UP_{\rm sym}f=\begin{cases} P_{\rm sym}U\,f & \text{if } T_1f=-abf,\\ 0 & \text{if } T_1f=-f.\end{cases}$$

Hence, for the basic representation of $\tilde{\mathfrak{H}}$ restricted to $S(\tilde{\mathfrak{H}})$, $\mathcal{A}_{\rm sym}$ is an invariant subspace. This representation of $S(\tilde{\mathfrak{H}})$ on $\mathcal{A}_{\rm sym}$ is faithful. Indeed, if $S(U)\,f=0$ for all $f\in\mathcal{A}_{\rm sym}$ then $S(U)\,f=0$ for all $f\in\mathcal{A}$, so $S(U)=0$ by the faithfulness of the basic representation of $\tilde{\mathfrak{H}}$ on $\mathcal{A}$.

$Z_{\tilde{\mathfrak{H}}}(T_1)$, the centralizer of T_1 in $\tilde{\mathfrak{H}}$, is a subalgebra of $\tilde{\mathfrak{H}}$. It has $P_{\rm sym}$ as a central element. Hence

$$S(U)=UP_{\rm sym}\qquad\text{and}\qquad S(UV)=S(U)\,S(V)\qquad(U,V\in Z_{\tilde{\mathfrak{H}}}(T_1)).\tag{3.1}$$

So S restricted to $Z_{\tilde{\mathfrak{H}}}(T_1)$ is an algebra homomorphism.

The algebra $\widetilde{AW}(3,Q_0)$ was defined by Definition 2.5. By Theorem 2.6 there is an algebra isomorphism from $\widetilde{AW}(3,Q_0)$ into $\tilde{\mathfrak{H}}$ such that $K_0\mapsto Y+q^{-1}abcdY^{-1}$, $K_1\mapsto Z+Z^{-1}$,

$T_1 \mapsto T_1$. This isomorphism embeds $\widetilde{AW}(3, Q_0)$ into $Z_{\tilde{\mathfrak{H}}}(T_1)$. So we may consider $\widetilde{AW}(3, Q_0)$ as a subalgebra of $Z_{\tilde{\mathfrak{H}}}(T_1)$ and (3.1) will hold for $U, V \in \widetilde{AW}(3, Q_0)$.

By (2.4), (2.5), (2.9) and (2.6), the algebra $AW(3, Q_0)$ can be presented by the same generators and relations as for $\widetilde{AW}(3, Q_0)$ but with additional relation $T_1 = -ab$.

By Theorem 2.6 $\widetilde{AW}(3, Q_0)$ has a basis consisting of the elements

$$K_0^n (K_1 K_0)^i K_1^m (T_1 + 1), \qquad K_0^n (K_1 K_0)^i K_1^m (T_1 + ab) \qquad (m, n = 0, 1, 2, \dots, \quad i = 0, 1).$$

Hence $S(\widetilde{AW}(3, Q_0))$ has a basis consisting of

$$(1 - ab)^{-1} K_0^n (K_1 K_0)^i K_1^m (T_1 + 1) \qquad (m, n = 0, 1, 2, \dots, \quad i = 0, 1).$$

By Theorem 2.2 $AW(3, Q_0)$ has a basis consisting of

$$K_0^n (K_1 K_0)^i K_1^m \qquad (m, n = 0, 1, 2, \dots, \quad i = 0, 1). \tag{3.2}$$

Hence the map which sends a basis element $K_0^n (K_1 K_0)^i K_1^m$ of $AW(3, Q_0)$ to a basis element $(1 - ab)^{-1} K_0^n (K_1 K_0)^i K_1^m (T_1 + 1)$ of $S(\widetilde{AW}(3, Q_0))$ extends linearly to a linear bijection from $AW(3, Q_0)$ onto $S(\widetilde{AW}(3, Q_0))$. In fact, this map remains well-defined if we write it as

$$U \mapsto (1 - ab)^{-1} \widetilde{U} (T_1 + 1), \tag{3.3}$$

where $U \mapsto \widetilde{U}$ sends words U in $AW(3, Q_0)$ to corresponding words $\widetilde{U}$ in $\widetilde{AW}(3, Q_0)$. Moreover, the map (3.3) can then be seen to be an algebra homomorphism. Indeed, consider the linear map $U \mapsto \widetilde{U}$ as a map to $\widetilde{AW}(3, Q_0)$ from the free algebra generated by K_0, K_1, T_1 with T_1 central such that it sends a word involving K_0, K_1, T_1 to the same word in $\widetilde{AW}(3, Q_0)$. This map is an algebra homomorphism. Composing it with S yields the map (3.3) which is again an algebra homomorphism. Now we have to check that R is sent to zero by the map (3.3) if $R = 0$ is a relation for $AW(3, Q_0)$. This is clearly the case for $R := T_1 + ab$, since $(T_1 + ab)(T_1 + 1) = 0$. It is also clear for the other relations $R = 0$ in $AW(3, Q_0)$ since these can be taken as the relations (2.14)–(2.16), which are also relations for $\widetilde{AW}(3, Q_0)$. So we have shown:

Proposition 3.1. *The map* (3.3)*, where $U \mapsto \widetilde{U}$ sends words U involving K_0, K_1 in $AW(3, Q_0)$ to the same words $\widetilde{U}$ in $\widetilde{AW}(3, Q_0)$, is a well defined algebra isomorphism from $AW(3, Q_0)$ onto $S(\widetilde{AW}(3, Q_0))$.*

Theorem 3.2. *$S(\tilde{\mathfrak{H}}) = S(\widetilde{AW}(3, Q_0))$, so the spherical subalgebra $S(\tilde{\mathfrak{H}})$ is isomorphic to the algebra $AW(3, Q_0)$ by the map* (3.3) *sending $AW(3, Q_0)$ to $S(\tilde{\mathfrak{H}})$.*

For the proof note first that $\tilde{\mathfrak{H}}$ has a basis consisting of the elements

$$Z^m Y^n (T_1 + 1), \qquad Z^m Y^n (T_1 + ab) \qquad (m, n \in \mathbb{Z}). \tag{3.4}$$

Hence $S(\tilde{\mathfrak{H}})$ is spanned by the elements

$$(T_1 + 1) Z^m Y^n (T_1 + 1) \qquad (m, n \in \mathbb{Z}).$$

Definition 3.3. Let $m, n \in \mathbb{Z}$. For an element in $\tilde{\mathfrak{H}}$ which is a linear combination of basis elements $Z^k Y^l$ we say that

$$\sum_{k,l \in \mathbb{Z}} c_{k,l} Z^k Y^l = o(Z^m Y^n)$$

if $c_{k,l} \neq 0$ implies $|k| \leq |m|$, $|l| \leq |n|$, $(|k|, |l|) \neq (|m|, |n|)$.

Theorem 3.2 will follow by induction with respect to $|m|+|n|$ from the following lemma:

Lemma 3.4. *Let $m,n\in\mathbb{Z}$. Then*

$$(T_1+1)Z^mY^n(T_1+1)\in(T_1+1)\Big(\widetilde{AW}(3,Q_0)+o(Z^mY^n)\Big)(T_1+1).$$

Proof. The procedure will be as follows:

1. Write $(T_1+1)Z^mY^n(T_1+1)$ as a linear combination of
$$\begin{aligned}&Z^{|m|}Y^{|n|}(T_1+1),\qquad Z^{-|m|}Y^{|n|}(T_1+1),\qquad Z^{|m|}Y^{-|n|}(T_1+1),\\ &Z^{-|m|}Y^{-|n|}(T_1+1)\qquad (\mathrm{mod}\ o(Z^{|m|}Y^{|n|})(T_1+1)).\end{aligned}\tag{3.5}$$
This is done by induction, starting with the $\tilde{\mathfrak{H}}$ relations for T_1Z, T_1Z^{-1}, T_1Y, T_1Y^{-1}.
2. Also write $K_1^mK_0^n(T_1+1)$ and $K_1^{m-1}K_0K_1K_0^{n-1}(T_1+1)$ $(m,n=0,1,\dots)$ as a linear combination of (3.5).
3. These latter linear combinations turn out to span the linear combinations obtained for $(T_1+1)Z^mY^n(T_1+1)$.

Step 1. We get the following expressions in terms of the elements (3.5) (here $m,n\in\{1,2,\dots\}$)

$$(T_1+1)Z^m(T_1+1)=\Big(Z^m+Z^{-m}+o(Z^m)\Big)(T_1+1),\tag{3.6}$$

$$(T_1+1)Z^{-m}(T_1+1)=-ab\Big(Z^m+Z^{-m}+o(Z^m)\Big)(T_1+1),\tag{3.7}$$

$$(T_1+1)Y^n(T_1+1)=-ab\Big(Y^n+(q^{-1}abcd)^nY^{-n}+o(Y^n)\Big)(T_1+1),\tag{3.8}$$

$$(T_1+1)Y^{-n}(T_1+1)=\Big((q^{-1}abcd)^{-n}Y^n+Y^{-n}+o(Y^n)\Big)(T_1+1),\tag{3.9}$$

$$(T_1+1)Z^mY^n(T_1+1)=\Big(Z^mY^n-ab(q^{-1}abcd)^nZ^{-m}Y^{-n}+o(Z^mY^n)\Big)(T_1+1),\tag{3.10}$$

$$\begin{aligned}(T_1+1)Z^{-m}Y^n(T_1+1)=\Big(&-(ab+1)Z^mY^n-ab(q^{-1}abcd)^nZ^mY^{-n}-abZ^{-m}Y^n\\&+o(Z^mY^n)\Big)(T_1+1),\end{aligned}\tag{3.11}$$

$$\begin{aligned}(T_1+1)Z^mY^{-n}(T_1+1)=\Big(&Z^mY^{-n}+(q^{-1}abcd)^{-n}Z^{-m}Y^n+(1+ab)Z^{-m}Y^{-n}\\&+o(Z^mY^n)\Big)(T_1+1),\end{aligned}\tag{3.12}$$

$$\begin{aligned}(T_1+1)Z^{-m}Y^{-n}(T_1+1)=\Big(&(q^{-1}abcd)^{-n}Z^mY^n-abZ^{-m}Y^{-n}\\&+o(Z^mY^n)\Big)(T_1+1).\end{aligned}\tag{3.13}$$

Step 2. We get the following expressions in terms of the elements (3.5) (here $m,n\in\{1,2,\dots\}$)

$$K_1^m(T_1+1)=\Big(Z^m+Z^{-m}+o(Z^m)\Big)(T_1+1),\tag{3.14}$$

$$K_0^n(T_1+1)=\Big(Y^n+(q^{-1}abcd)^nY^{-n}+o(Y^n)\Big)(T_1+1),\tag{3.15}$$

$$\begin{aligned}K_1^mK_0^n(T_1+1)=\Big(&Z^mY^n+Z^{-m}Y^n+(q^{-1}abcd)^n\big(Z^mY^{-n}+Z^{-m}Y^{-n}\big)\\&+o(Z^mY^n)\Big)(T_1+1),\end{aligned}\tag{3.16}$$

$$\begin{aligned}K_1^{m-1}K_0K_1K_0^{n-1}(T_1+1)=\Big(&qZ^mY^n+q^{-1}Z^{-m}Y^n+q^{-1}(q^{-1}abcd)^nZ^mY^{-n}\\&+q^{-1}(q^{-1}abcd)^n(1+ab-q^2ab)Z^{-m}Y^{-n}+o(Z^mY^n)\Big)(T_1+1).\end{aligned}\tag{3.17}$$

Step 3. The only cases which may not be immediately clear are for $(T_1+1)Z^{\pm m}Y^{\pm n}(T_1+1)$ ($m,n\in\{1,2,\ldots\}$). Then we have by comparing the identities in Step 1 and Step 2:

$$
\begin{aligned}
(T_1+1)Z^mY^n(T_1+1) &= \frac{1}{(1-q^2)}\Big(K_1^{m-1}(K_1K_0-qK_0K_1)K_0^{n-1}+o(Z^mY^n)\Big)(T_1+1),\\
(T_1+1)Z^{-m}Y^n(T_1+1) &= \frac{q}{(1-q^2)}\Big(K_1^{m-1}(-q^{-1}\big(1+ab-q^2ab)K_1K_0+K_0K_1\big)K_0^{n-1}\\
&\quad +o(Z^mY^n)\Big)(T_1+1),\\
(T_1+1)Z^mY^{-n}(T_1+1) &= \frac{q}{(1-q^2)(q^{-1}abcd)^n}\Big(K_1^{m-1}(-qK_1K_0+K_0K_1\big)K_0^{n-1}\\
&\quad +o(Z^mY^n)\Big)(T_1+1),\\
(T_1+1)Z^{-m}Y^{-n}(T_1+1) &= \frac{1}{(1-q^2)(q^{-1}abcd)^n}\Big(K_1^{m-1}(K_1K_0-qK_0K_1\big)K_0^{n-1}\\
&\quad +o(Z^mY^n)\Big)(T_1+1).
\end{aligned}
$$

Proofs for Step 1. We will use repeatedly that $(T_1+1)^2=(1-ab)(T_1+1)$. From the relation for T_1Z in (2.12) we obtain

$$
\begin{aligned}
&(T_1+1)Z = Z^{-1}(T_1+1)+Z+abZ^{-1}-(a+b), \qquad (3.18)\\
&(T_1+1)Z(T_1+1) = \big(Z+Z^{-1}-(a+b)\big)(T_1+1) = \big(Z+Z^{-1}+o(Z)\big)(T_1+1),
\end{aligned}
$$

i.e., (3.6) for $m=1$. Now we will prove (3.6) by induction. Suppose it holds for some positive integer m, then we will prove the identity with m replaced by $m+1$. By successively substituting (3.18) and the induction hypothesis we have:

$$
\begin{aligned}
(T_1+1)Z^{m+1}(T_1+1) &= Z^{-1}(T_1+1)Z^m(T_1+1)+\big(Z^{m+1}+o(Z^{m+1})\big)(T_1+1)\\
&= Z^{-1}\big(Z^m+Z^{-m}+o(Z^m)\big)(T_1+1)+\big(Z^{m+1}+o(Z^{m+1})\big)(T_1+1)\\
&= \big(Z^{m+1}+Z^{-m-1}+o(Z^{m+1})\big)(T_1+1).
\end{aligned}
$$

Similarly, from the relation for T_1Z^{-1} in (2.12) we prove (3.7) by induction:

$$
\begin{aligned}
&(T_1+1)Z^{-1} = Z(T_1+1)-Z-abZ^{-1}+(a+b), \quad \text{hence}\\
&(T_1+1)Z^{-1}(T_1+1) = \big(-ab(Z+Z^{-1})+(a+b)\big)(T_1+1)\\
&\qquad = -ab\big(Z+Z^{-1}+o(Z)\big)(T_1+1),\\
&(T_1+1)Z^{-m-1}(T_1+1) = Z(T_1+1)Z^{-m}(T_1+1)-ab\big(Z^{-m-1}+o(Z^{m+1})\big)(T_1+1)\\
&\quad = -abZ\big(Z^m+Z^{-m}+o(Z^m)\big)(T_1+1)-ab\big(Z^{-m-1}+o(Z^{m+1})\big)(T_1+1)\\
&\quad = -ab\big(Z^{m+1}+Z^{-m-1}+o(Z^{m+1})\big)(T_1+1).
\end{aligned}
$$

The proofs of (3.8), (3.9) are similar.

The proof of (3.10) is for fixed n by induction with respect to m. First we prove the case $m=1$. By (3.18) and (3.8) we obtain:

$$
\begin{aligned}
(T_1+1)ZY^n(T_1+1) &= \Big(Z^{-1}(T_1+1)Y^n+ZY^n+abZ^{-1}Y^n+o(ZY^n)\Big)(T_1+1)\\
&= \Big(ZY^n-ab(q^{-1}abcd)^nZ^{-1}Y^{-n}+o(ZY^n)\Big)(T_1+1).
\end{aligned}
$$

Now suppose (3.10) holds for some positive integer m, then we will prove the identity with m replaced by $m+1$. By (3.18) and the induction hypothesis we obtain:

$$
\begin{aligned}
(T_1+1)Z^{m+1}Y^n(T_1+1) &= \Big(Z^{-1}(T_1+1)Z^mY^n+Z^{m+1}Y^n+o(Z^{m+1}Y^n)\Big)(T_1+1)\\
&= \Big(Z^{m+1}Y^n-ab(q^{-1}abcd)^nZ^{-m-1}Y^{-n}+o(Z^{m+1}Y^n)\Big)(T_1+1).
\end{aligned}
$$

The proofs of (3.11), (3.12), (3.13) are similar.

Proofs for Step 2. Formulas (3.14), (3.15), (3.16) immediately follow by the substitutions $K_0 = Y + q^{-1}abcdY^{-1}$, $K_1 = Z + Z^{-1}$. As for (3.17) we first verify it for $m = n = 1$ by writing the left-hand side as $(1-ab)^{-1}(T_1+1)(Y + q^{-1}abcdY^{-1})(Z + Z^{-1})(T_1 + 1)$ and by using the last four relations in (2.12). Then we obtain for the case of general m, n:

$$\begin{aligned}
K_1^{m-1}K_0K_1K_0^{n-1}(T_1+1) &= (1-ab)^{-1}K_1^{m-1}K_0K_1(T_1+1)K_0^{n-1}(T_1+1)\\
&= (Z+Z^{-1})^{m-1}\Big(qZY + q^{-1}Z^{-1}Y + q^{-1}(q^{-1}abcd)ZY^{-1}\\
&\qquad + q^{-1}(q^{-1}abcd)(1+ab-q^2ab)Z^{-1}Y^{-1} + o(ZY)\Big)(Y + q^{-1}abcdY^{-1})^{n-1}(T_1+1)\\
&= \Big(qZ^mY^n + q^{-1}Z^{-m}Y^n + q^{-1}(q^{-1}abcd)^nZ^mY^{-n}\\
&\qquad + q^{-1}(q^{-1}abcd)^n(1+ab-q^2ab)Z^{-m}Y^{-n} + o(Z^mY^n)\Big)(T_1+1). \qquad \blacksquare
\end{aligned}$$

4 The antispherical subalgebra

In $\tilde{\mathfrak{H}}$ put

$$P^-_{\rm sym} := (ab-1)^{-1}(T_1 + ab).$$

Then

$$(P^-_{\rm sym})^2 = P^-_{\rm sym}.$$

In the basic representation of $\tilde{\mathfrak{H}}$ we have for $f \in \mathcal{A}$:

$$P^-_{\rm sym}f = \begin{cases} f & \text{if } T_1 f = -f,\\ 0 & \text{if } T_1 f = -abf.\end{cases}$$

Let $\mathcal{A}^-_{\rm sym}$ denote the eigenspace of T_1 acting on $\mathcal{A}$ for eigenvalue -1. Then $P^-_{\rm sym}$ projects $\mathcal{A}$ onto $\mathcal{A}^-_{\rm sym}$. Define the linear map $S^- : \tilde{\mathfrak{H}} \to \tilde{\mathfrak{H}}$ by

$$S^-(U) := P^-_{\rm sym}UP^-_{\rm sym} \qquad (U \in \tilde{\mathfrak{H}}).$$

Then

$$S^-(U)\,S^-(V) = S^-(UP^-_{\rm sym}V) \qquad (U, V \in \tilde{\mathfrak{H}}).$$

Hence the image $S^-(\tilde{\mathfrak{H}})$ is a subalgebra of $\tilde{\mathfrak{H}}$. We call it the *antispherical subalgebra* of $\tilde{\mathfrak{H}}$.

For $U \in \tilde{\mathfrak{H}}$ we have in the basic representation:

$$S^-(U)\,f = P^-_{\rm sym}UP^-_{\rm sym}f = \begin{cases} P^-_{\rm sym}U\,f & \text{if } T_1 f = -f,\\ 0 & \text{if } T_1 f = -abf.\end{cases}$$

Hence, for the basic representation of $\tilde{\mathfrak{H}}$ restricted to $S^-(\tilde{\mathfrak{H}})$, $\mathcal{A}^-_{\rm sym}$ is an invariant subspace.

Recall the algebra isomorphism from $\widetilde{AW}(3, Q_0)$ into $Z_{\tilde{\mathfrak{H}}}(T_1)$ given by Theorem 2.6. Since $S^-(U)\,S^-(V) = S^-(UV)$ for $U, V \in Z_{\tilde{\mathfrak{H}}}(T_1)$, we see that $\widetilde{AW}(3, Q_0)$, considered as a subalgabra of $\tilde{\mathfrak{H}}$ and hence of $Z_{\tilde{\mathfrak{H}}}(T_1)$, is mapped by S^- onto a subalgebra $S^-(\widetilde{AW}(3, Q_0))$ of $S^-(\tilde{\mathfrak{H}})$. From the basis (3.2) of $\widetilde{AW}(3, Q_0)$ we see that $S^-(\widetilde{AW}(3, Q_0))$ has basis

$$(ab-1)^{-1}K_0^n(K_1K_0)^iK_1^m(T_1 + ab) \qquad (m, n = 0, 1, 2, \ldots,\ i = 0, 1).$$

Compare this basis with the basis (3.2) of $AW(3,Q_0)$. Thus the map which sends a basis element $K_0^n(K_1K_0)^iK_1^m$ of $AW(3,Q_0)$ to a basis element $q^{n+i}(ab-1)^{-1}K_0^n(K_1K_0)^iK_1^m(T_1+ab)$ of $S^-(\widetilde{AW}(3,Q_0))$ extends linearly to a linear bijection from $AW(3,Q_0)$ onto $S^-(\widetilde{AW}(3,Q_0))$. In fact this map extends to an algebra isomorphism:

Proposition 4.1. *Let $AW(3,Q_0;qa,qb,c,d)$ be $AW(3,Q_0)$ with a, b replaced by qa, qb, respectively. Then there is a well-defined algebra isomorphism from $AW(3,Q_0;qa,qb,c,d)$ onto $S^-(\widetilde{AW}(3,Q_0))$ given by the map*

$$U\mapsto(ab-1)^{-1}\widetilde{U}(T_1+ab),\tag{4.1}$$

where $U\mapsto\widetilde{U}$ sends words U in $AW(3,Q_0;qa,qb,c,d)$ involving K_0, K_1 to corresponding words $\widetilde{U}$ in $\widetilde{AW}(3,Q_0)$ involving qK_0, K_1.

Proof. Consider the linear map $U\mapsto\widetilde{U}$ as a map to $\widetilde{AW}(3,Q_0)$ from the free algebra generated by K_0, K_1, T_1 with T_1 central such that it sends a word involving K_0, K_1, T_1 to the corresponding word involving qK_0, K_1, T_1 in $\widetilde{AW}(3,Q_0)$. This map is an algebra homomorphism. Composing it with S^- yields the map (4.1) which is again an algebra homomorphism. Now we have to check that R is sent to zero by the map (4.1) if $R=0$ is a relation for $AW(3,Q_0;qa,qb,c,d)$. This is clearly the case for $R:=T_1+1$, since $(T_1+1)(T_1+ab)=0$. To see this for the other relations, rewrite relations (2.14)–(2.16) for $\widetilde{AW}(3,Q_0)$ as:

$$\begin{aligned}
&(q+q^{-1})K_1K_0K_1-K_1^2K_0-K_0K_1^2-\big(B+(ab-1)E\big)K_1-C_0\,K_0-\big(D_0+(ab-1)F_0\big)\\
&\qquad-(E\,K_1+F_0)(T_1+1)=0,\\
&(q+q^{-1})K_0K_1K_0-K_0^2K_1-K_1K_0^2-\big(B+(ab-1)E\big)K_0-C_1\,K_1-\big(D_1+(ab-1)F_1\big)\\
&\qquad-(E\,K_0+F_1)(T_1+1)=0,\\
&K_1K_0K_1K_0-(q^2+1+q^{-2})K_0K_1K_0K_1+(q+q^{-1})K_0^2K_1^2-((q+q^{-1})(C_0K_0^2+C_1K_1^2)\\
&\qquad+\big(B+(ab-1)E\big)\big((q+1+q^{-1})K_0K_1+K_1K_0\big)+(q+1+q^{-1})\big(D_0+(ab-1)F_0\big)K_0\\
&\qquad+(q+1+q^{-1})\big(D_1+(ab-1)F_1\big)K_1+(ab-1)G-Q_0+(E\,K_1K_0+G)(T_1+1)\\
&\quad+(q+1+q^{-1})(E\,K_0K_1+F_0\,K_0+F_1\,K_1)(T_1+1)=0.
\end{aligned}$$

On multiplication with T_1+ab we see that the identities $R_i(T_1+ab)=0$ $(i=1,2,3)$ must be valid in $S^-(\widetilde{AW}(3,Q_0))$, where

$$\begin{aligned}
R_1:={}&(q+q^{-1})K_1K_0K_1-K_1^2K_0-K_0K_1^2-\big(B+(ab-1)E\big)K_1\\
&-C_0\,K_0-D_0-(ab-1)F_0,\\
R_2:={}&(q+q^{-1})K_0K_1K_0-K_0^2K_1-K_1K_0^2-(B+(ab-1)E\big)K_0\\
&-C_1\,K_1-D_1-(ab-1)F_1,\\
R_3:={}&K_1K_0K_1K_0-(q^2+1+q^{-2})K_0K_1K_0K_1+(q+q^{-1})K_0^2K_1^2\\
&-((q+q^{-1})(C_0K_0^2+C_1K_1^2)+\big(B+(ab-1)E\big)\big((q+1+q^{-1})K_0K_1+K_1K_0\big)\\
&+(q+1+q^{-1})\big(D_0+(ab-1)F_0\big)K_0+(q+1+q^{-1})\big(D_1+(ab-1)F_1\big)K_1\\
&+(ab-1)G-Q_0.
\end{aligned}$$

Now consider relations (2.4), (2.5) en (2.9) for $AW(3,Q_0;qa,qb,c,d)$ (so with a, b in the structure constants replaced by qa, qb). These can be written in the form $U_i=0$ $(i=1,2,3)$ by bringing everything to the left-hand side in the relations. Now consider U_i as elements of the free algebra generated by K_0, K_1. For the images under the map (4.1) we then obtain the following elements of $S^-(\widetilde{AW}(3,Q_0))$:

$$\widetilde{U}_1=(1-ab)^{-1}qR_1,\qquad\widetilde{U}_2=(1-ab)^{-1}q^2R_2,\qquad\widetilde{U}_3=(1-ab)^{-1}q^2R_3,$$

which are all zero. ∎

Theorem 4.2. *$S^-(\tilde{\mathfrak{H}}) = S(\widetilde{AW}(3,Q_0))$, so the antispherical subalgebra $S^-(\tilde{\mathfrak{H}})$ is isomorphic to the algebra $AW(3,Q_0;qa,qb,c,d)$ by the map (4.1) sending $AW(3,Q_0;qa,qb,c,d)$ to $S^-(\tilde{\mathfrak{H}})$.*

The proof is analogous to the proof of Theorem 3.2. Since $\tilde{\mathfrak{H}}$ has a basis (3.4), $S^-(\tilde{\mathfrak{H}})$ is spanned by the elements

$$(T_1+ab)Z^mY^n(T_1+ab) \qquad (m,n\in\mathbb{Z}).$$

Recall Definition 3.3. Theorem 4.2 will follow by induction with respect to $|m|+|n|$ from the following analogue of Lemma 3.4:

Lemma 4.3. *Let $m,n\in\mathbb{Z}$. Then*

$$(T_1+ab)Z^mY^n(T_1+ab)\in(T_1+ab)\Big(\widetilde{AW}(3,Q_0)+o(Z^mY^n)\Big)(T_1+ab).$$

Proof. We use the same procedure as in the proof of Lemma 3.4. I will only list the main formulas in the three steps. The reader can verify these formulas in an analogous way as in the proof of Lemma 3.4.

Step 1

$$\begin{aligned}
&(T_1+ab)Z^m(T_1+ab)=ab\Big(Z^m+Z^{-m}+o(Z^m)\Big)(T_1+ab),\\
&(T_1+ab)Z^{-m}(T_1+ab)=-\Big(Z^m+Z^{-m}+o(Z^m)\Big)(T_1+ab),\\
&(T_1+ab)Y^n(T_1+ab)=-\Big(Y^n+(q^{-1}abcd)^nY^{-n}+o(Y^n)\Big)(T_1+ab),\\
&(T_1+ab)Y^{-n}(T_1+ab)=ab\Big((q^{-1}abcd)^{-n}Y^n+Y^{-n}+o(Y^n)\Big)(T_1+ab),\\
&(T_1+ab)Z^mY^n(T_1+ab)=\Big(abZ^mY^n-(q^{-1}abcd)^nZ^{-m}Y^{-n}+o(Z^mY^n)\Big)(T_1+ab),\\
&(T_1+ab)Z^{-m}Y^n(T_1+ab)=\Big(-(ab+1)Z^mY^n-(q^{-1}abcd)^nZ^mY^{-n}-Z^{-m}Y^n\\
&\qquad\qquad\qquad\qquad\qquad\quad+o(Z^mY^n)\Big)(T_1+ab),\\
&(T_1+ab)Z^mY^{-n}(T_1+ab)=\Big(ab(q^{-1}abcd)^{-n}Z^{-m}Y^n+abZ^mY^{-n}+(1+ab)Z^{-m}Y^{-n}\\
&\qquad\qquad\qquad\qquad\qquad\quad+o(Z^mY^n)\Big)(T_1+ab),\\
&(T_1+ab)Z^{-m}Y^{-n}(T_1+ab)=\Big(ab(q^{-1}abcd)^{-n}Z^mY^n-Z^{-m}Y^{-n}+o(Z^mY^n)\Big)(T_1+ab).
\end{aligned}$$

Step 2

$$\begin{aligned}
&K_1^m(T_1+ab)=\Big(Z^m+Z^{-m}+o(Z^m)\Big)(T_1+ab),\\
&K_0^n(T_1+ab)=\Big(Y^n+(q^{-1}abcd)^nY^{-n}+o(Y^n)\Big)(T_1+ab),\\
&K_1^mK_0^n(T_1+ab)=\Big(Z^mY^n+Z^{-m}Y^n+(q^{-1}abcd)^n\big(Z^mY^{-n}+Z^{-m}Y^{-n}\big)\\
&\qquad\qquad\qquad\quad+o(Z^mY^n)\Big)(T_1+ab),\\
&K_1^{m-1}K_0K_1K_0^{n-1}(T_1+ab)=\Big(qZ^mY^n+q^{-1}Z^{-m}Y^n+q^{-1}(q^{-1}abcd)^nZ^mY^{-n}\\
&\qquad+(qab)^{-1}(q^{-1}abcd)^n(1+ab-q^2)Z^{-m}Y^{-n}+o(Z^mY^n)\Big)(T_1+ab).
\end{aligned}$$

Step 3

$$
\begin{aligned}
(T_1+ab)Z^mY^n(T_1+ab) &= \frac{ab}{1-q^2}\Big(K_1^{m-1}(K_1K_0-qK_0K_1)K_0^{n-1}+o(Z^mY^n)\Big)(T_1+ab),\\
(T_1+ab)Z^{-m}Y^n(T_1+ab) &= \frac{1}{1-q^2}\Big(K_1^{m-1}(-(1+ab-q^2)K_1K_0+qabK_0K_1)K_0^{n-1}\\
&\quad +o(Z^mY^n)\Big)(T_1+ab),\\
(T_1+ab)Z^mY^{-n}(T_1+ab) &= \frac{qab}{(1-q^2)(q^{-1}abcd)^n}\Big(K_1^{m-1}(-qK_1K_0+K_0K_1)K_0^{n-1}\\
&\quad +o(Z^mY^n)\Big)(T_1+ab),\\
(T_1+ab)Z^{-m}Y^{-n}(T_1+ab) &= \frac{ab}{(1-q^2)(q^{-1}abcd)^n}\Big(K_1^{m-1}(K_1K_0-qK_0K_1)K_0^{n-1}\\
&\quad +o(Z^mY^n)\Big)(T_1+ab). \qquad \blacksquare
\end{aligned}
$$

Remark 4.4. The same q-shift for the parameters as in Proposition 4.1 occurs in [7, (3.17)] (the eigenfunction of $Y+q^{-1}abcdY^{-1}$ in $\mathcal{A}^-_{\rm sym}$ expressed in terms of Askey–Wilson polynomials). Of course, these two results are very much related to each other.

Let $\tilde{\mathfrak{H}}(qa,qb,c,d)$ be $\tilde{\mathfrak{H}}$ with parameters a, b, c, d replaced by qa, qb, c, d, respectively. If we compare Theorems 3.2 and 4.2 then we can conclude that the spherical subalgebras $S^-(\tilde{\mathfrak{H}})$ and $S(\tilde{\mathfrak{H}}(qa,qb,c,d))$ are isomorphic. This result is an analogue of the result in [1, Proposition 4.11] for Cherednik algebras.

Remark 4.5. Just as we had in Step 1 of the proofs of Lemma 3.4 and Lemma 4.3, we can derive from the first, fourth and fifth relation in (2.12) that

$$
\begin{aligned}
&(T_1+1)(Y+q^{-1}a^2b^2cdY^{-1}-(q^{-1}abcd+ab))(T_1+1)=0,\\
&(T_1+ab)(Y+q^{-1}cdY^{-1}-(q^{-1}cd+1))(T_1+ab)=0.
\end{aligned}
$$

It follows that, in the basic representation of $\tilde{\mathfrak{H}}$, the operator

$$D^- := Y+q^{-1}a^2b^2cdY^{-1}-(q^{-1}abcd+ab)$$

maps $\mathcal{A}_{\rm sym}$ into $\mathcal{A}^-_{\rm sym}$, while

$$D^+ := Y+q^{-1}cdY^{-1}-(q^{-1}cd+1)$$

maps $\mathcal{A}^-_{\rm sym}$ into $\mathcal{A}_{\rm sym}$. Since both operators preserve the eigenspace of λ_n, we obtain

$$
\begin{aligned}
&D^-\,P_n=ab(q^{-1}cd-q^{-n})(q^n-1)\,Q_n,\\
&D^+\,Q_n=(q^{-1}cd-q^{-n}a^{-1}b^{-1})(q^nab-1)\,P_n,
\end{aligned}
$$

where the Askey–Wilson polynomial P_n and the shifted Askey–Wilson polynomial Q_n are given by (2.1) and (2.13), respectively, and the constant factors in the above identities follow by comparing coefficients of z^n. Thus the operators D^- and D^+ can be considered as *shift operators.*

The observations in this Remark were earlier made (in different notation) in [9, Lemma 12.2, Proposition 12.3].

5 Centralizers and centers

As a corollary of Theorem 3.2 and Theorem 4.2 we obtain:

Theorem 5.1. *The centralizer* $Z_{\tilde{\mathfrak{H}}}(T_1)$ *is equal to* $\widetilde{AW}(3, Q_0)$.

Proof. Write $U \in \tilde{\mathfrak{H}}$ as

$$U = (1 - ab)^{-1}U(T_1 + 1) + (ab - 1)^{-1}U(T_1 + ab). \tag{5.1}$$

Suppose that $U \in Z_{\tilde{\mathfrak{H}}}(T_1)$. Then

$$U(T_1 + 1) = (1 - ab)^{-1}(T_1 + 1)U(T_1 + 1),$$
$$U(T_1 + ab) = (ab - 1)^{-1}(T_1 + ab)U(T_1 + ab).$$

So $U(T_1+1) \in S(\tilde{\mathfrak{H}}) = S(\widetilde{AW}(3, Q_0)) \subset \widetilde{AW}(3, Q_0)$ and $U(T_1+ab) \in S^-(\tilde{\mathfrak{H}}) = S^-(\widetilde{AW}(3, Q_0)) \subset \widetilde{AW}(3, Q_0)$. ■

The following theorem is interesting in its own right, but it can also be used, in combination with Theorems 3.2 and 4.2, in order to show that the center of $\tilde{\mathfrak{H}}$ consists of the scalars (see Theorem 5.3). The proof is in the same spirit as the proof of the faithfulness of the basic representation of $AW(3, Q_0)$ (see [7, Theorem 2.2]).

Theorem 5.2. *The center of the algebra* $AW(3, Q_0)$ *consists of the scalars.*

Proof. Let U be in the center of $AW(3, Q_0)$. Because of Theorem 2.2 we may consider $AW(3, Q_0)$ in its faithful basic representation on $\mathcal{A}_{\mathrm{sym}}$. Then U can be uniquely expanded in terms of the basis of $AW(3, Q_0)$ in this representation:

$$U = \sum_{k,l} a_{k,l} D^l_{\mathrm{sym}}(Z + Z^{-1})^k + \sum_{k,l} b_{k,l} D^{l-1}_{\mathrm{sym}}(Z + Z^{-1})D_{\mathrm{sym}}(Z + Z^{-1})^{k-1}. \tag{5.2}$$

Since U is in the center, we have

$$U(Z + Z^{-1}) - (Z + Z^{-1})U = 0. \tag{5.3}$$

We will first show that if U given by (5.2) satisfies (5.3), then all coefficients $a_{k,l}$ and $b_{k,l}$ in (5.2) vanish except possibly for coefficients $a_{k,l}$ with $l = 0$. Indeed, suppose that this is not the case. Then there is a highest value m of k for which $a_{k,l} \neq 0$ or $b_{k,l} \neq 0$ for some $l \geq 1$. All terms in (5.2) with $k > m$ then will certainly commute with $Z + Z^{-1}$, so we may assume that the terms in (5.2) with $k > m$ vanish while (5.3) still holds. Let both sides of (5.3) act on the Askey–Wilson polynomial $P_j[z]$:

$$(U(Z + Z^{-1}) - (Z + Z^{-1})U)P_j[z] = 0.$$

Expand the left-hand side of the above equation in terms of Askey–Wilson polynomials $P_i[z]$. Then the highest occurring term will be for $i = j + m + 1$, so the coefficient of $P_{j+m+1}[z]$ in this expansion must be zero. This gives

$$\sum_l (a_{m,l}\lambda^l_{j+m+1} + b_{m,l}\lambda^{l-1}_{j+m+1}\lambda_{j+m}) - \sum_l (a_{m,l}\lambda^l_{j+m} + b_{m,l}\lambda^{l-1}_{j+m}\lambda_{j+m-1}) = 0. \tag{5.4}$$

We have, writing $x := q^{j+m}$ and $u := q^{-1}abcd$,

$$\lambda_{j+m+1} = q^{-1}x^{-1} + qux, \qquad \lambda_{j+m} = x^{-1} + ux, \qquad \lambda_{j+m-1} = qx^{-1} + q^{-1}ux.$$

We can consider the identity (5.4) as an identity for Laurent polynomials in x. Since the left-hand side vanishes for infinitely many values of x, it must be identically zero. Let n be the maximal $l > 0$ for which $a_{m,l} \neq 0$ or $b_{m,l} \neq 0$. Then, in particular, the coefficients of x^{-n} and x^n in the left-hand side of (5.4) must be zero. This gives explicitly:

$$a_{m,n}u^n(1-q^n) + q^{-1}b_{m,n}u^n(1-q^n) = 0, \qquad a_{m,n}(1-q^{-n}) + qb_{m,n}(1-q^{-n}) = 0.$$

Now $n > 0$, so $q^{\pm n} \neq 1$. Also $u \neq 0$. Hence,

$$a_{m,n} + q^{-1}b_{m,n} = 0, \qquad a_{m,n} + qb_{m,n} = 0.$$

Thus $a_{m,n} = 0 = b_{m,n}$, which is a contradiction.

So U in the center will have the form $U = \sum_k a_k(Z+Z^{-1})^k$ in the basic representation of $AW(3,Q_0)$. We have to show that $a_k = 0$ for $k > 0$. Suppose not. Then there is a highest value $m > 0$ of k for which $a_k \neq 0$. Then we have

$$UD_{\rm sym}(1) - D_{\rm sym}U(1) = 0.$$

Expand the left-hand side of the above equation in terms of Askey–Wilson polynomials $P_i[z]$. Then the coefficient of $P_m[z]$ will be zero. So $a_m\lambda_0 - a_m\lambda_m = 0$. Since $\lambda_m \neq \lambda_0$ if $m \neq 0$, we conclude that $a_m = 0$, a contradiction. ■

Theorem 5.3. *The center $Z(\tilde{\mathfrak{H}})$ of $\tilde{\mathfrak{H}}$ consists of the scalars.*

Proof. Let $U \in Z(\tilde{\mathfrak{H}})$. Then $U \in Z_{\tilde{\mathfrak{H}}}(T_1) = \widetilde{AW}(3,Q_0)$. So $U \in Z(\widetilde{AW}(3,Q_0))$. Write U as in (5.1). We have to show that $U(T_1+1)$ and $U(T_1+ab)$ are scalars. This follows from

$$U(T_1+1) = (1-ab)^{-1}S(U) \in Z(S(\tilde{\mathfrak{H}})),$$
$$U(T_1+ab) = (ab-1)^{-1}S^-(U) \in Z(S^-(\tilde{\mathfrak{H}})).$$

Now use the algebra isomorphisms from Theorems 3.2 and 4.2, and apply Theorem 5.2. ■

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