The Research of Thomas P. Branson*

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Abstract. The Midwest Geometry Conference 2007 was devoted to the substantial mathematical legacy of Thomas P. Branson who passed away unexpectedly the previous year. This contribution to the Proceedings briefly introduces this legacy. We also take the opportunity of recording his bibliography. Thomas Branson was on the Editorial Board of SIGMA and we are pleased that SIGMA is able to publish the Proceedings.

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In memory of Thomas P. Branson (1953–2006)

Tom's research interests ranged broadly; this is perhaps best indicated by the diverse group of collaborators listed implicitly in his bibliography, which follows this section. Tom was also known for his general knowledge in many areas of mathematics, and many of his colleagues will remember well his prompt and in-depth replies to email questions.

In the 1980s Tom's research on conformal invariance was significantly ahead of its time. An enduring theme of his research was the natural interplay between invariance and the underlying symmetry groups. Tom's work continues to motivate and inspire a thriving and impressive international research effort. Many of the conference speakers have contributed to the various research trends that Tom Branson started.

It is impossible in a written summary to do justice to Tom's research career. Here we shall outline just a few directions that we feel to be especially significant.

1 Q-curvature and extremal problems

Tom Branson is perhaps most well known for his definition, development, and application of a new curvature quantity in Riemannian geometry. For dimension 4 this first entered the public arena in his joint work [22] with Ørsted, but it was extended to all even dimensions and developed significantly in [29] and [36]; these days it is usually termed "Branson's Q-curvature". A key feature is its conformal transformation law in dimension n,

$$e^{-n\omega}\widehat{Q} = Q + P\omega,$$

*This paper is a contribution to the Proceedings of the 2007 Midwest Geometry Conference in honor of Thomas P. Branson. The full collection is available at http://www.emis.de/journals/SIGMA/MGC2007.html where $\widehat{g} = e^{2\omega}g$ for a smooth function ω and P is the celebrated conformal (i.e. conformally covariant) operator of the form $\Delta^{n/2}+$*lower order terms* due to Graham–Jenne–Mason–Sparling (and Paneitz in dimension 4). Because P turns out to be a divergence, this generalises the transformation of Gauss curvature in 2 dimensions and shows that on closed manifolds $\int Q$ is a global conformal invariant.

The article with Ørsted was motivated by the study of the functional determinants of integral powers of conformal operators and, in particular, the issue of extremising such quantities within the class of conformally related metrics of a fixed volume. This was part of a theme, extended in his work [27] with Chang and Yang, in [31] with Gilkey, and in [38], of developing results for dimension 4 and higher even dimensions, which paralleled the results from dimension 2 due to Polyakov and others. An excellent expository account of these directions, written by Tom Branson himself, is to be found in [76], which appears in this volume.

2 Conformal differential operators

Tom was also a pioneer of the systematic construction and study of conformal differential operators and related issues. The importance and use of ellipticity was an enduring theme and, for example in [46], he identified the elliptic operators within the class of second order formally self-adjoint operators arising as Stein–Weiss gradients. On the other hand, in [48] he provided a classification of second order linear conformal differential operators. The work [13], which constructs operators between differential forms on Minkowski space, was motivated by their rôle as representation intertwinors and links to physics. This partly followed some earlier work on differential forms in [11]. There he constructed, for example, new conformal differential operators of order 4 and 6 and some applications of these to variational problems. Notable is that on (unweighted) differential forms the operators he found took the form $L = dMd$, where d is the exterior derivative, which is itself conformal. Although it was not highlighted at the time it later became clear that, while conformal operators on forms with the symbol of L were to be expected, factorisations along these lines are rare, surprising, and valuable. This thread was picked up much later in the joint work [67] with Gover. There, using the Fefferman–Graham ambient metric and its links to conformal tractor calculus, it was found that on any k-forms for $k \leq n/2 - 1$ there are conformal differential operators generalising his earlier discoveries, but especially significant is what is captured in the details of the factorisation. The operators may be expressed in the form

$$L_k = \underbrace{d * \left\{ \overbrace{(d * d*)^{n/2-k-1} + \textit{lower order terms}}^{Q_{k+1}} \right\}}_{G_{k+1}} d, \tag{1}$$

where $*$ is the Hodge $*$-operator. As an operator on closed forms, Q_{k+1} generalises the Q curvature and the composition $d * Q_{k+1} = G_{k+1}$ forms a conformal gauge companion for d or alternatively for L_{k+1}; paired with either of these G_{k+1} yields an injectively elliptic conformal system. Another consequence of the factorisations (1) is that they lead to new elliptic conformal complexes. A driving motivation here is that such complexes admit torsions (or determinants), which generalise Cheeger's de Rham half torsion and, in particular, have Polyakov type conformal variation formulae. These ideas are sketched in Tom's fine survey article [66].

3 Spectrum generating functions and intertwining operators

Another enduring theme of Tom's work was exploiting the implications of invariance for spectral data. For example, on the round sphere the conformal covariance relation satisfied by the conformal Laplacian $D = \Delta + \frac{n(n-2)}{4}$ imposes relations among the eigenvalues of the Laplacian itself. In fact, in this case there are sufficient relations to deduce all the spectral data, and hence the term "spontaneous generation of eigenvalues" which titles the joint work of Branson with Ørsted [71]. Here, a heavy use is made of symmetry. The conformal covariance implies a simple formula for the commutator of D with the Lie derivative along conformal vector fields. But the sphere has a maximal dimension space of such fields and, via the commutation relation, one can show that the action of appropriately chosen conformal vector fields will shift eigenvectors to linear combinations of eigenvectors with adjacent eigenvalues.

An observation made much earlier by Tom is that, given the explicit spectral data for a basic Laplacian operator and again sufficient symmetry, one may deduce rather explicit formulae for a series of intertwinors, which are not generally differential. Of course, this uses that the span of the eigenfunctions for the given Laplacian is dense on the compact manifold concerned. Once again using the sphere as an example, one may suppose that some operator A_{2r}, acting between sections of the trivial bundle, is a function of Δ and satisfies an obvious generalisation of the conformal covariance relation enjoyed by D. Compressing to eigenspaces of the Laplacian, this implies characterising relations for the spectrum of the operator. For this series of operators, Tom was able to obtain the rather striking formula

$$A_{2r} = \frac{\Gamma(A_1 + \frac{1}{2} + r)}{\Gamma(A_1 + \frac{1}{2} - r)}, \qquad A_1 = \sqrt{\Delta + \left(\frac{n-1}{2}\right)^2}$$

for $r \in \mathbb{C}$ and $r \notin \{-n/2, -n/2 - 1, \dots\}$. These are intertwinors of the spherical principal series. As pointed out in [13], for $r \in \mathbb{Z}_+$ one obtains an explicit formula for a class of conformal differential operators, namely the conformal Laplacian operators on the sphere:

$$\prod_{p=1}^{r} \left\{ \Delta + \left(\frac{n}{2} + p - 1\right)\left(\frac{n}{2} - p\right) \right\},$$

(cf. Robin Graham's article in this volume[1]). In fact, in [13] Branson used similar ideas to construct the conformal differential operators on forms mentioned earlier. These are differential operators $D_{2l,k}$, on forms of all even orders $2l$, on differential forms of all orders k, on the double cover of the n-dimensional compactified Minkowski space.

Tom's early ideas were put into a representation theoretic framework and generalised in the significant work [37], with Ólafsson and Ørsted. There they used a spectrum generating operator, constructed from a combination of quadratic Casimirs and the development of ideas as above to construct intertwinors for representations induced from a maximal parabolic subgroup.

4 Heat kernel asymptotics

The so-called heat kernel asymptotics have long been of considerable interest in the study of an elliptic operator F. The coefficients of the small time expansion of such a kernel are spectral invariants; they encode information about the asymptotic properties of the spectrum of F.

Tom made substantial contributions in calculating such invariants. Perhaps his best known article in this direction is [17] written joint with Gilkey. They consider a rather general setting, namely a Riemannian manifold with possible boundary together with a smooth vector bundle V with connection ∇, a given section E of $\mathrm{End}(V)$, and an auxiliary smooth function f. The associated heat kernel is $\mathrm{Tr}(fe^{-tP})$, where $P = \nabla^*\nabla + E$. The inclusion of the function f allows for an efficient and secure calculation of the coefficients $a_n(f, P)$ for $n = 0$, 0.5, 1, 1.5, 2 under

[1]Graham C.R., Conformal powers of the Laplacian via stereographic projection, *SIGMA* **3** (2007), 121, 4 pages, arXiv:0711.4798.

both Dirichlet and Neumann boundary conditions. In particular, by setting $f = 1$ in $a_2(f, P)$, this fifth coefficient of the heat kernel aysymptotics is calculated for the first time. Another appealing feature of incorporating the function f is that its normal derivatives showing up in their formulae better record the distribution-like behaviour of the heat kernel near the boundary.

Another pioneering contribution of Tom's work in this area concerned the heat kernel asymptotics of operators not necessarily of Laplace type. This joint work with Gilkey and Fulling [23], and later Avramidi [57], assumes the second order operators concerned still have positive definite leading symbol but does not assume this coincides with (the inverse of) a background Riemannian metric. Results, in this general setting, include explicit formulae for the first two terms of the asymptotics.

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