Towards Unifying Structures in Higher Spin Gauge Symmetry*

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Abstract. This article is expository in nature, outlining some of the many still incompletely understood features of higher spin field theory. We are mainly considering higher spin gauge fields in their own right as free-standing theoretical constructs and not circumstances where they occur as part of another system. Considering the problem of introducing interactions among higher spin gauge fields, there has historically been two broad avenues of approach. One approach entails gauging a non-Abelian global symmetry algebra, in the process making it local. The other approach entails deforming an already local but Abelian gauge algebra, in the process making it non-Abelian. In cases where both avenues have been explored, such as for spin 1 and 2 gauge fields, the results agree (barring conceptual and technical issues) with Yang–Mills theory and Einstein gravity. In the case of an infinite tower of higher spin gauge fields, the first approach has been thoroughly developed and explored by M. Vasiliev, whereas the second approach, after having lain dormant for a long time, has received new attention by several authors lately. In the present paper we briefly review some aspects of the history of higher spin gauge fields as a backdrop to an attempt at comparing the gauging vs. deforming approaches. A common unifying structure of strongly homotopy Lie algebras underlying both approaches will be discussed. The modern deformation approach, using BRST-BV methods, will be described as far as it is developed at the present time. The first steps of a formulation in the categorical language of operads will be outlined. A few aspects of the subject that seems not to have been thoroughly investigated are pointed out.

Key words: higher spin; gauge field theory; BRST-BV methods

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1 Introduction

Back in 1978, Fang and Fronsdal, in a paper discussing the deformation theoretic approach to deriving a self-interacting massless spin-2 field theory, proposed what they called a "generalized Gupta program" to the effect of deriving self-interactions for higher spin gauge fields [1]. Fronsdal apparently soon realized that such a program was not likely to succeed for any single spin s. In a conference proceedings from 1979 he suggested that, what we now refer to as an *infinite tower* of higher spin gauge fields, is needed. Indeed he wrote [2]: "*For spins exceeding 2 it would seem to be very difficult to find a non-Abelian gauge algebra without including all spins or at least all integer spins. Thus the question calls for a single, unifying gauge algebra for all integer spins.*"

That something like this could be the case can surmised from an unsophisticated generalization from spin 1 and 2. For spin 1, commuting two "gauge generators" $\xi_a T^a$ and $\eta_a T^a$ yields

*This paper is a contribution to the Proceedings of the Seventh International Conference "Symmetry in Nonlinear Mathematical Physics" (June 24–30, 2007, Kyiv, Ukraine). The full collection is available at http://www.emis.de/journals/SIGMA/symmetry2007.html a new transformation of the same form. Likewise, by analogy, for spin 2, commuting $\xi^\mu\partial_\mu$ and $\eta^\mu\partial_\mu$, yields an object of the same form. The obvious generalization to spin 3 would be commuting objects of the form $\xi_a^{\mu\nu}T^a\partial_\mu\partial_\nu$. However, this operation does not close [3]. In the language of deformation theory, we discover an obstruction. The form of the non-closure terms in the commutator suggests the introduction of spin 5 gauge transformations [4]. The need for a "single unifying gauge algebra for all integer spins" can therefore be discerned.

Furthermore, the step to thinking in terms of infinite component objects Φ in some way maintaining fields of all spin, is now short. In the paper referred to above, Fronsdal outlined one such approach which will be described below.

Thus, adding in the fact that the massless representations of the Poincaré group can be realized on symmetric tensor fields (possibly modulo certain traces), these considerations together strongly hint at some kind of expansion

$$\Phi = \phi B + \phi^\mu B_\mu + \phi^{\mu\nu} B_{\mu\nu} + \phi^{\mu\nu\rho} B_{\mu\nu\rho} + \cdots ,$$

where the set $\{B_{\mu_1\mu_2\ldots\mu_s}\}_{s=0}^\infty = \{B_{(s)}\}_{s=0}^\infty$ can be thought of as an infinite basis or an infinite set of generators. Many such proposals can be found in the literature, indeed, this simple idea seems to be reinvented ever and anon. The question arises as to whether the B's obey an algebra, for instance

$$[B_{(s)}, B_{(t)}] = \sum_{i,j}^{s,t} c^{ij} B_{(i,j)}$$

in a vague, but hopefully suggestive notation, or if they simply commute. Of course, higher order brackets might occur, or be necessary, at least the Jacobiator must be considered. This opens up the whole field of abstract algebra.

Clearly, without guiding principles, we're groping about in the dark. However, after thirty years of groping about by many authors, the territory is glowing feebly. The purpose of the present paper is certainly not to attempt a full review, rather we will try to further illuminate some parts of the terrain. Lacking guiding principles, our work will very much be playing around with equations.

Since in my opinion, not very much is gained by working in higher dimensions, I will work in four spacetime dimensions unless otherwise stated.

2 Higher spin bases and generators

In this section, without any claims of being exhaustive, we will look at a few examples of higher spin bases/generators. Perhaps the simplest one, and certainly one that obviously might come to mind upon trying to generalize spin 1 and 2, is a derivative basis already alluded to in the introduction.

2.1 Back to zero

When two infinitesimal transformations δ_ξ and δ_η in some space are performed consecutively, the result is in general different depending on the order in which the transformations are done. This is one reason why it is interesting to study the commutator $[\delta_\xi, \delta_\eta]$ of the transformations. This could be anything, but it seems that the interesting cases arise when this difference can itself be expressed as a new transformation of the same type as the ones that are commuted. In any case, once the commutator is singled out as an interesting object of study, the Jacobi identities are forced upon the theory. As they are simply syntactical consequences of writing out $[[\delta_{\xi_1}, \delta_{\xi_2}], \delta_{\xi_3}] + \text{cyc. perm.}$ and assuming that the infinitesimal operations δ_ξ are associative, there is no way to get around them. Thus the semantics of the theory must obey them, i.e. whatever results from explicitly calculating the commutator brackets, it must obey the Jacobi identities.

In general, brackets $[\cdot,\cdot]$ are used as a notation for operations that are not commutators defined in terms of an underlying associative multiplication. In such cases the Jacobiator $[[\cdot,\cdot],\cdot]$ + cyc. perm. might only be zero up to higher order brackets, or *homotopies*. In any case, independent axioms are needed for the Jacobiator.

2.2 Derivative basis

Even on the weak inductive base step of comparing the sequence of natural numbers $s = 1, 2, 3, \ldots$ to the known sequence of spin 1 and 2 gauge parameters ξ^a, ξ^μ, it could have been guessed that the higher spin continuation of this sequence should be something like $\xi^{a\mu_1\mu_2}$, $\xi^{\mu_1\mu_2\mu_3}, \ldots$ with some internal index $a \in \{1, \ldots, n\}$ for odd spin and $s-1$ spacetime indices μ_i for spin s. The construction in 1983 of the cubic covariant vertex in Minkowski spacetime [5] for spin 3 and the light-front arbitrary s cubic vertices [6] would have corroborated such a guess.

Now let us see what, if anything, can be surmised from taking a derivative basis seriously. To begin with, assume that a spin 3 gauge generator is actually given by an object of the form $\xi_a^{\mu\nu}T^a\partial_\mu\partial_\nu$. Commuting two of these yield

$$[\xi_{1a}^{\mu\nu}T^a\partial_\mu\partial_\nu, \xi_{2b}^{\rho\sigma}T^b\partial_\rho\partial_\sigma] = \xi_{1a}^{\mu\nu}\xi_{2b}^{\rho\sigma}[T^a,T^b]\partial_\mu\partial_\nu\partial_\rho\partial_\sigma + 2(\xi_{1a}^{\mu\nu}\partial_\mu\xi_{2b}^{\rho\sigma} - \xi_{2a}^{\mu\nu}\partial_\mu\xi_{1b}^{\rho\sigma})T^aT^b\partial_\nu\partial_\rho\partial_\sigma + (\xi_{1a}^{\mu\nu}\partial_\mu\partial_\nu\xi_{2b}^{\rho\sigma} - \xi_{2a}^{\mu\nu}\partial_\mu\partial_\nu\xi_{1b}^{\rho\sigma})T^aT^b\partial_\rho\partial_\sigma. \tag{2.1}$$

The first term is easily interpreted as a spin 5 transformation by taking $[T_a, T_b] = f_{ab}{}^cT_c$ as we would be prejudiced to require of the matrices T. However, then the last term becomes problematic since it clearly must be a new spin 3 transformation. It the seems (provisionally) that we need a weaker set of equations for products of matrices, namely

$$T_aT_b = g_{ab}{}^cT_c, \tag{2.2}$$

with no conditions on the coefficients $g_{ab}{}^c$.

Next, the second term of (2.1) which corresponds to a spin 4 transformation, implies that also the even higher spin ($s > 2$) fields must carry internal indices. However, the light-front cubic interaction terms for even higher spin require no anti-symmetric internal indexing, indeed in that case the interactions would vanish. On the other hand, commuting two spin 4 generators of the form $\xi^{\mu\nu\rho}$, yield spin 3 generators without internal matrices T.

These considerations show that in order to make a simpleminded scheme like the one described here work, every higher spin field must be expanded over a set of matrices T with a product satisfying equation (2.2) as well as having a component along a unit matrix E, or

$$\phi = \varphi^aT_a + \varphi E.$$

Taking E to be T_0 we let the gauge fields be valued in a Clifford algebra with basis elements T_a, $a \in \{0, \ldots, N\}$. The equation (2.2) can then be satisfied with suitable choices for the $g_{ab}{}^c$ (such bases can be built using *matrix units* [7]).

Tentatively, the would-be gauge algebra is generated by operators

$$\Xi = \xi^aT_a + \sum_{k=1}^{\infty}\xi^{a\mu_{i_1}\cdots\mu_{i_k}}T_a\partial_{\mu_{i_1}}\cdots\partial_{\mu_{i_k}},$$

acting as

$$\delta_\xi\Phi = [\Xi, \Phi].$$

The crucial question is of course whether operators of this form generate a structure that can be considered to be a gauge algebra? This was studied in [8]. There it was shown that the operators Ξ obey the Jacobi identity (which they do since they associate as a consequence of the associativity of the matrices T and the Leibniz rule for the derivatives ∂) and that structure functions can be extracted according to the following scheme.

Using the Jacobi identity

$$[\delta_\lambda, \delta_\xi]\Phi = [\Lambda, [\Xi, \Phi]] - [\Xi, [\Lambda, \Phi]] = [[\Lambda, \Xi], \Phi],$$

we find that the commutator of two transformations can be written

$$[\delta_\lambda, \delta_\xi]\Phi = \delta_{[\lambda,\xi]}\Phi = \delta_\omega \Phi,$$

with structure thus extracted as $\Omega = [\Lambda, \Xi]$.

There are however some problems with this approach, the most serious that it does not seem to reproduce the lowest order spin 3 structure equations $[\lambda, \xi]^a_{\mu\nu}$ of [5, 4] (BBvD). I write *seem*, because as far as I know, this has not really been investigated. Reconsidering the whole setup, it is clear that, had the derivative basis scheme worked, we would have had a field independent higher spin gauge algebra. According to the analysis of (BBvD), they find field dependence once one goes beyond spin 2. They then perform a general analysis of the gauge algebra problem and find that the spin 3 gauge algebra cannot close on spin 3 fields. The non-closure terms however have a form that suggests that they correspond to spin 5 gauge transformations. Unfortunately it is not known whether in this way introducing a tower of higher spin gauge fields, the full algebra would become field independent. Clearly, there must be more interesting structure to extract here. There is a recent paper that alludes to this [9].

2.3 Fronsdal's symplectic basis

In the conference proceedings referred to in the introduction [2], Fronsdal briefly outlines an attempt to set up a non-Abelian higher spin gauge algebra. He considers a cotangent phase space over spacetime with the usual symplectic structure and coordinates (x^μ, π_ν). Then he considers a set of traceless symmetric gauge parameters $\{\xi^{\mu_1\cdots\mu_{s-1}}\}_{s=0}^\infty$ and the corresponding set of gauge fields $\{\phi^{\mu_1\cdots\mu_s}\}_{s=1}^\infty$. These are collected into formal power series

$$\Xi(\pi, x) = \sum_{s=1}^{\infty} (\pi^2)^{1-s/2} \pi_{\mu_1} \cdots \pi_{\mu_{s-1}} \xi^{\mu_1\cdots\mu_{s-1}},$$

$$\Phi(\pi, x) = \pi^2 + \sum_{s=1}^{\infty} (\pi^2)^{1-s/2} \pi_{\mu_1} \cdots \pi_{\mu_s} \phi^{\mu_1\cdots\mu_s}.$$

Transformations can now be calculated using the Poisson bracket $\{x_\mu, \pi_\nu\} = \eta_{\mu\nu}$ between x_μ and π_ν

$$\delta_\Xi \Phi = \{\Xi, \Phi\}.$$

The presence of the π^2 term in H is needed in order to get the free theory transformations

$$\{\Xi, \pi^2\} = 2 \sum_{s=1}^{\infty} (\pi^2)^{1-s/2} \pi_{\mu_1} \cdots \pi_{\mu_s} \partial^{\mu_1} \xi^{\mu_2\cdots\mu_s}.$$

Again, the Jacobi identity allows us to extract structure equations

$$[\delta_\Lambda, \delta_\Xi]\Phi = \{\Lambda, \{\Xi, \Phi\}\} - \{\Xi, \{\Lambda, \Phi\}\} = \delta_{\{\Lambda,\Xi\}}\Phi.$$

And once again we would get a field independent gauge algebra. Note that this construction is a form of Schouten brackets.

2.4 Oscillator basis

Formally introducing a covariant harmonic oscillator pair $(\alpha_\mu, \alpha_\mu^\dagger)$ with the usual commutator $[\alpha_\mu, \alpha_\mu^\dagger] = \eta_{\mu\nu}$ and a vacuum $|0\rangle$ satisfying $\alpha_\mu|0\rangle = 0$, one could consider collecting all higher spin gauge fields in the expansion

$$|\Phi\rangle = (\phi + \phi^\mu\alpha_\mu^\dagger + \phi^{\mu\nu}\alpha_\mu^\dagger\alpha_\nu^\dagger + \phi^{\mu\nu\rho}\alpha_\mu^\dagger\alpha_\nu^\dagger\alpha_\rho^\dagger + \cdots)|0\rangle. \tag{2.3}$$

This yields a concrete calculational scheme and one could hope to be able to rig some kind of kinetic operator K which when acting on the field $|\Phi\rangle$, generate field equations for the component fields. This can indeed be done as is well known. In practice it is done using BRST methods. Several equivalent such formulations were discovered in the mid 1980's [10, 11, 12, 13].

Formulations of this form has subsequently been rediscovered, developed and extended by a few research groups during the last ten years [14, 15, 16, 17]. Of course, expansions like (2.3) first occurred in the bosonic string theory, and the BRST formulations was partly inspired by string field theory and by taking the limit $\alpha' \to \infty$ (or the zero tension limit) [12]. The zero tension limit has been much studied lately (see for example [18, 19, 20]). For some recent reviews of these developments as well as more complete sets of references see for example [21, 22] and also [23, 24] which contain many references as well interesting discussions of these topics. This is closely tangled up with ideas about holography and the AdS/CFT duality [25] in a way that seems not fully understood. It is thus obvious that higher spin gauge fields occur in very many contexts. They are a prevalent feature of any model of extended and/or composite relativistic systems in various background geometries and dimensions. For discussions of these topics, see [26, 27]. But rather than study them in these contexts, I will continue to discuss higher spin symmetry and higher spin gauge theory as a free-standing theoretical construct. Thus let us return to the oscillator-BRST formulation and briefly review the simplest model.

Consider therefore a phase space spanned by bosonic variables (x_μ, p_μ) and $(\alpha_\mu, \alpha_\mu^\dagger)$ and ghost variables (c^+, b_+), (c^-, b_-) and (c^0, b_0) with commutation relations

$$[x_\mu, p_\nu] = i\eta_{\mu\nu}, \qquad [\alpha_\mu, \alpha_\nu^\dagger] = \eta_{\mu\nu}, \qquad \{c^+, b_+\} = \{c^-, b_-\} = \{c^0, b_0\} = 1.$$

The ghosts have the following properties under Hermitian conjugation

$$(c^-)^\dagger = c^+, \qquad (b_-)^\dagger = b_+, \qquad (c^0)^\dagger = c^0, \qquad (b_0)^\dagger = b_0.$$

The doubly degenerate vacuum states $|+\rangle$, $|-\rangle$ are annihilated by the operators α_μ, c^-, b^+ while the degeneracy is given by

$$b_0|+\rangle = 0, \qquad b_0|-\rangle = |+\rangle, \qquad c^0|-\rangle = 0, \qquad c^0|+\rangle = |-\rangle.$$

These equations relating the vacua, then implies that either one of the two vacua must be odd. I will choose $|-\rangle$ Grassmann odd.

The higher spin fields are collected into the ket $|\Phi\rangle$ with expansion

$$|\Phi\rangle = \Phi(p)|+\rangle + F(p)c^+b_-|+\rangle + H(p)b_-)|-\rangle,$$

where $\Phi(p)$ contains the symmetric higher spin gauge fields, and $F(p)$ and $H(p)$ are certain auxiliary fields. These fields are further expanded in terms of the oscillators

$$\begin{aligned}
\Phi &= \Phi_0 + i\Phi^\mu\alpha_\mu^\dagger + \Phi^{\mu\nu}\alpha_\mu^\dagger\alpha_\nu^\dagger + \cdots , \\
F &= F_0 + iF^\mu\alpha_\mu^\dagger + F^{\mu\nu}\alpha_\mu^\dagger\alpha_\nu^\dagger + \cdots , \\
H &= H_0 + iH^\mu\alpha_\mu^\dagger + H^{\mu\nu}\alpha_\mu^\dagger\alpha_\nu^\dagger + \cdots .
\end{aligned}$$

The gauge parameters are collected in

$$|\Xi\rangle = (\xi_0 - i\xi^\mu \alpha^\dagger_\mu + \xi^{\mu\nu}\alpha^\dagger_\mu \alpha^\dagger_\nu + \cdots) b_- |+\rangle.$$

The BRST operator Q is expressed in terms of the generators

$$G_0 = \tfrac{1}{2}p^2, \qquad G_- = \alpha \cdot p, \qquad G_+ = \alpha^\dagger \cdot p,$$

spanning the simple algebra

$$[G_-, G_+] = 2G_0,$$

with all other commutators zero.

In terms of these generators, the BRST operator reads

$$Q = c^0 G_0 - c^+ G_+ - c^- G_- - 2c^+ c^- b_0.$$

The action

$$A = \langle \Phi | Q | \Phi \rangle, \tag{2.4}$$

is invariant under the gauge transformations,

$$\delta_\Xi |\Phi\rangle = Q|\Xi\rangle, \tag{2.5}$$

as is the field equation

$$Q|\Phi\rangle = 0. \tag{2.6}$$

When expanding the equations (2.4), (2.5) and (2.6), everything works out nicely for the component fields, except for the fact that the theory contains auxiliary fields which cannot be solved for without introducing a further constraint. This constraint is applied to both the field and the gauge parameter

$$T|\Phi\rangle = 0, \qquad T|\Xi\rangle = 0,$$

where T is the operator

$$T = \tfrac{1}{2}\alpha \cdot \alpha + b_+ c^-.$$

When expanded, the constraint equations yield the double tracelessness constraint for component fields of spin $s \geq 4$ and the tracelessness constraint for the corresponding component gauge parameters. The constraints are needed in order to get the correct number of physical degrees of freedom. Including the constraints, this formulation precisely reproduces the Fronsdal equations [28] for higher spin gauge fields.

The free field theory is however still gauge invariant without imposing these constraints, and as discussed in [29], the constraints can actually be discarded with. The presence of the extra fields are then instead controlled by two global symmetries of the theory [30]. Other approaches to circumventing the tracelessness constraints have been proposed [31]. With some extra work this theory can be formulated in terms the Batalin–Vilkovisky field/antifield formalism. We will return to this issue in Section 4.

Perhaps just as interesting as the technical reconstruction of Fronsdal's free higher spin theory in terms of the BRST formalism, is the hint at an underlying mechanical model. In the first two schemes outlined above, it is not clear what the bases *are*. In the first case, generalizing from spin 1 $\xi_a T^a$ and spin 2 $\xi^\mu \partial_\mu$, (where for spin 2 we really should consider a full Poincaré generator $\xi^\mu\partial_\mu + \frac{1}{2}\xi^{ij}M_{ij}$) it is clear that we have no conceptual understanding what the higher derivative operators actually do. Perhaps somewhat vaguely we could say that we are re-using spacetime tangent space, not introducing anything new. The same goes for Fronsdal's approach. Although clever, it is clear that the scheme again is just re-using spacetime, but now its cotangent superstructure.

In contrast to this, introducing oscillators do add a something new, indeed new dimensions apart from spacetime itself. It is possible construct mechanical models that produces the first class constraints behind the above BRST construction of the field theory [32, 33]. During the 1960's and 70's there were other, parallel approaches to strong interactions apart from string theory (before QCD and asymptotic freedom swept the table). These also involved infinite towers of massless particles/fields and ideas about underlying composite mechanical models [34, 35], though at that time the interest was not primarily in the higher spin gauge fields.

3 Deformation vs. gauging

There are two main approaches to introducing interactions into a massless free spin 1 theory. Either gauge a global but non-Abelian Lie algebra, or deform local but Abelian gauge symmetries. It is of some interest to reconsider these approaches in order to get a hold on what is involved in generalizing to all spins.

One problem with the approaches described so far, apart from the fact that very few solid positive results on interactions have been achieved, is that they suffer from a weak conceptual underpinning. In my opinion this is also true for the much more successful Vasiliev approach, although in this case there are quite a few technical circumstances that lend some basic strength to the approach. We will now try to understand this in a simpleminded straightforward way, and in the process clarify some of the connections between the AdS and the Minkowski space (MiS) formulations of higher spin theory.

The Vasiliev approach is well described in the literature so I will not attempt yet another review, but instead focus some relevant points, perhaps in a somewhat idiosyncratic way.

3.1 What are the higher spin algebras?

The Vasiliev approach to higher spins seems simple enough. Just take a higher spin algebra and gauge it. Now, from where do we get the higher spin algebras? That's also easy, just take the algebra $so(3,2)$ or any of its higher dimensional and/or supersymmetric versions and form powers of its generators to any order. What results is an infinite dimensional associative algebra. The rest is technicalities. These are amply covered in the literature, what we want to do here is something more basic. The question is: why would this work? And why would it work in AdS and not in MiS.

Let T^a be the generators of an N-dimensional Lie algebra. Forming free powers of these generators and collecting these into the set

$$\mathbf{A} = \{T^{a_1}T^{a_2}\cdots T^{a_{i-1}}T^{a_i}\}_{i=1}^{\infty} \tag{3.1}$$

we can promote $\mathbf{A}$ to an associative algebra through the universal enveloping algebra construction. This involves forming equivalence classes of generators by modding out by the elements of the ideal generated by elements of the form $[T^a, T^b]$ and then taking care of automorphisms.

Non-compact Lie algebras have oscillator representations and these are easily seen to be infinite dimensional. Thus when we use a non-compact Lie algebra such as $so(3,2)$ to promote it to an infinite dimensional associative algebra using the construction (3.1), what we are really doing is to use its infinite-dimensional representations as a kind of basis for an infinite dimensional algebra. It is worthwhile to do this in some detail for the basic case of $so(3,2)$.

The higher spin algebra $hso(3,2)$

Let us then consider a pair of independent oscillators $(a, a^\dagger)$ and $(b, b^\dagger)$ with $[a, a^\dagger] = 1$ and $[b, b^\dagger] = 1$. Using these we can build low dimensional Lie algebras. Thus for the compact algebra $su(2)$ we define the generators

$$J_+ = b^\dagger a, \qquad J_- = a^\dagger b, \qquad J_3 = \tfrac{1}{2}(b^\dagger b - a^\dagger a).$$

spanning

$$[J_3, J_+] = J_+, \qquad [J_3, J_-] = -J_-, \qquad [J_+, J_-] = 2J_3.$$

To the $su(2)$ generators we can add the number operator

$$E = \tfrac{1}{2}(b^\dagger b + a^\dagger a + 1),$$

which commutes with the rest of the generators. This turns $su(2)$ into $u(2)$.

Next, using just one oscillator, we can build the non-compact $sp(2)$ algebra

$$S_+ = \tfrac{1}{2}a^\dagger a^\dagger, \qquad S_- = \tfrac{1}{2}aa, \qquad S_3 = \tfrac{1}{2}\big(a^\dagger a + \tfrac{1}{2}\big).$$

The algebra is

$$[S_3, S_+] = S_+, \qquad [S_3, S_-] = -S_-, \qquad [S_+, S_-] = -2S_3.$$

Starting with $sp(2)$ it is clear that we get an infinite dimensional representation over an oscillator ground state $|0\rangle$ with $a|0\rangle = 0$. Also, forming powers of the generators S_+, S_-, S_3 we get an infinite dimensional algebra represented over this same representation space. This is all well known including the necessary brush up details.

Now what happens in the $su(2)$ case? Define the double ground state $|0_a, 0_b\rangle \doteq |0_a\rangle|0_b\rangle$ with the obvious properties. For short we just write $|0,0\rangle$. Excited states are written $|n_a, n_b\rangle$.

The Fock space $\mathbf{F}_{su(2)}$ built upon the spin 0 ground state $|0,0\rangle$ gets a natural grading by spin. Using standard notation $|jm\rangle$ for angular momentum states we get

$$\begin{aligned}\mathbf{F}_{su(2)} &= \{|0,0\rangle\} \cup \{(|0,1\rangle, |1,0\rangle\} \cup \{|0,2\rangle, |1,1\rangle, |2,0\rangle)\} \cup \cdots \\ &= \bigcup_{n=0,1,2,\ldots} \big(\cup_{n_a+n_b=n} \{|n_a, n_b\rangle\}\big) = \bigcup_{j=0,\frac{1}{2},1,\ldots} (\oplus_{m=-j}^{m=j} |jm\rangle).\end{aligned} \tag{3.2}$$

We can now consider powers of the $su(2)$ generators, say $(J_+)^p$ and $(J_-)^q$. Then acting with these operators on spin $\frac{n}{2}$ states $|n_a, n_b\rangle$ we either get zero or a new spin $\frac{n}{2}$ state (when $p, q \leq n$), i.e. we stay within the same subspace, or we get zero (when $p, q > n$). Furthermore, since all states are eigenstates of J_3, powers of J_3 also stay within the same spin $\frac{n}{2}$ subspace. So in this way we cannot build an infinite dimensional algebra based on the compact $su(2)$ algebra that acts transitively on the full Fock space of states.

Let us now turn to the non-compact algebra $so(3,2)$. Its ten generators splits into three components according to $so(3,2) \mapsto g^{-1} \oplus g^0 \oplus g^{+1}$ where

$$g^{-1} = \{L^-_-, L^-_+, L^-_3\}, \qquad g^0 = \{E, J_+, J_-, J_3\}, \qquad g^{+1} = \{L^+_+, L^+_-, L^+_3\}.$$

The g^0 generators are precisely the already defined $u(2)$ generators. The rest are expressed in terms of the oscillators as

$$L^-_- = aa, \qquad L^-_+ = bb, \qquad L^-_3 = ab, \qquad L^+_+ = a^\dagger a^\dagger, \qquad L^+_- = b^\dagger b^\dagger, \qquad L^+_3 = a^\dagger b^\dagger.$$

Clearly, the g^{-1} are lowering operators and the g^{+1} are raising operators, and the overall structure of the algebra is

$$[g^m, g^m] \subseteq g^{m+n}, \qquad m, n = \{\pm 1, 0\}.$$

Using the raising and lowering operators we can now step up and down in the full Fock space $\mathbf{F}_{su(2)}$. This Fock space is actually the Di $\oplus$ Rac Fock space of states $|e, j\rangle$ with ground state $|\frac{1}{2}, 0\rangle$ and with the dispersion equation $e = j + \frac{1}{2}$ [36, 37, 38, 39, 40]. The subset of states $\{|n_a, n_b\rangle : n_a + n_b = n\}$ has $e = \frac{1}{2}(n_a + n_b + 1)$ and $m = \frac{1}{2}(n_a - n_b)$, where as in (3.2), m is the J_3 quantum number ranging between $-j$ and j.

Finally, modulo technical details, the higher spin algebra $hso(3, 2)$ is built by simply taking all positive powers of the $so(3, 2)$ generators. It becomes an infinite-dimensional Lie algebra transitively represented on the Di $\oplus$ Rac weight space (alternatively the Fock space $\mathbf{F}_{su(2)}$).

So what are the higher spin algebras? It seems that in four spacetime dimensions, the simplest higher spin algebra is that of an infinite-dimensional transformation algebra acting on the direct sum of all representations of the 3D angular momentum algebra. The crucial point is that the algebra connects states of different spin. What we get is a concrete realization of the universal enveloping algebra of $so(3, 2)$.

A question is now, can this be done for the Poincaré algebra? Analyzing how the Poincaré algebra is embedded in $so(3, 2)$ it becomes clear that the $so(3, 2)$ raising and lowering operators $L_a^{\pm}$ are linear combinations of Poincaré translations and boosts with the deformation parameter $\epsilon = \frac{1}{\sqrt{\Lambda}}$ as coefficient (a is a space index in the set $\{1, 2, 3\}$ or $\{+, -, 3\}$)

$$L_a^{\pm} = \epsilon P_a \pm i L_{a0}.$$

Thus when the Wigner–Inönü contraction $\Lambda \to 0$ is performed, the raising and lowering operators break up, and we lose much of the interesting AdS structure. Still there are remnants of the structure in MiS as is apparent from in the BRST oscillator expansion (2.3) of higher spin gauge fields. An interesting line of research would be to extend the MiS Lorentz algebra $so(3, 1)$ to an infinite algebra.

3.2 Gauging

Whereas the gauge theory approach to spin 1 interactions is fairly straightforward, indeed it is the paradigmatic example, gauging approaches to interacting spin 2 has always been plagued by conceptual (and technical) difficulties. As a backdrop, let us briefly run through the spin 1 case.

We have a non-Abelian Lie algebra represented by matrices T_a

$$[T_a, T_b] = f_{ab}{}^{c} T_c,$$

and a vector of matter fields φ transforming in some representation

$$\delta\varphi = \epsilon\varphi = \epsilon^a T_a \varphi,$$

and a matter Lagrangian which is invariant under these transformations. Making the parameters ϵ local: $\epsilon(x)$, we find the problem that spacetime derivatives of the fields transform in-homogeneously

$$\delta(\partial_\mu\varphi) = \partial_\mu(\delta\varphi) = \epsilon^a T_a \partial_\mu\varphi + (\partial_\mu\epsilon^a) T_a \varphi.$$

The remedy is introducing new gauge fields $A_\mu = A_\mu^a T_a$ transforming as

$$\delta A_\mu = [\epsilon, A_\mu] - \partial_\mu\epsilon.$$

Then the covariant derivative $D_\mu = \partial_\mu + A_\mu$ transforms homogeneously. Replacing ordinary derivatives with covariant ones then restores invariance to the matter Lagrangian. However, in our present context, the matter Lagrangian is just a crutch, what we are after is non-linear dynamics for the gauge field itself. The solution is quite simple and beautiful. Commuting covariant derivatives we find the field strengths $F_{\mu\nu}$ according to

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{3.3}$$

These transform as

$$\delta F_{\mu\nu} = [\epsilon, F_{\mu\nu}],$$

and it finally transpires that a non-linear, gauge-invariant Lagrangian can be written

$$\mathcal{L} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{\mu\nu}_a,$$

from which we read off that the interacting theory is a deformation of free spin 1 gauge theory running up to cubic and quartic order in the interaction. (Note that in the above brief outline the gauge coupling constant g has been absorbed into the fields and parameters, it can be restored by the substitutions $A \mapsto gA, \epsilon \mapsto g\epsilon$.)

What is the strong point of this? Clearly the fact that the correct non-linearity is forced upon us through the equation (3.3). It is like following a recipe. The weak point is finding the correct spin 1 Lagrangian, it looks simple enough here, but this step does not generalize easily to higher spin.

In the case of spin 2, the trouble starts at the very first step, choosing the gauge group. Having General Relativity in the back of our minds, we think about the local tangent spaces. In a frame, or vierbein, formulation this becomes very concrete as local Poincaré transformations. But we are running somewhat ahead of ourselves. Start with an (active) infinitesimal Poincaré transformation

$$\delta x^\mu = \epsilon^\mu{}_\nu x^\nu + \epsilon^\mu,$$

and correspondingly, a set of matter fields φ transforming as

$$\delta\varphi = \tfrac{1}{2}\epsilon^{ij} S_{ij}\varphi - \epsilon^\mu \partial_\mu \varphi. \tag{3.4}$$

Many things have slipped in here. In the first term we have changed indices from (curved) spacetime indices μ, ν to tangent space Lorentz indices i, j. So far we can see that we are really just generalizing the spin 1 recipe in a straightforward manner. The role of the T_a matrices are now played by the Lorentz $so(3,1)$ matrices S_{ij}. In the second term we see that corresponding to the translation part of the Poincaré transformations, we get generators that are spacetime derivatives. Anyone who has played around with these equations for a while is likely to suffer from some confusion, and it gets worse. As soon as the parameters ϵ^{ij} and ϵ^μ are made into local functions of the coordinates x^μ the distinction between local translations and local Lorentz transformations becomes blurred. Translations with a local $\epsilon^\mu(x)$ already contains all local coordinate transformations generated by the vector field $\epsilon^\mu(x)\partial_\mu$. However, if the local Lorentz transformations $\epsilon^{ij}S_{ij}$ are discarded, the transformations on the fields (3.4) become ambiguous.

Correspondingly, there are various approaches in the literature, gauging the Lorentz group [41], gauging the translations only, or gauging the full Poincaré group. A few more papers from the heydays of this approach to gravity are [42, 43, 44, 45, 46].

Following the review article [47] and proceeding to gauge the full Poincaré group, we again find that the derivate of a matter field transforms inhomogeneously

$$\delta\partial_\mu\varphi = \tfrac{1}{2}\epsilon^{ij} S_{ij}\partial_\mu\varphi - \epsilon^\nu\partial_\nu\partial_\mu\varphi + \tfrac{1}{2}(\partial_\mu\epsilon^{ij})S_{ij}\varphi - (\partial_\mu\epsilon^\nu)\partial_\nu\varphi.$$

The third term can be taken care of by introducing a gauge field ω_μ^{ij} with an inhomogeneous transformation term $-\partial_\mu \epsilon^{ij}$, and correspondingly we have a covariant derivative

$$\nabla_\mu = \partial_\mu + \tfrac{1}{2}\omega_\mu{}^{ij}.$$

The last term must be treated in a different way since it involves derivatives ∂_μ instead of matrices. Here we do a multiplicative gauging introducing the vierbeins $e_i{}^\mu$

$$D_i = e_i{}^\mu \nabla_\mu.$$

The rest of the story from here on involves commuting the covariant derivatives to find curvature $R_{ij}{}^{kl}$ and torsion tensors $T_{ij}{}^k$. Due to the multiplicative gauging, the curvature tensor is second order in spacetime derivatives (in contrast to the spin 1 field strengths which are first order in derivatives), and this is reflected in the gravity Lagrangian being expressed as

$$\mathcal{L} \simeq R_{ij}{}^{ij}.$$

The theory so obtained is almost Einstein gravity, the difference is the presence of torsion [41, 46].

This very (indeed) brief review of the gauging approaches to spin 1 and 2, shows disturbing differences between the two cases. There certainly isn't any standard recipe to generalize. In a very loose language we could say that Yang–Mills theory is basically very algebraic whereas Gravity is very geometric. One way to iron out the differences would be to geometrize Yang–Mills, this is precisely what one does in the modern fiber bundle approach. Another way would be to make Gravity more algebraic.

As we saw, one source to the problems came from the translation part of the Poincaré group which gives rise to derivatives as gauge generators. However in the AdS group $SO(3,2)$ the translation generators P_μ are just a subset of the $so(3,2)$ generators according to $P_\mu = \sqrt{\Lambda} M_{\mu 4}$ with $[P_\mu, P_\nu] = -i\Lambda M_{\mu\nu}$. So one could attempt to set up an $SO(3,2)$ gauge theory of gravity. This was done by Kibble and Stelle [48, 47] (see also [43]). Standard Einstein gravity is recovered upon spontaneously breaking $SO(3,2)$ to Poincaré corresponding to a Wigner–Inönü contraction $\Lambda \to 0$. This could be thought of as making gravity more algebraic. The Vasiliev approach to higher spin follow this road.

3.3 The core of the Vasiliev theory

What the Vasiliev theory of higher spins does is basically that it extends the gauge algebra $so(3,2)$ to the higher spin algebra $hso(3,2)$ (and correspondingly in other dimensions with or without supersymmetry) and then by generalizing the techniques of MacDowell and Mansouri [43]) and Stelle and West [48], it manages to derive field equations for interacting higher spin gauge fields coupled to gravity and Yang–Mills. This is a very impressive achievement and it involves a lot of technical details. Still it has so far not been possible to write down a unifying Lagrangian for this theory, and partly for this reason it must be said that the approach is still not entirely understood.

The field equations of the Vasiliev approach are written as a generalization of the Maurer–Cartan equations for a Lie algebra (for good reviews, see [49, 50] which we follow here)

$$R^a \doteq d\sigma^a + \tfrac{1}{2} f_{bc}{}^a \sigma^b \wedge \sigma^c = 0, \tag{3.5}$$

and where the σ^a are 1-forms on the Lie group manifold.

The Jacobi identity follows from the integrability conditions $dR^a = 0$ using $dd = 0$. This formulation is completely equivalent to the more common formulation in terms of vector fields (or generators) T_a satisfying the usual Lie brackets $[T_a, T_b] = f_{ab}{}^c T_c$ and with the inner product $\langle \sigma^a, T_b \rangle = \delta^a_b$.

The Maurer–Cartan equations are invariant under gauge transformations

$$\delta\sigma^a = D\epsilon^a \doteq d\epsilon^a + f_{bc}{}^a \sigma^b \wedge \epsilon^c.$$

This is then generalized by introducing, a possibly infinite, collection of $(p+1)$-form curvatures R^α defined as

$$R^\alpha \doteq d\Sigma^a + F^\alpha(\Sigma^\beta) = 0, \tag{3.6}$$

where

$$F^\alpha(\Sigma^\beta) \doteq \sum_{k=1}^{\infty} f^\alpha{}_{\beta_1 \dots \beta_k} \Sigma^{\beta_1} \wedge \dots \wedge \Sigma^{\beta_k}, \tag{3.7}$$

where the Σ^β are p-forms on the Lie group manifold. Here $f^\alpha_{\beta_1 \dots \beta_k}$ are the structure constants of a, possibly infinite-dimensional, algebra. The form degrees must match, i.e. $\sum_{i=1}^{k} p_{\beta_i} = p_\alpha + 1$.

Now the integrability conditions read $dF^\alpha(\Sigma^\beta) = 0$, or

$$F^\beta \wedge \frac{\partial F^\alpha}{\partial \Sigma^\beta} = 0, \tag{3.8}$$

yielding generalized Jacobi identities for the structure constants $f^\alpha_{\beta_1 \dots \beta_k}$. The gauge transformations also generalize to

$$\delta\Sigma^\alpha = d\epsilon^\alpha - \epsilon^\beta \wedge \frac{\partial F^\alpha}{\partial \Sigma^\beta},$$

under which the generalized curvatures of (3.6) stays invariant.

This kind of structure was initially developed in the context supergravity and termed Cartan Integrable Systems [51], later renamed as Free Differential Algebra in accordance with the corresponding mathematical constructs [52].

The actual field equations are formulated in terms of two differential forms, one zero-form $\Phi = \Sigma^0$ and one one-form $A = \Sigma^1$ which in their turn are infinite expansions in terms of oscillators, much like the expansions above, but with important differences. Vasiliev's expansions are in terms of both creation and annihilation operators and will therefore span infinite-dimensional algebras (given some restrictions), indeed the higher spin algebras.

This is the abstract scheme employed in the Vasiliev approach. Of course, we haven't even touched on the mass of technicalities involved in actually setting up higher spin gauge theory in this framework. Some helpful papers are [49, 50, 53] where also many further references can be found. Here we will restrict ourselves to pointing out how strongly homotopy Lie algebras emerge.

Consider the operator

$$K = F^\alpha(\Sigma) \frac{\partial}{\partial \Sigma^\alpha}. \tag{3.9}$$

Then calculating the square of this operator we get

$$K^2 = \tfrac{1}{2} K \wedge K = \left(F^\beta \wedge \frac{\partial F^\alpha}{\partial \Sigma^\beta} \right) \frac{\partial}{\partial \Sigma^a} = 0,$$

which is zero by (3.8). From here the step to the algebraic structure of a L_∞ algebra is short. Already in the generalization of (3.5) to (3.6) and the definition (3.7) we see a hint of the higher order brackets of an sh-Lie algebra. This becomes more clear if we write the Maurer–Cartan equations in an abstract, coordinate-free way as $d\sigma = -\frac{1}{2}[\sigma, \sigma]$. Generalizing this recklessly, we would consider multibrackets $[\Sigma, \Sigma, \ldots, \Sigma]$ and the Jacobi identities for the ordinary Lie bracket $[\cdot, \cdot]$ would generalize to the sh-Lie identities. Let us step back and do this in some more detail.

3.4 Strongly homotopy Lie algebras and higher order brackets

There are a few variants of the basic definitions of strongly homotopy Lie algebras in the literature (see for example [54, 55]), but the following, mildly technical, is sufficient for our purpose to bring out the connection to both the Vasiliev formalism and to the product identities of the BRST approach described below.

Definition

Consider a $\mathbf{Z}_2$ graded vector space $V = V_0 \oplus V_1$ over some number field, and denote the elements by x. The grading is given by ϱ with $\varrho(x) = 0$ if $x \in V_0$ and $\varrho(x) = 1$ if $x \in V_1$. V is supposed to carry a sequence of n-linear products denoted by brackets. The graded n-linearity is expressed by

$$
\begin{aligned}
&[x_1, \ldots, x_n, x_{n+1}, \ldots, x_m] = (-)^{\varrho(x_n)\varrho(x_{n+1})}[x_1, \ldots, x_{n+1}, x_n, \ldots, x_m], \\
&[x_1, \ldots, a_n x_n + b_n x'_n \ \ldots, x_m] \\
&\qquad = a_n (-)^{\iota(a_n, n)}[x_1, \ldots, x_n, \ldots, x_m] + b_n (-)^{\iota(b_n, n)}[x_1, \ldots, x'_n, \ldots, x_m],
\end{aligned}
$$

where $\iota(a_n, n) = \varrho(a_n)(\varrho(x_1) + \cdots + \varrho(x_{n-1}))$.

The defining identities (or "main" identities) for the algebra are, for all $n \in \mathbf{N}$

$$
\sum_{\substack{k=0 \\ l=0}}^{k+l=n} \sum_{\pi(k,l)} \epsilon(\pi(k,l))[[x_{\pi(1)}, \ldots, x_{\pi(k)}], x_{\pi(k+1)}, \ldots, x_{\pi(k+l)}] = 0, \tag{3.10}
$$

where $\pi(k, l)$ stands for (k, l)-shuffles. A (k, l)-shuffle is a permutation π of the indices $1, 2, \ldots,$ $k + l$ such that $\pi(1) < \cdots < \pi(k)$ and $\pi(k + 1) < \cdots < \pi(k + l)$. $\epsilon(\pi(k, l))$ is the sign picked up during the shuffle as the points x_i with indices $0 \le i \le k$ are taken through the points x_j with indices $k + 1 \le j \le l$. This is just the normal procedure in graded algebras.

The bracket (or braces) notation is common. Note that the n-ary brackets are all independent and abstract, not to be thought of as deriving from some underlying product (just as the abstract Lie bracket $[\cdot, \cdot]$ need not derive from a product). As discussed in [56], other types of gradings can be considered. The $\mathbf{Z}_2$ grading employed here can be thought as providing room for a BRST-type algebra with even fields and odd gauge parameters. In the BRST-BV reformulation [29] (see Section 4 of the present paper) the mechanical and field theory ghosts conspire to make all objects even (fields, anti-fields, ghosts and anti-ghosts) so in that case we can in fact do with an ungraded algebra. In order to accommodate the FDA of Vasiliev, a grading with respect to form degree should be defined.

Running the risk of pointing out something that is obvious, let us note that even in the case where we have an underlying associative product, such as when we work with matrix algebras, nothing prevents us from calculating higher order brackets and see what we get. What we would get is roughly linear combinations of elements in the vector space that the algebra is built upon, and these linear combinations would define for us generalized structure coefficients. Such higher order Lie algebras has been investigated in [57]. They obey generalized Jacobi identities similar in structure to the sh-Lie identities (of which they are a special case), but then as a consequence of the underlying associative product.

Next following [56] we can introduce generating functions for the sh-Lie structure. As pointed out there, due to the symmetry and linearity properties, the full sh-Lie structure is fixed by the algebra of even and coinciding elements, Σ say. We can indeed think of the Σ as vectors in one huge graded vector space, for example graded by form degree, corresponding to the forms of the Vasiliev theory. Next we write equation (3.7) in terms of brackets

$$K = \sum_{k\geq 0} \frac{1}{k!} \underbrace{[\Sigma, \Sigma, \ldots, \Sigma]}_{k}.$$

K is itself an element of the vector space and we can consider it as a formal vector field

$$K = F^{\alpha}(\Sigma) \frac{\partial}{\partial \Sigma^{\alpha}},$$

which is what we had before (3.9). But now computing the square K^2 using its definition in terms of the brackets, we get Jacobiators

$$J^n(\Sigma, \ldots, \Sigma) = \sum_{l=0}^{n} \frac{n!}{l!(n-l)!} [\underbrace{[\Sigma, \ldots, \Sigma]}_{n-l}, \underbrace{\Sigma, \ldots, \Sigma}_{l}],$$

for even elements Σ. In terms of these Jacobiators the "generalized Jacobi identities" or simply "main" identities from the definition of an sh-Lie algebra become $J^n = 0$ for all n, or

$$J = \sum_{n\geq 0} \frac{1}{n!} J^n = K^2 = 0.$$

This, admittedly hand-waving, argument shows that there is a connection between sh-Lie algebras and FDA's. This is often alluded to in the literature (see for example [49] but I haven't seen any rigorous proof. But there cannot really be any doubt that the technical details can be supplied to make the connection precise. According to [53], any FDA can be obtained from an sh-Lie algebra by fixing the form degree p_α of the Σ^α. Implicitly, this is what we have done in the argument above.

The same formal structure can also be seen in abstract approaches to BRST-BV theory. As in [58] we can write an odd vector field K

$$K = \Omega^b{}_a \psi^a \frac{\partial}{\partial \psi^b} + U^c{}_{ab} \psi^a \psi^b \frac{\partial}{\partial \psi^c} + U^d{}_{abc} \psi^a \psi^b \psi^c \frac{\partial}{\partial \psi^d} + \cdots,$$

where K could be either a BRST-operator Q or a BRST-BV operator s depending on the context. In both case, the equation $K^2 = 0$ generates an sh-Lie algebra.

Contemplating all this one might wonder whether there are more abstract common algebraic structures underlying both the Vasiliev approach and for instance the constructions outlined in Section 2, and in particular the BRST-BV approach of [29] (see Section 4 in the present paper) which superficially look very different. It seems that the language of category theory, and in particular the language of operads, could furnish us with such an abstract framework. Some preliminary remarks on this can be found in Section 5. For more connections between the Vasiliev the BRST formalism, see [59], where it is shown that both the free Vasiliev theory and the BRST theory can be derived from a common "parent field theory".

3.5 Deforming

The deformation approach starts out with a free gauge theory. We have the action and the Abelian gauge transformations and the object is to deform the action and the gauge transformations by non linear terms, still retaining gauge invariance. This works nicely for spin 1 as was shown a long time ago in [60]. The analogous deformation approach to spin 2 is considerable more complicated, both technically and conceptually, and there is a long list of classical papers treating this problem [61, 62, 63, 64, 65, 66, 67, 68, 1, 69]. No wonder then, perhaps, that the higher spin interaction problem is even more convoluted. The list of references pertaining to this problem is quite long, for a reasonable subset, see [70].

4 Review of the BRST-BV Minkowski approach

As a modern example of the deformation method, let us describe the concrete Fock complex vertex implementation of the BRST-BV approach to higher spin gauge self interactions in MiS in a top-down fashion. The exposition here will be brief, a thorough description can be found in the recent paper [29], which also contains references to the relevant theoretical background. For a general discussion of the BV-deformation approach to interactions see [71, 72].

Thus we start by writing out the master action to all orders in formal perturbation theory

$$S = \langle\Psi|Q|\Psi\rangle|_{\mathrm{gh_m}=0} + \sum_{n=3}^{\infty} g^{n-2}\langle\Psi|^{\otimes n}|\mathcal{V}_n\rangle|_{\mathrm{gh_m}=0}. \tag{4.1}$$

Here $|\Psi\rangle$ denotes a formal sum of ghost, field, antifield, antighost components

$$|\Psi\rangle = |\mathcal{C}\rangle \oplus |\Phi\rangle \oplus |\Phi^{\#}\rangle \oplus |\mathcal{C}^{\#}\rangle.$$

The components $|\mathcal{C}\rangle$, $|\Phi\rangle$, $|\Phi^{\#}\rangle$, $|\mathcal{C}^{\#}\rangle$ live in the Fock complex of mechanics oscillators and ghosts described in Section 2.4. Actually, the field $|\Phi\rangle$ is precisely the higher spin field whereas the field theory ghost field $|\mathcal{C}\rangle$ replaces the gauge parameter $|\Xi\rangle$. The #-decorated components are the corresponding anti-field and anti-ghost. When we want to refer to the fields without having this concrete realization in mind, we just write Ψ.

In this formulation there are no trace constraint operators T. Instead, the theory is subject to two global symmetries

$$\delta_{\mathcal{P}}|\Psi\rangle = i\epsilon\mathcal{P}|\Psi\rangle, \qquad \delta_{\mathcal{T}}|\Psi\rangle = i\epsilon\mathcal{T}|\Psi\rangle,$$

where the definition of the operators $\mathcal{P}$ and $\mathcal{T}$ can be found in [29].

It then follows that $\delta_{\mathcal{P}}\langle\Psi|Q|\Psi\rangle = \delta_{\mathcal{T}}\langle\Psi|Q|\Psi\rangle = 0$. In this formalism there are, for a given primary spin s field, a further spin $s-2$ field (corresponding to the trace of the spin s field) that is not solved for by the trace constraint. For example, the primary spin 5 field will be accompanied by a new secondary spin 3 field. It is interesting to note that cubic spin 3 interactions with non-minimal number of derivatives have been found [73], i.e. with five derivatives instead of three. This is expected for a secondary spin 3 field accompanying the primary spin 5 field.

The vertex operators $|\mathcal{V}_n\rangle$, encode all interaction data, i.e. all n-point vertex data as well as all higher order structure function data.

The BRST invariance of the classical theory is now expressed by the master equation (S, S)=0 where $(\cdot, \cdot)$ is the anti-bracket of BV theory.

Explicitly calculating $(S, S) = 0$ order by order in vertex order n yields

$$\sum_{r=1}^{3} Q_r|\mathcal{V}_3\rangle = 0, \tag{4.2}$$

$$\sum_{r=1}^{n+1} Q_r|\mathcal{V}_{n+1}\rangle = -\sum_{p=0}^{\lfloor (n-3)/2\rfloor} |\mathcal{V}_{p+3}\rangle \diamond |\mathcal{V}_{n-p}\rangle, \tag{4.3}$$

where $\diamond$ denotes a certain symmetrized contraction in the mechanics Fock complex. These equations encode the structure of a L_∞ algebra as will be shown in Section 5.

Extending the global symmetries to the interacting theory yield the following conditions on the vertices

$$\sum_{r=0}^{n} \mathcal{P}_r|\mathcal{V}_n\rangle = 0, \qquad \sum_{r=0}^{n} \mathcal{T}_r|\mathcal{V}_n\rangle = 0.$$

In order to take into account field redefinitions, we introduce field redefinition vertices $|\mathcal{R}_n\rangle$ and write

$$|\Psi\rangle \to |\Psi_r\rangle = |\Psi\rangle + \sum_{n=2}^{\infty} g^{(n-1)}\langle\Psi|^{\otimes n}|\mathcal{R}_{n+1}\rangle.$$

Performing field transformations of this form on the free action produce fake interaction terms of the form

$$\sum_{n=3}^{\infty} g^{(n-2)}\langle\Psi|^{\otimes n}\sum_{r=1}^{n} Q_r|\mathcal{R}_n\rangle.$$

Comparing to the general form of the interactions given in equation (4.1), we see that fake interactions can be characterized by

$$|\mathcal{V}_n\rangle_{\text{fake}} = \sum_{r=1}^{n} Q_r|\mathcal{R}_n\rangle,$$

which in the language of homology is to say that fake interaction vertices are exact.

Let us clarify one potentially confusing issue. The vertices $|\mathcal{V}_n\rangle$ provide us with products of fields, i.e. the product of n fields $\Psi_1, \dots, \Psi_n$, abstractly denoted by $\mathbf{pr}(\Psi_1, \dots, \Psi_n)$, can be calculated as

$$\mathbf{pr}(\Psi_1, \dots, \Psi_n) \hookrightarrow \langle\Psi_1|\cdots\langle\Psi_n|\mathcal{V}_{n+1}\rangle \to |\Psi_{n+1}\rangle,$$

i.e. contracting the $(n+1)$ vertex with n bra fields produces a new ket field. The index numbers are precisely the numbering of the mechanics Fock spaces including the field momenta. The vertices enforce momentum conservation. In this context it is natural to chose the abstract products $\mathbf{pr}$ to be fully symmetric in the abstract fields Ψ, and indeed, the Fock fields $|\Psi\rangle$ can be chosen to be Grassmann even.

Now, these products corresponds to what in the mathematics literature are generally denoted by n-ary brackets $[\cdot, \cdot, \dots, \cdot]$, generalizing the two-bracket $[\cdot, \cdot]$ of a differential graded Lie algebra. Such a bracket is skew symmetric, rather than symmetric like our 2-products $\mathbf{pr}(\cdot, \cdot)$ implemented by the three-vertex $|\mathcal{V}_3\rangle$. This issue is resolved by noting that our product is in fact alternatingly Grassmann odd/even in the mechanics Fock complex due to the presence of n odd vacua $|-\rangle$. Thus the two-product is intrinsically odd, and instead of the conventional grading

$$\mathbf{pr}(\Psi_1, \Psi_2) = -(-)^{\epsilon(\Psi_1)\cdot\epsilon(\Psi_2)}\mathbf{pr}(\Psi_2, \Psi_1), \qquad \epsilon([\Psi_1, \Psi_2]) = \epsilon(\Psi_1) + \epsilon(\Psi_2),$$

we have

$$\mathbf{pr}(\Psi_1, \Psi_2) = (-)^{\epsilon(\Psi_1)\cdot\epsilon(\Psi_2)}\mathbf{pr}(\Psi_2, \Psi_1), \qquad \epsilon([\Psi_1, \Psi_2]) = \epsilon(\Psi_1) + \epsilon(\Psi_2) + 1,$$

and correspondingly for higher order products. This is also consistent with the recursive equations (4.3) since $\epsilon(Q) = 1$.

5 Unifying structures

It seems quite clear that the algebraic structure of strongly homotopy Lie algebras naturally crop up in various formulations of higher spin gauge theory. How can this be understood from a formalism independent way? According to the literature on the subject [74], L_∞ structures was first spotted by Stasheff in the BBvD analysis [75] of the general higher spin interaction problem. Subsequently it was proved that given that the BBvD higher spin gauge algebra exists, it must be sh-Lie [76]. Influenced by this, and inspired by computer science thinking, I proved that the sh-Lie structure is grounded already in the syntax of any formal power series formulation of gauge theory. Could this be a hint that sh-Lie algebras, rather than being deep features of higher spins, are just superficial aspects of the formalism? As will be argued in the next section, category theory can throw light on this.

In category theory we do not seek structure by peeking into the objects, but instead build structure on the outside, so to speak. Which is of course precisely what we do when we build models of physical systems. The complex objects so formed can then be re-analyzed, i.e. peeked into.

5.1 Abstraction, categories and operads

In my opinion, and emphasized in [70], one of the main problems in higher spin theory is to control the inherent complexity. In that paper, we took a rather simpleminded syntax-semantics approach to the problem and were then able to see that the somewhat elusive connections between gauge invariance, BRST-BV formulations and strongly homotopy algebras that has often been referred to in the literature, is grounded already in the syntax of the theory. Now the theory of categories in general and the theory of operads in particular offer a solid mathematical framework for precisely this kind of situation. This should perhaps not come as a surprise since category theory is routinely applied to the syntax-semantics duality in theoretical computer science.

The first problem one is confronted with upon trying to apply category theory to field theory [77] is: what are the objects and what are the morphisms? As briefly discussed in [29], there is no one unique answer to that question, so let us start in another place. Any formulation of interacting higher spin gauge field theory will involve multi-field interaction vertices of some sort. The natural categorical correspondence then ought to be the n-ary multi-operations $\mathcal{P}(n)$ of an operad [78]. Now the operad in itself only provides the axioms for the $\mathcal{P}(n)$'s, that is, they provide a syntax for the interactions. To get a concrete model we need a semantics, i.e. in computer science terminology: an evaluation, or in category theory language: an algebra for the operad. Indeed an algebra for an operad precisely furnishes an evaluation of the operad. Let us make this more exact.

5.2 Operads and algebras for operads

The concept of an operad captures the idea of abstract n-ary operations $\mathcal{P}(n)$ with n input lines and one output line. The $\mathcal{P}(n)$'s (one set for each $n \geq 1$) can be taken to be vector spaces which is just to say that they can be added and multiplied with numbers from some ground field $\mathbf{k}$. It is then natural to string these n-operations together to from new n-operations. There are then some issues to contemplate such as associativity of the compositions and permutations of input lines. This is controlled by the axioms of operads (see for example [79]).

A potentially confusing issue is the type of objects figuring at the inputs of the n-operations. Having field theory application in mind, there are two basic options. On the one hand, aiming at modeling configuration space fields, we have just one single object Φ. This then provide input to the n-operations and we have the rudiments of a non-polynomial field theory. On the other hand, aiming at modeling momentum space fields, the objects come labeled by an index i (or p_i in concrete field theory). Then we have a countable set $\{\Phi_i\}_{i=1}^{\infty}$ of objects. In this case we have a slight generalization of operads into multi-categories where the n-operations take inputs from the set of objects $\{\Phi_i\}$. Specializing to unary operations $\mathcal{P}(1)$ we have an ordinary category with a countable set of objects and with the $\mathcal{P}$'s playing the role of arrows.

Another confusing issue is how an abstraction like this can really capture the details of field theory? How are the n-operations related to the concrete higher spin vertices of the BRST-BV approach? Consider the vector space $\mathcal{P}(n)$ of n-operations. Using the axioms of the multi-category (or the operad) these can be deconstructed into terms of simpler constituents, eventually coming down to a set of basic operations (classically, at the tree level). So, in the set of $\mathcal{P}(n)$ there is one special element (one for each n) which is the abstraction of the concrete vertex operator $|\mathcal{V}_n\rangle$. Let us denote the corresponding abstract element by $\mathcal{V}(n)$.

Thus, the objects of the multi-category are abstract fields Φ_i. We then consider the vertex operators $|\mathcal{V}_n\rangle$ as maps V_n providing evaluations

$$V_n : \mathcal{V}(n) \otimes \Phi^{\otimes n} \to \Phi. \tag{5.1}$$

On general non-elementary operations $\mathcal{P}(n)$, the action of the maps V_n are defined recursively in the standard way of defining maps on recursive data structures.

5.3 Semantic mapping of the sh-Lie structure

To see the strength of this simple formalism let us see how the main identities of sh-Lie structure directly maps to the concrete vertex equations (4.2), (4.3) of Section 4.

Let us first write down the product identities of the sh-Lie algebra as they were derived in [70] in the case of even abstract fields Φ. Taking into account that the product maps are fully symmetric in all arguments, we can write the product identities as

$$\sum_{\substack{k=0,l=0 \\ \text{cycl. perm.}}}^{k+l=n} \mathbf{pr}(\Phi^k, \mathbf{pr}(\Phi^l)) = 0,$$

where we write the multi-categorical morphism as $\mathbf{pr}(\Phi^n)$ which we think of as the abstract representation of the product of n fields mapping to a new field

$$\Phi^{\otimes n} \to \Phi : \mathbf{pr}(\Phi_1, \ldots, \Phi_n) \to \Phi_{n+1}. \tag{5.2}$$

These identities are just special cases of the sh-Lie identities (3.10), now written in terms of products instead of brackets.

It will be convenient to introduce a special provision to deal with the one-product $\mathbf{pr}(\Phi)$ which is naturally interpreted as a linear transformation $K\Phi$. Straining the formalism a little, the one-product can be made to conform to (5.2), if we write

$$K_2\Phi_2 = K_2\Phi_1\delta_{12} = \mathbf{pr}(\Phi_1) \to \Phi_2$$

by which we mean that in whatever way we compute the one-product (or linear transformation), we change index on route. This is practical when we do the same in the Fock complex

$$Q_2|\Phi_2\rangle = Q_1|\Phi_2\rangle\delta_{12} = \langle\Phi_1|\mathcal{V}_2\rangle.$$

In this way we can think of the action of the BRST operator Q in terms of a 2-vertex $|\mathcal{V}_2\rangle$.

The semantic map can now be defined. The abstract fields Φ are simply mapped to Fock complex fibers $|\Phi\rangle$, and the products $\mathbf{pr}_n$ are mapped to vertices $|\mathcal{V}_{n+1}\rangle$. Indeed using an arrow $\hookrightarrow$ to denote the semantic map, we have

$$\begin{aligned}
&\Phi_k \hookrightarrow |\Phi_k\rangle \qquad \text{for} \quad k \in N,\\
&\mathbf{pr}(\Phi_1, \ldots, \Phi_n) \hookrightarrow \langle\Phi_1|\cdots\langle\Phi_n|\mathcal{V}_{n+1}\rangle \qquad \text{for} \quad n \geq 1.
\end{aligned}$$

This is our concrete realization of the *evaluation* (5.1) (in computer science vernacular), or *algebra of the operad* as would be the operadic notion.

Note that the last equation precisely utilizes our special provisions for the one-product, i.e. the case $n = 1$.We are bit lenient with the formalism here, as the $(n+1)$-th Fock space in the vertex $|\mathcal{V}_{n+1}\rangle$ should really be switched to a bra. As it stands, the last equation produces a ket. It could be fixed at the cost of a more cumbersome notation.

With the ground so prepared we can finally apply the semantic map $\hookrightarrow$ to the left hand side of the product identities to get

$$\sum_{\substack{k=0,l=0\\ \text{cycl. perm.}}}^{k+l=n} \mathbf{pr}(\Phi^k, \mathbf{pr}(\Phi^l)) \hookrightarrow \sum_{\substack{k=0,l=0\\ \text{cycl. perm.}}}^{k+l=n} \langle\Phi|^{\otimes k}(\langle\Phi|^{\otimes l}|\mathcal{V}_{l+1}\rangle)\cdot|\mathcal{V}_{k+2}\rangle.$$

In writing this equation, we are freely switching bra $\leftrightarrow$ ket Fock spaces as need arise to do the contractions. With $l = n-k$ we now have

$$\sum_{\substack{k=0\\ \text{cycl.perm.}}}^{n-1} \langle\Phi|^{\otimes k}\langle\Phi|^{\otimes n-k}|\mathcal{V}_{k+2}\rangle\cdot|\mathcal{V}_{n-k+1}\rangle = 0. \tag{5.3}$$

The sum stops at $k = n-1$ since the last term $k = n$ is zero, in the abstract product identities corresponding to $\mathbf{pr}(\,) = 0$, which in the implementation would be $|\mathcal{V}_1\rangle = 0$, i.e. there is no 1-vertex.

Then focusing on the first $(k = 0)$ and next to last $(k = n-1)$ terms, we see, using the conventions introduced for the 2-vertex, that they simply give us

$$\sum_{r=1}^{n+1} Q_r|\mathcal{V}_{n+1}\rangle.$$

Then the rest of the terms in (5.3) pair off nicely in a similar way. Thus the $\mathbf{pr}(\Phi^k, (\mathbf{pr}\Phi^{n-k}))$ term for $k \geq 1$ maps to precisely $\binom{n}{k}$ terms containing the vertex combination $|\mathcal{V}_{k+2}\rangle\cdot|\mathcal{V}_{n-k+1}\rangle$, while the $\mathbf{pr}(\Phi^{n-k-1}, (\mathbf{pr}\Phi^{k+1}))$ term maps to precisely $\binom{n}{n-k-1}$ terms also containing the vertex combination $|\mathcal{V}_{k+2}\rangle\cdot|\mathcal{V}_{n-k+1}\rangle$. Since $\binom{n}{k} + \binom{n}{n-k-1} = \binom{n+1}{n-k}$ we see that we get precisely the correct number of terms to fully symmetrize $|\mathcal{V}_{k+2}\rangle\cdot|\mathcal{V}_{n-k+1}\rangle$. This is so because this contraction of vertices has $n+1$ free non-contracted indices (two of the $n+3$ being contracted). Doing the algebra carefully yields

$$\sum_{k=1}^{\lfloor(n-1)/2\rfloor} |\mathcal{V}_{k+2}\rangle \diamond |\mathcal{V}_{n-k+1}\rangle,$$

or, upon re-indexing the sum with $p = k - 1$

$$\sum_{p=0}^{\lfloor (n-3)/2 \rfloor} |\mathcal{V}_{p+3}\rangle \diamond |\mathcal{V}_{n-p}\rangle.$$

Hence, collecting all the terms, we get

$$\sum_{r=1}^{n+1} Q_r |\mathcal{V}_{n+1}\rangle = - \sum_{p=0}^{\lfloor (n-3)/2 \rfloor} |\mathcal{V}_{p+3}\rangle \diamond |\mathcal{V}_{n-p}\rangle,$$

which is exactly what we got before from the explicit $(S, S) = 0$ calculation to all orders in g and antighost number.

In conclusion, the syntactically derived product identities of the sh-Lie algebra maps semantically to equations for the vertices in the Fock complex implementation. This result lends considerable strength to our framework.

If higher spin gauge fields have anything to do with physical reality, then we would eventually have to do numerical calculations. Given the complexity of the theory discerned so far, these calculations would undoubtedly have to be computerized. In that case, all "objects" of the theory will have to be mapped to countable infinite data structures (truncated to finite data structures in practice). One might speculate whether in that case formulations that more easily translate into recursive and algebraic data structures of, for example, a functional programming language, might be more useful than the ordinary "pen-and-paper" mathematical formalism of field theory?

6 A note on references

The literature on higher spin field theory is enormous and growing rapidly. The present paper is not intended to be a full review and leaves out many interesting developments or just mentions them in passing. The reason for this is at least twofold: (i) the limits of my own knowledge, (ii) my wish to put forth a certain point of view, perhaps stressing what has gone un-noticed in other works. Thus I hope the present paper can serve as a complement to other excellent reviews of higher spin gauge theory. My referencing necessarily reflects these limitations and choices. I do apologize for any inadvertent omissions.

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