Fine Gradings of Low-Rank Complex Lie Algebras and of Their Real Forms*

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Abstract. In this review paper, we treat the topic of fine gradings of Lie algebras. This concept is important not only for investigating the structural properties of the algebras, but, on top of that, the fine gradings are often used as the starting point for studying graded contractions or deformations of the algebras. One basic question tackled in the work is the relation between the terms 'grading' and 'group grading'. Although these terms have originally been claimed to coincide for simple Lie algebras, it was revealed later that the proof of this assertion was incorrect. Therefore, the crucial statements about one-to-one correspondence between fine gradings and MAD-groups had to be revised and re-formulated for fine group gradings instead. However, there is still a hypothesis that the terms 'grading' and 'group grading' coincide for simple complex Lie algebras. We use the MAD-groups as the main tool for finding fine group gradings of the complex Lie algebras $A_3 \cong D_3$, $B_2 \cong C_2$, and D_2. Besides, we develop also other methods for finding the fine (group) gradings. They are useful especially for the real forms of the complex algebras, on which they deliver richer results than the MAD-groups. Systematic use is made of the faithful representations of the three Lie algebras by 4×4 matrices: $A_3 = sl(4, \mathbb{C})$, $C_2 = sp(4, \mathbb{C})$, $D_2 = o(4, \mathbb{C})$. The inclusions $sl(4, \mathbb{C}) \supset sp(4, \mathbb{C})$ and $sl(4, \mathbb{C}) \supset o(4, \mathbb{C})$ are important in our presentation, since they allow to employ one of the methods which considerably simplifies the calculations when finding the fine group gradings of the subalgebras $sp(4, \mathbb{C})$ and $o(4, \mathbb{C})$.

Key words: Lie algebra; real form; MAD-group; automorphism; grading; group grading; fine grading

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1 Introduction

Gradings of Lie algebras have been explicitly used for more than fifty years. Probably the most notorious application is the $\mathbb{Z}_2$-grading from the work of E. Inönü and E. Wigner [16]. Another well known fact is that $\mathbb{Z}_2$-gradings play an important role for classification of real forms of simple Lie algebras.

It was in 1989 that systematic studying of gradings of Lie algebras has started in an article by J. Patera and H. Zassenhaus [24]. In that article they have introduced the terms of fine grading and group grading, and investigated the role of automorphisms for construction of gradings. A number of works followed [8, 9, 10, 11, 12, 13], using the theoretical results of that article for applications on concrete Lie algebras. In recent years, gradings were intensively studied not only on the classical finite-dimensional Lie algebras in [1, 2], but also on the exceptional Lie algebras in [4, 5, 6].

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The area where usage of gradings has led to the most fruitful results is construction of contractions of Lie algebras [3, 14, 15]. Apart from the classical (i.e. continuous) contractions, the gradings of Lie algebras enabled to construct new contractions of a different type. These contractions are called discrete, since they cannot be obtained by means of continuous process. Interesting results are also obtained when using the fine gradings for construction of deformations of the algebras.

The main results of that fundamental article by J. Patera and H. Zassenhaus comprise a statement about equivalence of the terms grading and group grading, and a statement about one-to-one correspondence between fine gradings of finite-dimensional simple complex Lie algebras and MAD-groups. Therefrom, the crucial consequence was derived that for description of all fine gradings of a finite-dimensional simple complex Lie algebra it is sufficient to classify all the MAD-groups, which was done in the article [10] in 1998. On the basis of this classification, fine gradings for several low-rank Lie algebras were found [19, 20, 21].

A break-through moment has come when A. Elduque revealed that the proof in [24] of the one-to-one correspondence between fine gradings and MAD-groups was not correct. He gave an example in [7] of a 16-dimensional complex non-simple Lie algebra whose grading subspaces cannot be indexed by group neither semigroup elements. It was just the coincidence of the terms grading and group grading, on which the proof of the one-to-one correspondence between the fine gradings and the MAD-groups on finite-dimensional simple complex Lie algebras was based. Therefore, the relation between the MAD-groups and the fine gradings remains an open problem. Nevertheless, the efforts to find a counterexample on a finite-dimensional simple complex Lie algebra that would contradict that statement (of one-to-one correspondence) were unsuccessful so far.

In reaction to this revelation, the results in our articles [19, 20, 21] need to be revised. We have proved a statement about one-to-one correspondence between the MAD-groups and the fine group gradings. This statement holds for all finite-dimensional (not only simple) complex Lie algebras. The results in [19, 20, 21] thus remain valid when we replace the term fine grading by fine group grading.

In parallel to complex Lie algebras, we have studied also fine gradings of real Lie algebras. For them, no statement about one-to-one correspondence between the fine gradings and the MAD-groups has ever been asserted, and therefore the mistake explained above has not affected the results. The article [22] describes some fine group gradings of real forms of the algebras $sl(4,\mathbb{C})$, $o(4,\mathbb{C})$, and $sp(4,\mathbb{C})$. We have developed several methods for constructing these gradings, and we managed to show that there exist fine group gradings of real forms which are not generated by any MAD-group of the respective real form.

In this whole work, we only devote our attention to fine gradings, but we do not investigate their coarsenings. In this context, we appreciate the result of [4], proving a theorem by use of which all coarsenings of group gradings of finite-dimensional simple complex Lie algebras can be obtained.

2 Gradings of Lie algebras

2.1 Definition of a grading

A *grading* of a Lie algebra L is a decomposition Γ of the vector space L into vector subspaces $L_j \neq \{0\}$, $j \in \mathcal{J}$ such that L is a direct sum of these subspaces L_j, and, for any pair of indices $j,k \in \mathcal{J}$, there exists $l \in \mathcal{J}$ such that $[L_j, L_k] \subseteq L_l$. We denote the grading by

$$\Gamma : L = \bigoplus_{j\in\mathcal{J}} L_j.$$

Clearly, for any Lie algebra $L \neq \{0\}$, there exists the trivial grading $\Gamma : L = L$, i.e. the Lie algebra is not split up at all. The opposite extreme of splitting the Lie algebra into as many subspaces L_j as possible is more interesting and useful, though.

We call a grading $\widetilde{\Gamma} : L = \oplus_{j\in\mathcal{J}} \oplus_{i\in\mathcal{J}_j} L_{ji}$ a *refinement* of the grading $\Gamma : L = \oplus_{j\in\mathcal{J}} L_j$, when $\oplus_{i\in\mathcal{J}_j} L_{ji} = L_j$ for each $j \in \mathcal{J}$. A grading Γ is called *fine* if each refinement $\widetilde{\Gamma}$ of Γ is equal to Γ itself (i.e. Γ does not have any proper refinement $\widetilde{\Gamma} \neq \Gamma$). A grading is necessarily fine when all its grading subspaces are one-dimensional. Such gradings are called *finest*, and they have the practical effect of defining immediately a basis of the Lie algebra.

In the opposite direction, we can obtain any grading of L from some fine grading of L, by merging some grading subspaces together. Such process is called a *coarsening* of the grading. The relations of refinement and coarsening define an ordering on the set of all gradings of a given Lie algebra L. Practically, we can illustrate the ordering in a hierarchy, whose bottom nodes represent the fine gradings of L and whose top node is the trivial grading (the whole Lie algebra L itself). If a grading Γ_r is connected in the hierarchy by an edge with a grading Γ_s on a lower level, it means that Γ_s is a refinement of Γ_r, and that Γ_r is a coarsening of Γ_s.

There are three special grading types (Cartan, Pauli, and orthogonal), which are known on the infinite series of classical complex Lie algebras; Cartan gradings on all of the series A_n, B_n, C_n, D_n, Pauli gradings and orthogonal gradings on A_n only:

- Cartan grading is the most notorious example. It is derived from the theory of root decomposition, and thus it is often referred to as the root grading.
- Pauli grading is a decomposition into powers of generalized matrices $P_m, Q_m \in \mathbb{C}^{m\times m}$, as introduced in [23]:

$$Q_m = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}, \qquad P_m = \operatorname{diag}\left(1, \omega, \omega^2, \dots, \omega^{m-1}\right), \quad \text{where } \ \omega = \exp\left(\frac{2\pi i}{m}\right).$$

- The orthogonal grading, introduced in [17], is referred to as orthogonal, since any pair of the grading subspaces is mutually orthogonal with respect to the scalar product defined by $(A, B) = \operatorname{tr}(AB^+)$.

All gradings of these special types are fine, with the only exception of the Cartan grading on the non-simple algebra $D_2 = o(4, \mathbb{C})$.

Let us demonstrate the basic grading terminology on the well explored algebra $sl(2, \mathbb{C})$. This algebra has four gradings; two of them are fine, and even finest (Υ_1 – the Cartan grading, Υ_2 – the Pauli grading coinciding with the orthogonal grading), one is the trivial grading $sl(2, \mathbb{C})$, and the last one (Υ_0) is neither trivial, nor fine:

$$\begin{aligned}
\Upsilon_0 &: sl(2,\mathbb{C}) = \operatorname{span}^{\mathbb{C}}\left\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right\} \oplus \operatorname{span}^{\mathbb{C}}\left\{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\right\}, \\
\Upsilon_1 &: sl(2,\mathbb{C}) = \operatorname{span}^{\mathbb{C}}\left\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right\} \oplus \operatorname{span}^{\mathbb{C}}\left\{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\right\} \oplus \operatorname{span}^{\mathbb{C}}\left\{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\right\}, \\
\Upsilon_2 &: sl(2,\mathbb{C}) = \operatorname{span}^{\mathbb{C}}\left\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right\} \oplus \operatorname{span}^{\mathbb{C}}\left\{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right\} \oplus \operatorname{span}^{\mathbb{C}}\left\{\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right\}.
\end{aligned}$$

The notion $\operatorname{span}^{\mathbb{F}}\{M\}$ stands for the linear hull of the set M over the field $\mathbb{F}$.

2.2 Equivalence of gradings

The basic property of each automorphism $h \in \operatorname{Aut} L$ is that it preserves the commutation relations between the grading subspaces L_j of any grading Γ of L: Clearly, for $[L_j, L_k] \subseteq L_l$, we have $[h(L_j), h(L_k)] = h([L_j, L_k]) \subseteq h(L_l)$, and thus the grading $\Gamma : L = \oplus_{j\in\mathcal{J}} L_j$ transformed by the automorphism $h \in \operatorname{Aut} L$ gives rise to a new grading $\widetilde{\Gamma} : L = \oplus_{j\in\mathcal{J}} h(L_j)$. This grading $\widetilde{\Gamma}$ is generally different from Γ, however, its structure (meaning the commutation relations between the grading subspaces, dimensions of the grading subspaces, etc.) is the same as in Γ. Therefore, from the structural perspective, the two gradings $\widetilde{\Gamma}$ and Γ equivalent. More precisely, two gradings $\Gamma : L = \oplus_{j\in\mathcal{J}} L_j$ and $\widetilde{\Gamma} : L = \oplus_{k\in\mathcal{K}} M_k$ are called *equivalent*, when there exist an automorphism $h \in \operatorname{Aut} L$ and a bijection $\pi : \mathcal{J} \mapsto \mathcal{K}$ such that $h(L_j) = M_{\pi(j)}$ for each $j \in \mathcal{J}$. The equivalence is denoted by $\Gamma \cong \widetilde{\Gamma}$.

2.3 Group gradings

Now we describe a specific type of a grading, namely a so-called group grading. The most notorious case of a group grading is the $\mathbb{Z}_2$-grading introduced by E. Inönü and E. Wigner when decomposing a Lie algebra L into two non-zero grading subspaces L_0 and L_1, where

$$[L_0, L_0] \subseteq L_0, \qquad [L_0, L_1] \subseteq L_1, \qquad [L_1, L_1] \subseteq L_0.$$

A grading $\Gamma : L = \oplus_{j\in\mathcal{J}} L_j$ is called a *group grading* if the index set $\mathcal{J}$ is a subset of an Abelian group G (whose binary operation is denoted by $*$), and, for any pair of indices $j, k \in \mathcal{J}$, it holds that

$$[L_j, L_k] \neq \{0\} \quad \Rightarrow \quad [L_j, L_k] \subseteq L_{j*k}. \tag{1}$$

Such group grading is often called a *G-grading* of the Lie algebra L. If a refinement of a group grading is again a group grading, then we call it a *group refinement*.

For a G-grading of a finite-dimensional Lie algebra L, we can assume, without loss of generality, that the group G is finitely generated. If not, then G can be replaced by its subgroup generated by the elements of the finite index set $\mathcal{J}$.

Two basic questions arise with respect to the group gradings:

- Does the group G exist for each grading? In other words, is each grading also a group grading?
- In case the group G exists, is it determined uniquely for the respective group grading?

The latter question has an easy answer: the choice of the group G is not unique. However, there is one significant case among all the possible index groups generated by $\mathcal{J}$, called the *universal* (grading) *group*. It has the interesting property (shown in [4]) that the index set of any coarsening of the original grading is embedable into an image of the universal group by some group epimorphism.

The question whether each grading is also a group grading seemed to be positively answered in [24]. However, A. Elduque in [7] gave an example of a grading (on a 16-dimensional complex non-simple algebra), whose grading subspaces cannot be indexed by elements of any Abelian group neither semigroup while satisfying the commutation relations (1).

The distinction between group and semigroup is important in these considerations. Another example of a grading whose grading subspaces cannot be indexed by group elements is the Cartan-graded $D_2 = o(4, \mathbb{C})$, in which we further split the two-dimensional Cartan subspace into two one-dimensional subspaces while retaining the grading properties. This fine grading cannot be indexed by any group, but it can be indexed by a semigroup. Analogous situation as in this example would repeat for any semisimple non-simple complex Lie algebra, wherein we further refine its non-fine Cartan grading into a fine grading, whose grading subspaces can be indexed by a semigroup, but not by any group.

It has been proved in [4] for gradings of simple complex Lie algebras that if we embed the grading indices into a semigroup, then they can be also embedded into a group.

3 Automorphisms and group gradings

In this section we show how group gradings can be constructed by means of automorphisms of the Lie algebra.

For a diagonalizable automorphism $g \in \operatorname{Aut} L$, the eigensubspaces L_j corresponding to different eigenvalues λ_j of the automorphism g compose a group grading of the algebra L. Indeed, for $X_j \in L_j$ and $X_k \in L_k$, we have $g([X_j, X_k]) = [g(X_j), g(X_k)] = [\lambda_j X_j, \lambda_k X_k] = \lambda_j \lambda_k [X_j, X_k]$. Thus the commutator of X_j and X_k is either zero, or it lies in the eigensubspace of g corresponding to the eigenvalue $\lambda_j \lambda_k$. As g is diagonalizable, the sum of all its eigensubspaces makes up the whole Lie algebra L, and therefore, the decomposition $\Gamma : L = \oplus_j L_j$ into eigensubspaces L_j of g is a grading of L. We denote $\Gamma = \operatorname{Gr}(g)$. Since each automorphism is a non-singular mapping, the spectrum of g does not contain 0. Let us define G as the smallest subgroup of the multiplicative group $\mathbb{C} \setminus \{0\}$ such that G contains the whole spectrum $\sigma(g)$ of the automorphism g. Then we can take the spectrum of g for the index set $\mathcal{J} \subseteq G$, thus obtaining the group grading $\Gamma : L = \oplus_{j \in \mathcal{J}} L_j$.

Now let us consider a set $\mathcal{G} = (g_p)_{p \in \mathcal{P}}$ of diagonalizable mutually commuting automorphisms $g_p \in \operatorname{Aut} L$; the set $\mathcal{G}$ can be finite or infinite. We can find a basis of L consisting of simultaneous eigenvectors of all the automorphisms $g_p \in \mathcal{G}$. Taking a direct sum of the simultaneous eigensubspaces of all the automorphisms $g_p \in \mathcal{G}$, we obtain a group grading of L, which we denote by $\operatorname{Gr}(\mathcal{G})$. The grading subspaces are indexed by eigenvalues corresponding to the automorphisms g_p. Even if the set $\mathcal{G}$ is infinite, we can restrict ourselves to a finite number of elements $g_p \in \mathcal{G}$, $p = 1, \ldots, \ell$, which is sufficient for splitting the vector space into the simultaneous eigensubspaces, as well as for indexing these (grading) subspaces. (That is thanks to the finite dimension of L.) Again, the index set is a subset of the smallest subgroup G of the group $(\mathbb{C} \setminus \{0\})^\ell$, such that G contains $\sigma(g_1) \times \cdots \times \sigma(g_\ell)$. In other words, $\operatorname{Gr}(\mathcal{G})$ has the form $\operatorname{Gr}(\mathcal{G}) = \Gamma : L = \oplus_{j^1 \in \mathcal{J}^1} \cdots \oplus_{j^\ell \in \mathcal{J}^\ell} (L^1_{j^1} \cap \cdots \cap L^\ell_{j^\ell})$, where $\operatorname{Gr}(g_p) = \Gamma^p : L = \oplus_{j^p \in \mathcal{J}^p} L^p_{j^p}$ are the group gradings of L generated by the individual automorphisms g_p. For any subset $\tilde{\mathcal{G}} \subset \mathcal{G}$, the group grading $\operatorname{Gr}(\mathcal{G})$ is a group refinement of the group grading $\operatorname{Gr}(\tilde{\mathcal{G}})$.

Having described how a group grading of a Lie algebra L can be obtained by means of a given set of automorphisms, let us now approach the problem from the opposite direction, namely investigate the automorphisms related to a given grading. Let $\Gamma : L = \oplus_{j \in \mathcal{J}} L_j$ be a grading of L (not necessarily a group grading). The following notion will play an important role in our considerations:

$$\operatorname{Diag}(\Gamma) = \{g \in \operatorname{Aut} L \mid g/_{L_j} = \alpha_j \mathrm{Id} \text{ for any } j \in \mathcal{J}\}.$$

Directly from this definition we can derive the following:

- An automorphism g belongs to $\operatorname{Diag}(\Gamma)$ if and only if $g(X) = \lambda_j X$ for all $X \in L_j$, $j \in \mathcal{J}$, where $\lambda_j \neq 0$ depends only on $g \in \operatorname{Aut} L$ and on $j \in \mathcal{J}$.
- $\operatorname{Diag}(\Gamma)$ is a subgroup of $\operatorname{Aut} L$, all automorphisms in $\operatorname{Diag}(\Gamma)$ are diagonalizable and mutually commute.
- For a group grading $\Gamma = \operatorname{Gr}(\mathcal{G})$ generated by an arbitrary set $\mathcal{G}$ of mutually commuting diagonalizable automorphisms in $\operatorname{Aut} L$, it holds that $\operatorname{Diag}(\Gamma) \supseteq \mathcal{G}$.
- Let Γ be a grading of L (not necessarily a group grading). Then either $\operatorname{Gr}(\operatorname{Diag}(\Gamma)) = \Gamma$, or Γ is a proper refinement of $\operatorname{Gr}(\operatorname{Diag}(\Gamma))$.

There indeed do exist cases of such grading Γ which is a proper refinement of $\operatorname{Gr}(\operatorname{Diag}(\Gamma))$. The following theorem proves, however, that this case cannot occur for group gradings of finite-dimensional complex Lie algebras, and, moreover, that all group gradings of these algebras can be obtained by means of automorphisms.

Theorem 1. *Let* $\Gamma : L = \oplus_{j\in\mathcal{J}} L_j$ *be a group grading of a finite-dimensional complex Lie algebra. Then*

- *there exists a set of automorphisms* $\mathcal{G} \subseteq \operatorname{Aut} L$ *such that* $\operatorname{Gr}(\mathcal{G}) = \Gamma$*, and*
- $\operatorname{Gr}(\operatorname{Diag}(\Gamma)) = \Gamma$*.*

Remark 1. The fact that during the course of the proof of the Theorem 1 in [26] we define the set $\mathcal{G}$ in such a way that it is a group of automorphisms does not mean that we necessarily need whole groups of automorphisms for splitting the algebra into fine group gradings. On the contrary, in practice the set $\mathcal{G}$ fulfilling the role as stated in the Theorem 1 is a finite set of automorphisms containing just a very few elements.

Let us now direct our attention to fine group gradings. Remember that we may refine the grading by enlarging the set of mutually commuting diagonalizable automorphisms applied on the algebra. That is why we make use of the term MAD-group introduced by J. Patera and H. Zassenhaus: Let $\mathcal{G}$ be a subset of $\operatorname{Aut} L$ fulfilling the following properties:

- any $g \in \mathcal{G}$ is diagonalizable,
- $fg = gf$ for any $f, g \in \mathcal{G}$,
- if $h \in \operatorname{Aut} L$ is diagonalizable and $hg = gh$ for any $g \in \mathcal{G}$, then $h \in \mathcal{G}$.

Such $\mathcal{G}$ is called a *MAD-group* in $\operatorname{Aut} L$, which is an abbreviation for maximal Abelian group of diagonalizable automorphisms. (It follows obviously from the defining conditions imposed on the elements of $\mathcal{G}$ that the word ‘group’ in the notion is justified.)

Theorem 2. *Let* $\Gamma : L = \oplus_{j\in\mathcal{J}} L_j$ *be a group grading of a finite-dimensional complex Lie algebra* L*. Then* Γ *is a fine group grading if and only if the set* $\operatorname{Diag}(\Gamma)$ *is a MAD-group in* $\operatorname{Aut} L$*.*

It is this Theorem 2 to which we refer further in the text when talking about one-to-one correspondence between fine group gradings and MAD-groups of finite-dimensional complex Lie algebras. Both the Theorems 1 and 2 are proved in [26], or previously also in [18].

4 Methods for finding fine group gradings

Our aim is to find fine group gradings of the chosen Lie algebras, which are either complex or real (real forms of complex Lie algebras). In this process we use several methods, each of them applicable for the various algebras in a different way, and also with a difference in the strength of the result. In the sequel, we list all the various methods used and specify their applicability for the various Lie algebras in question.

4.1 MAD-group method

As stated in Theorem 2, there is a one-to-one correspondence between fine group gradings of a finite-dimensional complex Lie algebra L and MAD-groups of automorphisms in $\operatorname{Aut} L$.

One can expect that some subgroups of $\operatorname{Aut} L$ would generate equivalent gradings. That would of course be of no interest to us, since we are looking for gradings with different structural properties, i.e. non-equivalent. Luckily, there is an easy key between equivalent fine group gradings of L and the MAD-groups of automorphisms in $\operatorname{Aut} L$ that generate them; and it uses the term of conjugate sets: Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be subsets of $\operatorname{Aut} L$. We call these subsets *conjugate* when there exists an automorphism $h \in \operatorname{Aut} L$ such that $h\mathcal{G}_1 h^{-1} = \mathcal{G}_2$, and we denote conjugate subsets by $\mathcal{G}_1 \cong \mathcal{G}_2$.

Theorem 3. *Let $\mathcal{G}_1, \mathcal{G}_2 \in \operatorname{Aut} L$ be MAD-groups on a finite-dimensional complex Lie algebra L; let $\Gamma_1 = \mathrm{Gr}(\mathcal{G}_1)$ and $\Gamma_2 = \mathrm{Gr}(\mathcal{G}_2)$ be the fine group gradings of L generated by $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively. The gradings Γ_1 and Γ_2 are equivalent if and only if the MAD-groups $\mathcal{G}_1$ and $\mathcal{G}_2$ are conjugate.*

Remark 2. The equivalence in Theorem 3 is valid for fine group gradings and MAD-groups, but not for group gradings generated by any sets of automorphisms in general. There we would only have one implication in place, namely that conjugate subsets $\mathcal{G}_1, \mathcal{G}_2 \subseteq \operatorname{Aut} L$ generate equivalent group gradings $\mathrm{Gr}(\mathcal{G}_1) \cong \mathrm{Gr}(\mathcal{G}_2)$. However, if we consider the sets $\mathcal{G}_i$ not as just general subsets of $\operatorname{Aut} L$, but as the maximal sets of automorphisms that leave the gradings $\mathrm{Gr}(\mathcal{G}_i)$ invariant – i.e. $\mathcal{G}_i = \mathrm{Diag}(\mathrm{Gr}(\mathcal{G}_i))$, then we end up with equivalence again; namely that the (fine or non-fine) group gradings $\Gamma_i = \mathrm{Gr}(\mathcal{G}_i)$ are equivalent if and only if the sets $\mathcal{G}_i$ are conjugate.

Now we come to the applicability of this 'MAD-group' method for the various Lie algebras in question (remember that we keep on limiting ourselves only to the finite-dimensional classical complex Lie algebras and their real forms).

MAD-groups and fine group gradings of the algebras A_n, B_n, C_n, D_n

This is the most comfortable case, because, with the exception of the algebra D_4, the MAD-groups were fully classified for all the classical complex Lie algebras [10]. Therefore, we are able to find all the fine group gradings, as explained in Theorem 2. Nevertheless, the main question, which still remains unsolved, is whether this way leads to all the possible fine gradings.

The algebra D_2 is exceptional among the classical complex Lie algebras, because it is not simple, but only semisimple. It has a fine group grading which is not fine grading.

On simple complex Lie algebras, there has so far not been found any fine group grading, which would not be fine grading at the same time. That leaves open the hypothesis that for finite-dimensional simple complex Lie algebras the terms 'grading' and 'group grading' could coincide.

MAD-groups and fine group gradings of real forms of the complex Lie algebras

For real algebras, we unfortunately do not possess any analogue to the Theorems 1 and 2. Nevertheless, as well as for the classical complex Lie algebras A_n, B_n, C_n, D_n, also for the real forms of these complex algebras all the MAD-groups were classified [11].

MAD-groups of the real forms can be obtained from MAD-groups of the complex algebras as follows: Let $\mathcal{G}$ be a MAD-group on the complex algebra L, and let $\mathcal{G}^{\mathbb{R}} = \{g \in \mathcal{G} \,|\, \sigma(g) \subset \mathbb{R}\}$ be its so-called *real part*, namely the subgroup of $\mathcal{G}$ containing all automorphisms with real spectrum. This set $\mathcal{G}^{\mathbb{R}}$ is said to be maximal if there exists no such MAD-group $\widetilde{\mathcal{G}}$ on L (non-conjugate to $\mathcal{G}$) that $\mathcal{G}^{\mathbb{R}}$ is conjugate to some proper subgroup of $\widetilde{\mathcal{G}}^{\mathbb{R}}$. For all the classical complex Lie algebras L, except for D_4, it was proved in [11] that each MAD-group $\mathcal{F}$ on a real form $L_{\mathbf{J}}$ of L is equal to the maximal real part $\mathcal{G}^{\mathbb{R}}$ of some MAD-group $\mathcal{G}$ on L restricted onto the real form $L_{\mathbf{J}}$.

For general Lie algebras over $\mathbb{R}$ it has not been proved that MAD-groups generate fine group gradings. However, for real forms of the classical complex Lie algebras, it follows from the concrete construction of the MAD-groups that the gradings generated by these MAD-groups are already fine group gradings. The opposite direction is not true, though; i.e. not all of the fine group gradings of the real forms are generated by MAD-groups of these real forms, and not even on real forms of the classical simple complex Lie algebras. In [22] we have found several counterexamples, using some of the methods described below. All of them correspond to the cases where the universal group of the corresponding complex grading contains factors other than $\mathbb{Z}$ or $\mathbb{Z}_2$ (i.e. it contains e.g. $\mathbb{Z}_3$ or $\mathbb{Z}_4$ etc.); it is because the eigenvalues of orders higher than 2 are not elements of $\mathbb{R}$.

4.2 Displayed method

Searching for fine group gradings by means of MAD-groups is a laborious process. For the algebras $sl(m,\mathbb{C})$ we do not have any easier way, but for the other classical complex Lie algebras $o(m,\mathbb{C})$ and $sp(m,\mathbb{C})$, which are subalgebras of $sl(m,\mathbb{C})$, we can use the so-called 'displayed' method. This method is applicable not only for the complex Lie algebras $o(m,\mathbb{C})$ and $sp(m,\mathbb{C})$, but also for their real forms (which are subalgebras of the real forms of $sl(m,\mathbb{C})$).

The method consists in the following principle: A *subalgebra* $o_K(m,\mathbb{C})$ or $sp_K(m,\mathbb{C})$ of the Lie algebra $sl(m,\mathbb{C})$ is said to be *displayed by a grading* Γ of $sl(m,\mathbb{C})$, when $o_K(m,\mathbb{C})$ or $sp_K(m,\mathbb{C})$ respectively is equal to a direct sum of selected grading subspaces from Γ. Then, it necessarily holds that such a direct sum is also a grading of $o_K(m,\mathbb{C})$ or $sp_K(m,\mathbb{C})$ respectively.

It follows from the classification of MAD-groups on the classical complex Lie algebras that a fine grading Γ of $sl(m,\mathbb{C})$ displays $o_K(m,\mathbb{C})$ or $sp_K(m,\mathbb{C})$ if and only if the MAD-group $\mathcal{G} = \mathrm{Diag}(\Gamma)$ contains an outer automorphism Out_K with $K = K^T$ or $K = -K^T$ respectively. Let us explain this statement on the case of $o(m,\mathbb{C})$; the case of $sp(m,\mathbb{C})$ would be just analogous.

- Having the outer automorphism $\mathrm{Out}_K \in \mathcal{G}$ with $K = K^T$, the eigensubspace of Out_K corresponding to the eigenvalue $+1$ is the subalgebra $o_K(m,\mathbb{C})$.
- Any other element $g \in \mathcal{G}$ can then be restricted to $o_K(m,\mathbb{C})$ while preserving the $\mathbb{Z}_2$-grading generated by Out_K, because g and Out_K commute.
- The set $\{g|_{o_K(m,\mathbb{C})} \,|\, g \in \mathcal{G}\}$ is a MAD-group on $o_K(m,\mathbb{C})$, since every automorphism on $o_K(m,\mathbb{C})$ can be extended to an automorphism of $sl(m,\mathbb{C})$ commuting with Out_K.
- We can express the MAD-group $\mathcal{G}$ on $sl(m,\mathbb{C})$ in the form $\mathcal{G} = \mathcal{H} \cup \mathrm{Out}_K\mathcal{H}$, where $\mathcal{H}$ is the set of all inner automorphisms in $\mathcal{G}$. The corresponding MAD-group on $o_K(m,\mathbb{C})$ then has the form $\{g|_{o_K(m,\mathbb{C})} \,|\, g \in \mathcal{G}\} = \{g|_{o_K(m,\mathbb{C})} \,|\, g \in \mathcal{H}\}$. Thus, the subgroup $\mathcal{H}$, upon restriction onto $o_K(m,\mathbb{C})$, has the same splitting effect on the subalgebra $o_K(m,\mathbb{C})$ as the original group $\mathcal{G}$.

The 'displayed' method gives the same result in producing the fine group gradings as the 'MAD-group' method in cases of the complex subalgebras of $sl(m,\mathbb{C})$; in other words, we obtain all the fine group gradings for these complex algebras.

For real forms of $o(m,\mathbb{C})$ and $sp(m,\mathbb{C})$, the 'displayed' method in fact turns into making an intersection of the complex subalgebra $o(m,\mathbb{C})$ and $sp(m,\mathbb{C})$ respectively with a fine group grading of the relevant real form of $sl(m,\mathbb{C})$. This is a consequence of the fact that real forms of the subalgebras $o(m,\mathbb{C})$ and $sp(m,\mathbb{C})$ of the complex algebra $sl(m,\mathbb{C})$ are subalgebras of the real forms of $sl(m,\mathbb{C})$.

The gradings of the real forms of the subalgebras obtained by this 'displayed' method are fine group gradings, however, we have no certainty of getting all the fine group gradings of the respective real forms. Nevertheless, this method is still quite powerful and provides a high number of fine group gradings of the real forms.

4.3 Fundamental method

As can be seen from above, the 'MAD-group' method and the 'displayed' method are excellent when searching for fine group gradings of the complex Lie algebras, but not so strong in case of the real forms. Thus, we still need to broaden our scope by another method, which is specialized on (and applicable only for) the real forms. We call this method 'fundamental', because it derives directly from the definition of the real form. It assumes that we already dispose of the fine group gradings of the respective complex algebra.

Theorem 4. *Let $\Gamma : L = \oplus_{j\in\mathcal{J}} L_j$ be a fine group grading of a classical complex Lie algebra L. Let $\mathbf{J}$ be an involutive antiautomorphism on L, let $L_{\mathbf{J}}$ be the real form of L defined by $\mathbf{J}$, and let $Z_{j,l}$ be elements of L_j fulfilling the following properties:*

- *$(Z_{j,1}, \dots, Z_{j,l_j})$ is a basis of the grading subspace L_j for each $j \in \mathcal{J}$; i.e. $L_j = \operatorname{span}^{\mathbb{C}}(Z_{j,1}, \dots, Z_{j,l_j})$; and*
- *$\mathbf{J}(Z_{j,l}) = Z_{j,l}$; i.e. $Z_{j,l} \in L_{\mathbf{J}}$ for all $Z_{j,l} \in L_j$ for each $j \in \mathcal{J}$.*

Then the decomposition $\Gamma^{\mathbf{J}} : L_{\mathbf{J}} = \oplus_{j\in\mathcal{J}} L_j^{\mathbb{R}}$ into subspaces $L_j^{\mathbb{R}} = \operatorname{span}^{\mathbb{R}}(Z_{j,1}, \dots, Z_{j,l_j})$ is a fine group grading of the real form $L_{\mathbf{J}}$.

We then say that the fine group grading Γ of the complex Lie algebra L determines *the fine group grading $\Gamma^{\mathbf{J}}$ of the real form $L_{\mathbf{J}}$.*

This method in practice turns into rather laborious calculations, namely into looking for a suitable antiautomorphism $\mathbf{J}$ and a suitable basis of the complex algebra L, such that the basis vectors form not only the bases of the grading subspaces L_j of the complex grading Γ of L, but also lie in (and thus form the basis of) the real form $L_{\mathbf{J}}$.

Not even this method ensures finding all the fine group gradings of the real forms, however, we at least get as much as we can from the complex fine group gradings, and it is the strongest method we dispose of for the real forms. This 'fundamental' method as well as the 'real-basis' method, which follows below, were in detail described and proved in [25].

4.4 Real basis method

Finally, we describe a simplified version of the 'fundamental' method, called the 'real basis' method. It applies only for the real forms of the complex Lie algebras $sl(m,\mathbb{C})$, it is less powerful than the 'fundamental' method, but, on the other hand, much easier for practical application.

Theorem 5. *Let $\mathcal{G}$ be a MAD-group on the complex Lie algebra $sl(m,\mathbb{C})$ and let $\Gamma : sl(m,\mathbb{C}) = \oplus_{j\in\mathcal{J}} L_j$ be the fine group grading of $sl(m,\mathbb{C})$ generated by $\mathcal{G}$, such that all the subspaces $L_j = \operatorname{span}^{\mathbb{C}}(X_{j,1}, \dots, X_{j,l_j})$ have real basis vectors $X_{j,l} \in sl(m,\mathbb{R})$. Let h be an automorphism in $\operatorname{Aut} sl(m,\mathbb{C})$, such that $\mathbf{J} = \mathbf{J}_0 h$, where $\mathbf{J}_0$ acts as complex conjugation on elements of $sl(m,\mathbb{C})$, is an involutive antiautomorphism on $sl(m,\mathbb{C})$. Then the fine group grading Γ of $sl(m,\mathbb{C})$ determines a fine group grading of the real form $L_{\mathbf{J}} = L_{\mathbf{J}_0 h}$ if and only if the automorphism h is an element of the MAD-group $\mathcal{G}$.*

The simplicity of using this method follows from the fact that we do not have to occupy ourselves with the elements of the Lie algebra, but we only investigate the MAD-group $\mathcal{G}$, in order to find out whether it contains a convenient automorphism h. Obviously, the set of solutions provided by this method is generally not complete.

Throughout the task to find the fine group gradings of complex Lie algebras and of their real forms, we use the above mentioned methods alternatively, depending on their applicability and strength for the respective algebra. The concrete results are summarized in Section 5.

5 Results

As announced earlier, our main aim was to find fine group gradings of certain classical complex Lie algebras and their real forms. Naturally, one proceeds from those with the lowest ranks, since low rank implies low dimension of the algebra, and thus (relative) simplicity in the calculations.

Table 1. Fine group gradings of $o(5,\mathbb{C}) \cong sp(4,\mathbb{C})$ and of their real forms.

complex algebras $B_2 \cong C_2$	
$o(5,\mathbb{C}) \cong sp(4,\mathbb{C})$	Γ_1: $1 \times$ 2-dim $+ 8 \times$ 1-dim (Cartan) Γ_2: $10 \times$ 1-dim Γ_3: $10 \times$ 1-dim
real forms	
$so(5,0) \cong usp(4,0)$	$\Gamma_3^{\mathbb{R}}$
$so(4,1) \cong usp(2,2)$	$\Gamma_1^{\mathbb{R}}$, $\Gamma_2^{\mathbb{R}}$, $\Gamma_3^{\mathbb{R}}$
$so(3,2) \cong sp(4,\mathbb{R})$	$\Gamma_1^{\mathbb{R}}$, $\Gamma_2^{\mathbb{R}}$, $\Gamma_3^{\mathbb{R}}$

And last but not least argument is the fact that the low-rank Lie algebras are most widely used in the practice of physics.

On $A_1 = sl(2,\mathbb{C})$ the two fine group gradings (Cartan and Pauli) have been known for long time already. The four fine group gradings of the Lie algebra $A_2 = sl(3,\mathbb{C})$ were firstly published in [9], together with the fine group gradings of its three real forms.

In our work from the area of fine gradings, we have naturally started with the lowest rank where the results were missing, namely rank two and algebras B_2, C_2, and D_2. The two algebras $B_2 = o(5,\mathbb{C})$ and $C_2 = sp(4,\mathbb{C})$ are isomorphic, and thus their grading properties are the same. Then we continue with the algebra $D_2 = o(4,\mathbb{C})$, the only non-simple classical complex Lie algebra. Lastly, we move to the algebras of rank three, namely two isomorphic algebras $A_3 = sl(4,\mathbb{C})$ and $D_3 = o(6,\mathbb{C})$.

We have tried to combine all the various methods described in Section 4, in order to find as many fine group gradings as possible. The isomorphisms $B_2 \cong C_2$ and $A_3 \cong D_3$ are not the only auxiliary relations we dispose of. Additionally, we were able to benefit from the fact that $C_2 = sp(4,\mathbb{C})$ and $D_2 = o(4,\mathbb{C})$ are subalgebras of $A_3 = sl(4,\mathbb{C})$. That allows us to use the 'displayed' method, which is especially effective in the case of real forms.

The results were published in a series of articles [19, 20, 21, 22]. Let us recall that, in all these works, the lists of fine group gradings of the complex Lie algebras are complete (according to our Theorem 2), whereas for real forms, where no such statement has been proved, we cannot claim our results to be exhaustive solutions to the problem of fine group gradings.

$B_2 = o(5,\mathbb{C})$, $C_2 = sp(4,\mathbb{C})$

These (mutually isomorphic) algebras are 10-dimensional. There are three non-conjugate MAD-groups on the complex algebras, and thus three non-equivalent fine group gradings, which were found by the 'MAD-group' method in [19] and then confirmed for $C_2 = sp(4,\mathbb{C})$ also by the 'displayed' method in [20] (where several representations with different defining matrices K of $sp_K(4,\mathbb{C})$ appear).

Through coincidence, the number of real forms of these algebras is three, too. The detailed results on the real forms are in both the articles [19] and [22]; found by means of the 'MAD-group' method in the former and by means of the 'displayed' method in the latter. Both of the methods by definition have to provide the same result, only in different representations. It is only by mistake that the Cartan fine group grading of $usp(2,2)$ is missing in [19] – the respective MAD-group was forgotten there; hence the result in [22] is richer by this one fine group grading.

$D_2 = o(4,\mathbb{C})$

The six-dimensional algebra $D_2 = o(4,\mathbb{C})$ is the only non-simple case among the classical complex Lie algebras. It has six non-conjugate MAD-groups, one of them (Cartan) generating

Table 2. Fine group gradings of $o(4,\mathbb{C})$ and of its real forms.

complex algebra D_2	
$o(4,\mathbb{C})$	Γ_1: $1\times$ 2-dim + $4\times$ 1-dim (Cartan $\times$ Cartan) Γ_2: $6\times$ 1-dim (Cartan $\times$ Pauli) Γ_3: $6\times$ 1-dim (Pauli $\times$ Pauli) Γ_4: $6\times$ 1-dim Γ_5: $6\times$ 1-dim Γ_6: $6\times$ 1-dim
real forms	
$so^*(4)$	$\Gamma_2^{\mathbb{R}}$, $\Gamma_3^{\mathbb{R}}$, $\Gamma_6^{\mathbb{R}}$
$so(4,0)$	$\Gamma_3^{\mathbb{R}}$, $\Gamma_4^{\mathbb{R}}$
$so(3,1)$	$\Gamma_4^{\mathbb{R}}$, $\Gamma_5^{\mathbb{R}}$, $\Gamma_6^{\mathbb{R}}$
$so(2,2)$	$\Gamma_1^{\mathbb{R}}$, $\Gamma_2^{\mathbb{R}}$, $\Gamma_3^{\mathbb{R}}$, $\Gamma_4^{\mathbb{R}}$, $\Gamma_5^{\mathbb{R}}$, $\Gamma_6^{\mathbb{R}}$

Table 3. Fine group gradings of $sl(4,\mathbb{C}) \cong o(6,\mathbb{C})$ and of their real forms.

complex algebras $A_3 \cong D_3$	
$sl(4,\mathbb{C}) \cong o(6,\mathbb{C})$	Γ_1: $1\times$ 3-dim + $12\times$ 1-dim (Cartan) Γ_2: $15\times$ 1-dim (Pauli) Γ_3: $1\times$ 3-dim + $12\times$ 1-dim (orthogonal) Γ_4: $1\times$ 2-dim + $13\times$ 1-dim Γ_5: $1\times$ 2-dim + $13\times$ 1-dim Γ_6: $15\times$ 1-dim Γ_7: $15\times$ 1-dim Γ_8: $1\times$ 2-dim + $13\times$ 1-dim
real forms	
$sl(4,\mathbb{R}) \cong so(3,3)$	$\Gamma_1^{\mathbb{R}}$, $\Gamma_3^{\mathbb{R}}$, $\Gamma_4^{\mathbb{R}}$, $\Gamma_5^{\mathbb{R}}$, $\Gamma_6^{\mathbb{R}}$, $\Gamma_7^{\mathbb{R}}$, $\Gamma_8^{\mathbb{R}}$
$su^*(4) \cong so(5,1)$	$\Gamma_6^{\mathbb{R}}$, $\Gamma_7^{\mathbb{R}}$, $\Gamma_8^{\mathbb{R}}$
$su(4,0) \cong so(6,0)$	$\Gamma_3^{\mathbb{R}}$, $\Gamma_7^{\mathbb{R}}$
$su(3,1) \cong so^*(6)$	$\Gamma_2^{\mathbb{R}}$, $\Gamma_3^{\mathbb{R}}$, $\Gamma_4^{\mathbb{R}}$, $\Gamma_8^{\mathbb{R}}$
$su(2,2) \cong so(4,2)$	$\Gamma_2^{\mathbb{R}}$, $\Gamma_3^{\mathbb{R}}$, $\Gamma_4^{\mathbb{R}}$, $\Gamma_5^{\mathbb{R}}$, $\Gamma_6^{\mathbb{R}}$, $\Gamma_7^{\mathbb{R}}$, $\Gamma_8^{\mathbb{R}}$

a fine group grading, which is not fine grading. It splits the complex algebra into four one-dimensional grading subspaces and one two-dimensional subspace (the Cartan subalgebra). On simple classical complex Lie algebras of rank r, the r-dimensional Cartan subspace cannot be decomposed any further while preserving the grading properties within the Cartan grading. But the non-simple algebra $D_2 = o(4,\mathbb{C})$ is composed of two instances of $sl(2,\mathbb{C})$, and a direct sum of two instances of the Cartan-graded $sl(2,\mathbb{C})$ is a fine grading of $o(4,\mathbb{C})$, which is a non-group refinement of the group grading generated by the Cartan MAD-group on $o(4,\mathbb{C})$.

Note that also another two of the fine group gradings of $o(4,\mathbb{C})$ are composed of two fine group gradings of the algebra $sl(2,\mathbb{C})$, one is made up of two instances of the Pauli-graded $sl(2,\mathbb{C})$, and the other one consists of one Cartan-graded $sl(2,\mathbb{C})$ and one Pauli-graded $sl(2,\mathbb{C})$. The remaining three fine group gradings of $o(4,\mathbb{C})$ cannot be expressed in terms of fine group gradings of $sl(2,\mathbb{C})$. The full list of fine group gradings of the complex algebra D_2 are in [21].

The algebra $D_2 = o(4,\mathbb{C})$ has four real forms and we derive their fine group gradings by means of the 'displayed' method from the real forms of $sl(4,\mathbb{C})$, this set of solutions (provided in [22]) is not proved to be exhaustive, though.

$A_3 = sl(4, \mathbb{C})$, $D_3 = o(6, \mathbb{C})$

The last algebra whose fine gradings we have investigated is already of rank three and of dimension fifteen. We deal in fact again with two algebras that are isomorphic, namely $A_3 = sl(4, \mathbb{C})$ and $D_3 = o(6, \mathbb{C})$. The complex algebra has eight non-conjugate MAD-groups, and thus eight non-equivalent fine group gradings [20].

The number of real forms of the algebra $sl(4, \mathbb{C})$ is five, and again, we also try to deliver as many fine group gradings as possible for them. We apply all the methods we dispose of as explained in Section 4, starting from the easiest 'MAD-group' method, continuing with the 'real basis' method, and, lastly, turning to the 'fundamental' method. Each of these methods brings in additional results (see [22]), and those then enable, via the 'displayed' method, to obtain the richest possible results for the real forms of the subalgebras $sp(4, \mathbb{C})$ and $o(4, \mathbb{C})$.

6 Concluding remarks

Let us conclude by the open problems and hypotheses which need to be subject of further studies:

- The main question is the relationship between gradings and group gradings. Our hypothesis is that for finite-dimensional simple complex Lie algebras these two terms coincide.
- For the real forms of complex Lie algebras, it is to be clarified whether the 'fundamental' method provides all the fine group gradings. No counterexample has been found so far against this assumption.

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