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Branching Laws for Some Unitary Representations of $SL(4, \mathbb{R})$*

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Abstract. In this paper we consider the restriction of a unitary irreducible representation of type $A_{\mathfrak{q}}(\lambda)$ of $GL(4, \mathbb{R})$ to reductive subgroups H which are the fixpoint sets of an involution. We obtain a formula for the restriction to the symplectic group and to $GL(2, \mathbb{C})$, and as an application we construct in the last section some representations in the cuspidal spectrum of the symplectic and the complex general linear group. In addition to working directly with the cohmologically induced module to obtain the branching law, we also introduce the useful concept of pseudo dual pairs of subgroups in a reductive Lie group.

Key words: semisimple Lie groups; unitary representation; branching laws

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1 Introduction

Understanding a unitary representation π of a Lie groups G often involves understanding its restriction to suitable subgroups H. This is in physics referred to as breaking the symmetry, and often means exhibiting a nice basis of the representation space of π. Similarly, decomposing a tensor product of two representations of G is also an important branching problem, namely the restriction to the diagonal in $G \times G$. Generally speaking, the more branching laws we know for a given representation, the more we know the structure of this representation. For example, when G is semisimple and K a maximal compact subgroup, knowing the K-spectrum, i.e. the collection of K-types and their multiplicities, of π is an important invariant which serves to describe a good deal of its structure. It is also important to give good models of both π and its explicit K-types. There has been much progress in recent years (and of course a large number of more classical works, see for example [27, 6, 7, 8]), both for abstract theory as in [11, 12, 14, 13], and concrete examples of branching laws in [24, 25, 18, 4, 28].

In this paper, we shall study in a special case a generalization of the method applied in [19] and again in [8]; this is a method of Taylor expansion of sections of a vector bundle along directions normal to a submanifold. This works nicely when the original representation is a holomorphic discrete series for G, and the subgroup H also admits holomorphic discrete series and is embedded in a suitable way in G. The branching law is a discrete sum decomposition, even with finite multiplicities, so-called admissibility of the restriction to H; and the summands are themselves holomorphic discrete series representations for H. Since holomorphic discrete series

*This paper is a contribution to the Proceedings of the 2007 Midwest Geometry Conference in honor of Thomas P. Branson. The full collection is available at http://www.emis.de/journals/SIGMA/MGC2007.html representations are cohomologically induced representations in degree zero, it is natural to attempt a generalization to other unitary representations of similar type, namely cohomologically induced representations in higher degree. We shall focus on the line bundle case, i.e. the $A_{\mathfrak{q}}(\lambda)$ representations. In this case T. Kobayashi [13] obtained necessary and sufficient conditions that the restriction is discrete and that each representation appears with finite multiplicity, socalled *admissibility* of the representation relatively to the subgroup. Using explicit resolutions and filtrations associated with the imbedding of H in G, we analyze the derived functor modules and obtain an explicit decomposition into irreducible representations. It is perhaps not surprising, that with the appropriate conditions on the imbedding of the subgroup, the class of (in our case derived functor) modules is preserved in the restriction from H to G.

While the algebraic methods of derived functor modules, in particular the cohomologically induced representations, provide a very strong tool for the theoretical investigations of unitary representations of reductive Lie groups, it has been difficult to work with concrete models of these modules. It is however exactly these models that we use in this paper, as outlined above; the fact that one may consider these modules as Taylor expansions of appropriate differential forms, indicates that it should be natural to study these Taylor expansions along submanifolds – and when these submanifolds are natural for the subgroup for which one wants to do the branching law, there arises a useful link between the algebraic branching and the geometry of the imbedding of the subgroup. It is our hope, that this idea (that we carry out in some relatively small examples) will have a broader use in deciding the possible candidates for representations occuring in an admissible branching law.

Here is the general setting that we consider: Let G be a semisimple linear connected Lie group with maximal compact subgroup K and Cartan involution θ. Suppose that σ is another involution so that $\sigma\cdot\theta=\theta\cdot\sigma$ and let H be the fixpoint set of σ in G. Suppose that $L=L_x$ is the centralizer of an elliptic element $x\in i(\mathfrak{g}\cap\mathfrak{h})$ and let $\mathfrak{q}=\mathfrak{l}\oplus\mathfrak{u}$, $\mathfrak{q}^H=\mathfrak{q}\cap\mathfrak{h}$ be the corresponding θ-stable parabolic subgroups. Here we use as usual gothic letters for complex Lie algebras and subspaces thereof; a subscript will denote the real form, e.g. $\mathfrak{g}_o$. We say that pairs of parabolic subalgebras $\mathfrak{q}$, $\mathfrak{q}^H$ which are constructed this way are well aligned. For a unitary character λ of L we define following Vogan/Zuckerman the unitary representations $A_{\mathfrak{q}}(\lambda)$.

In this paper we consider the example of the group $G=SL(4,\mathbb{R})$. There are two G-conjugacy classes of skew symmetric matrices with representants $Q_1=\begin{pmatrix} J & 0\\ 0 & -J\end{pmatrix}$ and $Q_2=\begin{pmatrix} J & 0\\ 0 & J\end{pmatrix}$ where $J=\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}$. Let H_1 respectively H_2 be the symplectic subgroups defined by these matrices and H_1', H_2' the centralizer of Q_1, respectively Q_2. All these subgroups are fix point sets of involutions σ_i, $i=1,2$ and σ_i', $i=1,2$ respectively.

The matrix Q_2 has finite order, it is contained in all subgroups H_i and $iQ_2\in\mathfrak{g}$ defines a θ-stable parabolic subalgebra $\mathfrak{q}$ of $sl(4,\mathbb{C})$ and also θ-stable parabolic subalgebras $\mathfrak{q}_{\mathfrak{h}_1}=\mathfrak{q}\cap\mathfrak{h}_1$ of $\mathfrak{h}_1$, respectively $\mathfrak{q}_{\mathfrak{h}_2}=\mathfrak{q}\cap\mathfrak{h}_2$ of $\mathfrak{h}_2$. Its centralizer L in $SL(4,\mathbb{R})$ is isomorphic to $GL_1(2,\mathbb{C})=\{T\in GL(2,\mathbb{C})|\,|\det(T)|=1\}$. The parabolic subgroups $\mathfrak{q}$, $\mathfrak{q}_{\mathfrak{h}_1}$ as well as $\mathfrak{q}$, $\mathfrak{q}_{\mathfrak{h}_2}$ are well aligned.

We consider in this paper the unitary representation $A_{\mathfrak{q}}$ of G corresponding to trivial character λ. Its infinitesimal character is the same as that of the trivial representation. The representation $A_{\mathfrak{q}}$ was studied from an analytic point of view by S. Sahi [21]. Since the $A_{\mathfrak{q}}$ has nontrivial $(\mathfrak{g},K)$-cohomology and is isomorphic to a representation in the residual spectrum, this representation is also interesting from the point of view of automorphic forms. See for example [23]. We show in this paper that the restriction of $A_{\mathfrak{q}}$ to H_1 and H_1' is a direct sum of irreducible unitary representations, where as the restriction to H_2 and H_2' has continuous spectrum. We also determine explicitly the restriction of $A_{\mathfrak{q}}$ to the subgroups H_1 and H_1' and conclude that for all unitary $(\mathfrak{h}_1,H_1\cap K)$-modules V the dimension of $\mathrm{Hom}_{(\mathfrak{h}_1,H_1\cap K)}(A_{\mathfrak{q}},V)$ is at most 1.

If we interpret $SL(4,\mathbb{R})$ and $Sp(2,\mathbb{R})$ as Spin(3,3) and Spin(2,3) then these branching laws can in some sense be considered as supporting the conjectures by B. Gross and D. Prasad [3] for the restriction of Vogan packets of representations of $SO(n,n)$ to representations of $SO(n-1,n)$.

The paper is organized as follows: After introducing all the notation in Section 2 we prove in Section 3 using a result of T. Kobayashi, that the restriction of $A_{\mathfrak{q}}$ to H_1 and H_1' is a direct sum of irreducible unitary representations, whereas the restriction to H_2 and H_2' does have a continuous spectrum. This discrete/continuous alternative, see [12], is one of the deep results that we invoke for symmetric subgroups.

We do not attempt in this paper to say anything about the continuous spectrum, and we mainly focus on the admissible situation, so the alternative is really admissible/non-admissible.

In Sections 4 and 5 we determine the representations of H_1 respectively of H_1' that appear in the restriction of $A_{\mathfrak{q}}$ to H_1 respectively H_1' and show that it is a direct sum of unitary representations of the form $A_{\mathfrak{q}\cap\mathfrak{h}_1}(\mu)$ respectively $A_{\mathfrak{q}\cap\mathfrak{h}_1'}(\mu')$, each appearing with multiplicity one. The main point is here, that we find a natural model in which to do the branching law, based on the existence results of T. Kobayashi; and also following experience from some of his examples, where indeed derived functor modules decompose as derived functor modules (for the smaller group).

In Section 6 we introduce pseudo dual pairs. This allows us to find another interpretation of the restrictions of $A_{\mathfrak{q}}$ to the pseudo dual pair H_1, H_2. This notion turns out to be extremely useful for analyzing the spectrum in the admissible situation, and combined with our idea of restricting a cohomologically induced module gives the complete branching law. In Section 7 we recall some more examples of branching laws.

In Section 8 we formulate a conjecture about the multiplicity of representations in the restriction of representations $A_{\mathfrak{q}}$ of semisimple Lie groups G to subgroups H, which are centralizers of involutions. If the restriction of $A_{\mathfrak{q}}$ to H is a direct sum of irreducible representation of H we expect that there is a θ-stable parabolic subalgebra $\mathfrak{q}^H$ of H so that all representations which appear in the restriction are of the form $A_{\mathfrak{q}^H}(\mu)$ and that a Blattner-type formula holds. See the precise conjecture at the end of Section 8, where we introduce a natural generalization of previously known Blattner-type formulas for the maximal compact subgroup.

In Section 9 we see how these results may be used to construct automorphic representations of $Sp(2,\mathbb{R})$ and $GL(2,\mathbb{C})$ which are in the discrete spectrum for some congruence subgroup. For $Sp(2,\mathbb{R})$ these representations are in the residual spectrum, whereas for $GL(2,\mathbb{C})$ these representations are in the cuspidal spectrum. We expect that our methods extend to other situations with similar applications to automorphic representations; and we hope the point of view introduced here also will help to understand in a more explicit way the branching laws for semisimple Lie groups with respect to reductive subgroups.

2 Notation and generalities

2.1. Let G be a connected linear semisimple Lie group. We fix a maximal compact subgroup K and Cartan involution θ. Let H be a θ-stable connected semisimple subgroup with maximal compact subgroup $K^H = K \cap H$. We pick a fundamental Cartan subgroup $C^H = T^H \cdot A^H$ of H. It is contained in a fundamental Cartan subgroup $C = T \cdot A$ of G so that $T^H = T \cap H$ and $A^H = A \cap H$. The complex Lie algebra of a Lie group (as before) is denoted by small letters and its real Lie algebra by a subscript o. We denote the Cartan decomposition by $\mathfrak{g}_o = \mathfrak{k}_o \oplus \mathfrak{p}$.

Definition. Let $\mathfrak{q}$ and $\mathfrak{q}^H$ be θ-stable parabolic subalgebras of $\mathfrak{g}$, respectively $\mathfrak{h}$. We say that they are *well aligned* if $\mathfrak{q}^H = \mathfrak{q} \cap \mathfrak{h}$.

We fix x_o in $\mathfrak{t}^H$. Then $i\ x_o$ defines well aligned θ-stable parabolic subalgebras $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ and $\mathfrak{q}^H = \mathfrak{l}^H \oplus \mathfrak{u}^H = \mathfrak{q} \cap \mathfrak{h}$ of $\mathfrak{g}$ respectively $\mathfrak{h}$; for details see page 274 in [10].

We write L and L^H for the centralizer of x_0 in G and in H respectively. For a unitary character λ of L we write λ^H for the restriction of λ to L^H.

2.2. For later reference we recall the construction of the representations $A_{\mathfrak{q}}(V)$, V an irreducible $(\mathfrak{q}, L\cap K)$ module. We follow conventions of the book by Knapp and Vogan [10] (where much more detail on these derived functor modules is to be found – this is our standard reference) and will always consider representations of L and not of the metaplectic cover of L as some other authors. We consider $U(\mathfrak{g})$ as right $U(\mathfrak{q})$ module and write $V^\sharp = V \otimes \wedge^{\mathrm{top}}\mathfrak{u}$. Let $\mathfrak{p}_L$ be a $L\cap K$-invariant complement of $\mathfrak{l}\cap\mathfrak{k}$ in $\mathfrak{l}$. We write $r_G = \mathfrak{p}_L \oplus \mathfrak{u}$. Now we introduce the derived functor modules as on page 167 in [10], recalling that this formalizes Taylor expansions of certain differential forms. Since all the groups considered in the paper are connected we use the original definition of the Zuckerman functor [26] and do not use the the Hecke algebra $R(\mathfrak{g}, K)$ to define the representations $A_{\mathfrak{q}}(V)$. Consider the complex

$$
\begin{aligned}
0 &\to \mathrm{Hom}_{L\cap K}(U(\mathfrak{g}), \mathrm{Hom}(\wedge^0 r_G, V^\sharp))_K \to \\
&\to \mathrm{Hom}_{L\cap K}(U(\mathfrak{g}), \mathrm{Hom}(\wedge^1 r_G, V^\sharp))_K \to \mathrm{Hom}_{L\cap K}(U(\mathfrak{g}), \mathrm{Hom}(\wedge^2 r_G, V^\sharp))_K \to \cdots .
\end{aligned}
$$

Here the subscript K denotes the subspace of K-finite vectors. We denote by $T(x, U(\cdot))$ an element in $\mathrm{Hom}_{L\cap K}(U(\mathfrak{g}), \mathrm{Hom}_{\mathbb{C}}(\wedge^{n-1} r_G, V^\sharp))_K$. The differential d is defined by

$$
\begin{aligned}
d\,T(x, U(X_1\wedge X_2\wedge\cdots\wedge X_n)) &= \sum_{i=1}^{n}(-1)^i T(X_i x, U(X_1\wedge X_2\wedge\cdots\hat{X}_i\cdots\wedge X_n)) \\
&+ \sum_{i=1}^{n}(-1)^{i+1} T(x, X_i U(X_1\wedge X_2\wedge\cdots\hat{X}_i\cdots\wedge X_n)) \\
&+ \sum_{i<j}(-1)^{i+j} T(x, U(P_{r_G}[X_i, X_j]\wedge X_1\wedge X_2\wedge\cdots\hat{X}_i\cdots\hat{X}_j\cdots\wedge X_n)),
\end{aligned}
$$

where $x \in U(\mathfrak{g})$, $X_j \in r_G$ and P_{r_G} is the projection onto r_G along $\mathfrak{l}\cap\mathfrak{k}$. Let $s = \dim(\mathfrak{u}\cap\mathfrak{k})$ and let χ be the infinitesimal character of V. If

$$
\frac{2\langle \chi + \rho(\mathfrak{u}), \alpha\rangle}{|\alpha|^2} \notin \{0, -1, -2, -3, \dots\} \qquad \text{for} \quad \alpha \in \Delta(\mathfrak{u}),
$$

where $\langle\ ,\ \rangle$ denotes the Killing form of $\mathfrak{g}$, then the cohomology is zero except in degree s and if V is irreducible this defines an irreducible $((U(\mathfrak{g}), K)$-module $A_{\mathfrak{q}}(V)$ in degree s (8.28 in [10]). By (5.24 in [10], see also the remark/example on page 344) the infinitesimal character of $A_{\mathfrak{q}}(V)$ is $\chi + \rho(\mathfrak{u})$ (usual shift of the half-sum of all positive roots in $\mathfrak{u}$).

If V is trivial the infinitesimal character of $A_{\mathfrak{q}}(V)$ is ρ_G and we write simply $A_{\mathfrak{q}}$. Two representations $A_{\mathfrak{q}}$ and $A_{\mathfrak{q}'}$ are equivalent if $\mathfrak{q}$ and $\mathfrak{q}'$ are conjugate under the compact Weyl group W_K.

For an irreducible finite dimensional $(\mathfrak{q}^H, L^H\cap K)$-module V^{L^H} we define similarly the $(U(\mathfrak{h}), K^H)$-modules $A_{\mathfrak{q}^H}(V^{L^H})$.

2.3. Let H be the fix point set of an involutive automorphism σ of G which commutes with the Cartan involution θ. We write $\mathfrak{g}_o = \mathfrak{h}_o \oplus \mathfrak{s}_o$ for the induced decomposition of the Lie algebra. T. Kobayashi proved [12] that the restriction of $A_{\mathfrak{q}}$ to H decomposes as direct sum of irreducible representations of H if $A_{\mathfrak{q}}$ is K^H-admissible, i.e. if every K^H-type has finite multiplicity. If $A_{\mathfrak{q}}$ is discretely decomposable as an $(\mathfrak{h}_o, K\cap H)$-module we call an irreducible $(\mathfrak{h}_o, H\cap K)$-module π^H an *H-type* of $A_{\mathfrak{q}}$ if

$$
\mathrm{Hom}_{(\mathfrak{h}_o, K^H)}(\pi^H, A_{\mathfrak{q}}) \neq 0
$$

and the dimension of $\mathrm{Hom}_{(\mathfrak{h}_o, K^H)}(\pi^H, A_{\mathfrak{q}})$ its *multiplicity.*

We have $\mathfrak{l} = \mathfrak{l}^H \oplus \mathfrak{l} \cap \mathfrak{s}$. Put $\mathfrak{u}^H = \mathfrak{u} \cap \mathfrak{h}$. The representation of $\mathfrak{l}^H$ on $\mathfrak{u}$ is reducible and as $\mathfrak{l}^H$-module $\mathfrak{u} = \mathfrak{u}^H \oplus (\mathfrak{u} \cap \mathfrak{s})$. Let $\overline{\mathfrak{q}} = \mathfrak{l} \oplus \overline{\mathfrak{u}}$ be the opposite parabolic subgroup. Then $\mathfrak{h} = \mathfrak{k}^H \oplus \mathfrak{u}^H \oplus \overline{\mathfrak{u}}^H$ and $(\mathfrak{u} \cap \mathfrak{s}) \oplus \overline{\mathfrak{u}} \cap \mathfrak{s}$ is a $\mathfrak{l}^H$-module. As an $\mathfrak{l}^H$-module $\mathfrak{g} = \mathfrak{h} \oplus (\mathfrak{u} \cap \mathfrak{s}) \oplus (\mathfrak{l} \cap \mathfrak{s}) \oplus (\overline{\mathfrak{u}} \cap \mathfrak{s})$.

2.4. Now let $G = SL(4,\mathbb{R})$. The skew symmetric matrices

$$Q_1 = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix} \qquad \text{and} \qquad Q_2 = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$$

with $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ represent the conjugacy classes of skew symmetric matrices under G. They define symplectic forms also denoted by Q_1 and Q_2.

Let H_1, respectively H_2, be the θ-stable symplectic subgroups defined by Q_1, respectively Q_2. These subgroups are fix points of the involutions

$$\sigma_i(g) = Q_i \cdot (g^{-1})^{\mathrm{tr}} \cdot Q_i^{-1}, \qquad i = 1, 2.$$

Since Q_1 and Q_2 are conjugate in $GL(4,\mathbb{R})$, but not in $SL(4,\mathbb{R})$, the symplectic groups H_1 and H_2 are not conjugate in $SL(4,\mathbb{R})$.

Let H_1' and H_2' be the fix points of the involutions

$$\sigma_i'(g) = Q_i \cdot g \cdot Q_i^{-1}.$$

Both groups H_1' and H_2' are isomorphic to

$$GL_1(2,\mathbb{C}) = \{T \in GL(2,\mathbb{C}) |\, |\det(T)| = 1\},$$

but they are not conjugate in $SL(4,\mathbb{R})$.

2.5. We fix $x_0 = Q_2$. It has finite order and is contained in $\bigcap_{i=1}^2 H_i$ and in $\bigcap_{i=1}^2 H_i'$. Now $ix_0 \in i\mathfrak{g}$ defines a θ stable parabolic subalgebra $\mathfrak{q}$ of $sl(4,\mathbb{C})$ and also θ-stable well aligned parabolic subalgebras $\mathfrak{q}^H$ of the subalgebras $\mathfrak{h}$. Its centralizer $L = L_{x_0}$ in $SL(4,\mathbb{R})$, the Levi subgroup, is isomorphic to $GL(2,\mathbb{C}) = H_2'$. For a precise description of the parabolic see page 586 in [10].

We have

$$L = H_2', \qquad K^{H_2} = K \cap H_2 = K \cap L, \qquad K^{H_2'} = K \cap H_2' = K \cap L$$

and

$$K^{H_1} = K^{H_1'}.$$

Let $A_{\mathfrak{q}}$ be the representation holomorphically induced from $\mathfrak{q}$ which has a trivial infinitesimal character. This representations is a subrepresentation of a degenerate series representation induced from a one dimensional representation of the parabolic subgroup with Levi factor $S(GL(2,\mathbb{R}) \times GL(2,\mathbb{R}))$ and thus all its K-types have multiplicity one. See [21] for details.

The next proposition demonstrates how different imbeddings of the same subgroup (symplectic res. general linear complex) gives radically different branching laws.

Proposition 2.1.

1. *The restriction of $A_{\mathfrak{q}}$ to H_1 and to H_1' is a direct sum of irreducible representation each appearing with finite multiplicity.*
2. *The restriction of $A_{\mathfrak{q}}$ to H_2 and to H_2' is not admissible and has continuous spectrum.*

Proof. Since $K^{H_1} = K^{H_1'}$ and $K^{H_2} = K^{H_2'}$ it suffices by T. Kobayashi's Theorem 4.2 in [12] to show that $A_{\mathfrak{q}}$ is K^{H_1} admissible. We will prove this in the next section.

To prove (2) it suffices to show by Theorem 4.2 in [12] that $A_{\mathfrak{q}}$ is not K^{H_2} admissible. This is proved also in the next section. ∎

Remark. We will prove later in the paper that the representations of H_1 have at most multiplicity one in the restriction of $A_{\mathfrak{q}}$.

3 The restriction of $A_{\mathfrak{q}}$ to $K \cap H_i$, $i = 1, 2$

We use in this section the notation introduced on pages 586–588 in [10].

3.1. The Cartan algebra $\mathfrak{t}_o$ of $\mathrm{so}(4, \mathbb{R})$ consists of 2 by 2 blocks $\begin{pmatrix} 0 & \theta_j \\ -\theta_j & 0 \end{pmatrix}$ down the diagonal. We have a θ-stable Cartan subalgebra $\mathfrak{h}_o = \mathfrak{t}_o \oplus \mathfrak{a}_o$ where $\mathfrak{a}_o$ consists of the 2 by 2 blocks $\begin{pmatrix} x_j & 0 \\ 0 & x_j \end{pmatrix}$, also down the diagonal. We define $e_j \in \mathfrak{h}^*$ by

$$e_j \begin{pmatrix} x_j & -iy_j \\ iy_j & x_j \end{pmatrix} = y_j$$

and $f_j \in \mathfrak{h}^*$ by

$$f_j \begin{pmatrix} x_j & -iy_j \\ iy_j & x_j \end{pmatrix} = x_j.$$

Then the roots $\Delta(\mathfrak{u})$ of $(\mathfrak{h}, \mathfrak{u})$ are

$$e_1 + e_2 + (f_1 - f_2), \qquad e_1 + e_2 - (f_1 - f_2), \qquad 2e_1, \qquad 2e_2$$

and a compatible set of positive roots $\Delta^+(\mathfrak{l})$ of $(\mathfrak{h}, \mathfrak{l})$ are

$$e_1 - e_2 + (f_1 - f_2), \qquad e_1 - e_2 - (f_1 - f_2).$$

The roots $\alpha_1 = e_1 + e_2$, $\alpha_2 = e_1 - e_2$ are compatible positive roots of the Lie algebra $\mathfrak{k}$ with respect to $\mathfrak{t}$.

The highest weight of the minimal K-type of $A_{\mathfrak{q}}$ is $\Lambda = 3(e_1 + e_2)$. See page 588 in [10]. All other K-types are of the form

$$\Lambda + m_1(e_1 + e_2) + 2m_2 e_1, \qquad m_1, m_2 \in \mathbb{N}.$$

$K^{H_2} = K \cap L$ is isomorphic to $U(2)$. We assume that the positive root of $(\mathfrak{t}, \mathfrak{k}^{H_2})$ is α_2 whereas the α_1 can be identified with a character of the center of K^{H_2}. So the restriction of each K-type with highest weight $(m+3)\alpha_1$ to K^{H_2} is a sum of characters $d\alpha_1$ with $-(m+3) \le d \le (m+3)$. Hence $A_{\mathfrak{q}}$ is not K^{H_2}-finite and thus by T. Kobayashi's theorem $A_{\mathfrak{q}}$ is not a discrete sum of irreducible representations of H_2. This proves the second claim of Proposition 2.1.

The groups H_1 and H_2 are conjugate under the outer automorphism which changes the sign of e_2. Hence the simple positive root of $(\mathfrak{t}, \mathfrak{k}^{H_1})$ can be identified with α_1 and α_2 can be identified with a character of the center of K^{H_1} (see Fig. 1). A K-type with highest weight $\Lambda + m_1(e_1 + e_2) + 2m_2\, e_1$ is a tensor product of a representation with highest weight $(3+m_1+m_2, 3+m_1+m_2)$ and a representation with highest weight $(m_2, -m_2)$. Its restriction to K^{H_1} is a direct sum of representations with highest weights $(3 + m_1 + m_2 + i, 3 + m_1 + m_2 - i)$, $-m_2 \le i \le m_2$. Fig. 2 shows the highest weights of the K^{H_1}-types for the restriction of $A_{\mathfrak{q}}$ to K^{H_1}. Their multiplicities are indicated by a number.

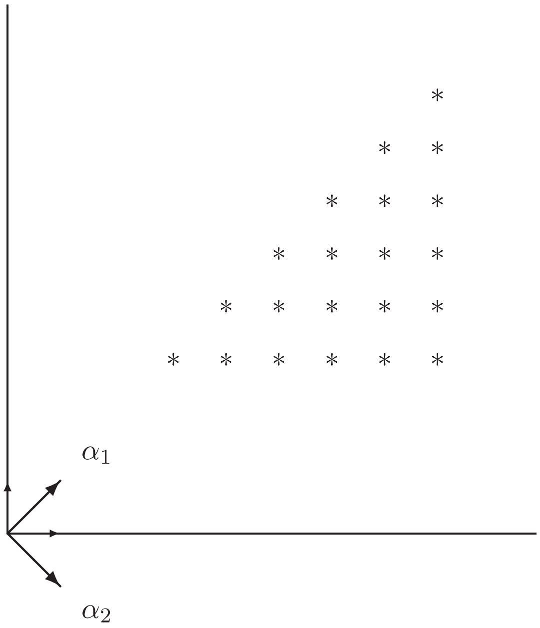

Figure 1.

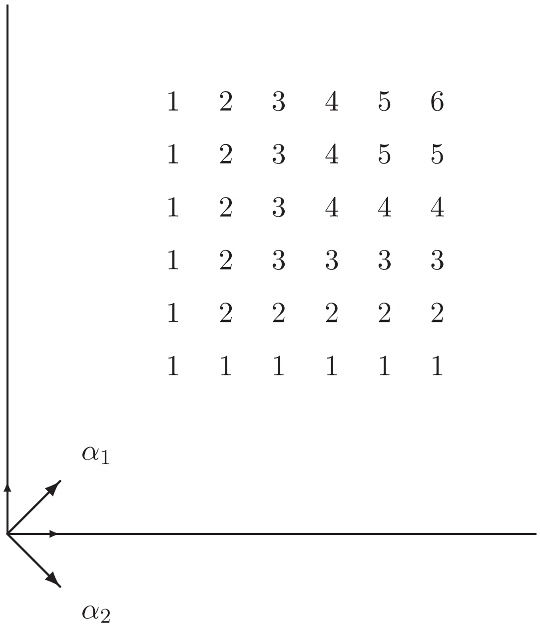

Figure 2.

Thus $A_\mathfrak{q}$ is K^{H_1}-finite. This completes the proof of the first assertion of Proposition 2.1.

Remark. A second series of representations is obtained if we define another θ-stable parabolic subalgebra $\mathfrak{q}'$ using the matrix $Q_1 \in \mathfrak{g}$ instead of Q_2. We obtain a representation $A_{\mathfrak{q}'}$ which is not equivalent to $A_\mathfrak{q}$. The same arguments as in the previous case prove that restriction of $A_{\mathfrak{q}'}$ to H_1 does not have a purely discrete spectrum whereas the restriction to H_2 is a direct sum of irreducible unitary representations.

The representation $\mathcal{A}_\mathfrak{q}$ of $SL^{+/-}(4,\mathbb{R})$ (determinant ± 1) obtained by inducing representation $A_\mathfrak{q}$ is irreducible and its restriction to $SL(4,\mathbb{R})$ is equal to $A_\mathfrak{q} \oplus A_{\mathfrak{q}'}$. Hence the restriction of $\mathcal{A}_\mathfrak{q}$ to H_1 does not have discrete spectrum.

4 The restriction of $A_\mathfrak{q}$ to the symplectic group H_1

In this section we determine H_1-types of $A_\mathfrak{q}$. Our techniques are based on homological algebra and the construction of an "enlarged complex" whose cohomology computes the restriction. We introduce it in **4.1** for semisimple connected Lie groups H and connected reductive subgroups H. Then we will compute the restriction to H_1 by restricting $A_\mathfrak{q}$ to a subgroup conjugate to H_1. The motivation for this "enlarged complex" or "branching complex" is the same as when one is restricting holomorphic functions to a complex submanifold, and identifying the functions with their normal derivatives along the submanifold. In our case we are working with (formalizations of) differential forms satisfying a similar differential equation, so it is natural to try to identify them with their "normal derivatives"; this is what is formalized in our definition. As it turns out, with the appropriate conditions (well aligned parabolic subgroups, vanishing of the cohomology in many degrees, and the non-vanishing of explicit classes corresponding to small K-types) we can indeed make the calculation of the branching law effective, at least in the examples at hand.

4.1. We define another complex for a semisimple connected Lie group G and a connected reductive subgroup H satisfying the assumptions of **2.3**. Let $\mathbb{C}_{\lambda_H}$ be the one dimensional representation of L^H defined by $\lambda \otimes \wedge^{\mathrm{top}}(\mathfrak{u} \cap \mathfrak{s})$. Then

$$\mathbb{C}^\sharp_\lambda = \mathbb{C}_\lambda \otimes \wedge^{\mathrm{top}}\mathfrak{u} = \mathbb{C}_{\lambda_H} \otimes \wedge^{\mathrm{top}}\mathfrak{u}^H = \mathbb{C}^\sharp_{\lambda_H}.$$

Consider the complex L_H^*

$$(\mathrm{Hom}_{L\cap K\cap H}(U(\mathfrak{g}), \mathrm{Hom}(\wedge^i r_H, \mathbb{C}^\sharp_{\lambda_H}))_{K\cap H}, d_H). \tag{4.1}$$

Here $r_H = r_G \cap \mathfrak{h}$ and d_H is defined analogously to the differential d in **2.2**. As a left $U(\mathfrak{l}^H)$-module

$$U(\mathfrak{g}) = Q \otimes U(\mathfrak{h}),$$

where Q is the symmetric algebra $S(\mathfrak{s})$. (See [10, 2.56].) We have

$$\begin{aligned}\mathrm{Hom}_{L\cap K\cap H}(U(\mathfrak{g}), \mathrm{Hom}(\wedge^i r_H, \mathbb{C}^\sharp_{\lambda_H}))_{K\cap H} &= \mathrm{Hom}_{L\cap K\cap H}(Q\otimes U(\mathfrak{h}), \mathrm{Hom}(\wedge^i r_H, \mathbb{C}^\sharp_{\lambda_H}))_{K\cap H}\\ &= \mathrm{Hom}_{L\cap K\cap H}(U(\mathfrak{h}), \mathrm{Hom}(\wedge^i r_H, Q^*\otimes \mathbb{C}^\sharp_{\lambda_H}))_{K\cap H}.\end{aligned}$$

$U(\mathfrak{g})$ acts on the enlarged complex from the right and a quick check shows that d_H also commutes with this action and therefore we have an action of $U(\mathfrak{g})$ on the cohomology of the complex.

We also consider the "large complex" L^*

$$(\mathrm{Hom}_{K\cap L\cap H}(U(\mathfrak{g}), \mathrm{Hom}(\wedge^i r_G, \mathbb{C}^\sharp_{\lambda}))_{K\cap H},\ d).$$

We have $r_G = r_H \oplus (\mathfrak{u}\cap\mathfrak{s}) \oplus (p_L\cap\mathfrak{s})$, and so

$$\wedge^i r_G = \oplus_{l+k=i} \wedge^k r_H \otimes \wedge^l(\mathfrak{u}\cap\mathfrak{s}\oplus p_L\cap\mathfrak{s}).$$

Using this we can define a "pull back" map of forms

$$\begin{aligned}\mathbf{pb}^i_H:\ &\mathrm{Hom}_{L\cap K\cap H}(U(\mathfrak{h}), \mathrm{Hom}(\wedge^i r_H, Q^*\otimes\mathbb{C}^\sharp_{\lambda_H}))_{K\cap H}\\ &\to \mathrm{Hom}_{K\cap L\cap H}(U(\mathfrak{g}), \mathrm{Hom}(\wedge^i r_G, \mathbb{C}^\sharp_{\lambda}))_{K\cap H}.\end{aligned}$$

The pullback map commutes with the right action of $U(\mathfrak{h})$ and induces a map of complexes. This is the main observation we use in order to analyze the action of H on the cohomologically induced module.

4.2. For the rest of this section we assume that $G = SL(4,\mathbb{R})$. We will show that there exists a symplectic subgroup, which we denote by H_1^w conjugate to H_1 by an element w, so that the pullback induces a nontrivial map in cohomology. Since the restriction of $A_\mathfrak{q}$ depends only on the conjugacy class of H_1 this determines the restriction.

Since

$$\begin{pmatrix}1&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&1\end{pmatrix} Q_1 \begin{pmatrix}1&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&1\end{pmatrix} = \begin{pmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix},$$

by abuse of notation we will also write H_1 and H_1' for the groups defined by the skew symmetric form

$$\begin{pmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix}.$$

Thus

$$\mathfrak{h}_1 = \begin{pmatrix}A & X\\ Y & -A^{\mathrm{tr}}\end{pmatrix}$$

for symmetric matrices X and Y.

Recall that $\mathfrak{q}$ is defined by

$$i\,Q_2 = \begin{pmatrix} i\,J & 0 \\ 0 & i\,J \end{pmatrix} \in \mathfrak{g}$$

and that $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{s}_1$. We need the fine structure of the parabolic relative to the symmetric subgroup, in order to compare the cohomology of these complexes during the branching.

Lemma 4.1. *Under the above assumptions*

a) $\mathfrak{l} \cap \mathfrak{h}_1$ *is isomorphic to* $sl(2,\mathbb{R}) \oplus i\mathbb{R}$ *and* $\dim(\mathfrak{u} \cap \mathfrak{h}_1) = 3$*;*

b) the representation of $L \cap H_1$ *acts by a nontrivial character* μ_1 *with differential* $(e_1 + e_2)$ *on the one dimensional space* $\mathfrak{u} \cap \mathfrak{s}_1$*;*

c) $\mathfrak{l} \cap \mathfrak{s}_1$ *is a direct sum of the trivial representation and the adjoint representation of* $\mathfrak{l} \cap \mathfrak{h}_1$*.*

d) $\mathfrak{u} \cap \mathfrak{k} = \mathfrak{u} \cap \mathfrak{k} \cap \mathfrak{h}_1$ *has dimension* 1*.*

Proof. We have

$$\mathfrak{l}_o \cap \mathfrak{h}_1 = \begin{pmatrix} a & b & x & 0 \\ -b & a & 0 & x \\ y & 0 & -a & b \\ 0 & y & -b & -a \end{pmatrix}.$$

The nilradical of a parabolic subalgebra with this Levi subalgebra has dimension 3.

The dimension of $\mathfrak{l} \cap \mathfrak{h}_1 \cap \mathfrak{k}$ is 2. Hence the dimension of $\mathfrak{u} \cap \mathfrak{k} \cap \mathfrak{h}_1$ is 1. On the other hand the dimension of $\mathfrak{l} \cap \mathfrak{k}$ is 4. So the dimension of $\mathfrak{u} \cap \mathfrak{k}$ is 1. Since $\mathfrak{u} \cap \mathfrak{k} \cap \mathfrak{h}_1 \subset \mathfrak{u} \cap \mathfrak{k}$ we have equality.

$\mathfrak{u} \cap \mathfrak{s}_1$ is in the roots spaces for roots $e_1 + e_2 + (f_1 - f_2)$ and $e_1 + e_2 - (f_1 - f_2)$. Hence $\mathfrak{l} \cap \mathfrak{h}_1 \cap \mathfrak{k}$ acts on $\mathfrak{u} \cap \mathfrak{s}_1$ by $e_1 + e_2$.

$\mathfrak{l} \cap \mathfrak{h}_1$ acts on the 4 dimensional space $\mathfrak{l} \cap \mathfrak{s}_1$ via the adjoint representation. ■

The representation of $L \cap H_1$ on the symmetric algebra $S((\mathfrak{u} \cap \mathfrak{s}_1) \oplus (\overline{\mathfrak{u}} \cap \mathfrak{s}_1) \oplus (\mathfrak{l} \cap \mathfrak{s}_1))$ is isomorphic to a direct sum of representations $\mu_1^{n_1} \otimes \mu_1^{-m_1} \otimes \mathrm{ad}^{r_1}$ with $n_1, m_1, r_1 \in \mathbb{N}$. These powers of μ_1 will label the constituents in the branching law; it will also be sometimes convenient to think of their differentials in additive notation.

Now it is important to note, that the parameter $\lambda_{H_1} \otimes \mu_1^{n_1}$, $0 \le n_1$ is in the good range [10] and thus the representation on the cohomology in degree $1 = \dim(\mathfrak{u} \cap \mathfrak{k} \cap \mathfrak{h}_1)$ of the complex $L^*_{H_1}$ has composition factors isomorphic to

$$A_{\mathfrak{q} \cap \mathfrak{h}_1}(\lambda_{H_1} \otimes \mu_1^{n_1}),$$

where $0 \le n_1$. In particular $A_{\mathfrak{q} \cap \mathfrak{h}_1}(\lambda_{H_1})$ is an $(\mathfrak{h}_1, K \cap H_1)$-submodule module of the cohomology of $L^*_{H_1}$.

Proposition 4.2. $A_{\mathfrak{q} \cap \mathfrak{h}_1}(\lambda_{H_1})$ *is a composition factor of the restriction of* $A_{\mathfrak{q}}$ *to* $(\mathfrak{h}_1, K \cap H_1)$*.*

Proof. Note that $\dim(\mathfrak{u} \cap \mathfrak{k}) = \dim(\mathfrak{u} \cap \mathfrak{k} \cap \mathfrak{h}_1)$ and that $1 = \dim(\mathfrak{u} \cap \mathfrak{k})$ is the degree in which the complexes defining the representations $A_{\mathfrak{q}}$ respectively $A_{\mathfrak{q} \cap \mathfrak{h}_1}(\lambda_{H_1})$ both have nontrivial cohomology [10]. Considering the complex defining $A_{\mathfrak{q}}$ as a subcomplex of the "large complex" L^* the pullback $\mathbf{pb}^i_{H_1}$ of forms defines a $(\mathfrak{h}_1, K \cap H_1)$-equivariant map

$$A_{\mathfrak{q}} \to \oplus_{n_1=0}^{\infty} A_{\mathfrak{q} \cap \mathfrak{h}_1}(\mu_1^{n_1} \otimes \lambda_{H_1}).$$

Recall the definition of the K-module $\mathcal{R}^s_K(\lambda)$ from V.5.70 in [10]. We have bottom layer maps of $\mathfrak{k}_o$-modules.

$$\mathcal{B}(\lambda) : A_{\mathfrak{q}} \to \mathcal{R}^1_K(\lambda_0)$$

and

$$\mathcal{B}(\lambda_{H_1}) : A_{q\cap\mathfrak{h}_1}(\lambda_{H_1}) \to \mathcal{R}^1_{K\cap H_1}(\lambda_{H_1}),$$

where λ_0 is the trivial character of $L \cap K$. These maps are defined by the inclusion of of complexes and hence of forms. See Theorem V.5.80 and its proof in [10]. The minimal K-types of $A_{\mathfrak{q}}$, respectively K^{H_1}-type of $A_{\mathfrak{q}\cap\mathfrak{h}_1}(\lambda_{H_1})$ are in the bottom layer.

On the other hand we have an inclusion of complexes (the notation in analogy with the case in **4.1**, now for the case where we take $G = K$)

$$\begin{aligned}\mathbf{pb}^i_{H_1\cap K} :\ & \mathrm{Hom}_{K\cap L\cap H_1}(U(\mathfrak{k}), \mathrm{Hom}(\wedge^i(r_G \cap \mathfrak{k} \cap \mathfrak{h}_1), \mathbb{C}^\sharp_{\lambda_{H_1}}))_{K\cap H_1} \\ & \to \mathrm{Hom}_{K\cap L\cap H_1}(U(\mathfrak{k}), \mathrm{Hom}(\wedge^i(r_G \cap \mathfrak{k}), \mathbb{C}^\sharp_{\lambda_0}))_{K\cap H_1}.\end{aligned}$$

But $\mathbb{C}^\sharp_{\lambda_0} = \mathbb{C}^\sharp_{\lambda_{H_1}}$ and $r_G \cap \mathfrak{k} \cap \mathfrak{h}_1 = r_G \cap \mathfrak{k}$ and so using a forgetful functor we may consider

$$\mathrm{Hom}_{K\cap L}(U(\mathfrak{k}), \mathrm{Hom}(\wedge^i(r_G \cap \mathfrak{k}), \mathbb{C}^\sharp_{\lambda_0}))_K$$

as a subspace, respectively subcomplex, of

$$\begin{aligned}&\mathrm{Hom}_{K\cap L\cap H_1}(U(\mathfrak{k}), \mathrm{Hom}(\wedge^i(r_G \cap \mathfrak{k} \cap \mathfrak{h}_1), \mathbb{C}^\sharp_{\lambda_0}))_{K\cap H_1} \\ &\quad = \mathrm{Hom}_{K\cap L\cap H_1}(U(\mathfrak{k} \cap \mathfrak{h}_1), \mathrm{Hom}(\wedge^i(r_G \cap \mathfrak{k} \cap \mathfrak{h}_1), Q_H \otimes \mathbb{C}^\sharp_{\lambda_0}))_{K\cap H_1},\end{aligned}$$

where Q_H is the symmetric algebra of the complement of $\mathfrak{h}_1 \cap \mathfrak{k}$ in $\mathfrak{k}$. Note that $K \cap H_1$ and K is again a symmetric a pair and so we have a bottom-layer map for the representation

$$\mathcal{R}^1_K(\lambda_0) \to \mathcal{R}^1_{K\cap H}(\lambda_{H_1}).$$

Since the representation $\mathcal{R}^1_K(\lambda_0)$ is irreducible restricted to $K \cap H_1$ this map is an isomorphism. ■

Definition. We call $A_{\mathfrak{q}\cap\mathfrak{h}_1}(\lambda_{H_1})$ the minimal H_1-type of $A_{\mathfrak{q}}$.

Theorem 4.3. *The representation $A_{\mathfrak{q}}$ restricted to H_1 is the direct sum of the representations each occuring with multiplicity one, namely*

$$A_{\mathfrak{q}|H_1} = \oplus_{n_1=0}^{\infty} A_{\mathfrak{q}\cap\mathfrak{h}_1}(\mu_1^{n_1} \otimes \lambda_{H_1}).$$

Proof. By the proof of the lemma $A_{\mathfrak{q}\cap\mathfrak{h}_1}(\lambda_{H_1})$ is a submodule of the restriction of $A_{\mathfrak{q}}$ to the symplectic group H_1. Its minimal K^{H_1}-type is also a minimal K-type of $A_{\mathfrak{q}}$ and hence occurs with multiplicity one. Hence $A_{\mathfrak{q}\cap\mathfrak{h}_1}(\lambda_{H_1})$ is a H_1-type of $A_{\mathfrak{q}}$ with multiplicity one.

The minimal K^{H_1}-type of $A_{\mathfrak{q}\cap\mathfrak{h}_1}(\lambda)$ has highest weight $\lambda + 3e_1 + 3e_2$. The roots of $\mathfrak{u} \cap \mathfrak{h}_1 \cap p$ are $2e_1$, $2e_2$. Applying successively the root vectors to the highest weight vector of the minimal K^{H_1}-type of $A_{\mathfrak{q}\cap\mathfrak{h}_1}(\lambda)$ we deduce that $A_{\mathfrak{q}\cap\mathfrak{h}_1}(\lambda)$ contains the K^{H_1}-types with highest weight $((3 + 2r_1)e_1 + (3 + 2r_2)e_1 + \lambda)$, $r_1, r_2 \in \mathbb{N}$. Theorem 8.29 in [10] show that all these K^{H_1}-types have multiplicity one. Fig. 3 shows the K^{H_1}-type multiplicities of $A_{\mathfrak{q}}(\lambda_{H_1})$.

Note that we are here using quite a bit of a priori information about the derived functor modules for the smaller group; on the other hand, the branching problem has essentially been reduced to one for compact groups, K-type by K-type.

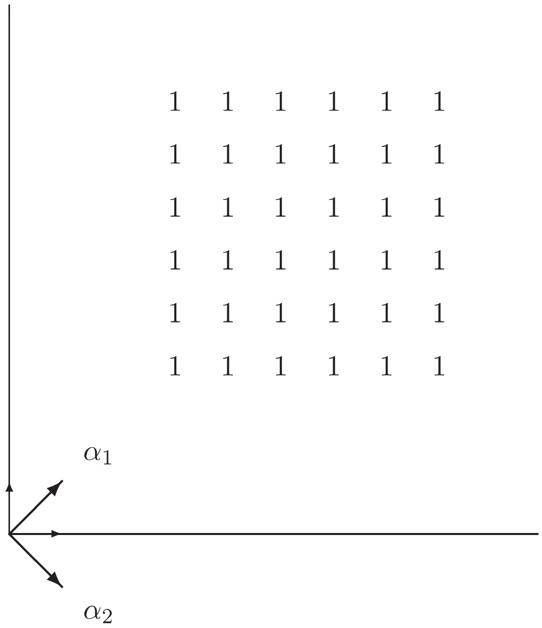

Figure 3.

The Borel subalgebra of $\mathfrak{k}\cap\mathfrak{h}_1$ acts on the one dimensional space $\mathfrak{u}\cap\mathfrak{s}_1$ by a character μ_1 with differential (e_1+e_2). Let $Y\neq 0$ be in $\mathfrak{u}\cap\mathfrak{s}_1$ and $v\neq 0$ a highest weight vector of the minimal K-types of $A_{\mathfrak{q}}$. Then $Y^n\cdot v\neq 0$ is also the highest weight of an K^{H_1}-type of highest weight $(3+n)e_1+(3+n)e_2$ of $A_{\mathfrak{q}}$.

Let $X_k\neq 0$ be in $\mathfrak{u}\cap\mathfrak{k}$. The linear map

$$T_s : U(\mathfrak{g}) \to \wedge^1 r_G \otimes \mathbb{C}^\sharp_{\lambda_0}$$

which maps 1 to $X_k\otimes\mathbb{C}^\sharp_{\lambda_0}$ is non-zero in cohomology and its class $[T_s]$ is the highest weight vector of the minimal K-type. But

$$Y\cdot T_s \quad\in\quad \mathrm{Hom}_{L\cap K\cap H_1}(\mathfrak{s}_1\otimes U(\mathfrak{h}_1), \mathrm{Hom}(\wedge^s r_{H_1}, \mathbb{C}^\sharp_{\lambda_{H_1}})),$$

so may consider

$$Y\cdot T_s \quad\in\quad \mathrm{Hom}_{L\cap K\cap H_1}(U(\mathfrak{h}_1), \mathrm{Hom}(\wedge^i r_{H_1}, \mathfrak{s}_1^*\otimes\mathbb{C}^\sharp_{\lambda_{H_1}}))_{K\cap H_1}.$$

Hence $0\neq[Y\cdot T_s]=Y\cdot[T_s]\in A_{\mathfrak{q}\cap\mathfrak{h}_1}(\mu_1\otimes\lambda_{H_1})$ and thus $A_{\mathfrak{q}\cap\mathfrak{h}_1}(\lambda_{H_1}\otimes\mu_1)$ is an H_1-type of $A_{\mathfrak{q}}$. The same argument shows that $A_{\mathfrak{q}\cap\mathfrak{h}_1}(\lambda_{H_1}\otimes\mu_1^n),\ n\in\mathbb{N}$, is a H_1-type of $A_{\mathfrak{q}}$.

Now every K-type with highest weight (n,n) has multiplicity $n-2$ and is contained in exactly $n-2$ composition factors. The multiplicity computations in Section 3 now show that every composition factor is equal to $A_{\mathfrak{q}\cap\mathfrak{h}_1}(\lambda_{H_1}\otimes\mu_1^n)$ for some n. See Fig. 4. ■

Remark. Another proof of Theorem 4.3 can be obtained using Proposition 5.1 and the ideas of **6.1**.

By Proposition 8.11 in [10] for any $(\mathfrak{g},K)$-module X we have

$$\mathrm{Hom}_{\mathfrak{g},K}(X,A_{\mathfrak{q}})=\mathrm{Hom}_{\mathfrak{l},K\cap L}(H_s(\mathfrak{u},X),\mathbb{C}^\sharp),$$

where $H_s(\mathfrak{u},X)$ is the Lie algebra homology as defined in [10] and $s=\dim(\mathfrak{u}\cap\mathfrak{k})$. Thus we have a "Blattner type formula" for the H_1-types of $A_{\mathfrak{q}}$.

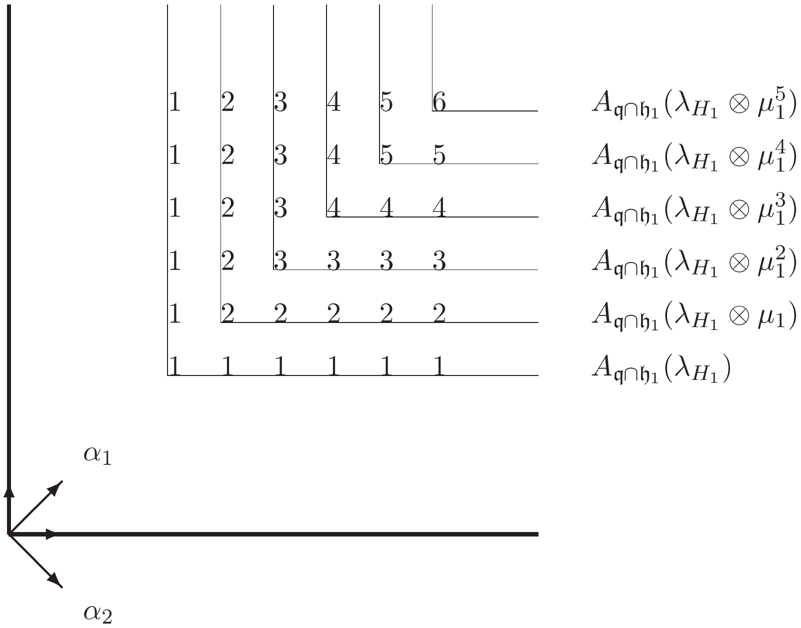

Figure 4.

Corollary 4.4. *Let V be an irreducible $(\mathfrak{h}_1, K^{H_1})$-module. Then*

$$\dim \operatorname{Hom}_{\mathfrak{h}_1, K\cap H_1}(V, A_{\mathfrak{q}}) = \sum_i \dim \operatorname{Hom}_{(\mathfrak{l}\cap\mathfrak{h}_1, K^{H_1}\cap L)}(H_1(\mathfrak{u}\cap\mathfrak{h}_1, V), S^i(\mathfrak{u}\cap\mathfrak{s}_1)\otimes\mathbb{C}^{\sharp}_{H_1}).$$

Remark. The $H_1(\cdot,\cdot)$ on the right refers to homology in degree one. The maximal Abelian split subalgebra $\mathfrak{a}_1$ in $\mathfrak{l}\cap\mathfrak{h}_1$ are the diagonal matrices. So the parabolic subgroup of the Langlands parameter of the H_1-types of $A_{\mathfrak{q}}$ is the so called "mirabolic", i.e. the maximal parabolic subgroup with Abelian nilradical. The other parts of Langlands parameter can de determined using the algorithm in [10].

We consider in Theorem 4.3 the restriction of a "small" representation $A_{\mathfrak{q}}$ of Spin(3,3) to Spin(2,3) similar to the restriction of "small" discrete series representations of $SO(n+1,n)$ to $SO(n,n)$ considered by B. Gross and N. Wallach in [4]. It would be interesting to see if their techniques could be adapted to the problem discussed in the paper.

5 Restriction of $A_{\mathfrak{q}}$ to the group H_1'

In this section we describe H_1'-types of $A_{\mathfrak{q}}$ using the same techniques as in the previous section.

5.1. For H_1' we consider the complex

$$(\operatorname{Hom}_{L\cap K\cap H_1'}(U(\mathfrak{g}), \operatorname{Hom}(\wedge^i r_{H_1'}, \mathbb{C}^{\sharp}_{\lambda_{H_1'}}))_{K\cap H_1'}, d_{H_1'})$$

and the map

$$\begin{aligned}\mathbf{pb}^i_{H_1'}:\quad & \operatorname{Hom}_{L\cap K\cap H_1'}(U(\mathfrak{h}_1'), \operatorname{Hom}(\wedge^i r_{H_1'}, Q^*\otimes\mathbb{C}^{\sharp}_{\lambda_H})_{K\cap H_1'})_{K\cap H_1'}\\ & \to \operatorname{Hom}_{K\cap L\cap H_1'}(U(\mathfrak{g}), \operatorname{Hom}(\wedge^i r_G, \mathbb{C}^{\sharp}_{\lambda}))_{K\cap H_1'}.\end{aligned}$$

We write $\mathfrak{g} = \mathfrak{h}_1' \oplus \mathfrak{s}_1'$. The intersection $\mathfrak{u}\cap\mathfrak{s}_1'$ is 2-dimensional and the representation of the group of $L\cap H_1'$ on $\mathfrak{u}\cap\mathfrak{s}_1'$ is reducible and thus a sum of 2 one dimensional representations

$\chi_1 \oplus \chi_2$. The weights of these characters are $2e_1$ and $2e_2$. So the symmetric algebra $S(\mathfrak{u}\cap\mathfrak{s}'_1)$ is a direct sum of one dimensional representations of $L\cap H'_1$ with weights $2m_1e_1+2m_2e_2$.

In the cohomology in degree 1 of the complex $L^*_{H'_1}$ we have composition factors

$$A_{\mathfrak{q}\cap\mathfrak{h}'_1}(\lambda_{H'_1}\otimes\chi_1^{n_1}\otimes\chi_2^{n_2})$$

with $0\le n_1,n_2$. In particular $A_{\mathfrak{q}\cap\mathfrak{h}'_1}(\lambda_{H'_1})$ is an $(\mathfrak{h}'_1, K\cap H'_1)$-submodule module of the cohomology in degree 1.

Proposition 5.1. *$A_{\mathfrak{q}\cap\mathfrak{h}'_1}(\lambda_{H'_1})$ is a composition factor of the restriction of $A_{\mathfrak{q}}$ to $(\mathfrak{h}'_1, H'_1\cap K)$.*

Proof. The maximal compact subgroups of H_1 and H'_1 are identical. Thus $\dim\mathfrak{u}\cap\mathfrak{k}\cap\mathfrak{h}_1 = \dim\mathfrak{u}\cap\mathfrak{k}\cap\mathfrak{h}_1 = 1$ and the minimal K-type is irreducible under restriction to $K^{H'_1}$. Thus the same argument as in Lemma 4.1 completes the proof. ∎

Definition. We call $A_{\mathfrak{q}\cap\mathfrak{h}'_1}(\lambda_{H'_1})$ the minimal H'_1 type of $A_{\mathfrak{q}}$.

Theorem 5.2. *The representation $A_{\mathfrak{q}}$ restricted to H'_1 is the direct sum of the representations each occurring with multiplicity one, namely*

$$A_{\mathfrak{q}|H'_1} = \oplus_{n_1,n_2=0}^{\infty} A_{\mathfrak{q}\cap\mathfrak{h}'_1}(\lambda_{H'_1}\otimes\chi_1^{n_1}\otimes\chi_2^{n_2}).$$

Their minimal $K\cap H'_1$ -types have highest weights $(3+m_1+m_2+i, 3+m_1+m_2-i)$, $-m_2\le i\le m_2$.

Proof. The proof is the same in the previous section where we proved that the representations $A_{\mathfrak{q}\cap\mathfrak{h}'_1}(\lambda_{H'_1}\otimes\chi_1^{n_1}\otimes\chi_2^{n_2})$, $n_1,n_2\in\mathbb{N}$ appear in the restriction of the $A_{\mathfrak{q}}$ to H'_1

The $K\cap H'_1$ -types of all unitary representations of $GL(2,\mathbb{C})$ have multiplicity one. If the minimal $K\cap H'_1$-type has highest weight $l_1e_1+l_2e_2-2$, then the highest weights of the other $K\cap H'_1$-types are $(l_1+j)e_1+(l_2+j)e_2$. Multiplicity considerations of $K\cap H'_1$-types of $A_{\mathfrak{q}}$ conclude the proof. ∎

Remark. Using Proposition 4.2, **4.1** and the ideas of **6.1** we can obtain another proof of this theorem.

Fig. 5 shows the decomposition into irreducible representations. The highest weights of the $K\cap H'_1$-types of a composition factors lie on the lines. For each highest weight there is exactly one composition factor which has a $K\cap H'_1$-type with this weight as a minimal $K\cap H'_1$-type.

We have again a “Blattner-type formula” for the H'_1 types.

Corollary 5.3. *Let V be an irreducible $(\mathfrak{h}'_1,(K\cap H'_1))$-module, then*

$$\dim\operatorname{Hom}_{\mathfrak{h}'_1,K\cap H'_1}(V,A_{\mathfrak{q}}) = \sum_i \dim\operatorname{Hom}_{(\mathfrak{l}\cap\mathfrak{h}'_1),K\cap H'_1\cap L}(H_1(\mathfrak{u}\cap\mathfrak{h}'_1,V), S^i(\mathfrak{u}\cap\mathfrak{s}'_1)\otimes\mathbb{C}^{\sharp}_{H'_1}).$$

5.2. All the H'_1-types $A_{\mathfrak{q}\cap\mathfrak{h}'_1}(\lambda_{H'_1}\otimes\chi_1^{n_1}\otimes\chi_2^{n_2})$ of $A_{\mathfrak{q}}$ are simply unitarily induced principal series representations of $GL(2,\mathbb{C})$.

6 Pseudo dual pairs

6.1. Suppose now that G is a reductive connected Lie group with maximal compact subgroup K, Cartan involution θ of $\mathfrak{g}$. For an involution

$$\tau : G\to G$$

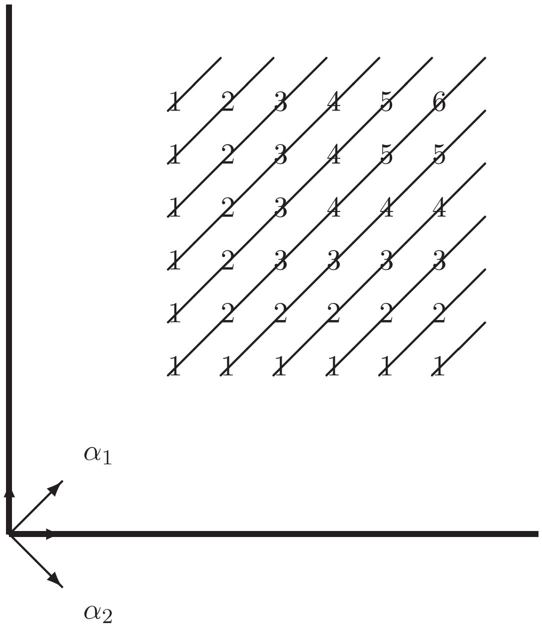

Figure 5.

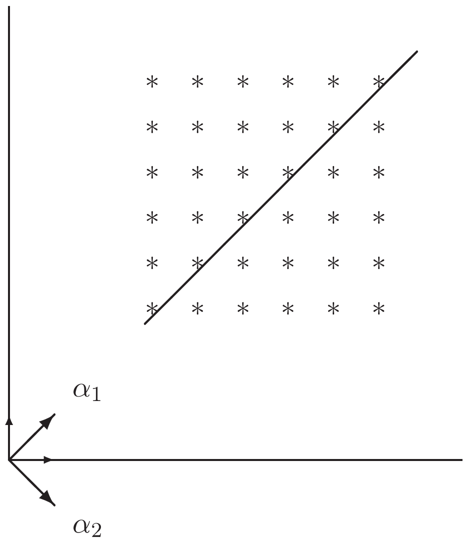

Figure 6.

commuting with θ we define

$$\tau' = \tau \circ \theta \qquad \text{and} \qquad H = G^{\tau}, \qquad H' = G^{\tau'}.$$

Definition. We call H and H' a *pseudo dual pair.*

Since θ, τ and τ' commute

Lemma 6.1. *Suppose that H and H' are a pseudo dual pair in G. Then*

1) $K \cap H = K \cap H'$;
2) *we have* $\mathfrak{p} = \mathfrak{p}^{\tau} \oplus \mathfrak{p}^{\tau'}$;
3) $U(\mathfrak{g}) \cong S(\mathfrak{p}^{\tau'})S(\mathfrak{p}^{\tau})U(\mathfrak{k})$ *and so we can write every element in $U(\mathfrak{g})$ as a linear combination of terms of the form $\omega_2\omega_1\omega_k$ where $\omega_k \in U(\mathfrak{k})$, $\omega_1 \in U(\mathfrak{h})$ and $\omega_2 \in U(\mathfrak{h}')$.*

Suppose now that π be an irreducible unitary $(\mathfrak{g}, K)$-module. Then π is $K \cap H$-admissible if and only if it is $K \cap H'$-admissible and by Theorem 4.2 of [12] its restriction to $\mathfrak{h}$ is a direct sum of irreducible $(\mathfrak{h}, K \cap H)$-modules if and only if π is a direct sum of irreducible $(\mathfrak{h}', K \cap H')$-modules.

Suppose that π is $H \cap K$ admissible and that $V_k \subset \pi$ is a minimal K-type of π. If π_o is a $(\mathfrak{h}, K \cap H)$-module, which occurs in the restriction of π to $\mathfrak{h}$ then

$$\pi_o \cap U(\mathfrak{h}')V_k \neq 0.$$

6.2. These observations allow us to better understand the restriction of $A_{\mathfrak{q}}$ to the pseudo dual pair $H_1 = Sp(2, \mathbb{R})$, $H_1' = GL(2, \mathbb{C})$ in $SL(4, \mathbb{R})$. The minimal K-type V_K of $A_{\mathfrak{q}}$ has highest weight $3\alpha_1$ and is also irreducible under $K \cap H_1 = K \cap H_1'$. The $(\mathfrak{h}_1', K \cap H_1')$-submodule generated by V_K is the minimal H_1' type and is isomorphic to a spherical principal series representation. We draw a diagram of its $K \cap H_1'$-types using the same conventions as in the previous sections. The $K \cap H_1'$-types are on the black line in Fig. 6.

Each $K \cap H_1 = K \cap H_1'$-type of this representation is the minimal $K \cap H_1$-type of an irreducible $(\mathfrak{h}_1, K \cap H_1)$-module as indicated in Fig. 7.

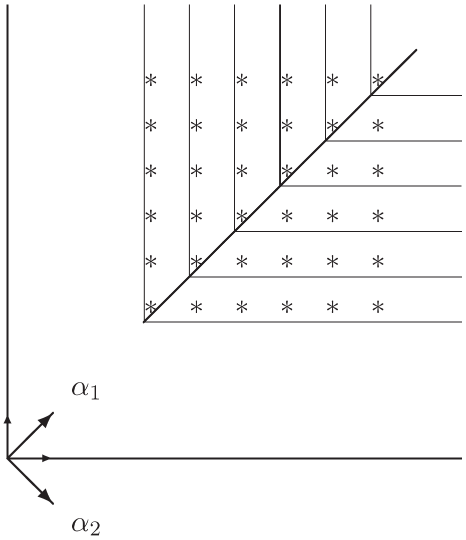

Figure 7.

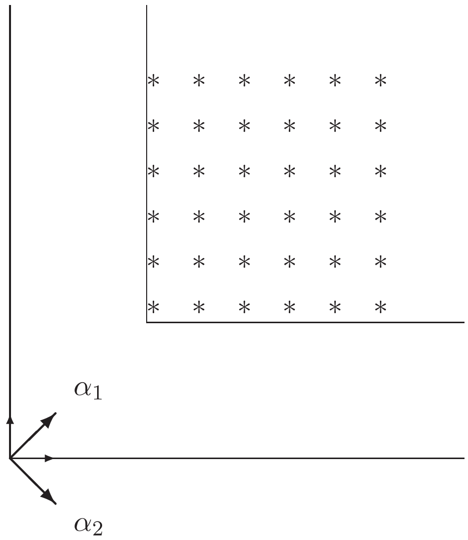

Figure 8.

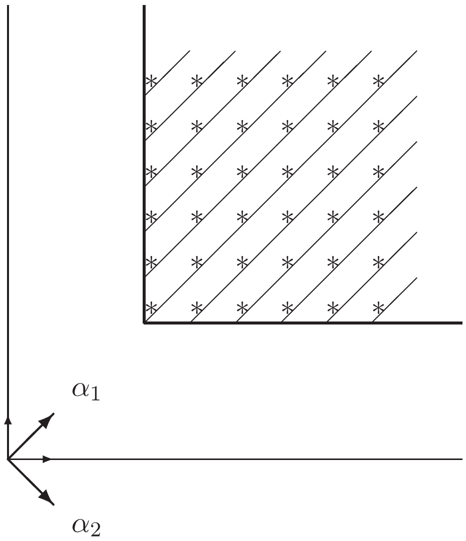

Figure 9.

The $K \cap H_1$ types of the minimal $(\mathfrak{h}_1, K \cap H_1)$-module generated by the minimal K-type have multiplicity one and are indicated by the dots in Fig. 8.

Each of the $K \cap H_1$-types of this $(\mathfrak{h}_1, K \cap H_1)$-module is the minimal $K \cap H_1 = K \cap H_1'$-type of a $(\mathfrak{h}_1', K \cap H_1')$-module in the restriction of $A_\mathfrak{q}$ to H_1', as illustrated in Fig. 9.

7 More branching

In this section we sketch the restriction to H_1 and H_1' of a representation $A_\mathfrak{q}(\lambda_1)$ with a parameter λ which is no longer in the weakly fair range and hence the representation is no longer irreducible and has a composition series of length 2. We indicate a procedure to use our previous techniques to compute the restriction of both composition factors of $A_\mathfrak{q}(\lambda_1)$ to H_1 and H_1'. Using the local isomorphisms

$$SL(4,\mathbb{R}) \sim SO(3,3), \quad Sp(2,\mathbb{R}) \sim SO(2,3), \quad SL(4,\mathbb{R}) \cap GL(2,\mathbb{C}) \sim SO(2) \times SO(1,3)$$

we obtain a different proof of a result by T. Kobayashi and B. Ørsted of the branching of the minimal representation π of $SO(3,3)$ to $SO(3,1) \times SO(2)$ and to $SO(3,2)$ [17].

7.1. We use the notation introduced in **3.1**. Using the conventions on page 586 in [10] we denote the character of L by $\lambda = m(e_1+e_2+e_3+e_4)$. With this parametrization the representation $A_{\mathfrak{q}}(\lambda)$ is irreducible and unitary for $m > -3$ (see page 588 in [10]) and $A_{\mathfrak{q}}$ corresponds to the parameter $m = 0$. The representation $A_{\mathfrak{q}}(\lambda)$ for $m = 3$ is nonzero.

To simplify the notation we denote the representation $A_{\mathfrak{q}}(\lambda)$ for $\lambda = m(e_1 + e_2 + e_3 + e_4)$ by $A_{\mathfrak{q}}(m)$.

We consider now the representation $A_{\mathfrak{q}}(-3)$. This representation is nonzero and has a trivial K-type. It is not irreducible, but has 2 composition factors. One composition factor is $A_{\mathfrak{q}}(-2)$. The other composition factor is a unitarily induced representation π_0 from one dimensional representation of a maximal parabolic subgroup. It is a ladder representation and the highest weights of its K-types are multiples of e_1. See [17] for $SO(3,3)$ or [1] for $GL(4,\mathbb{R})$.

The restriction of $A_{\mathfrak{q}}(-3)$ to $H_1 \cap K$ are $m_1(e_1 + e_2) + 2m_2e_1$, $m_1, m_2 \in \mathbb{N}$.

7.2. The same arguments as in Sections 4 and 5. allows us to compute the restriction of $A_{\mathfrak{q}}(-3)$ to H_1 and H_1'. In this case the minimal H_1 type is a spherical representation $\pi_0^{H_1}$ and comparing the multiplicities of $K \cap H_1$-types of $\pi_0^{H_1}$ and of π_0 we deduce that the restriction of π_0 to H^1 is irreducible and equal to $\pi_0^{H_1}$. A similar argument show that the restriction to H_1' is direct sum of principal series representations of $GL(2,\mathbb{C})$.

Using the local isomorphism $SL(4,\mathbb{R})$ and $SO(3,3)$ we obtain a new proof of the branching of π_0 determined by B. Ørsted and T. Kobayashi in [17].

8 A conjecture

8.1. The examples in the previous section and the calculations in [14] support the following conjecture: Let H be the connected fixpoint set of an involution σ. We write again $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$. Let $A_{\mathfrak{q}}(\lambda)$ be a representation, which is $K \cap H$-admissible and thus decomposes discretely, when restricted to H. Suppose $\mathfrak{q}$, $\mathfrak{q}^h$ are defined by $x_o \in T^H$. Since $A_{\mathfrak{q}} = A_{\mathfrak{p}}$ if $\mathfrak{q}$ and $\mathfrak{p}$ are conjugate under the compact Weyl group W_K we use the following

Definition. Let $y_o \in T^H$ and let $\mathfrak{p}$, $\mathfrak{p}^H$ be well aligned parabolic subalgebras defined by y_o. We call the well aligned parabolic subalgebras $\mathfrak{p}$, $\mathfrak{p}^H$ *related* to $\mathfrak{q}$, $\mathfrak{q}^H$, if x_o and y_o are conjugate by an element in the compact Weyl group W_K of K with respect to T.

If x_o and y_o are not conjugate by an element in the Weyl group $W_{K\cap H}$ of (K^H, T^H) then the parabolic subalgebras $\mathfrak{q}^H$, $\mathfrak{p}^H$ of H are not conjugate in H and thus we have up to conjugacy at most $W_K/W_{H\cap K}$ different *pairs of well aligned pairs of θ-stable invariant parabolic subalgebras which are related to* $\mathfrak{q}$, $\mathfrak{q}^H$. If $G = SL(4,\mathbb{R})$, $H = H_1$ and $(\mathfrak{q}, \mathfrak{q} \cap \mathfrak{h}_1)$ is the pair of well aligned parabolic subalgebras defined by $x_0 = Q_2$, there there are at most 2 related pairs of well aligned parabolic subalgebras.

We expect the following Blattner-type formula to hold for the restriction to H:

Conjecture. *There exists a pair $\mathfrak{p}$, $\mathfrak{p}^H$ of well aligned θ-stable parabolic subgroups related to $\mathfrak{q}$, $\mathfrak{q}^H$ so that every H-type V of $A_{\mathfrak{q}}$ is of the form $A_{\mathfrak{p}^H}(\mu)$ for a character μ of L^H and that*

$$\dim \operatorname{Hom}_{\mathfrak{h},K^H}(V, A_{\mathfrak{p}}) = \sum_i \sum_j (-1)^{s-j} \dim \operatorname{Hom}_{L\cap H}(H_j(\mathfrak{u} \cap \mathfrak{h}, V), S^i(\mathfrak{u} \cap \mathfrak{s}) \otimes \mathbb{C}_{\lambda_H}).$$

Remark. Some of the characters μ in this formula may be out of the fair range as defined in [10] and hence reducible.

If H is the maximal compact subgroup K, then $|W_K/W_{H\cap K}| = 1$, all related pairs of well aligned parabolic subalgebra are conjugate to $\mathfrak{q}$, $\mathfrak{q}^H$ and hence we get the usual Blattner formula (5.108b on page 376 in [10]).

In the example discussed in this paper, $G = SL(4,\mathbb{R})$, the representation $A_{\mathfrak{q}}$ and H the symplectic group considered in Section 4, all related pairs of well aligned parabolic subalgebras are conjugate to $\mathfrak{q}$, $\mathfrak{q}_{\mathfrak{h}_1}$ and thus we obtain the Blattner type formula in Corollary 4.4.

9 An application to automorphic representations

We use here our results to give different constructions of some known automorphic representations of $Sp(2,\mathbb{R})$ and $GL(2,\mathbb{C})$. We first explain the ideas in **7.1** in a more general setting. Again we may consider restrictions, this time in the obvious way of restricting functions on locally symmetric spaces to locally symmetric subspaces.

9.1. Assume first that G is a semisimple matrix group and Γ an arithmetic subgroup, H a semisimple subgroup of G. Then $\Gamma_H = \Gamma \cap H$ is an arithmetic subgroup of H. Let $V_\pi \subset L^2(G/\Gamma)$ be an irreducible $(\mathfrak{g}, K)$-submodule of $L^2(G/\Gamma)$. If $f \in V_\pi$ then f is a C^∞-function and so we define f_H as the restriction of f to H/Γ_H.

Lemma 9.1. *The map*

$$\begin{array}{rccc} \mathrm{RES}_H : & V_\pi & \to & C^\infty(H/\Gamma_H) \\ & f & \to & f_H \end{array}$$

is an $(\mathfrak{h}, K\cap H)$*-map.*

Proof. Let $h_t = \exp(tX_H)$, $h_o \in H$. Then

$$\rho(X_H)f(h_o) = \frac{d}{dt}f(h_f^{-1}h_0)_{t=0} = \frac{d}{dt}f_H(h_t^{-1}h_0)_{t=0} = \rho(X_H)f_H(h_o). \qquad \blacksquare$$

Suppose that the irreducible unitary $(\mathfrak{g}, K)$-module π is a submodule of $L^2(G/\Gamma)$ and that its restriction to H is a direct sum of unitary irreducible representations.

Proposition 9.2. *Under the above assumptions* $\mathrm{RES}_H(\pi)$ *is nonzero and its image is contained in the automorphic functions on* H/Γ_H*.*

Proof. Let f_H be a function in $\mathrm{RES}_H(\pi)$. Then by Section 2 it is $K\cap H$-finite and we may assume that it is an eigenfunction of the center of $U(\mathfrak{h})$.

Let $||g||^2 = \mathrm{tr}(g^*g)$. Since $\sup_{g\in G}|f(g)|\ ||g||^{-r} < \infty$, the same is true for f_H and so f_H is an automorphic function on H/Γ_H.

The functions in the $(\mathfrak{g}, K)$-module $\pi \subset L^2(G/\Gamma)$ are eigenfunctions of the center of the enveloping algebra $U(\mathfrak{g})$ and are K-finite, hence analytic. Thus if f is a K-finite function in $\pi \subset L^2(G/\Gamma)$ then there exists $W \in U(\mathfrak{g})$ so that $Wf(e) \neq 0$. Hence $\mathrm{RES}_H(Wf) \neq 0$. $\blacksquare$

Instead of restricting the automorphic function f to the orbit of e/Γ under H we may also consider the restriction to an orbit of γ/Γ, for rational γ. Since the rational elements are dense at least one of the restrictions is not zero. So following Oda we consider the restriction correspondence for functions on G/Γ to functions on $\prod_g H/H\cap g\Gamma g^{-1}$. For rational g the intersection $\Gamma \cap g\Gamma g^{-1}$ contains an arithmetic group Γ' and $\Gamma' \cap H$ is an arithmetic subgroup. For more details see for example page 55 in [2].

9.2. Now we assume that $G = GL(4,\mathbb{R})$ and that $\Gamma \subset GL(4,\mathbb{Z})$ is a congruence subgroup. The groups $\Gamma_1 = \Gamma \cap H_1$ and $\Gamma_1' = \Gamma \cap H_1'$ are arithmetic subgroups of $Sp(4,\mathbb{R})$, respectively $GL(2,\mathbb{C})$.

Recall the definition of the $(\mathfrak{g}, K)$-module $\mathcal{A}_\mathfrak{q}$ from **3.1**. It is a submodule of $L^2(Z\backslash G/\Gamma)$ for Γ small enough where Z the connected component of the center of $GL(4,\mathbb{R})$. We will for the remainder of this sections consider it as an automorphic representation in the residual spectrum [23]. Then $\mathrm{RES}_{H_1}(\mathcal{A}_\mathfrak{q})$ and $\mathrm{RES}_{H_1'}(\mathcal{A}_\mathfrak{q})$ are nonzero. Its discrete summands are contained in the space of automorphic forms.

Theorem 9.3. *The discrete summands of the two representations* $\mathrm{RES}_{H_1}(\mathcal{A}_\mathfrak{q})$ *respectively* $\mathrm{RES}_{H_1'}(\mathcal{A}_\mathfrak{q})$ *are subrepresentations of the discrete spectrum of* $L^2(H_1/\Gamma_{H_1})$*, respectively* $L^2(H_1'/\Gamma'_{H_1})$*.*

Proof. All the functions in $\mathcal{A}_\mathfrak{q}$ decay rapidly at the cusps. Since the cusps of H_1/Γ_{H_1} are contained in the cusps of G/Γ this is true for the functions in $\mathrm{RES}_{H_1}(\mathcal{A}_\mathfrak{q})$. Thus they are also contained in the discrete spectrum. ■

For $Sp(2,\mathbb{R})$ the representations constructed in the previous theorem were first described by H. Kim (see [9] and [22]). For $GL(2,\mathbb{C})$ we obtain the stronger result

Theorem 9.4. *The representations in the discrete spectrum of* $\mathrm{RES}_{H_1'}(\mathcal{A}_\mathfrak{q})$ *are in the cuspidal spectrum of* $L^2(H_1'/\Gamma_{H_1'})$*.*

Proof. By **5.2** the representations in the discrete spectrum of $\mathrm{RES}_{H_1'}(\mathcal{A}_\mathfrak{q})$ are unitarily induced principal series representations and so by a result of Wallach they are in fact cuspidal representations. ■

The embedding of $H_1' = GL(2,\mathbb{C})$ into $SL(4,\mathbb{R})$ is defined as follows: Write $g = A + iB$ with real matrices A, B. Then

$$g \to \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

Thus $\Gamma_{H_1'}$ is isomorphic to a congruence subgroup of $GL(2, Z[i])$.

Since all the representations in the discrete spectrum of the restriction of $A_\mathfrak{q}$ do have nontrivial $(\mathfrak{h}, K^H)$-cohomology with respect to some irreducible finite dimensional nontrivial representation F we obtained the well known result [5, 20].

Corollary 9.5. *There exists a congruence subgroup* $\Gamma \subset GL(2, Z[i])$ *and a finite dimensional non-trivial representation* F *of* $GL(2,\mathbb{C})$ *so that*

$$H^i(\Gamma, F) \neq 0 \qquad \textit{for} \quad i = 1, 2.$$

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