Symmetry, Integrability and Geometry: Methods and Applications SIGMA 4 (2008), 029, 30 pages

Reductions of Multicomponent mKdV Equations on Symmetric Spaces of DIII-Type*

Vladimir S. GERDJIKOV and Nikolay A. KOSTOV

Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences,
72 Tsarigradsko chaussee, 1784 Sofia, Bulgaria
E-mail: *gerjikov@inrne.bas.bg, nakostov@inrne.bas.bg*

Received December 14, 2007, in final form February 27, 2008; Published online March 11, 2008
Original article is available at http://www.emis.de/journals/SIGMA/2008/029/

Abstract. New reductions for the multicomponent modified Korteweg–de Vries (MMKdV) equations on the symmetric spaces of **DIII**-type are derived using the approach based on the reduction group introduced by A.V. Mikhailov. The relevant inverse scattering problem is studied and reduced to a Riemann–Hilbert problem. The minimal sets of scattering data $\mathcal{T}_i$, $i = 1, 2$ which allow one to reconstruct uniquely both the scattering matrix and the potential of the Lax operator are defined. The effect of the new reductions on the hierarchy of Hamiltonian structures of MMKdV and on $\mathcal{T}_i$ are studied. We illustrate our results by the MMKdV equations related to the algebra $\mathfrak{g} \simeq so(8)$ and derive several new MMKdV-type equations using group of reductions isomorphic to $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$.

Key words: multicomponent modified Korteweg–de Vries (MMKdV) equations; reduction group; Riemann–Hilbert problem; Hamiltonian structures

2000 Mathematics Subject Classification: 37K20; 35Q51; 74J30; 78A60

1 Introduction

The modified Korteweg–de Vries equation [1]

$$q_t + q_{xxx} + 6\epsilon q_x q^2(x,t) = 0, \qquad \epsilon = \pm 1,$$

has natural multicomponent generalizations related to the symmetric spaces [2]. They can be integrated by the ISM using the fact that they allow the following Lax representation:

$$\begin{gathered}
L\psi \equiv \left(i\frac{d}{dx} + Q(x,t) - \lambda J \right) \psi(x,t,\lambda) = 0, \qquad (1.1)\\
Q(x,t) = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix},\\
M\psi \equiv \left(i\frac{d}{dt} + V_0(x,t) + \lambda V_1(x,t) + \lambda^2 V_2(x,t) - 4\lambda^3 J \right) \psi(x,t,\lambda) = \psi(x,t,\lambda) C(\lambda),\\
V_2(x,t) = 4Q(x,t), \qquad V_1(x,t) = 2iJQ_x + 2JQ^2, \qquad V_0(x,t) = -Q_{xx} - 2Q^3,
\end{gathered}$$

where J and $Q(x,t)$ are $2r\times 2r$ matrices: J is a block diagonal and $Q(x,t)$ is a block-off-diagonal matrix. The corresponding MMKdV equations take the form

$$\frac{\partial Q}{\partial t} + \frac{\partial^3 Q}{\partial x^3} + 3\left(Q_x Q^2 + Q^2 Q_x \right) = 0.$$

*This paper is a contribution to the Proceedings of the Seventh International Conference "Symmetry in Nonlinear Mathematical Physics" (June 24–30, 2007, Kyiv, Ukraine). The full collection is available at http://www.emis.de/journals/SIGMA/symmetry2007.html

The analysis in [2, 3, 4] reveals a number of important results. These include the corresponding multicomponent generalizations of KdV equations and the generalized Miura transformations interrelating them with the generalized MMKdV equations; two of their most important reductions as well as their Hamiltonians.

Our aim in this paper is to explore new types of reductions of the MMKdV equations. To this end we make use of the reduction group introduced by Mikhailov [5, 6] which allows one to impose algebraic reductions on the coefficients of $Q(x,t)$ which will be automatically compatible with the evolution of MMKdV. Similar problems have been analyzed for the N-wave type equations related to the simple Lie algebras of rank 2 and 3 [7, 8] and the multicomponent NLS equations [9, 10]. Here we illustrate our analysis by the MMKdV equations related to the algebras $\mathfrak{g} \simeq so(2r)$ which are linked to the **DIII**-type symmetric spaces series. Due to the fact that the dispersion law for MNLS is proportional to λ^2 while for MMKdV it is proportional to λ^3 the sets of admissible reductions for these two NLEE equations differ substantially.

In the next Section 2 we give some preliminaries on the scattering theory for L, the reduction group and graded Lie algebras. In Section 3 we construct the fundamental analytic solutions of L, formulate the corresponding Riemann–Hilbert problem and introduce the minimal sets of scattering data $\mathcal{T}_i$, $i = 1, 2$ which define uniquely both the scattering matrix and the solution of the MMKdV $Q(x,t)$. Some of these facts have been discussed in more details in [10], others had to be modified and extended so that they adequately take into account the peculiarities of the **DIII** type symmetric spaces. In particular we modified the definition of the fundamental analytic solution which lead to changes in the formulation of the Riemann–Hilbert problem. In Section 4 we show that the ISM can be interpreted as a generalized Fourier [10] transform which maps the potential $Q(x,t)$ onto the minimal sets of scattering data $\mathcal{T}_i$. Here we briefly outline the hierarchy of Hamiltonian structures for the generic MMKdV equations. The next Section 5 contains two classes of nontrivial reductions of the MMKdV equations related to the algebra $so(8)$. The first class is performed with automorphisms of $so(8)$ that preserve J; the second class uses automorphisms that map J into $-J$. While the reductions of first type can be applied both to MNLS and MMKdV equations, the reductions of second type can be applied only to MMKdV equations. Under them 'half' of the members of the Hamiltonian hierarchy become degenerated [11, 2]. For both classes of reductions we find examples with groups of reductions isomorphic to $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_4$. We also provide the corresponding reduced Hamiltonians and symplectic forms and Poisson brackets. At the end of Section 5 we derive the effects of these reductions on the scattering matrix and on the minimal sets of scattering data. In Section 6 following [3] we analyze the classical r-matrix for the corresponding NLEE. The effect of reductions on the classical r-matrix is discussed. The last Section contains some conclusions.

2 Preliminaries

2.1 Cartan–Weyl basis and Weyl group for $so(2r)$

Here we fix the notations and the normalization conditions for the Cartan–Weyl generators of $\mathfrak{g} \simeq so(2r)$, see e.g. [12]. The root system Δ of this series of simple Lie algebras consists of the roots $\Delta \equiv \{\pm(e_i - e_j), \pm(e_i + e_j)\}$ where $1 \leq i < j \leq r$. We introduce an ordering in Δ by specifying the set of positive roots $\Delta^+ \equiv \{e_i - e_j, e_i + e_j\}$ for $1 \leq i < j \leq r$. Obviously all roots have the same length equal to 2.

We introduce the basis in the Cartan subalgebra by $h_k \in \mathfrak{h}$, $k = 1, \dots, r$ where $\{h_k\}$ are the Cartan elements dual to the orthonormal basis $\{e_k\}$ in the root space $\mathbb{E}^r$. Along with h_k we introduce also

$$H_\alpha = \sum_{k=1}^{r} (\alpha, e_k) h_k, \qquad \alpha \in \Delta,$$

where (α, e_k) is the scalar product in the root space $\mathbb{E}^r$ between the root α and e_k. The basis in $so(2r)$ is completed by adding the Weyl generators E_α, $\alpha \in \Delta$.

The commutation relations for the elements of the Cartan–Weyl basis are given by [12]

$$[h_k, E_\alpha] = (\alpha, e_k)E_\alpha, \qquad [E_\alpha, E_{-\alpha}] = H_\alpha,$$
$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha,\beta}E_{\alpha+\beta} & \text{for } \alpha+\beta \in \Delta, \\ 0 & \text{for } \alpha+\beta \notin \Delta \cup \{0\}. \end{cases}$$

We will need also the typical $2r$-dimensional representation of $so(2r)$. For convenience we choose the following definition for the orthogonal algebras and groups

$$X \in so(2r) \longrightarrow X + S_0 X^T \hat{S}_0 = 0, \qquad T \in SO(2r) \longrightarrow S_0 T^T \hat{S}_0 = \hat{T}, \tag{2.1}$$

where by 'hat' we denote the inverse matrix $\hat{T} \equiv T^{-1}$ and

$$S_0 \equiv \sum_{k=1}^{r} (-1)^{k+1} \left(E_{k,\bar{k}} + E_{\bar{k},k}\right) = \begin{pmatrix} 0 & \boldsymbol{s}_0 \\ \hat{\boldsymbol{s}}_0 & 0 \end{pmatrix}, \qquad \bar{k} = 2r+1-k. \tag{2.2}$$

Here and below by E_{jk} we denote a $2r \times 2r$ matrix with just one non-vanishing and equal to 1 matrix element at j,k-th position: $(E_{jk})_{mn} = \delta_{jm}\delta_{kn}$. Obviously $S_0^2 = \mathbb{1}$. In order to have the Cartan generators represented by diagonal matrices we modified the definition of orthogonal matrix, see (2.1). Using the matrices E_{jk} defined by equation (2.2) we get

$$h_k = E_{kk} - E_{\bar{k}\bar{k}}, \qquad E_{e_i-e_j} = E_{ij} - (-1)^{i+j} E_{\bar{j}\bar{i}},$$
$$E_{e_i+e_j} = E_{i\bar{j}} - (-1)^{i+j} E_{j\bar{i}}, \qquad E_{-\alpha} = E_\alpha^T,$$

where $\bar{k} = 2r+1-k$.

We will denote by $\vec{a} = \sum_{k=1}^{r} e_k$ the r-dimensional vector dual to $J \in \mathfrak{h}$; obviously $J = \sum_{k=1}^{r} h_k$. If the root $\alpha \in \Delta_+$ is positive (negative) then $(\alpha, \vec{a}) \geq 0$ ($(\alpha, \vec{a}) < 0$ respectively). The normalization of the basis is determined by

$$E_{-\alpha} = E_\alpha^T, \qquad \langle E_{-\alpha}, E_\alpha \rangle = 2, \qquad N_{-\alpha,-\beta} = -N_{\alpha,\beta}.$$

The root system Δ of $\mathfrak{g}$ is invariant with respect to the Weyl reflections S_α; on the vectors $\vec{y} \in \mathbb{E}^r$ they act as

$$S_\alpha \vec{y} = \vec{y} - \frac{2(\alpha, \vec{y})}{(\alpha, \alpha)}\alpha, \qquad \alpha \in \Delta.$$

All Weyl reflections S_α form a finite group $W_\mathfrak{g}$ known as the Weyl group. On the root space this group is isomorphic to $\mathcal{S}_r \otimes (\mathbb{Z}_2)^{r-1}$ where $\mathcal{S}_r$ is the group of permutations of the basic vectors $e_j \in \mathbb{E}^r$. Each of the $\mathbb{Z}_2$ groups acts on $\mathbb{E}^r$ by changing simultaneously the signs of two of the basic vectors e_j.

One may introduce also an action of the Weyl group on the Cartan–Weyl basis, namely [12]

$$S_\alpha(H_\beta) \equiv A_\alpha H_\beta A_\alpha^{-1} = H_{S_\alpha \beta},$$
$$S_\alpha(E_\beta) \equiv A_\alpha E_\beta A_\alpha^{-1} = n_{\alpha,\beta} E_{S_\alpha \beta}, \qquad n_{\alpha,\beta} = \pm 1.$$

The matrices A_α are given (up to a factor from the Cartan subgroup) by

$$A_\alpha = e^{E_\alpha} e^{-E_{-\alpha}} e^{E_\alpha} H_A, \tag{2.3}$$

where H_A is a conveniently chosen element from the Cartan subgroup such that $H_A^2 = \mathbb{1}$. The formula (2.3) and the explicit form of the Cartan–Weyl basis in the typical representation will be used in calculating the reduction condition following from (4.16).

2.2 Graded Lie algebras

One of the important notions in constructing integrable equations and their reductions is the one of graded Lie algebra and Kac–Moody algebras [12]. The standard construction is based on a finite order automorphism $C \in \operatorname{Aut}\mathfrak{g}$, $C^N = \mathbb{1}$. The eigenvalues of C are ω^k, $k = 0, 1, \ldots, N-1$, where $\omega = \exp(2\pi i/N)$. To each eigenvalue there corresponds a linear subspace $\mathfrak{g}^{(k)} \subset \mathfrak{g}$ determined by

$$\mathfrak{g}^{(k)} \equiv \left\{X : X \in \mathfrak{g},\ C(X) = \omega^k X\right\}.$$

Then $\mathfrak{g} = \mathop{\oplus}\limits_{k=0}^{N-1} \mathfrak{g}^{(k)}$ and the grading condition holds

$$\left[\mathfrak{g}^{(k)}, \mathfrak{g}^{(n)}\right] \subset \mathfrak{g}^{(k+n)}, \tag{2.4}$$

where $k+n$ is taken modulo N. Thus to each pair $\{\mathfrak{g}, C\}$ one can relate an infinite-dimensional algebra of Kac–Moody type $\widehat{\mathfrak{g}}_C$ whose elements are

$$X(\lambda) = \sum_k X_k \lambda^k, \qquad X_k \in \mathfrak{g}^{(k)}. \tag{2.5}$$

The series in (2.5) must contain only finite number of negative (positive) powers of λ and $\mathfrak{g}^{(k+N)} \equiv \mathfrak{g}^{(k)}$. This construction is a most natural one for Lax pairs; we see that due to the grading condition (2.4) we can always impose a reduction on $L(\lambda)$ and $M(\lambda)$ such that both $U(x,t,\lambda)$ and $V(x,t,\lambda) \in \widehat{\mathfrak{g}}_C$. In the case of symmetric spaces $N = 2$ and C is the Cartan involution. Then one can choose the Lax operator L in such a way that

$$Q \in \mathfrak{g}^{(1)}, \qquad J \in \mathfrak{g}^{(0)}$$

as it is the case in (1.1). Here the subalgebra $\mathfrak{g}^{(0)}$ consists of all elements of $\mathfrak{g}$ commuting with J. The special choice of $J = \sum\limits_{k=1}^{r} h_k$ taken above allows us to split the set of all positive roots Δ^+ into two subsets

$$\Delta^+ = \Delta_0^+ \cup \Delta_1^+, \qquad \Delta_0^+ = \{e_i - e_j\}_{i<j}, \qquad \Delta_1^+ = \{e_i + e_j\}_{i<j}.$$

Obviously the elements $\alpha \in \Delta_1^+$ have the property $\alpha(J) = (\alpha, \vec{a}) = 2$, while the elements $\beta \in \Delta_0^+$ have the property $\beta(J) = (\beta, \vec{a}) = 0$. In this section we outline some of the well known facts about the spectral theory of the Lax operators of the type (1.1).

2.3 The scattering problem for L

Here we briefly outline the basic facts about the direct and the inverse scattering problems [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] for the system (1.1) for the class of potentials $Q(x,t)$ that are smooth enough and fall off to zero fast enough for $x \to \pm\infty$ for all t. In what follows we treat **DIII**-type symmetric spaces which means that $Q(x,t)$ is an element of the algebra $so(2r)$. In the examples below we take $r = 4$ and $\mathfrak{g} \simeq so(8)$.

The main tool for solving the direct and inverse scattering problems are the Jost solutions which are fundamental solutions defined by their asymptotics at $x \to \pm\infty$

$$\lim_{x\to\infty} \psi(x,\lambda) e^{i\lambda J x} = \mathbb{1}, \qquad \lim_{x\to-\infty} \phi(x,\lambda) e^{i\lambda J x} = \mathbb{1}.$$

Along with the Jost solutions we introduce

$$\xi(x,\lambda) = \psi(x,\lambda) e^{i\lambda J x}, \qquad \varphi(x,\lambda) = \phi(x,\lambda) e^{i\lambda J x},$$

which satisfy the following linear integral equations

$$\xi(x,\lambda) = \mathbb{1} + \mathrm{i}\int_{\infty}^{x} \mathrm{d}y \mathrm{e}^{-\mathrm{i}\lambda J(x-y)} Q(y)\xi(y,\lambda)\mathrm{e}^{\mathrm{i}\lambda J(x-y)}, \tag{2.6}$$

$$\varphi(x,\lambda) = \mathbb{1} + \mathrm{i}\int_{-\infty}^{x} \mathrm{d}y \mathrm{e}^{-\mathrm{i}\lambda J(x-y)} Q(y)\varphi(y,\lambda)\mathrm{e}^{\mathrm{i}\lambda J(x-y)}. \tag{2.7}$$

These are Volterra type equations which, have solutions providing one can ensure the convergence of the integrals in the right hand side. For λ real the exponential factors in (2.6) and (2.7) are just oscillating and the convergence is ensured by the fact that $Q(x,t)$ is quickly vanishing for $x \to \infty$.

Remark 1. It is an well known fact that if the potential $Q(x,t) \in so(2r)$ then the corresponding Jost solutions of equation (1.1) take values in the corresponding group, i.e. $\psi(x,\lambda), \phi(x,\lambda) \in SO(2r)$.

The Jost solutions as whole can not be extended for $\operatorname{Im}\lambda \neq 0$. However some of their columns can be extended for $\lambda \in \mathbb{C}_+$, others – for $\lambda \in \mathbb{C}_-$. More precisely we can write down the Jost solutions $\psi(x,\lambda)$ and $\phi(x,\lambda)$ in the following block-matrix form

$$\psi(x,\lambda) = \left(|\psi^-(x,\lambda)\rangle, |\psi^+(x,\lambda)\rangle\right), \qquad \phi(x,\lambda) = \left(|\phi^+(x,\lambda)\rangle, |\phi^-(x,\lambda)\rangle\right),$$

$$|\psi^{\pm}(x,\lambda)\rangle = \begin{pmatrix} \boldsymbol{\psi}_1^{\pm}(x,\lambda) \\ \boldsymbol{\psi}_2^{\pm}(x,\lambda) \end{pmatrix}, \qquad |\phi^{\pm}(x,\lambda)\rangle = \begin{pmatrix} \boldsymbol{\phi}_1^{\pm}(x,\lambda) \\ \boldsymbol{\phi}_2^{\pm}(x,\lambda) \end{pmatrix},$$

where the superscript $+$ and (resp. $-$) shows that the corresponding $r \times r$ block-matrices allow analytic extension for $\lambda \in \mathbb{C}_+$ (resp. $\lambda \in \mathbb{C}_-$).

Solving the direct scattering problem means given the potential $Q(x)$ to find the scattering matrix $T(\lambda)$. By definition $T(\lambda)$ relates the two Jost solutions

$$\phi(x,\lambda) = \psi(x,\lambda)T(\lambda), \qquad T(\lambda) = \begin{pmatrix} \boldsymbol{a}^+(\lambda) & -\boldsymbol{b}^-(\lambda) \\ \boldsymbol{b}^+(\lambda) & \boldsymbol{a}^-(\lambda) \end{pmatrix} \tag{2.8}$$

and has compatible block-matrix structure. In what follows we will need also the inverse of the scattering matrix

$$\psi(x,\lambda) = \phi(x,\lambda)\hat{T}(\lambda), \qquad \hat{T}(\lambda) \equiv \begin{pmatrix} \boldsymbol{c}^-(\lambda) & \boldsymbol{d}^-(\lambda) \\ -\boldsymbol{d}^+(\lambda) & \boldsymbol{c}^+(\lambda) \end{pmatrix},$$

where

$$\begin{aligned}
\boldsymbol{c}^-(\lambda) &= \hat{\boldsymbol{a}}^+(\lambda)(\mathbb{1} + \rho^-\rho^+)^{-1} = (\mathbb{1} + \tau^+\tau^-)^{-1}\hat{\boldsymbol{a}}^+(\lambda), \\
\boldsymbol{d}^-(\lambda) &= \hat{\boldsymbol{a}}^+(\lambda)\rho^-(\lambda)(\mathbb{1} + \rho^+\rho^-)^{-1} = (\mathbb{1} + \tau^+\tau^-)^{-1}\tau^+(\lambda)\hat{\boldsymbol{a}}^-(\lambda), \\
\boldsymbol{c}^+(\lambda) &= \hat{\boldsymbol{a}}^-(\lambda)(\mathbb{1} + \rho^+\rho^-)^{-1} = (\mathbb{1} + \tau^-\tau^+)^{-1}\hat{\boldsymbol{a}}^-(\lambda), \\
\boldsymbol{d}^+(\lambda) &= \hat{\boldsymbol{a}}^-(\lambda)\rho^+(\lambda)(\mathbb{1} + \rho^-\rho^+)^{-1} = (\mathbb{1} + \tau^-\tau^+)^{-1}\tau^-(\lambda)\hat{\boldsymbol{a}}^+(\lambda).
\end{aligned} \tag{2.9}$$

The diagonal blocks of $T(\lambda)$ and $\hat{T}(\lambda)$ allow analytic continuation off the real axis, namely $\boldsymbol{a}^+(\lambda)$, $\boldsymbol{c}^+(\lambda)$ are analytic functions of λ for $\lambda \in \mathbb{C}_+$, while $\boldsymbol{a}^-(\lambda)$, $\boldsymbol{c}^-(\lambda)$ are analytic functions of λ for $\lambda \in \mathbb{C}_-$. We introduced also $\rho^{\pm}(\lambda)$ and $\tau^{\pm}(\lambda)$ the multicomponent generalizations of the reflection coefficients (for the scalar case, see [27, 17, 28])

$$\rho^{\pm}(\lambda) = \boldsymbol{b}^{\pm}\hat{\boldsymbol{a}}^{\pm}(\lambda) = \hat{\boldsymbol{c}}^{\pm}\boldsymbol{d}^{\pm}(\lambda), \qquad \tau^{\pm}(\lambda) = \hat{\boldsymbol{a}}^{\pm}\boldsymbol{b}^{\mp}(\lambda) = \boldsymbol{d}^{\mp}\hat{\boldsymbol{c}}^{\pm}(\lambda).$$

The reflection coefficients do not have analyticity properties and are defined only for $\lambda \in \mathbb{R}$.

From Remark 1 one concludes that $T(\lambda) \in SO(2r)$, therefore it must satisfy the second of the equations in (2.1). As a result we get the following relations between $\boldsymbol{c}^{\pm}$, $\boldsymbol{d}^{\pm}$ and $\boldsymbol{a}^{\pm}$, $\boldsymbol{b}^{\pm}$

$$\begin{aligned} &\boldsymbol{c}^{+}(\lambda) = \hat{\boldsymbol{s}}_0 \boldsymbol{a}^{+,T}(\lambda)\boldsymbol{s}_0, \qquad \boldsymbol{c}^{-}(\lambda) = \boldsymbol{s}_0 \boldsymbol{a}^{-,T}(\lambda)\hat{\boldsymbol{s}}_0, \\ &\boldsymbol{d}^{+}(\lambda) = -\hat{\boldsymbol{s}}_0 \boldsymbol{b}^{+,T}(\lambda)\boldsymbol{s}_0, \qquad \boldsymbol{d}^{-}(\lambda) = -\boldsymbol{s}_0 \boldsymbol{b}^{-,T}(\lambda)\hat{\boldsymbol{s}}_0, \end{aligned} \tag{2.10}$$

and in addition we have

$$\begin{aligned} &\rho^{+}(\lambda) = -\hat{\boldsymbol{s}}_0 \rho^{+,T}(\lambda)\boldsymbol{s}_0, \qquad \rho^{-}(\lambda) = -\boldsymbol{s}_0 \rho^{-,T}(\lambda)\hat{\boldsymbol{s}}_0, \\ &\tau^{+}(\lambda) = -\boldsymbol{s}_0 \tau^{+,T}(\lambda)\hat{\boldsymbol{s}}_0, \qquad \tau^{-}(\lambda) = -\hat{\boldsymbol{s}}_0 \tau^{-,T}(\lambda)\boldsymbol{s}_0. \end{aligned} \tag{2.11}$$

Next we need also the asymptotics of the Jost solutions and the scattering matrix for $\lambda \to \infty$

$$\begin{aligned} &\lim_{\lambda\to-\infty} \phi(x,\lambda)\mathrm{e}^{\mathrm{i}\lambda Jx} = \lim_{\lambda\to\infty} \psi(x,\lambda)\mathrm{e}^{\mathrm{i}\lambda Jx} = \mathbb{1}, \qquad \lim_{\lambda\to\infty} T(\lambda) = \mathbb{1}, \\ &\lim_{\lambda\to\infty} \boldsymbol{a}^{+}(\lambda) = \lim_{\lambda\to\infty} \boldsymbol{c}^{-}(\lambda) = \mathbb{1}, \qquad \lim_{\lambda\to\infty} \boldsymbol{a}^{-}(\lambda) = \lim_{\lambda\to\infty} \boldsymbol{c}^{+}(\lambda) = \mathbb{1}. \end{aligned}$$

The inverse to the Jost solutions $\hat{\psi}(x,\lambda)$ and $\hat{\phi}(x,\lambda)$ are solutions to

$$\mathrm{i}\frac{\mathrm{d}\hat{\psi}}{\mathrm{d}x} - \hat{\psi}(x,\lambda)(Q(x) - \lambda J) = 0, \tag{2.12}$$

satisfying the conditions

$$\lim_{x\to\infty} \mathrm{e}^{-\mathrm{i}\lambda Jx}\hat{\psi}(x,\lambda) = \mathbb{1}, \qquad \lim_{x\to-\infty} \mathrm{e}^{-\mathrm{i}\lambda Jx}\hat{\phi}(x,\lambda) = \mathbb{1}. \tag{2.13}$$

Now it is the collections of rows of $\hat{\psi}(x,\lambda)$ and $\hat{\phi}(x,\lambda)$ that possess analytic properties in λ

$$\begin{aligned} &\hat{\psi}(x,\lambda) = \begin{pmatrix} \langle\hat{\psi}^{+}(x,\lambda)| \\ \langle\hat{\psi}^{-}(x,\lambda)| \end{pmatrix}, \qquad \hat{\phi}(x,\lambda) = \begin{pmatrix} \langle\hat{\boldsymbol{\phi}}^{-}(x,\lambda)| \\ \langle\hat{\boldsymbol{\phi}}^{+}(x,\lambda)| \end{pmatrix}, \\ &\langle\hat{\psi}^{\pm}(x,\lambda)| = \left(\boldsymbol{s}_0^{\pm 1}\boldsymbol{\psi}_2^{\pm}, \boldsymbol{s}_0^{\pm 1}\boldsymbol{\psi}_1^{\pm}\right)(x,\lambda), \qquad \langle\hat{\phi}^{\pm}(x,\lambda)| = \left(\boldsymbol{s}_0^{\mp 1}\boldsymbol{\psi}_2^{\pm}, \boldsymbol{s}_0^{\mp 1}\boldsymbol{\psi}_1^{\pm}\right)(x,\lambda). \end{aligned} \tag{2.14}$$

Just like the Jost solutions, their inverse (2.14) are solutions to linear equations (2.12) with regular boundary conditions (2.13); therefore they can have no singularities on the real axis $\lambda \in \mathbb{R}$. The same holds true also for the scattering matrix $T(\lambda) = \hat{\psi}(x,\lambda)\phi(x,\lambda)$ and its inverse $\hat{T}(\lambda) = \hat{\phi}(x,\lambda)\psi(x,\lambda)$, i.e.

$$\boldsymbol{a}^{+}(\lambda) = \langle\hat{\psi}^{+}(x,\lambda)|\phi^{+}(x,\lambda)\rangle, \qquad \boldsymbol{a}^{-}(\lambda) = \langle\hat{\psi}^{-}(x,\lambda)|\phi^{-}(x,\lambda)\rangle,$$

as well as

$$\boldsymbol{c}^{+}(\lambda) = \langle\hat{\phi}^{+}(x,\lambda)|\psi^{+}(x,\lambda)\rangle, \qquad \boldsymbol{c}^{-}(\lambda) = \langle\hat{\phi}^{-}(x,\lambda)|\psi^{-}(x,\lambda)\rangle,$$

are analytic for $\lambda \in \mathbb{C}_{\pm}$ and have no singularities for $\lambda \in \mathbb{R}$. However they may become degenerate (i.e., their determinants may vanish) for some values $\lambda_j^{\pm} \in \mathbb{C}_{\pm}$ of λ. Below we briefly analyze the structure of these degeneracies and show that they are related to discrete spectrum of L.

3 The fundamental analytic solutions and the Riemann–Hilbert problem

3.1 The fundamental analytic solutions

The next step is to construct the fundamental analytic solutions (FAS) $\chi^{\pm}(x,\lambda)$ of (1.1). Here we slightly modify the definition in [10] to ensure that $\chi^{\pm}(x,\lambda)\in SO(2r)$. Thus we define

$$\begin{aligned}
&\chi^{+}(x,\lambda)\equiv\left(|\phi^{+}\rangle,|\psi^{+}\hat{c}^{+}\rangle\right)(x,\lambda)=\phi(x,\lambda)\boldsymbol{S}^{+}(\lambda)=\psi(x,\lambda)\boldsymbol{T}^{-}(\lambda)D^{+}(\lambda),\\
&\chi^{-}(x,\lambda)\equiv\left(|\psi^{-}\hat{c}^{-}\rangle,|\phi^{-}\rangle\right)(x,\lambda)=\phi(x,\lambda)\boldsymbol{S}^{-}(\lambda)=\psi(x,\lambda)\boldsymbol{T}^{+}(\lambda)D^{-}(\lambda),
\end{aligned}\tag{3.1}$$

where the block-triangular functions $\boldsymbol{S}^{\pm}(\lambda)$ and $\boldsymbol{T}^{\pm}(\lambda)$ are given by

$$\begin{aligned}
&\boldsymbol{S}^{+}(\lambda)=\begin{pmatrix}\mathbb{1} & \boldsymbol{d}^{-}\hat{\boldsymbol{c}}^{+}(\lambda)\\ 0 & \mathbb{1}\end{pmatrix},\qquad \boldsymbol{T}^{-}(\lambda)=\begin{pmatrix}\mathbb{1} & 0\\ \boldsymbol{b}^{+}\hat{\boldsymbol{a}}^{+}(\lambda) & \mathbb{1}\end{pmatrix},\\
&\boldsymbol{S}^{-}(\lambda)=\begin{pmatrix}\mathbb{1} & 0\\ -\boldsymbol{d}^{+}\hat{\boldsymbol{c}}^{-}(\lambda) & \mathbb{1}\end{pmatrix},\qquad \boldsymbol{T}^{+}(\lambda)=\begin{pmatrix}\mathbb{1} & -\boldsymbol{b}^{-}\hat{\boldsymbol{a}}^{-}(\lambda)\\ 0 & \mathbb{1}\end{pmatrix}.
\end{aligned}\tag{3.2}$$

The matrices $D^{\pm}(\lambda)$ are block-diagonal and equal

$$D^{+}(\lambda)=\begin{pmatrix}\boldsymbol{a}^{+}(\lambda) & 0\\ 0 & \hat{\boldsymbol{c}}^{+}(\lambda)\end{pmatrix},\qquad D^{-}(\lambda)=\begin{pmatrix}\hat{\boldsymbol{c}}^{-}(\lambda) & 0\\ 0 & \boldsymbol{a}^{-}(\lambda)\end{pmatrix}.$$

The upper scripts $\pm$ here refer to their analyticity properties for $\lambda\in\mathbb{C}_{\pm}$.

In view of the relations (2.10) it is easy to check that all factors $\boldsymbol{S}^{\pm}$, $\boldsymbol{T}^{\pm}$ and $D^{\pm}$ take values in the group $SO(2r)$. Besides, since

$$\begin{aligned}
&T(\lambda)=\boldsymbol{T}^{-}(\lambda)D^{+}(\lambda)\hat{\boldsymbol{S}}^{+}(\lambda)=\boldsymbol{T}^{+}(\lambda)D^{-}(\lambda)\hat{\boldsymbol{S}}^{-}(\lambda),\\
&\hat{T}(\lambda)=\boldsymbol{S}^{+}(\lambda)\hat{D}^{+}(\lambda)\hat{\boldsymbol{T}}^{-}(\lambda)=\boldsymbol{S}^{-}(\lambda)\hat{D}^{-}(\lambda)\hat{\boldsymbol{T}}^{+}(\lambda),
\end{aligned}\tag{3.3}$$

we can view the factors $\boldsymbol{S}^{\pm}$, $\boldsymbol{T}^{\pm}$ and $D^{\pm}$ as generalized Gauss decompositions (see [12]) of $T(\lambda)$ and its inverse.

The relations between $\boldsymbol{c}^{\pm}(\lambda)$, $\boldsymbol{d}^{\pm}(\lambda)$ and $\boldsymbol{a}^{\pm}(\lambda)$, $\boldsymbol{b}^{\pm}(\lambda)$ in equation (2.9) ensure that equations (3.3) become identities. From equations (3.1), (3.2) we derive

$$\chi^{+}(x,\lambda)=\chi^{-}(x,\lambda)G_0(\lambda),\qquad \chi^{-}(x,\lambda)=\chi^{+}(x,\lambda)\hat{G}_0(\lambda),\tag{3.4}$$

$$G_0(\lambda)=\begin{pmatrix}\mathbb{1} & \tau^{+}\\ \tau^{-} & \mathbb{1}+\tau^{-}\tau^{+}\end{pmatrix},\qquad \hat{G}_0(\lambda)=\begin{pmatrix}\mathbb{1}+\tau^{+}\tau^{-} & -\tau^{+}\\ -\tau^{-} & \mathbb{1}\end{pmatrix}\tag{3.5}$$

valid for $\lambda\in\mathbb{R}$. Below we introduce

$$X^{\pm}(x,\lambda)=\chi^{\pm}(x,\lambda)e^{i\lambda Jx}.$$

Strictly speaking it is $X^{\pm}(x,\lambda)$ that allow analytic extension for $\lambda\in\mathbb{C}_{\pm}$. They have also another nice property, namely their asymptotic behavior for $\lambda\to\pm\infty$ is given by

$$\lim_{\lambda\to\infty}X^{\pm}(x,\lambda)=\mathbb{1}.\tag{3.6}$$

Along with $X^{\pm}(x,\lambda)$ we can use another set of FAS $\tilde{X}^{\pm}(x,\lambda)=X^{\pm}(x,\lambda)\hat{D}^{\pm}$, which also satisfy equation (3.6) due to the fact that

$$\lim_{\lambda\to\infty}D^{\pm}(\lambda)=\mathbb{1}.$$

The analyticity properties of $X^{\pm}(x,\lambda)$ and $\tilde{X}^{\pm}(x,\lambda)$ for $\lambda\in\mathbb{C}_{\pm}$ along with equation (3.6) are crucial for our considerations.

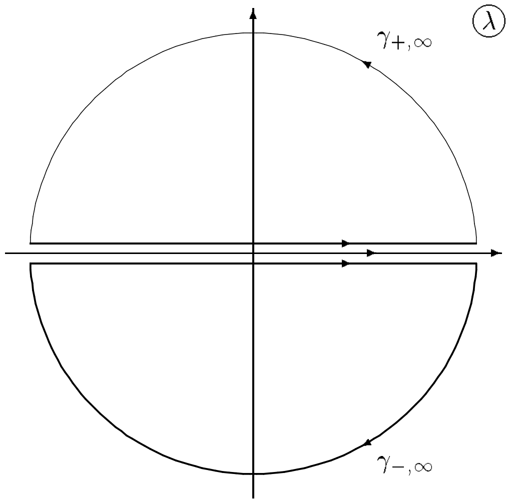

Figure 1. The contours $\gamma_\pm = \mathbb{R} \cup \gamma_{\pm\infty}$.

3.2 The Riemann–Hilbert problem

The equations (3.4) and (3.5) can be written down as

$$X^+(x,\lambda) = X^-(x,\lambda)G(x,\lambda), \qquad \lambda \in \mathbb{R}, \tag{3.7}$$

where

$$G(x,\lambda) = \mathrm{e}^{-\mathrm{i}\lambda Jx} G_0(\lambda) \mathrm{e}^{\mathrm{i}\lambda Jx}.$$

Likewise the second pair of FAS satisfy

$$\tilde{X}^+(x,\lambda) = \tilde{X}^-(x,\lambda)\tilde{G}(x,\lambda), \qquad \lambda \in \mathbb{R} \tag{3.8}$$

with

$$\tilde{G}(x,\lambda) = \mathrm{e}^{-\mathrm{i}\lambda Jx} \tilde{G}_0(\lambda) \mathrm{e}^{\mathrm{i}\lambda Jx} \qquad \tilde{G}_0(\lambda) = \begin{pmatrix} \mathbb{1} + \rho^-\rho^+ & \rho^- \\ \rho^+ & \mathbb{1} \end{pmatrix}.$$

Equation (3.7) (resp. equation (3.8)) combined with (3.6) is known in the literature [24] as a Riemann–Hilbert problem (RHP) with canonical normalization. It is well known that RHP with canonical normalization has unique regular solution; the matrix-valued solutions $X_0^+(x,\lambda)$ and $X_0^-(x,\lambda)$ of (3.7), (3.6) is called regular if $\det X_0^\pm(x,\lambda)$ does not vanish for any $\lambda \in \mathbb{C}_\pm$.

Let us now apply the contour-integration method to derive the integral decompositions of $X^\pm(x,\lambda)$. To this end we consider the contour integrals

$$\mathcal{J}_1(\lambda) = \frac{1}{2\pi\mathrm{i}} \oint_{\gamma_+} \frac{\mathrm{d}\mu}{\mu-\lambda} X^+(x,\mu) - \frac{1}{2\pi\mathrm{i}} \oint_{\gamma_-} \frac{\mathrm{d}\mu}{\mu-\lambda} X^-(x,\mu),$$

and

$$\mathcal{J}_2(\lambda) = \frac{1}{2\pi\mathrm{i}} \oint_{\gamma_+} \frac{\mathrm{d}\mu}{\mu-\lambda} \tilde{X}^+(x,\mu) - \frac{1}{2\pi\mathrm{i}} \oint_{\gamma_-} \frac{\mathrm{d}\mu}{\mu-\lambda} \tilde{X}^-(x,\mu),$$

where $\lambda \in \mathbb{C}_+$ and the contours $\gamma_\pm$ are shown on Fig. 1.

Each of these integrals can be evaluated by Cauchy residue theorem. The result for $\lambda \in \mathbb{C}_+$ are

$$\mathcal{J}_1(\lambda) = X^+(x,\lambda) + \sum_{j=1}^{N} \operatorname*{Res}_{\mu=\lambda_j^+} \frac{X^+(x,\mu)}{\mu-\lambda} + \sum_{j=1}^{N} \operatorname*{Res}_{\mu=\lambda_j^-} \frac{X^-(x,\mu)}{\mu-\lambda}, \tag{3.9}$$

$$\mathcal{J}_2(\lambda) = \tilde{X}^+(x,\lambda) + \sum_{j=1}^{N} \operatorname*{Res}_{\mu=\lambda_j^+} \frac{\tilde{X}^+(x,\mu)}{\mu-\lambda} + \sum_{j=1}^{N} \operatorname*{Res}_{\mu=\lambda_j^-} \frac{\tilde{X}^-(x,\mu)}{\mu-\lambda}. \tag{3.10}$$

The discrete sums in the right hand sides of equations (3.9) and (3.10) naturally provide the contribution from the discrete spectrum of L. For the sake of simplicity we assume that L has a finite number of simple eigenvalues $\lambda_j^\pm \in \mathbb{C}_\pm$; for additional details see [10]. Let us clarify the above statement. For the 2×2 Zakharov–Shabat problem it is well known that the discrete eigenvalues of L are provided by the zeroes of the transmission coefficients $a^\pm(\lambda)$, which in that case are scalar functions. For the more general $2r \times 2r$ Zakharov–Shabat system (1.1) the situation becomes more complex because now $a^\pm(\lambda)$ are $r\times r$ matrices. The discrete eigenvalues $\lambda_j^\pm$ now are the points at which $a^\pm(\lambda)$ become degenerate and their inverse develop pole singularities. More precisely, we assume that in the vicinities of $\lambda_j^\pm$ $\boldsymbol{a}^\pm(\lambda)$, $\boldsymbol{c}^\pm(\lambda)$ and their inverse $\hat{\boldsymbol{a}}^\pm(\lambda)$, $\hat{\boldsymbol{c}}^\pm(\lambda)$ have the following decompositions in Laurent series

$$\begin{aligned}
&\boldsymbol{a}^\pm(\lambda) = \boldsymbol{a}_j^\pm + (\lambda-\lambda_j^\pm)\dot{\boldsymbol{a}}_j^\pm + \cdots, &&\boldsymbol{c}^\pm(\lambda) = \boldsymbol{c}_j^\pm + (\lambda-\lambda_j^\pm)\dot{\boldsymbol{c}}_j^\pm + \cdots,\\
&\hat{\boldsymbol{a}}^\pm(\lambda) = \frac{\hat{\boldsymbol{a}}_j^\pm}{\lambda-\lambda_j^\pm} + \hat{\dot{\boldsymbol{a}}}_j^\pm + \cdots, &&\hat{\boldsymbol{c}}^\pm(\lambda) = \frac{\hat{\boldsymbol{c}}_j^\pm}{\lambda-\lambda_j^\pm} + \hat{\dot{\boldsymbol{a}}}_j^\pm + \cdots,
\end{aligned}$$

where all the leading coefficients $\boldsymbol{a}_j^\pm$, $\hat{\boldsymbol{a}}_j^\pm$ $\boldsymbol{c}_j^\pm$, $\hat{\boldsymbol{c}}_j^\pm$ are degenerate matrices such that

$$\hat{\boldsymbol{a}}_j^\pm \boldsymbol{a}_j^\pm = \boldsymbol{a}_j^\pm \hat{\boldsymbol{a}}_j^\pm = 0, \qquad \hat{\boldsymbol{c}}_j^\pm \boldsymbol{c}_j^\pm = \boldsymbol{c}_j^\pm \hat{\boldsymbol{c}}_j^\pm = 0.$$

In addition we have more relations such as

$$\hat{\boldsymbol{a}}_j^\pm \dot{\boldsymbol{a}}_j^\pm + \hat{\dot{\boldsymbol{a}}}_j^\pm \boldsymbol{a}_j^\pm = \mathbb{1}, \qquad \hat{\boldsymbol{c}}_j^\pm \dot{\boldsymbol{c}}_j^\pm + \hat{\dot{\boldsymbol{c}}}_j^\pm \boldsymbol{c}_j^\pm = \mathbb{1},$$

that are needed to ensure that the identities $\hat{\boldsymbol{a}}^\pm(\lambda)\boldsymbol{a}^\pm(\lambda) = \mathbb{1}$, $\hat{\boldsymbol{c}}^\pm(\lambda)\boldsymbol{c}^\pm(\lambda) = \mathbb{1}$ etc hold true for all values of λ.

The assumption that the eigenvalues are simple here means that we have considered only first order pole singularities of $\hat{\boldsymbol{a}}_j^\pm(\lambda)$ and $\hat{\boldsymbol{c}}_j^\pm(\lambda)$. After some additional considerations we find that the 'halfs' of the Jost solutions $|\psi^\pm(x,\lambda)\rangle$ and $|\phi^\pm(x,\lambda)\rangle$ satisfy the following relationships for $\lambda = \lambda_j^\pm$

$$|\psi_j^\pm(x)\hat{\boldsymbol{c}}_j^\pm\rangle = \pm|\phi_j^\pm(x)\tau_j^\pm\rangle, \qquad |\phi_j^\pm(x)\hat{\boldsymbol{a}}_j^\pm\rangle = \pm|\psi_j^\pm(x)\rho_j^\pm\rangle,$$

where

$$\begin{aligned}
&|\psi_j^\pm(x)\rangle = |\psi^\pm(x,\lambda_j^\pm)\rangle, &&|\phi_j^\pm(x)\rangle = |\phi^\pm(x,\lambda_j^\pm)\rangle,\\
&\rho_j^\pm = \hat{\boldsymbol{c}}_j^\pm \boldsymbol{d}_j^\pm = \boldsymbol{b}_j^\pm \hat{\boldsymbol{a}}_j^\pm, &&\tau_j^\pm = \hat{\boldsymbol{a}}_j^\pm \boldsymbol{b}_j^\pm = \boldsymbol{d}_j^\pm \hat{\boldsymbol{c}}_j^\pm
\end{aligned}$$

and the additional coefficients $\boldsymbol{b}_j^\pm$ and $\boldsymbol{d}_j^\pm$ are constant $r\times r$ nondegenerate matrices which, as we shall see below, are also part of the minimal sets of scattering data needed to determine the potential $Q(x,t)$.

These considerations allow us to calculate explicitly the residues in equations (3.9), (3.10) with the result

$$\operatorname*{Res}_{\mu=\lambda_j^+} \frac{X^+(x,\mu)}{\mu-\lambda} = \frac{(|\mathbf{0}\rangle, |\phi_j^+(x)\tau_j^+\rangle)}{\lambda_j^+-\lambda}, \qquad \operatorname*{Res}_{\mu=\lambda_j^+} \frac{\tilde{X}^+(x,\mu)}{\mu-\lambda} = \frac{(|\psi_j^+(x)\rho_j^+\rangle, |\mathbf{0}\rangle)}{\lambda_j^+-\lambda},$$

$$\operatorname*{Res}_{\mu=\lambda_j^+} \frac{X^-(x,\mu)}{\mu-\lambda} = -\frac{(|\phi_j^-(x)\tau_j^-\rangle, |\mathbf{0}\rangle)}{\lambda_j^+-\lambda}, \qquad \operatorname*{Res}_{\mu=\lambda_j^+} \frac{\tilde{X}^-(x,\mu)}{\mu-\lambda} = -\frac{(|\mathbf{0}\rangle, |\psi_j^-(x)\tau_j^-\rangle)}{\lambda_j^+-\lambda},$$

where $|\mathbf{0}\rangle$ stands for a collection of r columns whose components are all equal to 0.

We can also evaluate $\mathcal{J}_1(\lambda)$ and $\mathcal{J}_2(\lambda)$ by integrating along the contours. In integrating along the infinite semi-circles of $\gamma_{\pm,\infty}$ we use the asymptotic behavior of $X^\pm(x,\lambda)$ and $\tilde{X}^\pm(x,\lambda)$ for $\lambda\to\infty$. The results are

$$\mathcal{J}_1(\lambda) = \mathbb{1} + \frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty} \frac{\mathrm{d}\mu}{\mu-\lambda}\phi(x,\mu)\mathrm{e}^{\mathrm{i}\mu Jx}K(x,\mu), \tag{3.11}$$

$$\mathcal{J}_2(\lambda) = \mathbb{1} + \frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty} \frac{\mathrm{d}\mu}{\mu-\lambda}\psi(x,\mu)\mathrm{e}^{\mathrm{i}\mu Jx}\tilde{K}(x,\mu), \tag{3.12}$$

$$K(x,\mu) = \mathrm{e}^{-\mathrm{i}\mu Jx}K_0(\mu)\mathrm{e}^{\mathrm{i}\mu Jx}, \qquad \tilde{K}(x,\mu) = \mathrm{e}^{-\mathrm{i}\mu Jx}\tilde{K}_0(\mu)\mathrm{e}^{\mathrm{i}\mu Jx},$$

$$K_0(\mu) = \begin{pmatrix} 0 & \tau^+(\mu) \\ \tau^-(\mu) & 0 \end{pmatrix}, \qquad \tilde{K}_0(\mu) = \begin{pmatrix} 0 & \rho^+(\mu) \\ \rho^-(\mu) & 0 \end{pmatrix},$$

where in evaluating the integrands we made use of equations (2.8), (2.9), (3.7) and (3.8).

Equating the right hand sides of (3.9) and (3.11), and (3.10) and (3.12) we get the following integral decomposition for $X^\pm(x,\lambda)$:

$$X^+(x,\lambda) = \mathbb{1} + \frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty} \frac{\mathrm{d}\mu}{\mu-\lambda}X^-(x,\mu)K_1(x,\mu) + \sum_{j=1}^{N}\frac{X_j^-(x)K_{1,j}(x)}{\lambda_j^- - \lambda}, \tag{3.13}$$

$$X^-(x,\lambda) = \mathbb{1} + \frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty} \frac{\mathrm{d}\mu}{\mu-\lambda}X^-(x,\mu)K_2(x,\mu) - \sum_{j=1}^{N}\frac{X_j^+(x)K_{2,j}(x)}{\lambda_j^+ - \lambda}, \tag{3.14}$$

where $X_j^\pm(x) = X^\pm(x,\lambda_j^\pm)$ and

$$K_{1,j}(x) = \mathrm{e}^{-\mathrm{i}\lambda_j^- Jx}\begin{pmatrix} 0 & \rho_j^+ \\ \tau_j^- & 0 \end{pmatrix}\mathrm{e}^{\mathrm{i}\lambda_j^- Jx}, \qquad K_{2,j}(x) = \mathrm{e}^{-\mathrm{i}\lambda_j^+ Jx}\begin{pmatrix} 0 & \tau_j^+ \\ \rho_j^- & 0 \end{pmatrix}\mathrm{e}^{\mathrm{i}\lambda_j^+ Jx}.$$

Equations (3.13), (3.14) can be viewed as a set of singular integral equations which are equivalent to the RHP. For the MNLS these were first derived in [25].

We end this section by a brief explanation of how the potential $Q(x,t)$ can be recovered provided we have solved the RHP and know the solutions $X^\pm(x,\lambda)$. First we take into account that $X^\pm(x,\lambda)$ satisfy the differential equation

$$\mathrm{i}\frac{\mathrm{d}X^\pm}{\mathrm{d}x} + Q(x,t)X^\pm(x,\lambda) - \lambda[J, X^\pm(x,\lambda)] = 0 \tag{3.15}$$

which must hold true for all λ. From equation (3.6) and also from the integral equations (3.13), (3.14) one concludes that $X^\pm(x,\lambda)$ and their inverse $\hat{X}^\pm(x,\lambda)$ are regular for $\lambda\to\infty$ and allow asymptotic expansions of the form

$$X^\pm(x,\lambda) = \mathbb{1} + \sum_{s=1}^{\infty}\lambda^{-s}X_s(x), \qquad \hat{X}^\pm(x,\lambda) = \mathbb{1} + \sum_{s=1}^{\infty}\lambda^{-s}\hat{X}_s(x).$$

Inserting these into equation (3.15) and taking the limit $\lambda\to\infty$ we get

$$Q(x,t) = \lim_{\lambda\to\infty}\lambda(J - X^\pm(x,\lambda)J\hat{X}^\pm(x,\lambda)]) = [J, X_1(x)]. \tag{3.16}$$

4 The generalized Fourier transforms

It is well known that the ISM can be interpreted as a generalized Fourier [10] transform which maps the potential $Q(x,t)$ onto the minimal sets of scattering data $\mathcal{T}_i$. Here we briefly formulate these results and in the next Section we will analyze how they are modified under the reduction conditions.

The generalized exponentials are the 'squared solutions' which are determined by the FAS and the Cartan–Weyl basis of the corresponding algebra as follows

$$\boldsymbol{\Psi}_\alpha^\pm = \chi^\pm(x,\lambda)E_\alpha\hat{\chi}^\pm(x,\lambda), \qquad \boldsymbol{\Phi}_\alpha^\pm = \chi^\pm(x,\lambda)E_{-\alpha}\hat{\chi}^\pm(x,\lambda), \qquad \alpha\in\Delta_1^+.$$

4.1 Expansion over the 'squared solutions'

The 'squared solutions' are complete set of functions in the phase space [10]. This allows one to expand any function over the 'squared solutions'.

Let us introduce the sets of 'squared solutions'

$$\begin{aligned}
&\{\boldsymbol{\Psi}\} = \{\boldsymbol{\Psi}\}_{\rm c}\cup\{\boldsymbol{\Psi}\}_{\rm d}, \qquad \{\boldsymbol{\Phi}\} = \{\boldsymbol{\Phi}\}_{\rm c}\cup\{\boldsymbol{\Phi}\}_{\rm d},\\
&\{\boldsymbol{\Psi}\}_{\rm c} \equiv \left\{\boldsymbol{\Psi}_\alpha^+(x,\lambda), \quad \boldsymbol{\Psi}_{-\alpha}^-(x,\lambda), \quad i<r, \quad \lambda\in\mathbb{R}\right\},\\
&\{\boldsymbol{\Psi}\}_{\rm d} \equiv \left\{\boldsymbol{\Psi}_{\alpha;j}^+(x), \quad \dot{\boldsymbol{\Psi}}_{\alpha;j}^+(x), \quad \boldsymbol{\Psi}_{-\alpha;j}^-(x), \quad \dot{\boldsymbol{\Psi}}_{-\alpha;j}^-(x)\right\}_{j=1}^N,\\
&\{\boldsymbol{\Phi}\}_{\rm c} \equiv \left\{\boldsymbol{\Phi}_{-\alpha}^+(x,\lambda), \quad \boldsymbol{\Phi}_\alpha^-(x,\lambda), \quad i<r, \quad \lambda\in\mathbb{R}\right\},\\
&\{\boldsymbol{\Phi}\}_{\rm d} \equiv \left\{\boldsymbol{\Phi}_{-\alpha;j}^+(x), \quad \dot{\boldsymbol{\Phi}}_{-\alpha;j}^+(x), \quad \boldsymbol{\Phi}_{\alpha;j}^-(x), \quad \dot{\boldsymbol{\Phi}}_{\alpha;j}^-(x)\right\}_{j=1}^N,
\end{aligned}$$

where the subscripts 'c' and 'd' refer to the continuous and discrete spectrum of L. The 'squared solutions' in bold-face $\boldsymbol{\Psi}_\alpha^+, \ldots$ are obtained from $\Psi_\alpha^+, \ldots$ by applying the projector P_{0J}, i.e. $\boldsymbol{\Psi}_\alpha^+(x,\lambda) = P_{0J}\Psi_\alpha^+(x,\lambda)$.

Using the Wronskian relations one can derive the expansions over the 'squared solutions' of two important functions. Skipping the calculational details we formulate the results [10]. The expansion of $Q(x)$ over the systems $\{\boldsymbol{\Phi}^\pm\}$ and $\{\boldsymbol{\Psi}^\pm\}$ takes the form

$$\begin{aligned}
Q(x) &= \frac{\rm i}{\pi}\int_{-\infty}^\infty {\rm d}\lambda \sum_{\alpha\in\Delta_1^+}\left(\tau_\alpha^+(\lambda)\boldsymbol{\Phi}_\alpha^+(x,\lambda) - \tau_\alpha^-(\lambda)\boldsymbol{\Phi}_{-\alpha}^-(x,\lambda)\right)\\
&\quad + 2\sum_{k=1}^N\sum_{\alpha\in\Delta_1^+}\left(\tau_{\alpha;j}^+\boldsymbol{\Phi}_{\alpha;j}^+(x) + \tau_{\alpha;j}^-\boldsymbol{\Phi}_{-\alpha;j}^-(x)\right), \qquad (4.1)\\
Q(x) &= -\frac{\rm i}{\pi}\int_{-\infty}^\infty {\rm d}\lambda \sum_{\alpha\in\Delta_1^+}\left(\rho_\alpha^+(\lambda)\boldsymbol{\Psi}_{-\alpha}^+(x,\lambda) - \rho_\alpha^-(\lambda)\boldsymbol{\Psi}_\alpha^-(x,\lambda)\right)\\
&\quad - 2\sum_{k=1}^N\sum_{\alpha\in\Delta_1^+}\left(\rho_{\alpha;j}^+\boldsymbol{\Psi}_{-\alpha;j}^+(x) + \rho_{\alpha;j}^-\boldsymbol{\Psi}_{\alpha;j}^-(x)\right). \qquad (4.2)
\end{aligned}$$

The next expansion is of $\operatorname{ad}_J^{-1}\delta Q(x)$ over the systems $\{\boldsymbol{\Phi}^\pm\}$ and $\{\boldsymbol{\Psi}^\pm\}$

$$\begin{aligned}
\operatorname{ad}_J^{-1}\delta Q(x) &= \frac{\rm i}{2\pi}\int_{-\infty}^\infty {\rm d}\lambda \sum_{\alpha\in\Delta_1^+}\left(\delta\tau_\alpha^+(\lambda)\boldsymbol{\Phi}_\alpha^+(x,\lambda) + \delta\tau_\alpha^-(\lambda)\boldsymbol{\Phi}_{-\alpha}^-(x,\lambda)\right)\\
&\quad + \sum_{k=1}^N\sum_{\alpha\in\Delta_1^+}\left(\delta W_{\alpha;j}^+(x) - \delta' W_{-\alpha;j}^-(x)\right), \qquad (4.3)
\end{aligned}$$

$$\operatorname{ad}_J^{-1}\delta Q(x) = \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda \sum_{\alpha\in\Delta_1^+} \left(\delta\rho_\alpha^+(\lambda)\boldsymbol{\Psi}_{-\alpha}^+(x,\lambda) + \delta\rho_\alpha^-(\lambda)\boldsymbol{\Psi}_\alpha^-(x,\lambda)\right)$$
$$+ \sum_{k=1}^N \sum_{\alpha\in\Delta_1^+} \left(\delta\tilde{W}_{-\alpha;j}^+(x) - \delta\tilde{W}_{\alpha;j}^-(x)\right), \tag{4.4}$$

where

$$\delta W_{\pm\alpha;j}^\pm(x) = \delta\lambda_j^\pm \tau_{\alpha;j}^\pm \dot{\boldsymbol{\Phi}}_{\pm\alpha;j}^\pm(x) + \delta\tau_{\alpha;j}^\pm \boldsymbol{\Phi}_{\pm\alpha;j}^\pm(x),$$
$$\delta\tilde{W}_{\mp\alpha;j}^\pm(x) = \delta\lambda_j^\pm \rho_{\alpha;j}^\pm \dot{\boldsymbol{\Psi}}_{\mp\alpha;j}^\pm(x) + \delta\rho_{\alpha;j}^\pm \boldsymbol{\Psi}_{\mp\alpha;j}^\pm(x)$$

and $\boldsymbol{\Phi}_{\pm\alpha;j}^\pm(x) = \boldsymbol{\Phi}_{\pm\alpha}^\pm(x,\lambda_j^\pm)$, $\dot{\boldsymbol{\Phi}}_{\pm\alpha;j}^\pm(x) = \partial_\lambda \boldsymbol{\Phi}_{\pm\alpha}^\pm(x,\lambda)|_{\lambda=\lambda_j^\pm}$.

The expansions (4.1), (4.2) is another way to establish the one-to-one correspondence between $Q(x)$ and each of the minimal sets of scattering data $\mathcal{T}_1$ and $\mathcal{T}_2$ (4.6). Likewise the expansions (4.3), (4.4) establish the one-to-one correspondence between the variation of the potential $\delta Q(x)$ and the variations of the scattering data $\delta\mathcal{T}_1$ and $\delta\mathcal{T}_2$.

The expansions (4.3), (4.4) have a special particular case when one considers the class of variations of $Q(x,t)$ due to the evolution in t. Then

$$\delta Q(x,t) \equiv Q(x,t+\delta t) - Q(x,t) = \frac{\partial Q}{\partial t}\delta t + (O)((\delta t)^2).$$

Assuming that δt is small and keeping only the first order terms in δt we get the expansions for $\operatorname{ad}_J^{-1} Q_t$. They are obtained from (4.3), (4.4) by replacing $\delta\rho_\alpha^\pm(\lambda)$ and $\delta\tau_\alpha^\pm(\lambda)$ by $\partial_t\rho_\alpha^\pm(\lambda)$ and $\partial_t\rho_\alpha^\pm(\lambda)$.

4.2 The generating operators

To complete the analogy between the standard Fourier transform and the expansions over the 'squared solutions' we need the analogs of the operator $D_0 = -\mathrm{i}\mathrm{d}/\mathrm{d}x$. The operator D_0 is the one for which $\mathrm{e}^{\mathrm{i}\lambda x}$ is an eigenfunction: $D_0\mathrm{e}^{\mathrm{i}\lambda x} = \lambda\mathrm{e}^{\mathrm{i}\lambda x}$. Therefore it is natural to introduce the generating operators $\Lambda_\pm$ through

$$(\Lambda_+ - \lambda)\boldsymbol{\Psi}_{-\alpha}^+(x,\lambda) = 0, \qquad (\Lambda_+ - \lambda)\boldsymbol{\Psi}_\alpha^-(x,\lambda) = 0, \qquad (\Lambda_+ - \lambda_j^\pm)\boldsymbol{\Psi}_{\mp\alpha;j}^+(x) = 0,$$
$$(\Lambda_- - \lambda)\boldsymbol{\Phi}_\alpha^+(x,\lambda) = 0, \qquad (\Lambda_- - \lambda)\boldsymbol{\Phi}_{-\alpha}^-(x,\lambda) = 0, \qquad (\Lambda_+ - \lambda_j^\pm)\boldsymbol{\Phi}_{\pm\alpha;j}^+(x) = 0,$$

where the generating operators $\Lambda_\pm$ are given by

$$\Lambda_\pm X(x) \equiv \operatorname{ad}_J^{-1}\left(\mathrm{i}\frac{\mathrm{d}X}{\mathrm{d}x} + \mathrm{i}\left[Q(x), \int_{\pm\infty}^x \mathrm{d}y\,[Q(y),X(y)]\right]\right). \tag{4.5}$$

The rest of the squared solutions are not eigenfunctions of neither Λ_+ nor Λ_-

$$(\Lambda_+ - \lambda_j^+)\dot{\boldsymbol{\Psi}}_{-\alpha;j}^+(x) = \boldsymbol{\Psi}_{-\alpha;j}^+(x), \qquad (\Lambda_+ - \lambda_j^-)\dot{\boldsymbol{\Psi}}_{\alpha;j}^-(x) = \boldsymbol{\Psi}_{\alpha;j}^-(x),$$
$$(\Lambda_- - \lambda_j^+)\dot{\boldsymbol{\Phi}}_{ir;j}^+(x) = \boldsymbol{\Phi}_{\alpha;j}^+(x), \qquad (\Lambda_- - \lambda_j^-)\dot{\boldsymbol{\Phi}}_{\alpha;j}^-(x) = \boldsymbol{\Phi}_{\alpha;j}^-(x),$$

i.e., $\dot{\boldsymbol{\Psi}}_{\alpha;j}^+(x)$ and $\dot{\boldsymbol{\Phi}}_{\alpha;j}^+(x)$ are adjoint eigenfunctions of Λ_+ and Λ_-. This means that $\lambda_j^\pm$, $j = 1,\dots,N$ are also the discrete eigenvalues of $\Lambda_\pm$ but the corresponding eigenspaces of $\Lambda_\pm$ have double the dimensions of the ones of L; now they are spanned by both $\boldsymbol{\Psi}_{\mp\alpha;j}^\pm(x)$ and $\dot{\boldsymbol{\Psi}}_{\mp\alpha;j}^\pm(x)$. Thus the sets $\{\Psi\}$ and $\{\Phi\}$ are the complete sets of eigen- and adjoint functions of Λ_+ and Λ_-.

4.3 The minimal sets of scattering data

Obviously, given the potential $Q(x)$ one can solve the integral equations for the Jost solutions which determine them uniquely. The Jost solutions in turn determine uniquely the scattering matrix $T(\lambda)$ and its inverse $\hat{T}(\lambda)$. But $Q(x)$ contains $r(r-1)$ independent complex-valued functions of x. Thus it is natural to expect that at most $r(r-1)$ of the coefficients in $T(\lambda)$ for $\lambda \in \mathbb{R}$ will be independent; the rest must be functions of those. The set of independent coefficients of $T(\lambda)$ are known as the minimal set of scattering data.

The completeness relation for the 'squared solutions' ensure that there is one-to-one correspondence between the potential $Q(x,t)$ and its expansion coefficients. Thus we may use as minimal sets of scattering data the following two sets $\mathcal{T}_i \equiv \mathcal{T}_{i,\mathrm{c}} \cup \mathcal{T}_{i,\mathrm{d}}$

$$\begin{aligned} &\mathcal{T}_{1,\mathrm{c}} \equiv \{\rho^+(\lambda), \rho^-(\lambda), \quad \lambda \in \mathbb{R}\}, \qquad &&\mathcal{T}_{1,\mathrm{d}} \equiv \left\{\rho_j^\pm, \lambda_j^\pm\right\}_{j=1}^N, \\ &\mathcal{T}_{2,\mathrm{c}} \equiv \{\tau^+(\lambda), \tau^-(\lambda), \quad \lambda \in \mathbb{R}\}, \qquad &&\mathcal{T}_{1,\mathrm{d}} \equiv \left\{\tau_j^\pm, \lambda_j^\pm\right\}_{j=1}^N, \end{aligned} \tag{4.6}$$

where the reflection coefficients $\rho^\pm(\lambda)$ and $\tau^\pm(\lambda)$ were introduced in equation (2.9), $\lambda_j^\pm$ are (simple) discrete eigenvalues of L and $\rho_j^\pm$ and $\tau_j^\pm$ characterize the norming constants of the corresponding Jost solutions.

Remark 2. A consequence of equation (2.11) is the fact that $\boldsymbol{S}^\pm(\lambda), \boldsymbol{T}^\pm(\lambda) \in SO(2r)$. These factors can be written also in the form

$$\boldsymbol{S}^\pm(\lambda) = \exp\left(\sum_{\alpha\in\Delta_1^+} \tau_\alpha^\pm(\lambda) E_{\pm\alpha}\right), \qquad \boldsymbol{T}^\pm(\lambda) = \exp\left(\sum_{\alpha\in\Delta_1^+} \rho_\alpha^\pm(\lambda) E_{\pm\alpha}\right).$$

Taking into account that in the typical representation we have $E_{\pm\alpha}E_{\pm\beta} = 0$ for all roots $\alpha, \beta \in \Delta_1^+$ we find that

$$\begin{aligned} &\sum_{\alpha\in\Delta_1^+} \tau_\alpha^+(\lambda) E_{\pm\alpha} = \begin{pmatrix} 0 & \tau^+(\lambda) \\ 0 & 0 \end{pmatrix}, \qquad &&\sum_{\alpha\in\Delta_1^+} \tau_\alpha^-(\lambda) E_{-\alpha} = \begin{pmatrix} 0 & 0 \\ \tau^-(\lambda) & 0 \end{pmatrix}, \\ &\sum_{\alpha\in\Delta_1^+} \rho_\alpha^+(\lambda) E_{\pm\alpha} = \begin{pmatrix} 0 & \rho^+(\lambda) \\ 0 & 0 \end{pmatrix}, \qquad &&\sum_{\alpha\in\Delta_1^+} \rho_\alpha^-(\lambda) E_{-\alpha} = \begin{pmatrix} 0 & 0 \\ \rho^-(\lambda) & 0 \end{pmatrix}, \end{aligned} \tag{4.7}$$

where Δ_1^+ is a subset of the positive roots of $so(2r)$ defined in Subsection 2.2. The formulae (4.7) ensure that the number of independent matrix elements of $\tau^+(\lambda)$ and $\tau^-(\lambda)$ (resp., $\rho^+(\lambda)$ and $\rho^-(\lambda)$) equals $2|\Delta_1^+| = r(r-1)$ which coincides with the number of independent functions of $Q(x)$.

An important consequence of the expansions is the theorem [10]

Theorem 1. *Any nonlinear evolution equation (NLEE) integrable via the inverse scattering method applied to the Lax operator L (1.1) can be written in the form*

$$\mathrm{i}\,\mathrm{ad}_J^{-1} \frac{\partial Q}{\partial t} + 2f(\Lambda) Q(x,t) = 0, \tag{4.8}$$

where the function $f(\lambda)$ is known as the dispersion law of this NLEE. The generic MMKdV equation is a member of this class and is obtained by choosing $f(\lambda) = -4\lambda^3$. If $Q(x,t)$ is a solution to (4.8) then the corresponding scattering matrix satisfy the linear evolution equation

$$\mathrm{i}\frac{dT}{dt} + f(\lambda)[J, T(\lambda,t)] = 0, \tag{4.9}$$

or equivalently

$$\mathrm{i}\frac{\mathrm{d}\rho^{\pm}}{\mathrm{d}t} \mp 2f_0(\lambda)\rho^{\pm} = 0, \qquad \frac{\mathrm{d}\lambda_j^{\pm}}{\mathrm{d}t} = 0, \qquad \mathrm{i}\frac{\mathrm{d}\rho_{;j}^{\pm}}{\mathrm{d}t} \mp 2f_0(\lambda_j^{\pm})\rho_{;j}^{\pm} = 0,$$
$$\mathrm{i}\frac{d\tau^{\pm}}{\mathrm{d}t} \pm 2f_0(\lambda)\tau^{\pm} = 0, \qquad \frac{\mathrm{d}\lambda_j^{\pm}}{\mathrm{d}t} = 0, \qquad i\frac{\mathrm{d}\tau_{;j}^{\pm}}{\mathrm{d}t} \pm 2f_0(\lambda_j^{\pm})\tau_{;j}^{\pm} = 0,$$

and vice versa. In particular from (4.9) there follows that $\mathbf{a}^{\pm}(\lambda)$ and $\mathbf{c}^{\pm}(\lambda)$ are time-independent and therefore can be considered as generating functionals of integrals of motion for the NLEE.

Let us, before going into the non-trivial reductions, briefly discuss the Hamiltonian formulations for the generic (i.e., non-reduced) MMKdV type equations. It is well known (see [10] and the numerous references therein) that the class of these equations is generated by the so-called recursion operator $\Lambda = 1/2(\Lambda_+ + \Lambda_-)$ which is defined by equation (4.5).

If no additional reduction is imposed one can write each of the equations in (4.8) in Hamiltonian form. The corresponding Hamiltonian and symplectic form for the MMKV equation are given by

$$H^{(0)}_{\mathrm{MMKdV}} = \frac{1}{4}\int_{-\infty}^{\infty} dx\ \left(\mathrm{tr}\,(JQ_xQ_{xx}) - 3\mathrm{tr}\,(JQ^3Q_x)\right), \tag{4.10}$$
$$\Omega^{(0)} = \frac{1}{\mathrm{i}}\int_{-\infty}^{\infty} dx\ \mathrm{tr}\ \left(\mathrm{ad}_J^{-1}\delta Q(x) \mathop{\wedge}_{,} \left[J, \mathrm{ad}_J^{-1}\delta Q(x)\right]\right) = \frac{1}{2\mathrm{i}}\int_{-\infty}^{\infty} dx\ \mathrm{tr}\,(J\delta Q(x) \mathop{\wedge}_{,} \delta Q(x)).$$

The Hamiltonian can be identified as proportional to the fourth coefficient I_4 in the asymptotic expansion of $A^+(\lambda)$ (5.15) over the negative powers of λ

$$A^+(\lambda) = \sum_{k=1}^{\infty} \mathrm{i}I_k\lambda^{-k}.$$

This series of integrals of motion is known as the principal one. The first three of these integrals take the form

$$I_1 = \frac{1}{4}\int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{tr}\,(Q^2(x,t)), \qquad I_2 = -\frac{\mathrm{i}}{4}\int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{tr}\,(Q\mathrm{ad}_J^{-1}Q_x),$$
$$I_3 = -\frac{1}{8}\int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{tr}\,(QQ_{xx} + 2Q^4), \qquad I_4 = \frac{1}{32}\int_{-\infty}^{\infty} dx\ \left(\mathrm{tr}\,(JQ_xQ_{xx}) - 3\mathrm{tr}\,(JQ^3Q_x)\right).$$

We will remind also another important result, namely that the gradient of I_k is expressed through Λ as

$$\nabla_{Q^T(x)}I_k = -\frac{1}{2}\Lambda^{k-1}Q(x,t).$$

Then the Hamiltonian equations written through $\Omega^{(0)}$ and the Hamiltonian vector field $X_{H^{(0)}}$ in the form

$$\Omega^{(0)}(\cdot, X_{H^{(0)}}) + \delta H^{(0)} = 0 \tag{4.11}$$

for $H^{(0)}$ given by (4.10) coincides with the MMKdV equation.

An alternative way to formulate Hamiltonian equations of motion is to introduce along with the Hamiltonian the Poisson brackets on the phase space $\mathcal{M}$ which is the space of smooth functions taking values in $\mathfrak{g}^{(0)}$ and vanishing fast enough for $x \to \pm\infty$, see (5.1). These brackets can be introduced by

$$\{F, G\}_{(0)} = \mathrm{i}\int_{-\infty}^{\infty} dx\ \mathrm{tr}\ \left(\nabla_{Q^T(x)}F, \left[J, \nabla_{Q^T(x)}G\right]\right).$$

Then the Hamiltonian equations of motions

$$\frac{dq_{ij}}{dt} = \{q_{ij}, H^{(0)}\}_{(0)}, \qquad \frac{dp_{ij}}{dt} = \{p_{ij}, H^{(0)}\}_{(0)} \tag{4.12}$$

with the above choice for $H^{(0)}$ again give the MMKdV equation.

Along with this standard Hamiltonian formulation there exist a whole hierarchy of them. This is a special property of the integrable NLEE. The hierarchy is generated again by the recursion operator and has the form

$$H^{(m)}_{\mathrm{MMKdV}} = -8I_{4+m}, \qquad \Omega^{(m)} = \frac{1}{\mathrm{i}} \int_{-\infty}^{\infty} dx \, \mathrm{tr} \left(\mathrm{ad}_J^{-1} \delta Q(x) \mathop{\wedge}_{,} \left[J, \Lambda^m \mathrm{ad}_J^{-1} \delta Q(x) \right] \right).$$

Of course there is also a hierarchy of Poisson brackets

$$\{F, G\}_{(m)} = \mathrm{i} \int_{-\infty}^{\infty} dx \, \mathrm{tr} \left(\nabla_{Q^T(x)} F, \left[J, \Lambda^{-m} \nabla_{Q^T(x)} G \right] \right).$$

For a fixed value of m the Poisson bracket $\{\cdot,\cdot\}_{(m)}$ is dual to the symplectic form $\Omega^{(m)}$ in the sense that combined with a given Hamiltonian they produce the same equations of motion. Note that since Λ is an integro-differential operator in general it is not easy to evaluate explicitly its negative powers. Using this duality one can avoid the necessity to evaluate negative powers of Λ.

Then the analogs of (4.11) and (4.12) take the form:

$$\Omega^{(m)}(\cdot, X_{H^{(m)}}) + \delta H^{(m)} = 0, \tag{4.13}$$

$$\frac{dq_{ij}}{dt} = \{q_{ij}, H^{(-m)}\}_{(m)}, \qquad \frac{dp_{ij}}{dt} = \{p_{ij}, H^{(-m)}\}_{(m)}, \tag{4.14}$$

where the hierarchy of Hamiltonians is given by:

$$H^{(m)} = -4 \sum_k f_k I_{k+1-m}. \tag{4.15}$$

The equations (4.13) and (4.14) with the Hamiltonian $H^{(m)}$ given by (4.15) will produce the NLEE (4.8) with dispersion law $f(\lambda) = \sum_k f_k \lambda^k$ for any value of m.

Remark 3. It is a separate issue to prove that the hierarchies of symplectic structures and Poisson brackets have all the necessary properties. This is done using the spectral decompositions of the recursion operators $\Lambda_\pm$ which are known also as the expansions over the 'squared solutions' of L. We refer the reader to the review papers [26, 10] where he/she can find the proof of the completeness relation for the 'squared solutions' along with the proof that any two of the symplectic forms introduced above are compatible.

4.4 The reduction group of Mikhailov

The reduction group G_R is a finite group which preserves the Lax representation (1.1), i.e. it ensures that the reduction constraints are automatically compatible with the evolution. G_R must have two realizations: i) $G_R \subset \mathrm{Aut}\,\mathfrak{g}$ and ii) $G_R \subset \mathrm{Conf}\,\mathbb{C}$, i.e. as conformal mappings of the complex λ-plane. To each $g_k \in G_R$ we relate a reduction condition for the Lax pair as follows [5]

$$C_k(L(\Gamma_k(\lambda))) = \eta_k L(\lambda), \qquad C_k(M(\Gamma_k(\lambda))) = \eta_k M(\lambda), \tag{4.16}$$

where $C_k \in \mathrm{Aut}\,\mathfrak{g}$ and $\Gamma_k(\lambda) \in \mathrm{Conf}\,\mathbb{C}$ are the images of g_k and $\eta_k = 1$ or -1 depending on the choice of C_k. Since G_R is a finite group then for each g_k there exist an integer N_k such that $g_k^{N_k} = \mathbb{1}$.

More specifically the automorphisms C_k, $k = 1, \ldots, 4$ listed above lead to the following reductions for the potentials $U(x,t,\lambda)$ and $V(x,t,\lambda)$ of the Lax pair

$$U(x,t,\lambda) = Q(x,t) - \lambda J, \qquad V(x,t,\lambda) = \sum_{k=0}^{2} \lambda^k V_k(x,t) - 4\lambda^3 J,$$

of the Lax representation

$$\begin{aligned}
&1) \quad C_1(U^\dagger(\kappa_1(\lambda))) = U(\lambda), && C_1(V^\dagger(\kappa_1(\lambda))) = V(\lambda), && (4.17)\\
&2) \quad C_2(U^T(\kappa_2(\lambda))) = -U(\lambda), && C_2(V^T(\kappa_2(\lambda))) = -V(\lambda), && (4.18)\\
&3) \quad C_3(U^*(\kappa_1(\lambda))) = -U(\lambda), && C_3(V^*(\kappa_1(\lambda))) = -V(\lambda), && (4.19)\\
&4) \quad C_4(U(\kappa_2(\lambda))) = U(\lambda), && C_4(V(\kappa_2(\lambda))) = V(\lambda), && (4.20)
\end{aligned}$$

where

$$\text{a)} \quad \kappa_1(\lambda) = \lambda^*, \qquad \text{b)} \quad \kappa_2(\lambda) = -\lambda.$$

The condition (4.16) is obviously compatible with the group action.

5 Finite order reductions of MMKdV equations

In order that the potential $Q(x,t)$ be relevant for a **DIII** type symmetric space it must be of the form

$$Q(x,t) = \sum_{\alpha\in\Delta_1^+} \left(q_\alpha(x,t)E_\alpha + p_\alpha(x,t)E_{-\alpha}\right),$$

or, equivalently

$$Q(x,t) = \sum_{1\leq i<j\leq r} \left(q_{ij}(x,t)E_{e_i+e_j} + p_{ij}(x,t)E_{-e_i-e_j}\right). \tag{5.1}$$

In the next two subsections we display new reductions of the MMKdV equations.

5.1 Class A Reductions preserving J

The class A reductions can be applied also to the MMKdV type equations. The corresponding automorphisms C preserve J, i.e. $C^{-1}JC = J$ and are of the form

$$C^{-1}U^\dagger(x,\lambda^*)C = U(x,\lambda), \qquad U(x,\lambda) = Q(x,t) - \lambda J,$$

where J is an element of the Cartan subalgebra dual to the vector $e_1 + e_2 + e_3 + e_4$. In the typical representation of $so(8)$ $U(x,\lambda)$ takes the form

$$U(x,t,\lambda) = \begin{pmatrix} \lambda \mathbb{1} & q(x,t) \\ p(x,t) & -\lambda\mathbb{1} \end{pmatrix}, \qquad q(x,t) = \begin{pmatrix} q_{14} & q_{13} & q_{12} & 0 \\ q_{24} & q_{23} & 0 & q_{12} \\ q_{34} & 0 & q_{23} & -q_{13} \\ 0 & q_{34} & -q_{24} & q_{14} \end{pmatrix},$$

$$p(x,t) = \begin{pmatrix} p_{14} & p_{24} & p_{34} & 0 \\ p_{13} & p_{23} & 0 & p_{34} \\ p_{12} & 0 & p_{23} & -p_{24} \\ 0 & p_{12} & -p_{13} & p_{14} \end{pmatrix}.$$

Remark 4. The automorphisms that satisfy $C^{-1}JC = J$ naturally preserve the eigensubspaces of ad_J; in other words their action on the root space maps the subsets of roots $\Delta_1^{\pm}$ onto themselves: $C\Delta_1^{\pm} = \Delta_1^{\pm}$.

We list here several inequivalent reductions of the Zakharov–Shabat system. In the first one we choose $C = C_0$ to be an element of the Cartan subgroup

$$C_0 = \exp\left(\pi \mathrm{i} \sum_{k=1}^{4} s_k h_k\right),$$

where s_k take the values 0 and 1. This condition means that $C_0^2 = \mathbb{1}$, so this will be a $\mathbb{Z}_2$-reduction, or involution. Then the first example of $\mathbb{Z}_2$-reduction is

$$C_0^{-1} Q^\dagger(x,t) C_0 = Q(x,t),$$

or in components

$$p_{ij} = \epsilon_{ij} q_{ij}^*, \qquad \epsilon_{ij} = \epsilon_i \epsilon_j, \qquad \epsilon_j = e^{\pi i s_j} = \pm 1.$$

Obviously ϵ_j takes values ± 1 depending on whether s_j equals 0 or 1.

The next examples of $\mathbb{Z}_2$-reduction correspond to several choices of C as elements of the Weyl group eventually combined with the Cartan subgroup element C_0

$$C_1 = S_{e_1-e_2} S_{e_3-e_4} C_0,$$

where $S_{e_i-e_j}$ is the Weyl reflection related to the root $e_i - e_j$. Again we have a $\mathbb{Z}_2$-reduction, or an involution

$$C_1^{-1} Q^\dagger(x,t) C_1 = Q(x,t).$$

Written in components it takes the form ($\epsilon_{12} = \epsilon_{34} = 1$)

$$\begin{aligned}
&p_{12} = -q_{12}^*, \qquad p_{24} = -\epsilon_{23} q_{13}^*, \qquad p_{23} = -\epsilon_{13} q_{14}^*,\\
&p_{14} = -\epsilon_{13} q_{23}^*, \qquad p_{13} = -\epsilon_{23} q_{24}^*, \qquad p_{34} = -q_{34}^*.
\end{aligned}$$

The corresponding Hamiltonian and symplectic form take the form

$$\begin{aligned}
8I_3 = {} & -\int_{-\infty}^{\infty} \Big(\partial_x q_{12}^* \partial_x q_{12} + \partial_x q_{34}^* \partial_x q_{34} + \epsilon_{13}(q_{23}^* \partial_x q_{14} + \partial_x q_{14}^* \partial_x q_{23})\\
& + \epsilon_{23}\big(\partial_x q_{24}^* \partial_x q_{13} + \partial_x q_{13}^* \partial_x q_{24})\Big) dx + \int_{-\infty}^{\infty} \Big(\epsilon_{12} q_{12}^* q_{12} + q_{34}^* q_{34}\\
& + \epsilon_{13}(q_{23}^* q_{14} + q_{24}^* q_{13}) + \epsilon_{23}(q_{14}^* q_{23} + q_{13}^* q_{24})\Big)^2 dx\\
& + \int_{-\infty}^{\infty} |q_{13} q_{24} + q_{12} q_{34} - q_{14} q_{23}|^2 \, dx,\\
\Omega^{(0)} = {} & \frac{1}{\mathrm{i}} \int_{-\infty}^{\infty} \Big(\delta q_{12}^* \wedge \delta q_{12} + \epsilon_{13}(q_{23}^* \wedge \delta q_{14} + \delta q_{14}^* \wedge \delta q_{23})\\
& + \epsilon_{23}(\delta q_{24}^* \wedge \delta q_{13} + \delta q_{13}^* \wedge \delta q_{24}) + \delta q_{34}^* \wedge \delta q_{34}\Big) dx.
\end{aligned}$$

Another inequivalent examples of $\mathbb{Z}_2$-reduction corresponds to

$$C_2 = S_{e_1-e_2} C_0.$$

The involution is $C_2^{-1}Q^\dagger(x,t)C_2 = Q(x,t)$, or in components it takes the form

$$\begin{aligned}
&p_{12} = -\epsilon_{12}q_{12}^*, \qquad && p_{24} = -\epsilon_{13}q_{14}^*, \qquad && p_{23} = -\epsilon_{14}q_{13}^*,\\
&p_{14} = -\epsilon_{23}q_{24}^*, && p_{13} = -\epsilon_{24}q_{23}^*, && p_{34} = -\epsilon_{34}q_{34}^*.
\end{aligned}$$

As a consequence we get

$$\begin{aligned}
8I_3 = &-\int_{-\infty}^{\infty}\Big(\epsilon_{12}\partial_x q_{12}^*\partial_x q_{12} + \epsilon_{34}\partial_x q_{34}^*\partial_x q_{34} + \epsilon_{14}\partial_x q_{13}^*\partial_x q_{23}\\
&+ \epsilon_{23}\partial_x q_{24}^*\partial_x q_{14} + \epsilon_{24}\partial_x q_{23}^*\partial_x q_{13} + \epsilon_{13}\partial_x q_{14}^*\partial_x q_{24}\Big)dx\\
&+\int_{-\infty}^{\infty}\Big(\epsilon_{12}|q_{12}|^2 + \epsilon_{34}|q_{34}|^2 + \epsilon_{23}q_{24}^*q_{14} + \epsilon_{24}q_{23}^*q_{13} + \epsilon_{14}q_{13}^*q_{23} + \epsilon_{13}q_{14}^*q_{24}\Big)^2 dx\\
&+\epsilon_{12}\epsilon_{34}\int_{-\infty}^{\infty}|q_{13}q_{24} + q_{12}q_{34} - q_{14}q_{23}|^2\,dx,\\
\Omega^{(0)} = &\frac{1}{\mathrm{i}}\int_{-\infty}^{\infty}\Big(\epsilon_{12}\delta q_{12}^*\wedge\delta q_{12} + \epsilon_{34}q_{34}^*\wedge\delta q_{34} + \epsilon_{23}\delta q_{24}^*\wedge\delta q_{14}\\
&+\epsilon_{14}\delta q_{13}^*\wedge\delta q_{23} + \epsilon_{24}\delta q_{23}^*\wedge\delta q_{13} + \epsilon_{13}\delta q_{14}^*\wedge\delta q_{24}\Big)dx.
\end{aligned}$$

Next we consider a $\mathbb{Z}_3$-reduction generated by $C_3 = S_{e_1-e_2}S_{e_2-e_3}$ which also maps J into J. It splits each of the sets $\Delta_1^\pm$ into two orbits which are

$$\begin{aligned}
&(O)_1^\pm = \{\pm(e_1+e_2)\ \pm(e_2+e_3)\ \pm(e_1+e_3)\},\\
&(O)_2^\pm = \{\pm(e_1+e_4)\ \pm(e_2+e_4)\ \pm(e_3+e_4)\}.
\end{aligned}$$

In order to be more efficient we make use of the following basis in $\mathfrak{g}^{(0)}$

$$\mathcal{E}_\alpha^{(k)} = \sum_{p=0}^{2}\omega^{-kp}C_3^{-k}E_\alpha C_3^k, \qquad \mathcal{F}_\alpha^{(k)} = \sum_{p=0}^{2}\omega^{-kp}C_3^{-k}F_\alpha C_3^k,$$

where $\omega = \exp(2\pi i/3)$ and α takes values e_1+e_2 and e_1+e_4. Obviously

$$C_3^{-1}\mathcal{E}_\alpha^{(k)}C_3 = \omega^k\mathcal{E}_\alpha^{(k)}, \qquad C_3^{-1}\mathcal{F}_\alpha^{(k)}C_3 = \omega^k\mathcal{F}_\alpha^{(k)}. \tag{5.2}$$

In addition, since $\omega^* = \omega^{-1}$ we get $(\mathcal{E}_\alpha^{(0)})^\dagger = \mathcal{F}_\alpha^{(0)}$ and $(\mathcal{E}_\alpha^{(k)})^\dagger = \mathcal{F}_\alpha^{(3-k)}$ for $k = 1,2$. Then we introduce the potential

$$Q(x,t) = \sum_{k=0}^{3}\sum_\alpha\left(q_\alpha^{(k)}(x,t)\mathcal{E}_\alpha^{(k)} + p_\alpha^{(k)}(x,t)\mathcal{F}_\alpha^{(k)}\right).$$

In view of equation (5.2) the reduction condition (4.17) leads to the following relations between the coefficients

$$\begin{aligned}
&p_{12}^{(0)} = (q_{12}^{(0)})^*, \qquad && p_{12}^{(k)} = \omega^k(q_{12}^{(3-k)})^*, \qquad && q_{12}^{(k)} = \omega^k(p_{12}^{(3-k)})^*,\\
&p_{14}^{(0)} = (q_{14}^{(0)})^*, && p_{14}^{(k)} = \omega^k(q_{14}^{(3-k)})^*, && q_{14}^{(k)} = \omega^k(p_{14}^{(3-k)})^*,
\end{aligned} \tag{5.3}$$

where $k = 1,2$. It is easy to check that from the conditions (5.3) there follows $p_{12}^{(k)} = q_{12}^{(k)} = p_{14}^{(k)} = q_{14}^{(k)} = 0$. So we are left with only one pair of independent functions $q_{12}^{(0)}$ and $q_{14}^{(0)}$ and their complex conjugate $p_{14}^{(0)}$, $q_{14}^{(0)}$.

Similarly the reduction (4.18) leads to

$$
\begin{aligned}
q_{12}^{(0)} &= -(q_{12}^{(0)})^*, \qquad q_{12}^{(k)} = -\omega^{3-k}(q_{12}^{(3-k)})^*, \qquad q_{14}^{(k)} = -\omega^{3-k}(q_{14}^{(3-k)})^*, \\
p_{12}^{(0)} &= -(p_{12}^{(0)})^*, \qquad p_{12}^{(k)} = -\omega^{3-k}(p_{12}^{(3-k)})^*, \qquad p_{14}^{(k)} = -\omega^{3-k}(p_{14}^{(3-k)})^*,
\end{aligned} \tag{5.4}
$$

where $k = 1, 2$. Again from the conditions (5.4) there follows $p_{12}^{(k)} = q_{12}^{(k)} = p_{14}^{(k)} = q_{14}^{(k)} = 0$. So we are left with two pairs of purely imaginary independent functions: $q_{12}^{(0)}$, $q_{14}^{(0)}$ and $p_{12}^{(0)}$, $q_{14}^{(0)}$.

The corresponding Hamiltonian and symplectic form are obtained from the slightly more general formulae below by imposing the constraints (5.3) and (5.4). Here for simplicity we skip the upper zeroes in q_{ij} and p_{ij}

$$
\begin{aligned}
H_{\mathrm{MMKdV}} &= \frac{1}{6}\int_{-\infty}^{\infty} dx\, \left(\partial_x^2 q_{12}\partial_x p_{12} - \partial_x q_{12}\partial_x^2 p_{12} + \partial_x^2 q_{14}\partial_x p_{14} - \partial_x q_{14}\partial_x^2 p_{14}\right) \\
&\quad - \frac{1}{12}\int_{-\infty}^{\infty} \left(p_{12}^2 q_{12}^2 \partial_x - p_{12}^2\partial_x q_{12}^2 + q_{14}^2\partial_x p_{14}^2 - p_{12}^2\partial_x q_{12}^2\right) dx, \\
8I_3 &= \frac{4}{3}\int_{-\infty}^{\infty} dx\, \left(\partial_x q_{12}\partial_x p_{12} + \partial_x q_{14}\partial_x p_{14}\right) - \frac{8}{9}\int_{-\infty}^{\infty} \left(q_{14}^2 p_{14}^2 + q_{12}^2 p_{12}^2\right) dx, \\
\Omega^{(0)} &= \frac{4}{3}\int_{-\infty}^{\infty} dx\, \left(\delta q_{14}\wedge\delta p_{14} + \delta q_{12}\wedge\delta p_{12}\right),
\end{aligned}
$$

i.e. in this case we get two decoupled mKdV equations.

The $\mathbb{Z}_4$-reduction generated by $C_4 = S_{e_1-e_2}S_{e_2-e_3}S_{e_3-e_4}$ also maps J into J. It splits each of the sets $\Delta_1^\pm$ into two orbits which are

$$
\begin{aligned}
(O)_1^\pm &= \{\pm(e_1+e_2) \pm (e_2+e_3) \pm (e_3+e_4) \pm (e_1+e_4)\}, \\
(O)_2^\pm &= \{\pm(e_1+e_3) \pm (e_2+e_4)\}.
\end{aligned}
$$

Again we make use of a convenient basis in $\mathfrak{g}^{(0)}$

$$
\mathcal{E}_\alpha^{(k)} = \sum_{p=0}^{3} i^{-kp} C_4^{-k} E_\alpha C_4^k, \qquad \mathcal{F}_\alpha^{(k)} = \sum_{p=0}^{3} i^{-kp} C_4^{-k} F_\alpha C_4^k,
$$

where α takes values $e_1 + e_2$ and $e_1 + e_3$. Obviously

$$
C_4^{-1}\mathcal{E}_\alpha^{(k)} C_4 = i^k \mathcal{E}_\alpha^{(k)}, \qquad C_4^{-1}\mathcal{F}_\alpha^{(k)} C_4 = i^k \mathcal{F}_\alpha^{(k)} \tag{5.5}
$$

and in addition, $(\mathcal{E}_\alpha^{(0)})^\dagger = \mathcal{F}_\alpha^{(0)}$, $(\mathcal{E}_\alpha^{(k)})^\dagger = \mathcal{F}_\alpha^{(4-k)}$ and $(\mathcal{E}_\alpha^{(k)})^* = \mathcal{E}_\alpha^{(4-k)}$ for $k = 1, 2, 3$. Then we introduce the potential

$$
Q(x,t) = \sum_{k=0}^{3}\sum_\alpha \left(q_\alpha^{(k)}(x,t)\mathcal{E}_\alpha^{(k)} + p_\alpha^{(k)}(x,t)\mathcal{F}_\alpha^{(k)}\right).
$$

In view of equation (5.5) the reduction condition (4.17) leads to the following relations between the coefficients

$$
p_\alpha^{(0)} = (q_\alpha^{(0)})^*, \qquad p_\alpha^{(k)} = i^k (q_\alpha^{(4-k)})^*, \qquad q_\alpha^{(k)} = i^k (p_\alpha^{(4-k)})^* \tag{5.6}
$$

for $k = 1, 2, 3$. Here p_α, q_α coincide with p_{12}, q_{12} (resp. p_{13}, q_{13}) for $\alpha = e_1+e_2$ (resp. $\alpha = e_1+e_3$). Analogously the reduction (4.18) gives

$$
q_\alpha^{(0)} = -(q_\alpha^{(0)})^*, \qquad p_\alpha^{(0)} = -(p_\alpha^{(0)})^*, \qquad q_\alpha^{(k)} = -i^k (q_\alpha^{(4-k)})^*, \qquad p_\alpha^{(k)} = -i^k (p_\alpha^{(4-k)})^* \tag{5.7}
$$

for $k = 1,2,3$. Both conditions (5.6), (5.7) lead to $p_{12}^{(k)} = q_{12}^{(k)} = p_{14}^{(k)} = q_{14}^{(k)} = 0$ for $k = 1,3$. In addition it comes up that $\mathcal{E}_{13}^{(0)} = \mathcal{E}_{13}^{(2)} = \mathcal{F}_{13}^{(0)} = \mathcal{F}_{13}^{(2)}$. So we are left with only two pairs of independent functions $p_{12}^{(0)}$, $q_{12}^{(0)}$ and $p_{12}^{(2)}$, $q_{12}^{(2)}$. We provide below slightly more general formulae for the corresponding Hamiltonian and symplectic form which are obtained by imposing the constraints (5.6) or (5.7); again for simplicity of notations we skip the upper zeroes in $q_{ij}^{(0)}$ and $p_{ij}^{(0)}$ and replace $q_{ij}^{(2)}$ and $p_{ij}^{(2)}$ by $\tilde{q}_{ij}$ and $\tilde{p}_{ij}$

$$\begin{aligned} H_{\mathrm{MMKdV}} &= \frac{1}{4}\int_{-\infty}^{\infty} dx\, \left(\partial_x^2 q_{12}\partial_x p_{12} - \partial_x q_{12}\partial_x^2 p_{12} + \partial_x^2\tilde{q}_{12}\partial_x\tilde{p}_{12} - \partial_x\tilde{q}_{12}\partial_x^2\tilde{p}_{12}\right) \\ &\quad - \frac{3}{32}\int_{-\infty}^{\infty} \Big(\left(\partial_x(p_{12}^2) + \partial_x(\tilde{p}_{12}^2)\right)\left(q_{12}^2 + \tilde{q}_{12}^2\right) + p_{12}\tilde{p}_{12}\partial_x\left(q_{12}\tilde{q}_{12}\right) \\ &\quad + \left(\partial_x(q_{12}^2) + \partial_x(\tilde{q}_{12}^2)\right)\left(p_{12}^2 + \tilde{p}_{12}^2\right) + q_{12}\tilde{q}_{12}\partial_x\left(p_{12}\tilde{p}_{12}\right)\Big) dx, \\ 8I_3 &= 2\int_{-\infty}^{\infty} \left(\partial_x q_{12}\partial_x p_{12} + \partial_x\tilde{q}_{12}\partial_x\tilde{p}_{12}\right) dx \\ &\quad - \int_{-\infty}^{\infty} \left((q_{12}p_{12} + \tilde{q}_{12}\tilde{p}_{12})^2 + (q_{12}\tilde{p}_{12} + \tilde{q}_{12}p_{12})^2 \right) dx, \\ \Omega^{(0)} &= 2\int_{-\infty}^{\infty} dx\, \left(\delta q_{12}\wedge\delta p_{12} + \delta\tilde{q}_{12}\wedge\delta\tilde{p}_{12}\right). \end{aligned}$$

Now we get two specially coupled mKdV-type equations. Now we get four specially coupled mKdV-type equations given by

$$\begin{aligned} &\partial_t q_0 + \partial_x^3 q_0 + \frac{3}{2}(q_2p_2 + q_0p_0)\partial_x q_0 + \frac{3}{2}(q_2p_0 + q_0p_2)\partial_x q_2 = 0, \\ &\partial_t q_2 + \partial_x^3 q_2 + \frac{3}{2}(q_2p_0 + q_0p_2)\partial_x q_0 + \frac{3}{2}(q_2p_2 + q_0p_0)\partial_x q_2 = 0, \\ &\partial_t p_0 + \partial_x^3 p_0 + \frac{3}{2}(q_2p_2 + q_0p_0)\partial_x p_0 + \frac{3}{2}(q_0q_2 + q_2p_0)\partial_x p_2 = 0, \\ &\partial_t p_2 + \partial_x^3 p_2 + \frac{3}{2}(q_2p_0 + q_0p_2)\partial_x p_0 + \frac{3}{2}(q_0p_0 + q_2p_2)\partial_x p_2 = 0, \end{aligned}$$

where we use for simplicity

$$q_{12} = q_0, \qquad \tilde{q}_{12} = q_2, \qquad p_{12} = p_0, \qquad \tilde{p}_{12} = p_2$$

and with second reduction (5.7) $p_0 = q_0^*$ and $p_2 = -q_2^*$ we have

$$\begin{aligned} &\partial_t q_0 + \partial_x^3 q_0 + \frac{3}{2}(-q_2q_2^* + q_0q_0^*)\partial_x q_0 + \frac{3}{2}(q_2q_0^* - q_0q_2^*)\partial_x q_2 = 0, \\ &\partial_t q_2 + \partial_x^3 q_2 + \frac{3}{2}(q_2q_0^* - q_0q_2^*)\partial_x q_0 + \frac{3}{2}(q_2q_2^* - q_0q_2^*)\partial_x q_2 = 0. \end{aligned} \tag{5.8}$$

Obviously in the system (5.8) we can put both q_0, q_2 real with the result

$$\partial_t q_0 + \partial_x^3 q_0 + \frac{3}{2}(q_0^2 - q_2^2)\partial_x q_0 = 0, \qquad \partial_t q_2 + \partial_x^3 q_2 + \frac{3}{2}(q_0^2 - q_2^2)\partial_x q_2 = 0.$$

The reduction (4.18) means that $q_\alpha^{(0)} = iq_0^\vee$, $p_\alpha^{(0)} = ip_0^\vee$, $q_\alpha^{(2)} = q_2^\vee$, $p_\alpha^{(2)} = p_2^\vee$ with real valued $p_i^\vee$, $q_i^\vee$, $i = 0,2$. Thus we get

$$\partial_t q_0^\vee + \partial_x^3 q_0^\vee + \frac{3}{2}(q_2^\vee p_0^\vee + q_0^\vee p_2^\vee)\partial_x q_2^\vee + \frac{3}{2}(p_2^\vee q_2^\vee - q_0^\vee p_0^\vee)\partial_x q_0^\vee = 0,$$

$$\partial_t q_2^\vee + \partial_x^3 q_2^\vee + \frac{3}{2}(q_2^\vee p_2^\vee - q_0^\vee p_0^\vee)\partial_x q_2^\vee + \frac{3}{2}(q_2^\vee p_0^\vee + q_0^\vee p_2^\vee)\partial_x q_0^\vee = 0,$$
$$\partial_t p_0^\vee + \partial_x^3 p_0^\vee + \frac{3}{2}(q_2^\vee p_0^\vee - q_0^\vee p_2^\vee)\partial_x p_0^\vee + \frac{3}{2}(-q_0^\vee p_0^\vee + q_2^\vee p_2^\vee)\partial_x p_2^\vee = 0,$$
$$\partial_t p_2^\vee + \partial_x^3 p_2^\vee + \frac{3}{2}(q_2^\vee p_2^\vee - q_0^\vee p_0^\vee)\partial_x p_0^\vee + \frac{3}{2}(q_0^\vee p_2^\vee - q_2^\vee p_0^\vee)\partial_x p_2^\vee = 0.$$

5.2 Class B Reductions mapping J into $-J$

The class B reductions of the Zakharov–Shabat system change the sign of J, i.e. $C^{-1}JC = -J$; therefore we must have also $\lambda \to -\lambda$.

Remark 5. Note that ad_J has three eigensubspaces $\mathcal{W}_a$, $a = 0, \pm 1$ corresponding to the eigenvalues 0 and ± 2. The automorphisms that satisfy $C^{-1}JC = -J$ naturally preserve the eigensubspace $\mathcal{W}_0$ but map $\mathcal{W}_{-1}$ onto $\mathcal{W}_1$ and vice versa. In other words their action on the root space maps the subset of roots Δ_1^+ onto Δ_1^- and vice versa: $C\Delta_1^\pm = \Delta_1^\mp$.

Here we first consider

$$C_5 = S_{e_1-e_2} S_{e_1+e_2} S_{e_3-e_4} S_{e_3+e_4} C_0.$$

Obviously the product of the the above 4 Weyl reflections will change the sign of J. Its effect on Q in components reads

$$q_{ij} = -\epsilon_{ij} q_{ij}^*, \qquad p_{ij} = -\epsilon_{ij} p_{ij}^*, \qquad \epsilon_{ij} = \epsilon_i \epsilon_j,$$

i.e. some of the components of Q become purely imaginary, others may become real depending on the choice of the signs ϵ_j.

There is no $\mathbb{Z}_3$ reduction for the $so(8)$ MMKdV that maps J into $-J$. So we go directly to the $\mathbb{Z}_4$-reduction generated by $C_6 = S_{e_1+e_2} S_{e_2+e_3} S_{e_3+e_4}$ which maps J into $-J$. The orbits of C_3 are

$$(O)_1^\pm = \{\pm(e_1+e_2) \mp (e_2+e_3) \pm (e_3+e_4) \mp (e_1+e_4)\},$$
$$(O)_2^\pm = \{\pm(e_1+e_3) \mp (e_2+e_4)\}.$$

Again we make use of a convenient basis in $\mathfrak{g}^{(0)}$

$$\mathcal{E}_\alpha^{(k)} = \sum_{p=0}^{3} \mathrm{i}^{-kp} C_6^{-k} E_\alpha C_6^k, \qquad \mathcal{F}_\alpha^{(k)} = \sum_{p=0}^{3} \mathrm{i}^{-kp} C_6^{-k} F_\alpha C_6^k,$$

where α takes values e_1+e_2 and e_1+e_3. Obviously

$$C_6^{-1} \mathcal{E}_\alpha^{(k)} C_6 = \mathrm{i}^k \mathcal{E}_\alpha^{(k)}, \qquad C_6^{-1} \mathcal{F}_\alpha^{(k)} C_6 = \mathrm{i}^k \mathcal{F}_\alpha^{(k)}, \tag{5.9}$$

where again $(\mathcal{E}_\alpha^{(0)})^\dagger = \mathcal{F}_\alpha^{(0)}$, $(\mathcal{E}_\alpha^{(k)})^\dagger = \mathcal{F}_\alpha^{(4-k)}$ and $(\mathcal{E}_\alpha^{(k)})^* = \mathcal{E}_\alpha^{(4-k)}$ for $k = 1, 2, 3$. Then we introduce the potential

$$Q(x,t) = \sum_{k=0}^{3} \sum_\alpha \left(q_\alpha^{(k)}(x,t) \mathcal{E}_\alpha^{(k)} + p_\alpha^{(k)}(x,t) \mathcal{F}_\alpha^{(k)} \right).$$

In view of equation (5.9) the reduction condition (4.17) leads to the following relations between the coefficients

$$p_\alpha^{(0)} = (q_\alpha^{(0)})^*, \qquad p_\alpha^{(k)} = i^k (q_\alpha^{(4-k)})^*, \qquad q_\alpha^{(k)} = i^k (p_\alpha^{(4-k)})^*, \qquad k = 1, 2, 3, \tag{5.10}$$

while the reduction (4.18) gives

$$
\begin{aligned}
&q_\alpha^{(0)} = -(q_\alpha^{(0)})^*, \qquad q_\alpha^{(k)} = -i^k (q_\alpha^{(4-k)})^*, \qquad k=1,2,3,\\
&p_\alpha^{(0)} = -(p_\alpha^{(0)})^*, \qquad p_\alpha^{(k)} = -i^k (p_\alpha^{(4-k)})^*, \qquad k=1,2,3,
\end{aligned}
\tag{5.11}
$$

where α takes values e_1+e_2 and e_1+e_3. From the conditions (5.10) there follows $p_{12}^{(k)} = q_{12}^{(k)} = p_{13}^{(k)} = q_{13}^{(k)} = 0$ for $k=1,3$. In addition however, it comes up that $\mathcal{E}_{13}^{(0)} = \mathcal{E}_{13}^{(2)} = \mathcal{F}_{13}^{(0)} = \mathcal{F}_{13}^{(2)}$. So we are left with only two pairs of independent functions $p_{12}^{(0)}$, $q_{12}^{(0)}$ and $p_{12}^{(2)}$, $q_{12}^{(2)}$. We provide below slightly more general formulae for the corresponding Hamiltonian and symplectic form which are obtained by imposing the constraints (5.10) or (5.11) and again for simplicity we skip the upper zeroes in q_{ij} and p_{ij} and replace $q_{12}^{(2)}$ and $p_{12}^{(2)}$ by $\tilde{q}_{12}$ and $\tilde{p}_{12}$

$$
\begin{aligned}
H_{\mathrm{MMKdV}} = {}& \frac{1}{4}\int_{-\infty}^{\infty} dx\, \left(\partial_x^2 q_{12}\partial_x p_{12} - \partial_x q_{12}\partial_x^2 p_{12} + \partial_x^2 \tilde{q}_{12}\partial_x \tilde{p}_{12} - \partial_x \tilde{q}_{12}\partial_x^2 \tilde{p}_{12}\right)\\
&- \frac{3}{32}\int_{-\infty}^{\infty} \Big(\left(\partial_x(p_{12}^2) + \partial_x(\tilde{q}_{12}^2)\right)\left(q_{12}^2 + \tilde{p}_{12}^2\right) + \partial_x\left(q_{12}\tilde{p}_{12}\right) p_{12}\tilde{q}_{12}\\
&- \left(\partial_x(q_{12}^2) + \partial_x(\tilde{p}_{12}^2)\right)\left(p_{12}^2 + \tilde{q}_{12}^2\right) - \partial_x\left(p_{12}\tilde{q}_{12}\right) q_{12}\tilde{p}_{12}\Big) dx,\\
8I_3 = {}& 2\int_{-\infty}^{\infty} \left(\partial_x q_{12}\partial_x p_{12} + \partial_x \tilde{q}_{12}\partial_x \tilde{p}_{12}\right) dx\\
&- \int_{-\infty}^{\infty} \left(\left(q_{12}p_{12} + \tilde{q}_{12}\tilde{p}_{12}\right)^2 + \left(q_{12}\tilde{p}_{12} + \tilde{q}_{12}p_{12}\right)^2 \right) dx,\\
\Omega^{(0)} = {}& 2\int_{-\infty}^{\infty} dx\, \left(\delta q_{12}\wedge\delta p_{12} + \delta\tilde{q}_{12}\wedge\delta\tilde{p}_{12}\right).
\end{aligned}
$$

Now we get two specially coupled mKdV-type equations.

The class B reductions also render all the symplectic forms and Hamiltonians in the hierarchy real-valued. They allow to render the corresponding systems of MMKdV equations into ones involving only real-valued fields. 'Half' of the Hamiltonian structures do not survive these reductions and become degenerate. This holds true for all symplectic forms $\Omega^{(2m)}$ and integrals of motion I_{2m} with even indices. However the other 'half' of the hierarchy with $\Omega^{(2m+1)}$ and integrals of motion I_{2m+1} remains and provides Hamiltonian properties of the MMKdV.

Now we get four specially coupled mKdV-type equations given by

$$
\begin{aligned}
&\partial_t q_0 + \partial_x^3 q_0 + \frac{3}{2}(q_0q_2 + p_0p_2)\partial_x p_2 + \frac{3}{2}(p_2q_2 + q_0p_0)\partial_x q_0 = 0,\\
&\partial_t q_2 + \partial_x^3 q_2 + \frac{3}{2}(q_2p_2 + q_0p_0)\partial_x q_2 + \frac{3}{2}(q_0q_2 + p_0p_2)\partial_x p_0 = 0,\\
&\partial_t p_0 + \partial_x^3 p_0 + \frac{3}{2}(q_2p_2 + q_0p_0)\partial_x p_0 + \frac{3}{2}(q_0q_2 + p_0p_2)\partial_x q_2 = 0,\\
&\partial_t p_2 + \partial_x^3 p_2 + \frac{3}{2}(q_2q_0 + p_0p_2)\partial_x q_0 + \frac{3}{2}(q_0p_0 + q_2p_2)\partial_x p_2 = 0,
\end{aligned}
$$

where we use for simplicity

$$
q_{12} = q_0, \qquad \tilde{q}_{12} = q_2, \qquad p_{12} = p_0, \qquad \tilde{p}_{12} = p_2
$$

and with second reduction (5.10) $p_0 = q_0^*$ and $p_2 = -q_2^*$ we have

$$
\begin{aligned}
&\partial_t q_0 + \partial_x^3 q_0 - \frac{3}{2}(q_0q_2 - q_0^*q_2^*)\partial_x q_2^* - \frac{3}{2}(q_2^*q_2 - q_0q_0^*)\partial_x q_0 = 0,\\
&\partial_t q_2 + \partial_x^3 q_2 - \frac{3}{2}(q_2q_2^* - q_0q_0^*)\partial_x q_2 + \frac{3}{2}(q_0q_2 - q_0^*q_2^*)\partial_x q_0^* = 0.
\end{aligned}
\tag{5.12}
$$

Obviously in the system (5.12) we can put both q_0, q_2 real with the result

$$\partial_t q_0 + \partial_x^3 q_0 + \frac{3}{2}(q_0^2 - q_2^2)\partial_x q_0 = 0, \qquad \partial_t q_2 + \partial_x^3 q_2 + \frac{3}{2}(q_0^2 - q_2^2)\partial_x q_2 = 0.$$

The reduction (5.11) means that $q_\alpha^{(0)} = iq_0^\vee$, $p_\alpha^{(0)} = ip_0^\vee$, $q_\alpha^{(2)} = q_2^\vee$, $p_\alpha^{(2)} = p_2^\vee$ with real valued $p_i^\vee$, $q_i^\vee$, $i = 0, 2$. Thus we get

$$\begin{aligned}
&\partial_t q_0^\vee + \partial_x^3 q_0^\vee + \frac{3}{2}(q_0^\vee q_2^\vee + p_0^\vee p_2^\vee)\partial_x p_2^\vee + \frac{3}{2}(p_2^\vee q_2^\vee - q_0^\vee p_0^\vee)\partial_x q_0^\vee = 0,\\
&\partial_t q_2^\vee + \partial_x^3 q_2^\vee + \frac{3}{2}(q_2^\vee p_2^\vee - q_0^\vee p_0^\vee)\partial_x q_2^\vee - \frac{3}{2}(q_0^\vee q_2^\vee + p_0^\vee p_2^\vee)\partial_x p_0^\vee = 0,\\
&\partial_t p_0^\vee + \partial_x^3 p_0^\vee + \frac{3}{2}(q_2^\vee p_2^\vee - q_0^\vee p_0^\vee)\partial_x p_0^\vee + \frac{3}{2}(q_0^\vee q_2^\vee + p_0^\vee p_2^\vee)\partial_x q_2^\vee = 0,\\
&\partial_t p_2^\vee + \partial_x^3 p_2^\vee - \frac{3}{2}(q_2^\vee q_0^\vee + p_0^\vee p_2^\vee)\partial_x q_0^\vee - \frac{3}{2}(q_0^\vee p_0^\vee - q_2^\vee p_2^\vee)\partial_x p_2^\vee = 0.
\end{aligned}$$

The class B reductions also render all the symplectic forms and Hamiltonians in the hierarchy real-valued. They allow to render the corresponding systems of MMKdV equations into ones involving only real-valued fields. 'Half' of the Hamiltonian structures do not survive these reductions and become degenerate. This holds true for all symplectic forms $\Omega^{(2m)}$ and integrals of motion I_{2m} with even indices. However the other 'half' of the hierarchy with $\Omega^{(2m+1)}$ and integrals of motion I_{2m+1} remains and provides Hamiltonian properties of the MMKdV.

5.3 Effects of class A reductions on the scattering data

The reflection coefficients $\rho^\pm(\lambda)$ and $\tau^\pm(\lambda)$ are defined only on the real λ-axis, while the diagonal blocks $\boldsymbol{a}^\pm(\lambda)$ and $\boldsymbol{c}^\pm(\lambda)$ (or, equivalently, $D^\pm(\lambda)$) allow analytic extensions for $\lambda \in \mathbb{C}_\pm$. From the equations (2.9) there follows that

$$\boldsymbol{a}^+(\lambda)\boldsymbol{c}^-(\lambda) = (\mathbb{1} + \rho^-\rho^+(\lambda))^{-1}, \qquad \boldsymbol{a}^-(\lambda)\boldsymbol{c}^+(\lambda) = (\mathbb{1} + \rho^+\rho^-(\lambda))^{-1}, \tag{5.13}$$

$$\boldsymbol{c}^-(\lambda)\boldsymbol{a}^+(\lambda) = (\mathbb{1} + \tau^+\tau^-(\lambda))^{-1}, \qquad \boldsymbol{c}^+(\lambda)\boldsymbol{a}^-(\lambda) = (\mathbb{1} + \tau^-\tau^+(\lambda))^{-1}. \tag{5.14}$$

Given $\mathcal{T}_1$ (resp. $\mathcal{T}_2$) we determine the right hand sides of (5.13) (resp. (5.14)) for $\lambda \in \mathbb{R}$. Combined with the facts about the limits

$$\lim_{\lambda\to\infty} \boldsymbol{a}^+(\lambda) = \mathbb{1}, \qquad \lim_{\lambda\to\infty} \boldsymbol{c}^-(\lambda) = \mathbb{1}, \qquad \lim_{\lambda\to\infty} \boldsymbol{a}^-(\lambda) = \mathbb{1}, \qquad \lim_{\lambda\to\infty} \boldsymbol{c}^+(\lambda) = \mathbb{1}, \tag{5.15}$$

each of the relations (5.13), (5.14) can be viewed as a RHP with canonical normalization. Such RHP can be solved explicitly in the one-component case (provided we know the locations of their zeroes) by using the Plemelj–Sokhotsky formulae [24]. These zeroes are in fact the discrete eigenvalues of L. One possibility to make use of these facts is to take log of the determinants of both sides of (5.13) getting

$$A^+(\lambda) + C^-(\lambda) = -\ln\det(\mathbb{1} + \rho^-\rho^+(\lambda), \qquad \lambda \in \mathbb{R},$$

where

$$A^\pm(\lambda) = \ln\det \boldsymbol{a}^\pm(\lambda), \qquad C^\pm(\lambda) = \ln\det \boldsymbol{c}^\pm(\lambda).$$

Then Plemelj–Sokhotsky formulae allows us to recover $A^\pm(\lambda)$ and $C^\pm(\lambda)$

$$\mathcal{A}(\lambda) = \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty} \frac{d\mu}{\mu - \lambda} \ln\det(\mathbb{1} + \rho^-\rho^+(\mu)) + \sum_{j=1}^{N} \ln\frac{\lambda - \lambda_j^+}{\lambda - \lambda_j^-}, \tag{5.16}$$

where $\mathcal{A}(\lambda) = A^+(\lambda)$ for $\lambda \in \mathbb{C}_+$ and $\mathcal{A}(\lambda) = -C^-(\lambda)$ for $\lambda \in \mathbb{C}_-$. In deriving (5.16) we have also assumed that $\lambda_j^\pm$ are simple zeroes of $A^\pm(\lambda)$ and $C^\pm(\lambda)$.

Let us consider a reduction condition (4.17) with C_1 from the Cartan subgroup: $C_1 = \operatorname{diag}(B_+, B_-)$ where the diagonal matrices $B_\pm$ are such that $B_\pm^2 = \mathbb{1}$. Then we get the following constraints on the sets $\mathcal{T}_{1,2}$

$$\begin{aligned} &\rho^-(\lambda) = (B_-\rho^+(\lambda)B_+)^\dagger, &&\rho_j^- = (B_-\rho_j^+B_+)^\dagger, &&\lambda_j^- = (\lambda_j^+)^*,\\ &\tau^-(\lambda) = (B_+\tau^+(\lambda)B_-)^\dagger, &&\tau_j^- = (B_+\tau_j^+B_-)^\dagger, &&\lambda_j^- = (\lambda_j^+)^*, \end{aligned}$$

where $j = 1, \ldots, N$. For more details see Subsections 4.3 and 4.4.

Remark 6. For certain reductions such as, e.g. $Q = -Q^\dagger$ the generalized Zakharov–Shabat system $L(\lambda)\psi = 0$ can be written down as an eigenvalue problem $\mathcal{L}\psi = \lambda\psi(x,\lambda)$ where $\mathcal{L}$ is a self-adjoint operator. The continuous spectrum of $\mathcal{L}$ fills up the whole real λ-axis thus 'leaving no space' for discrete eigenvalues. Such Lax operators have no discrete spectrum and the corresponding MNLS or MMKdV equations do not have soliton solutions.

From the general theory of RHP [24] one may conclude that (5.13), (5.14) allow unique solutions provided the number and types of the zeroes $\lambda_j^\pm$ are properly chosen. Thus we can outline a procedure which allows one to reconstruct not only $T(\lambda)$ and $\hat{T}(\lambda)$ and the corresponding potential $Q(x)$ from each of the sets $\mathcal{T}_i$, $i = 1, 2$:

i) Given $\mathcal{T}_2$ (resp. $\mathcal{T}_1$) solve the RHP (5.13) (resp. (5.14)) and construct $\boldsymbol{a}^\pm(\lambda)$ and $\boldsymbol{c}^\pm(\lambda)$ for $\lambda \in \mathbb{C}_\pm$.

ii) Given $\mathcal{T}_1$ we determine $\boldsymbol{b}^\pm(\lambda)$ and $\boldsymbol{d}^\pm(\lambda)$ as

$$\boldsymbol{b}^\pm(\lambda) = \rho^\pm(\lambda)\boldsymbol{a}^\pm(\lambda), \qquad \boldsymbol{d}^\pm(\lambda) = \boldsymbol{c}^\pm(\lambda)\rho^\pm(\lambda),$$

or if $\mathcal{T}_2$ is known then

$$\boldsymbol{b}^\pm(\lambda) = \boldsymbol{a}^\pm(\lambda)\tau^\pm(\lambda), \qquad \boldsymbol{d}^\pm(\lambda) = \tau^\pm(\lambda)\boldsymbol{c}^\pm(\lambda).$$

iii) The potential $Q(x)$ can be recovered from $\mathcal{T}_1$ by solving the RHP (3.7) and using equation (3.16).

Another method for reconstructing $Q(x)$ from $\mathcal{T}_j$ uses the interpretation of the ISM as generalized Fourier transform, see [27, 28, 29].

Let in this subsection all automorphisms C_i are of class A. Therefore acting on the root space they preserve the vector $\sum\limits_{k=1}^{r} e_k$ which is dual to J, and as a consequence, the corresponding Weyl group elements map the subset of roots Δ_1^+ onto itself.

Remark 7. An important consequence of this is that C_i will map block-upper-triangular (resp. block-lower-triangular) matrices like in equation (3.2) into matrices with the same block structure. The block-diagonal matrices will be mapped again into block-diagonal ones.

From the reduction conditions (4.17)–(4.20) one gets, in the limit $x \to \infty$ that

$$\begin{aligned} &\text{a)}\quad C_1((\kappa_1(\lambda)J)^\dagger) = \lambda J, &&\text{b)}\quad C_2((\kappa_2(\lambda)J)^T) = -\lambda J,\\ &\text{c)}\quad C_3((\kappa_3(\lambda)J)^*) = \lambda J, &&\text{d)}\quad C_4((\kappa_4(\lambda)J)) = \lambda J. \end{aligned} \tag{5.17}$$

Using equation (5.17) and $C_i(J) = J$ one finds that:

$$\text{a)}\quad \kappa_1(\lambda) = \lambda^*, \qquad \text{b)}\quad \kappa_2(\lambda) = -\lambda, \qquad \text{c)}\quad \kappa_3(\lambda) = -\lambda^*, \qquad \text{d)}\quad \kappa_4(\lambda) = \lambda.$$

It remains to take into account that the reductions (4.17)–(4.20) for the potentials of L lead to the following constraints on the scattering matrix $T(\lambda)$

$$\begin{array}{llll} \text{a)} & C_1(T^\dagger(\lambda^*)) = \hat{T}(\lambda), & \text{b)} & C_2(T^T(-\lambda)) = \hat{T}(\lambda), \\ \text{c)} & C_3((T^*(-\lambda^*)) = T(\lambda), & \text{d)} & C_4(T(\lambda)) = T(\lambda). \end{array}$$

These results along with Remark 7 lead to the following results for the generalized Gauss factors of $T(\lambda)$

$$\begin{array}{lll} \text{a)} & C_1(\boldsymbol{S}^{+,\dagger}(\lambda^*)) = \hat{\boldsymbol{S}}^-(\lambda), & C_1(\boldsymbol{T}^{-,\dagger}(\lambda^*)) = \hat{\boldsymbol{T}}^+(\lambda), \\ \text{b)} & C_2(\boldsymbol{S}^{+,T}(-\lambda)) = \hat{\boldsymbol{S}}^-(\lambda), & C_2(\boldsymbol{T}^{-,T}(-\lambda)) = \hat{\boldsymbol{T}}^+(\lambda), \\ \text{c)} & C_3(\boldsymbol{S}^{\pm,*}(-\lambda^*)) = \boldsymbol{S}^\pm(\lambda), & C_3(\boldsymbol{T}^{\pm,*}(-\lambda^*)) = \boldsymbol{T}^\pm(\lambda), \\ \text{d)} & C_4(\boldsymbol{S}^\pm(\lambda)) = \boldsymbol{S}^\pm(\lambda), & C_4(\boldsymbol{T}^\pm(\lambda)) = \boldsymbol{T}^\pm(\lambda) \end{array}$$

and

$$\begin{array}{llll} \text{a)} & C_1(D^{+,\dagger}(\lambda^*)) = \hat{D}^-(\lambda), & \text{b)} & C_2(D^{+,T}(-\lambda)) = \hat{D}^-(\lambda), \\ \text{c)} & C_3(D^{\pm,*}(-\lambda)) = D^\pm(\lambda), & \text{d)} & C_4(D^\pm(\lambda)) = D^\pm(\lambda). \end{array}$$

5.4 Effects of class B reductions on the scattering data

In this subsection all automorphisms C_i are of class B. Therefore acting on the root space they map the vector $\sum\limits_{k=1}^{r} e_k$ dual to J into $-\sum\limits_{k=1}^{r} e_k$. As a consequence, the corresponding Weyl group elements map the subset of roots Δ_1^+ onto $\Delta_1^- \equiv -\Delta_1^+$.

Remark 8. An important consequence of this is that C_i will map block-upper-triangular into block-lower-triangular matrices like in equation (3.2) and vice versa. The block-diagonal matrices will be mapped again into block-diagonal ones.

Now equation (5.17) with $C_i(J) = -J$ leads to

$$\text{a)}\quad \kappa_1(\lambda) = -\lambda^*, \qquad \text{b)}\quad \kappa_2(\lambda) = \lambda, \qquad \text{c)}\quad \kappa_3(\lambda) = \lambda^*, \qquad \text{d)}\quad \kappa_4(\lambda) = -\lambda.$$

The reductions (4.17)–(4.20) for the potentials of L lead to the following constraints on the scattering matrix $T(\lambda)$:

$$\begin{array}{llll} \text{a)} & C_1(T^\dagger(-\lambda^*)) = \hat{T}(\lambda), & \text{b)} & C_2(T^T(\lambda)) = \hat{T}(\lambda), \\ \text{c)} & C_3((T^*(\lambda^*)) = T(\lambda), & \text{d)} & C_4(T(-\lambda)) = T(\lambda). \end{array}$$

Then along with Remark 8 we find the following results for the generalized Gauss factors of $T(\lambda)$

$$\begin{array}{lll} \text{a)} & C_1(\boldsymbol{S}^{\pm,\dagger}(-\lambda^*)) = \hat{\boldsymbol{S}}^\pm(\lambda), & C_1(\boldsymbol{T}^{\pm,\dagger}(-\lambda^*)) = \hat{\boldsymbol{T}}^\pm(\lambda), \\ \text{b)} & C_2(\boldsymbol{S}^{\pm,T}(\lambda)) = \hat{\boldsymbol{S}}^\pm(\lambda), & C_2(\boldsymbol{T}^{\pm,T}(\lambda)) = \hat{\boldsymbol{T}}^\pm(\lambda), \\ \text{c)} & C_3(\boldsymbol{S}^{+,*}(\lambda^*)) = \boldsymbol{S}^-(\lambda), & C_3(\boldsymbol{T}^{-,*}(\lambda^*)) = \boldsymbol{T}^+(\lambda), \\ \text{d)} & C_4(\boldsymbol{S}^+(-\lambda)) = \boldsymbol{S}^-(\lambda), & C_4(\boldsymbol{T}^-(-\lambda)) = \boldsymbol{T}^+(\lambda) \end{array}$$

and

$$\begin{array}{llll} \text{a)} & C_1(D^{\pm,\dagger}(-\lambda^*)) = \hat{D}^\pm(\lambda), & \text{b)} & C_2(D^{\pm,T}(\lambda)) = \hat{D}^\pm(\lambda), \\ \text{c)} & C_3(D^{+,*}(\lambda^*)) = D^-(\lambda), & \text{d)} & C_4(D^+(-\lambda)) = D^-(\lambda). \end{array}$$

6 The classical r-matrix and the NLEE of MMKdV type

One of the definitions of the classical r-matrix is based on the Lax representation for the corresponding NLEE. We will start from this definition, but before to state it will introduce the following notation

$$\left\{ U(x,\lambda) \underset{,}{\otimes} U(y,\mu) \right\},$$

which is an abbreviated record for the Poisson bracket between all matrix elements of $U(x,\lambda)$ and $U(y,\mu)$

$$\left\{ U(x,\lambda) \underset{,}{\otimes} U(y,\mu) \right\}_{ik,lm} = \{ U_{ik}(x,\lambda), U_{lm}(y,\mu) \}.$$

In particular, if $U(x,\lambda)$ is of the form

$$U(x,\lambda) = Q(x,t) - \lambda J, \qquad Q(x,t) = \sum_{i<r} (q_{ir} E_{ir} + p_{ri} E_{ri}),$$

and the matrix elements of $Q(x,t)$ satisfy canonical Poisson brackets

$$\{ q_{ks}, p_{ri} \} = \mathrm{i} \delta_{ik} \delta_{rs} \delta(x-y),$$

then

$$\left\{ U(x,\lambda) \underset{,}{\otimes} U(y,\mu) \right\} = \mathrm{i} \sum_{i<r} (E_{ir} \otimes E_{ri} - E_{ri} \otimes E_{ir}) \delta(x-y).$$

The classical r-matrix can be defined through the relation [19]

$$\left\{ U(x,\lambda) \underset{,}{\otimes} U(y,\mu) \right\} = \mathrm{i} \left[r(\lambda-\mu), U(x,\lambda) \otimes 1\!\!1 + 1\!\!1 \otimes U(y,\mu) \right] \delta(x-y), \tag{6.1}$$

which can be understood as a system of N^2 equation for the N^2 matrix elements of $r(\lambda-\mu)$. However, these relations must hold identically with respect to λ and μ, i.e., (6.1) is an overdetermined system of algebraic equations for the matrix elements of r. It is far from obvious whether such $r(\lambda-\mu)$ exists, still less obvious is that it depends only on the difference $\lambda-\mu$. In other words far from any choice for $U(x,\lambda)$ and for the Poisson brackets between its matrix elements allow r-matrix description. Our system (6.1) allows an r-matrix given by

$$r(\lambda-\mu) = -\frac{1}{2} \frac{P}{\lambda-\mu}, \tag{6.2}$$

where P is a constant $N^2 \times N^2$ matrix

$$P = \sum_{\alpha\in\Delta^+}^{N} (E_\alpha \otimes E_{-\alpha} + E_{-\alpha} \otimes E_\alpha) + \sum_{k=1}^{r} h_k \otimes h_k.$$

The matrix P possesses the following special properties

$$[P, X \otimes 1\!\!1 + 1\!\!1 \otimes X] = 0 \qquad \forall\, X \in \mathfrak{g}.$$

By using these properties of P we are getting

$$[P, Q(x) \otimes 1\!\!1 + 1\!\!1 \otimes Q(x)] = 0, \tag{6.3}$$

i.e., the r.h.s. of (6.1) does not contain $Q(x,t)$. Besides:

$$\begin{aligned}[P, \lambda J \otimes \mathbb{1} + \mu \mathbb{1} \otimes J] &= (\lambda - \mu)\,[P, J \otimes \mathbb{1}] \\ &= -2(\lambda-\mu)\left(\sum_{\alpha\in\Delta_1^+} (E_\alpha \otimes E_{-\alpha} - E_{-\alpha}\otimes E_\alpha)\right), \end{aligned} \tag{6.4}$$

where we used the commutation relations between the elements of the Cartan–Weyl basis. The comparison between (6.3), (6.4) and (6.1) leads us to the result, that $r(\lambda-\mu)$ (6.2) indeed satisfies the definition (6.1).

Remark 9. It is easy to prove that equation (6.1) is invariant under the reduction conditions (4.17) and (4.18) of class A, so the corresponding reduced equations have the same r-matrix. However (6.1) is not invariant under the class B reductions and the corresponding reduced NLEE do not allow classical r-matrix defined through equation (6.1).

Let us now show, that the classical r-matrix is a very effective tool for calculating the Poisson brackets between the matrix elements of $T(\lambda)$. It will be more convenient here to consider periodic boundary conditions on the interval $[-L, L]$, i.e. $Q(x-L) = Q(x+L)$ and to use the fundamental solution $T(x,y,\lambda)$ defined by

$$\mathrm{i}\frac{\mathrm{d}T(x,y,\lambda)}{\mathrm{d}x} + U(x,\lambda)T(x,y,\lambda) = 0, \qquad T(x,x,\lambda) = \mathbb{1}.$$

Skipping the details we just formulate the following relation for the Poisson brackets between the matrix elements of $T(x,y,\lambda)$ [19]

$$\left\{T(x,y,\lambda) \underset{,}{\otimes} T(x,y,\mu)\right\} = [r(\lambda-\mu), T(x,y,\lambda)\otimes T(x,y,\mu)]\,. \tag{6.5}$$

The corresponding monodromy matrix $T_L(\lambda)$ describes the transition from $-L$ to L and $T_L(\lambda) = T(-L, L, \lambda)$. The Poisson brackets between the matrix elements of $T_L(\lambda)$ follow directly from equation (6.5) and are given by

$$\left\{T_L(\lambda) \underset{,}{\otimes} T_L(\mu)\right\} = [r(\lambda-\mu), T_L(\lambda)\otimes T_L(\mu)]\,.$$

An elementary consequence of this result is the involutivity of the integrals of motion $I_{L,k}$ from the principal series which are from the expansions of

$$\begin{aligned} &\ln\det \boldsymbol{a}_L^+(\lambda) = \sum_{k=1}^{\infty} I_{L,k}\lambda^{-k}, \qquad -\ln\det \boldsymbol{c}_L^-(\lambda) = \sum_{k=1}^{\infty} I_{L,k}\lambda^{-k}, \\ &\ln\det \boldsymbol{c}_L^+(\lambda) = \sum_{k=1}^{\infty} J_{L,k}\lambda^{-k}, \qquad -\ln\det \boldsymbol{a}_L^-(\lambda) = \sum_{k=1}^{\infty} J_{L,k}\lambda^{-k}. \end{aligned} \tag{6.6}$$

An important property of the integrals $I_{L,k}$ and $J_{L,k}$ is their locality, i.e. their densities depend only on Q and its x-derivatives.

The simplest consequence of the relation (6.5) is the involutivity of $I_{L,k}$, $J_{L,k}$. Indeed, taking the trace of both sides of (6.5) shows that $\{\operatorname{tr} T_L(\lambda), \operatorname{tr} T_L(\mu)\} = 0$. We can also multiply both sides of (6.5) by $C\otimes C$ and then take the trace using equation (6.3); this proves

$$\{\operatorname{tr} T_L(\lambda)C, \operatorname{tr} T_L(\mu)C\} = 0.$$

In particular, for $C = \mathbb{1} + J$ and $C = \mathbb{1} - J$ we get the involutivity of

$$\begin{aligned} &\left\{\operatorname{tr} \boldsymbol{a}_L^+(\lambda), \operatorname{tr} \boldsymbol{a}_L^+(\mu)\right\} = 0, \qquad \left\{\operatorname{tr} \boldsymbol{a}_L^-(\lambda), \operatorname{tr} \boldsymbol{a}_L^-(\mu)\right\} = 0, \\ &\left\{\operatorname{tr} \boldsymbol{c}_L^+(\lambda), \operatorname{tr} \boldsymbol{c}_L^+(\mu)\right\} = 0, \qquad \left\{\operatorname{tr} \boldsymbol{c}_L^-(\lambda), \operatorname{tr} \boldsymbol{c}_L^-(\mu)\right\} = 0. \end{aligned}$$

Equation (6.5) was derived for the typical representation $V^{(1)}$ of $\mathfrak{G} \simeq SO(2r)$, but it holds true also for any other finite-dimensional representation of $\mathfrak{G}$. Let us denote by $V^{(k)} \simeq \wedge^k V^{(1)}$ the k-th fundamental representation of $\mathfrak{G}$; then the element $T_L(\lambda)$ will be represented in $V^{(k)}$ by $\wedge^k T_L(\lambda)$ – the k-th wedge power of $T_L(\lambda)$, see [12]. In particular, if we consider equation (6.5) in the representation $V^{(n)}$ and sandwich it between the highest and lowest weight vectors in $V^{(n)}$ we get [30]

$$\{\det \boldsymbol{a}_L^+(\lambda), \det \boldsymbol{a}_L^+(\mu)\} = 0, \qquad \{\det \boldsymbol{c}_L^-(\lambda), \det \boldsymbol{c}_L^-(\mu)\} = 0. \tag{6.7}$$

Likewise considering (6.5) in the representation $V^{(m)}$ and sandwich it between the highest and lowest weight vectors in $V^{(m)}$ we get

$$\{\det \boldsymbol{a}_L^-(\lambda), \det \boldsymbol{a}_L^-(\mu)\} = 0, \qquad \{\det \boldsymbol{c}_L^+(\lambda), \det \boldsymbol{c}_L^+(\mu)\} = 0. \tag{6.8}$$

Since equations (6.7) and (6.8) hold true for all values of λ and μ we can insert into them the expansions (6.6) with the result

$$\{I_{L,k}, I_{L,p}\} = 0, \qquad \{J_{L,k}, J_{L,p}\} = 0, \qquad k, p = 1, 2, \dots .$$

Somewhat more general analysis along this lines allows one to see that only the eigenvalues of $\boldsymbol{a}_L^\pm(\lambda)$ and $\boldsymbol{c}_L^\pm(\lambda)$ produce integrals of motion in involution.

Taking the limit $L \to \infty$ we are able to transfer these results also for the case of potentials with zero boundary conditions. Indeed, let us multiply (6.5) by $E(y, \lambda) \otimes E(y, \mu)$ on the right and by $E^{-1}(x, \lambda) \otimes E^{-1}(x, \mu)$ on the left, where $E(x, \lambda) = \exp(-\mathrm{i}\lambda J x)$ and take the limit for $x \to \infty$, $y \to -\infty$. Since

$$\lim_{x \to \pm\infty} \frac{\mathrm{e}^{\mathrm{i}x(\lambda-\mu)}}{\lambda - \mu} = \pm \mathrm{i}\pi\delta(\lambda - \mu),$$

we get

$$\left\{T(\lambda) \underset{,}{\otimes} T(\mu)\right\} = r_+(\lambda - \mu) T(\lambda) \otimes T(\mu) - T(\lambda) \otimes T(\mu) r_-(\lambda - \mu),$$

$$r_\pm(\lambda - \mu) = -\frac{1}{2(\lambda - \mu)} \left(\sum_{k=1}^{r} h_k \otimes h_k + \sum_{\alpha \in \Delta_0^+} (E_\alpha \otimes E_{-\alpha} + E_{-\alpha} \otimes E_\alpha) \right) \pm \mathrm{i}\pi\delta(\lambda - \mu)\Pi_{0J},$$

where Π_{0J} is defined by

$$\Pi_{0J} = \sum_{\alpha \in \Delta_1^+} (E_\alpha \otimes E_{-\alpha} - E_{-\alpha} \otimes E_\alpha).$$

Analogously we prove that i) the integrals $I_k = \lim\limits_{L\to\infty} I_{L,k}$ and $J_p = \lim\limits_{L\to\infty} J_{L,p}$ are in involution, i.e.

$$\{I_k, I_p\} = \{I_k, J_p\} = \{J_k, J_p\} = 0,$$

for all positive values of k and p; ii) only the eigenvalues of $\boldsymbol{a}^\pm(\lambda)$ and $\boldsymbol{c}^\pm(\lambda)$ produce integrals of motion in involution.

7 Conclusions

We showed that the interpretation of the ISM as a generalized Fourier transform holds true also for the generalized Zakharov–Shabat systems related to the symmetric spaces **DIII**. The expansions over the 'squared solutions' are natural tool to derive the fundamental properties not only of the MNLS type equations, but also of the NLEE with generic dispersion laws, in particular MMKdV equations. Some of these equations, besides the intriguing properties as dynamical systems allowing for boomerons, trappons etc., may also have interesting physical applications.

Another interesting area for further investigations is to study and classify the reductions of these NLEE. For results along this line for the MNLS equations see the reports [10] and [9]; reductions of other types of NLEE have been considered in [26, 7, 8, 6].

One can also treat generalized Zakharov–Shabat systems related to other symmetric spaces. The expansions over the 'squared solutions' can be closely related to the graded Lie algebras, and to the reduction group and provide an effective tool to derive and analyze new soliton equations. For more details and further reading see [11, 26].

In conclusion, we have considered the reduced multicomponent MKdV equations associated with the **DIII**-type symmetric spaces using A.V. Mikhailov reduction group. Several examples of such nontrivial reductions leading to new MMKdV systems related to the $so(8)$ Lie algebra are given. In particular we provide examples with reduction groups isomorphic to $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$ and derive their effects on the scattering matrix, the minimal sets of scattering data and on the hierarchy of Hamiltonian structures. These results can be generalized also for other types of symmetric spaces.

Acknowledgements

It is our pleasure to thank Professor Gaetano Vilasi for useful discussions. One of us (VSG) thanks the Physics Department of the University of Salerno for the kind hospitality and the Italian Istituto Nazionale di Fisica Nucleare for financial support. We also thank the Bulgarian Science Foundation for partial support through contract No. F-1410. One of us (VSG) is grateful to the organizers of the Kyiv conference for their hospitality and acknowledges financial support by an CEI grant.

References

[1] Wadati M., The modified Korteweg–de Vries equation, *J. Phys. Soc. Japan* **34** (1972), 1289–1296.

[2] Athorne C., Fordy A., Generalised KdV and MKDV equations associated with symmetric spaces, *J. Phys. A: Math. Gen.* **20** (1987), 1377–1386.

[3] Fordy A.P., Kulish P.P., Nonlinear Schrödinger equations and simple Lie algebras, *Comm. Math. Phys.* **89** (1983), 427–443.

[4] Ablowitz M.J., Prinari B., Trubatch A.D., Discrete and continuous nonlinear Schrödinger systems, *London Mathematical Society Lecture Notes Series*, Vol. 302, Cambridge University Press, 2004.

[5] Mikhailov A.V., The reduction problem and the inverse scattering problem, *Phys. D* **3** (1981), 73–117.

[6] Lombardo S., Mikhailov A.V., Reductions of integrable equations. Dihedral group, *J. Phys. A: Math. Gen.* **37** (2004), 7727–7742, nlin.SI/0404013.
Lombardo S., Mikhailov A.V., Reduction groups and automorphic Lie algebras, *Comm. Math. Phys.* **258** (2005), 179–202, math-ph/0407048.

[7] Gerdjikov V.S., Grahovski G.G., Kostov N.A., Reductions of N-wave interactions related to low-rank simple Lie algebras. I. $\mathbb{Z}_2$-reductions, *J. Phys. A: Math. Gen.* **34** (2001), 9425–9461, nlin.SI/0006001.

[8] Gerdjikov V.S., Grahovski G.G., Ivanov R.I., Kostov N.A., N-wave interactions related to simple Lie algebras. $\mathbb{Z}_2$-reductions and soliton solutions, *Inverse Problems* **17** (2001), 999–1015, nlin.SI/0009034.

[9] Gerdjikov V.S., Grahovski G.G., Kostov N.A., On the multi-component NLS-type models and their gauge equivalent, in Proceedings of the International Conference "Contemporary Aspects of Astronomy, Theoretical and Gravitational Physics" (May 20–22, 2004, Sofia), Heron Press, Sofia, 2005, 306–317.

[10] Gerdjikov V.S., Grahovski G.G., Kostov N.A., On the multi-component NLS-type equations on symmetric spaces: reductions and soliton solutions, in Proceedings of the Sixth International Conference "Geometry, Integrability and Quantization" (July 3–10, 2004, Varna, Bulgaria), Softex, Sofia, 2005, 203–217.

[11] Drinfel'd V., Sokolov V.V., Lie algebras and equations of Korteweg–de Vries type, *Sov. J. Math.* **30** (1985), 1975–2036.

[12] Helgasson S., Differential geometry, Lie groups and symmetric spaces, Academic Press, 1978.

[13] Zakharov V.E., Shabat A.B., A scheme for integrating nonlinear equations of mathematical physics by the method of the inverse scattering transform. I, *Funct. Annal. Appl.* **8** (1974), 226–235.
Zakharov V.E., Shabat A.B., A scheme for integrating nonlinear equations of mathematical physics by the method of the inverse scattering transform. II, *Funct. Anal. Appl.* **13** (1979), no. 3, 13–23.

[14] Shabat A.B., Inverse-scattering problem for a system of differential equations, *Funct. Anal. Appl.* **9** (1975), 244–247.

[15] Shabat A.B., An inverse scattering problem, *Differ. Equations* **15** (1979), 1299–1307.

[16] Zakharov V.E., Manakov S.V., Novikov S.P., Pitaevskii L.I., Theory of solitons: the inverse scattering method, Plenum, New York, 1984.

[17] Calogero F., Degasperis A., Nonlinear evolution equations solvable by the inverse spectral transform. I, *Nuovo Cimento B (11)* **32** (1976), 201–242.
Calogero F., Degasperis A., Nonlinear evolution equations solvable by the inverse spectral transform. II, *Nuovo Cimento B (11)* **39** (1976), 1–54.

[18] Calogero F., Degasperis A., Spectral transform and solitons, Vol. I. North Holland, Amsterdam, 1982.

[19] Faddeev L.D., Takhtadjan L.A., Hamiltonian methods in the theory of solitons, Springer Verlag, Berlin, 1987.

[20] Calogero F., Degasperis A., Coupled nonlinear evolution equations solvable via the inverse spectral transform, and solitons that come back: the boomeron, *Lett. Nuovo Cimento (2)* **16** (1976), 425–433.

[21] Degasperis A., Solitons, boomerons, trappons, in Nonlinear Evolution Equations Solvable by the Spectral Transform, Editor F. Calogero, Pitman, London, 1978, 97–126.

[22] Beals R., Sattinger D.H., On the complete integrability of completely integrable systems, *Comm. Math. Phys.* **138** (1991), 409–436.

[23] Gerdjikov V.S., On the spectral theory of the integro-differential operator Λ, generating nonlinear evolution equations, *Lett. Math. Phys.* **6** (1982), 315–324.

[24] Gakhov F.D., Boundary value problems, Oxford, Pergamon Press, 1966 (translated from Russian, Editor I.N. Sneddon).

[25] Manakov S.V., On the theory of two-dimensional stationary self-focusing of electromagnetic waves, *Zh. Eksp. Teor. Fiz.* **65** (1973), 505–516 (English transl.: *Sov. Phys. J. Exp. Theor. Phys.* **38** (1974), 248–253).

[26] Gerdjikov V.S., Algebraic and analytic aspects of soliton type equations, *Contemp. Math.* **301** (2002), 35–68, nlin.SI/0206014.

[27] Ablowitz M., Kaup D., Newell A., Segur H., The inverse scattering transform – Fourier analysis for nonlinear problems, *Studies in Appl. Math.* **53** (1974), 249–315.

[28] Kaup D.J., Newell A.C., Soliton equations, singular dispersion relations and moving eigenvalues, *Adv. Math.* **31** (1979), 67–100.

[29] Gerdjikov V.S., Generalized Fourier transforms for the soliton equations. Gauge covariant formulation, *Inverse Problems* **2** (1986), 51–74.
Gerdjikov V.S., Generating operators for the nonlinear evolution equations of soliton type related to the semisimple Lie algebras, Doctor of Sciences Thesis, JINR, Dubna, USSR, 1987 (in Russian).

[30] Gerdjikov V.S., The Zakharov–Shabat dressing method amd the representation theory of the semisimple Lie algebras, *Phys. Lett. A* **126** (1987), 184–188.