Classical R-Operators and Integrable Generalizations of Thirring Equations*

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Abstract. We construct different integrable generalizations of the massive Thirring equations corresponding loop algebras $\widetilde{\mathfrak{g}}^{\sigma}$ in different gradings and associated "triangular" R-operators. We consider the most interesting cases connected with the Coxeter automorphisms, second order automorphisms and with "Kostant–Adler–Symes" R-operators. We recover a known matrix generalization of the complex Thirring equations as a partial case of our construction.

Key words: infinite-dimensional Lie algebras; classical R-operators; hierarchies of integrable equations

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1 Introduction

A theory of hierarchies of integrable equations in partial derivatives is based on the possibility to represent each of the equations of the hierarchy in the so-called zero-curvature form:

$$\frac{\partial U(x,t,u)}{\partial t} - \frac{\partial V(x,t,u)}{\partial x} + [U(x,t,u), V(x,t,u)] = 0, \tag{1}$$

where $U(x,t,u)$, $V(x,t,u)$ are $\mathfrak{g}$-valued functions with dynamical variable coefficients, $\mathfrak{g}$ is simple (reductive) Lie algebra and u is an additional complex parameter usually called spectral. In order for the equation (1) to be consistent it is necessary that $U(x,t,u)$ and $V(x,t,u)$ belong to some closed infinite-dimensional Lie algebra $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$-valued functions of u.

There are several approaches to construction of zero-curvature equations (1) starting from Lie algebras $\widetilde{\mathfrak{g}}$. All of them are based on the so-called Kostant–Adler–Symes scheme. One of the most simple and general approaches is the approach of [3, 10] and [6, 7] that interprets equation (1) as a consistency condition for a two commuting Hamiltonian flows written in the Euler–Arnold or Lax form. In the framework of this approach elements $U(x,t,u)$ and $V(x,t,u)$ coincide with the algebra-valued gradients of the commuting Hamiltonians constructed with the help of the Kostant–Adler–Symes scheme, the cornerstone of which is a decomposition of the Lie algebra $\widetilde{\mathfrak{g}}$ (as a vector space) in a direct sum of its two subalgebras:

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_+ + \widetilde{\mathfrak{g}}_-.$$

The algebra-valued gradients of the commuting Hamiltonians coincide with the restrictions of the algebra-valued gradients of Casimir functions of $\widetilde{\mathfrak{g}}$ onto the subalgebras $\widetilde{\mathfrak{g}}_{\pm}$. Hence, such

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approach permits [14] to construct using Lie algebra $\widetilde{\mathfrak{g}}$ the three types of integrable equations: two types of equations with U-V pair belonging to the same Lie subalgebras $\widetilde{\mathfrak{g}}_\pm$ and the third type of equations with U-operator belonging to $\widetilde{\mathfrak{g}}_+$ and V-operator belonging to $\widetilde{\mathfrak{g}}_-$ (or vise verse). The latter equations are sometimes called "negative flows" of integrable hierarchies.

Nevertheless the approach of [3, 6, 7] does not cover all integrable equations. In particular, it was not possible to produce by means of this approach integrable equations possessing U-V pairs with U-operator belonging to $\widetilde{\mathfrak{g}}_+$ and V-operator belonging to $\widetilde{\mathfrak{g}}_-$ in the case $\widetilde{\mathfrak{g}}_+ \cap \widetilde{\mathfrak{g}}_- \neq 0$, i.e. in cases dropping out of the scope of Kostant–Adler–Symes method. An example of such situation is the Thirring integrable equations with the standard $sl(2)$-valued U-V pair [9].

In the present paper we generalize the method of [3, 10] of producing soliton equations and their U-V pairs, filling the gap described above, i.e. making the method to include all known soliton equations, and among them Thirring equation. We use the same idea as in [3] and [6], i.e. we interpret zero-curvature condition as a compatibility condition for a set of commuting Hamiltonian flows on $\widetilde{\mathfrak{g}}^*$ but constructed not with the help of Kostant–Adler–Symes method but with the help of its generalization – method of the classical R-operator, where $R : \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$, satisfies a modified Yang–Baxter equation [12]. In this case one can also define [12] two Lie subalgebras $\widetilde{\mathfrak{g}}_{R_\pm}$ such that $\widetilde{\mathfrak{g}}_{R_\pm} + \widetilde{\mathfrak{g}}_{R_-} = \widetilde{\mathfrak{g}}$, but in this case this sum is not a direct sum of two vector spaces, i.e. $\widetilde{\mathfrak{g}}_{R_\pm} \cap \widetilde{\mathfrak{g}}_{R_-} \neq 0$.

Hence, in order to achieve our goal it is necessary to construct with the help of the R-operator an algebra of mutually commuting functions. Contrary to the classical approach of [12] they should commute not with respect to the R-bracket $\{\ ,\ \}_R$ but with respect to the initial Lie–Poisson bracket $\{\ ,\ \}$ on $\widetilde{\mathfrak{g}}^*$. In our previous paper [13] the corresponding functions were constructed using the ring of Casimir functions $I^G(\widetilde{\mathfrak{g}}^*)$. In more detail, we proved that the functions $I_k^{R_\pm}(L) \equiv I_k((R^* \pm 1)(L))$, where $I_k(L), I_l(L) \in I^G(\widetilde{\mathfrak{g}}^*)$, constitute an Abelian subalgebra in $C^\infty(\widetilde{\mathfrak{g}}^*)$ with respect to the standard Lie–Poisson brackets $\{\ ,\ \}$ on $\widetilde{\mathfrak{g}}^*$. The algebra-valued gradients of functions the $I_k^{R_\pm}(L)$ belong to the subalgebras $\widetilde{\mathfrak{g}}_{R_\pm}$ correspondingly.

In the case when the R-operator is of Kostant–Adler–Symes type, i.e. $R = P_+ - P_-$, where $P_\pm$ are projection operators onto subalgebras $\widetilde{\mathfrak{g}}_{R_\pm} = \widetilde{\mathfrak{g}}_\pm$ and $\widetilde{\mathfrak{g}}_+ \cap \widetilde{\mathfrak{g}}_- = 0$ we re-obtain the results of [10] (see also [14]) as a partial case of our construction. In the cases of more complicated R-operators our scheme is new and generalizes the approach of [10]. In particular, the important class of U-V pairs satisfying zero-curvature equations that can be obtained by our method are connected with the so-called "triangular" R-operators. In more detail, if $\widetilde{\mathfrak{g}}$ possess a "triangular" decomposition: $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_+ + \mathfrak{g}_0 + \widetilde{\mathfrak{g}}_-$, where the sum is a direct sum of vector spaces, $\widetilde{\mathfrak{g}}_\pm$ and $\mathfrak{g}_0$ are closed Lie subalgebras and $\widetilde{\mathfrak{g}}_\pm$ are $\mathfrak{g}_0$-modules, R_0 is a solution of the modified Yang–Baxter equation on $\mathfrak{g}_0$, $P_\pm$ are the projection operators onto the subalgebras $\widetilde{\mathfrak{g}}_\pm$ then $R = P_+ + R_0 - P_-$ is a solution of the modified Yang–Baxter equation on $\widetilde{\mathfrak{g}}$ (see [13] for the detailed proof). The Lie subalgebras $\widetilde{\mathfrak{g}}_{R_\pm}$ have in this case the following form: $\widetilde{\mathfrak{g}}_{R_\pm} = \widetilde{\mathfrak{g}}_\pm + \mathfrak{g}_0$, i.e. $\widetilde{\mathfrak{g}}_{R_+} \cap \widetilde{\mathfrak{g}}_{R_-} \neq 0$. Such R-operators are connected with the Thirring-type integrable models. These R-operators were first considered in [4] and in [5], where the usual $sl(2)$-Thirring equation was obtained using "geometric" technique.

In order to obtain the Thirring integrable equation and its various generalizations in the framework of our algebraic approach we consider the case when $\widetilde{\mathfrak{g}}$ coincides with a loop algebra $\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathrm{Pol}(u^{-1}, u)$ or its "twisted" with the help of finite-order automorphism σ subalgebra $\widetilde{\mathfrak{g}}^\sigma$. The algebras $\widetilde{\mathfrak{g}}^\sigma$ possess the natural "triangular" decomposition with the algebra $\mathfrak{g}_0$ being a reductive subalgebra of $\mathfrak{g}$ stable under the action of σ. For each algebra $\widetilde{\mathfrak{g}}^\sigma$ with a natural triangular decomposition and for each classical R operator R_0 on $\mathfrak{g}_0$ we define an integrable equation of the hyperbolic type which we call the "non-Abelian generalized Thirring equation". We show that in the case $\mathfrak{g} = sl(2)$, $\sigma^2 = 1$ and $\mathfrak{g}_0 = \mathfrak{h}$, where $\mathfrak{h}$ is a Cartan subalgebra of $sl(2)$ our construction yields the usual Thirring equation and its standard $sl(2)$-valued U-V pair. We consider in some detail the cases of the generalized Thirring equations that correspond to the second order automorphism of $\mathfrak{g}$. For the case of such automorphisms and $\mathfrak{g} = gl(n)$ and special choice of R_0 we obtain the so-called matrix generalization of complex Thirring equations obtained by other method in [15]. After a reduction to real subalgebra $u(n)$ these equations read as follows:

$$i\partial_{x_-}\Psi_+ = \left(\frac{1}{\kappa_-}\Psi_+(\Psi_-^\dagger\Psi_-) + \kappa_+\Psi_-\right), \qquad i\partial_{x_+}\Psi_- = -\left(\frac{1}{\kappa_+}(\Psi_+\Psi_+^\dagger)\Psi_- + \kappa_-\Psi_+\right),$$

where $\Psi_\pm \in \mathrm{Mat}(p,q)$, $\kappa_\pm \in \mathbb{R}$ are constants, i.e. are exact matrix analogues of the usual massive Thirring equations.

We also consider in detail the case of the generalized Thirring equations that correspond to the Coxeter automorphisms of $\mathfrak{g}$. We call the generalized Thirring equations corresponding to the the Coxeter automorphisms the "generalized Abelian Thirring equations". We show that the number of independent fields in the generalized Abelian Thirring equation corresponding to Coxeter automorphism is equal to $2(\dim\mathfrak{g} - \operatorname{rank}\mathfrak{g})$ and the order of the equations grows with the growth of the rank of Lie algebra $\mathfrak{g}$. We consider in detail the generalized Abelian Thirring equations corresponding to the case $\mathfrak{g} = sl(3)$.

For the sake of completeness we also consider non-linear differential equations of hyperbolic type corresponding to $\widetilde{\mathfrak{g}}^\sigma$ and the Kostant–Adler–Symes R-operator. We show that the obtained equations are in a sence intermediate between the generalized Thirring and non-Abelian Toda equations. The cases of the second order and Coxeter automorphisms are considered. The $sl(2)$ and $sl(3)$ examples are worked out in detail.

The structure of the present paper is the following: in the second section we describe commutative subalgebras associated with the classical R operator. In the third section we obtain associated zero-curvature equations. At last in the fourth section we consider integrable hierarchies associated with loop algebras in different gradings and different R-operators.

2 Commutative subalgebras and classical R operator

Let $\widetilde{\mathfrak{g}}$ be a Lie algebra (finite or infinite-dimensional) with a Lie bracket $[\ ,\ ]$. Let $R : \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$ be some linear operator on $\widetilde{\mathfrak{g}}$. The operator R is called the classical R-operator if it satisfies modified Yang–Baxter equation:

$$R([R(X),Y] + [X,R(Y)]) - [R(X),R(Y)] = [X,Y] \qquad \forall\, X,Y \in \widetilde{\mathfrak{g}}. \tag{2}$$

We will use, in addition to the operator R, the following operators: $R_\pm \equiv R \pm 1$. As it is known [2, 12] the maps $R_\pm$ define Lie subalgebras $\widetilde{\mathfrak{g}}_{R_\pm} \subset \widetilde{\mathfrak{g}}$: $\widetilde{\mathfrak{g}}_{R_\pm} = \operatorname{Im} R_\pm$. It is easy to see from their definition $\widetilde{\mathfrak{g}}_{R_+} + \widetilde{\mathfrak{g}}_{R_-} = \widetilde{\mathfrak{g}}$, but, in general, this sum is not a direct sum of vector spaces, i.e., $\widetilde{\mathfrak{g}}_{R_+} \cap \widetilde{\mathfrak{g}}_{R_-} \neq 0$.

Let $\widetilde{\mathfrak{g}}^*$ be the dual space to $\widetilde{\mathfrak{g}}$ and $\langle\ ,\ \rangle : \widetilde{\mathfrak{g}}^* \times \widetilde{\mathfrak{g}} \to \mathbb{C}$ be a pairing between $\widetilde{\mathfrak{g}}^*$ and $\widetilde{\mathfrak{g}}$. Let $\{X_i\}$ be a basis in the Lie algebra $\widetilde{\mathfrak{g}}$, $\{X_i^*\}$ be a basis in the dual space $\widetilde{\mathfrak{g}}^*$: $\langle X_j^*, X_i\rangle = \delta_{ij}$, $L = \sum_i L_i X_i^* \in \widetilde{\mathfrak{g}}^*$ be the generic element of $\widetilde{\mathfrak{g}}^*$, L_i be the coordinate functions on $\widetilde{\mathfrak{g}}^*$.

Let us consider the standard Lie–Poisson bracket between $F_1, F_2 \in C^\infty(\widetilde{\mathfrak{g}}^*)$:

$$\{F_1(L), F_2(L)\} = \langle L, [\nabla F_1, \nabla F_2]\rangle,$$

where

$$\nabla F_k(L) = \sum_i \frac{\partial F_k(L)}{\partial L_i} X_i$$

is a so-called algebra-valued gradient of F_k. The summations hereafter are implied over all basic elements of $\widetilde{\mathfrak{g}}$.

Let R^* be the operator dual to R, acting in the space $\widetilde{\mathfrak{g}}^*$: $\langle R^*(Y), X\rangle \equiv \langle Y, R(X)\rangle$, $\forall Y \in \widetilde{\mathfrak{g}}^*$, $X \in \widetilde{\mathfrak{g}}$. Let $I^G(\widetilde{\mathfrak{g}}^*)$ be the ring of invariants of the coadjoint representation of $\widetilde{\mathfrak{g}}$.

We will consider the functions $I_k^{R_\pm}(L)$ on $\widetilde{\mathfrak{g}}^*$ defined by the following formulas:

$$I_k^{R_\pm}(L) \equiv I_k((R^* \pm 1)(L)) \equiv I_k(R^*_\pm(L)).$$

The following theorem holds true [13]:

Theorem 1. *Functions $\{I_k^{R_+}(L)\}$ and $\{I_l^{R_-}(L)\}$, where $I_k(L), I_l(L) \in I^G(\widetilde{\mathfrak{g}}^*)$, generate an Abelian subalgebra in $C^\infty(\widetilde{\mathfrak{g}}^*)$ with respect to the standard Lie–Poisson brackets $\{\ ,\ \}$ on $\widetilde{\mathfrak{g}}^*$:*

$$(i)\ \{I_k^{R_+}(L), I_l^{R_+}(L)\} = 0, \quad (ii)\ \{I_k^{R_-}(L), I_l^{R_-}(L)\} = 0, \quad (iii)\ \{I_k^{R_+}(L), I_l^{R_-}(L)\} = 0.$$

Remark 1. Note that the commutative subalgebras constructed in this theorem differ from the commutative subalgebras constructed using standard R-matrix scheme [12]. Indeed, our theorem states commutativity of functions $\{I_k^{R_+}(L)\}$ and $\{I_l^{R_-}(L)\}$ with respect to the initial Lie–Poisson bracket $\{\ ,\ \}$ on $\widetilde{\mathfrak{g}}$. The standard R-matrix scheme states commutativity of the functions $\{I_k(L)\}$ with respect to the so-called R-bracket $\{\ ,\ \}_R$, where:

$$\{F_1(L), F_2(L)\}_R = \langle L, [R(\nabla F_1), \nabla F_2] + [\nabla F_1, R(\nabla F_2)]\rangle.$$

Theorem 1 provides us a large Abelian subalgebra in the space $(C^\infty(\widetilde{\mathfrak{g}}^*), \{\ ,\ \})$. We will consider the following two examples of the R-operators and the corresponding Abelian subalgebras.

Example 1. Let us consider the case of the Lie algebras $\widetilde{\mathfrak{g}}$ with the so-called "Kostant–Adler–Symes" (KAS) decomposition into a direct sum of the two vector subspaces $\widetilde{\mathfrak{g}}_\pm$:

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_+ + \widetilde{\mathfrak{g}}_-,$$

where subspaces $\widetilde{\mathfrak{g}}_\pm$ are closed Lie subalgebras. Let $P_\pm$ be the projection operators onto the subalgebras $\widetilde{\mathfrak{g}}_\pm$ respectively. Then it is known [12] that in this case it is possible to define the so-called Kostant–Adler–Symes R-matrix:

$$R = P_+ - P_-.$$

It is easy to see that $R_+ = 1 + R = 2P_+$, $R_- = R - 1 = -2P_-$, are proportional to the projection operators onto the subalgebras $\widetilde{\mathfrak{g}}_\pm$. It follows that $\widetilde{\mathfrak{g}}_{R_\pm} \equiv \widetilde{\mathfrak{g}}_\pm$ and $\widetilde{\mathfrak{g}}_{R_+} \cap \widetilde{\mathfrak{g}}_{R_-} = 0$.

The Poisson commuting functions $I_k^{R_\pm}(L)$ acquire the following simple form:

$$I_k^{R_\pm}(L) \equiv I_k^\pm(L) \equiv I_k(L_\pm), \qquad \text{where} \quad L_\pm \equiv P^*_\pm L,$$

i.e. $I_k^\pm(L)$ are restrictions of the coadjoint invariants onto the dual spaces $\widetilde{\mathfrak{g}}^*_\pm$.

Example 2. Let us consider the case of Lie algebras $\widetilde{\mathfrak{g}}$ with the "triangular" decomposition:

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_+ + \mathfrak{g}_0 + \widetilde{\mathfrak{g}}_-,$$

where the sum is a direct sum of vector spaces, $\widetilde{\mathfrak{g}}_\pm$ and $\mathfrak{g}_0$ are closed subalgebras, and $\widetilde{\mathfrak{g}}_\pm$ are $\mathfrak{g}_0$-modules. As it is known [4] (see also [13] for the detailed proof), if R_0 is a solution of the modified Yang–Baxter equation (2) on $\mathfrak{g}_0$ then

$$R = P_+ + R_0 - P_- \tag{3}$$

is a solution of the modified Yang–Baxter (2) equation on $\mathfrak{g}$ if $P_\pm$ are the projection operators onto the subalgebras $\widetilde{\mathfrak{g}}_\pm$.

In the case when $R_0 = \pm Id_0$ (which are obviously the solutions of the equation (2) on $\mathfrak{g}_0$) we obtain that R-matrix (3) passes to the standard Kostant–Adler–Symes R-matrix. Nevertheless in the considered "triangular cases" there are other possibilities. For example, if a Lie subalgebra $\mathfrak{g}_0$ is Abelian then (3) is a solution of (2) for any tensor R_0 on $\mathfrak{g}_0$.

In the case of the R-matrix (3) we have: $R_+ = 2P_+ + (P_0 + R_0)$, $R_- = -(2P_- + (P_0 - R_0))$, $\widetilde{\mathfrak{g}}_{R_\pm} = \widetilde{\mathfrak{g}}_\pm + \mathrm{Im}\,(R_0)_\pm$, where $(R_0)_\pm = (R_0 \pm P_0)$ are the $R_\pm$-operators on $\mathfrak{g}_0$ and $\widetilde{\mathfrak{g}}_{R_+} \cap \widetilde{\mathfrak{g}}_{R_-} = \mathrm{Im}\,(R_0)_+ \cap \mathrm{Im}\,(R_0)_-$. The Poisson-commutative functions $I_k^{R_\pm}(L)$ acquire the following form:

$$I_k^{R_\pm}(L) \equiv I_k^{R_{0,\pm}}(L) = I_k\left(L_\pm + \frac{(1 \pm R_0^*)}{2}(L_0)\right),$$

where $L_\pm \equiv P_\pm^* L$, $L_0 \equiv P_0^* L$. We will use such functions constructing different generalizations of the Thirring model.

The Theorem 1 gives us a set of mutually commutative functions on $\widetilde{\mathfrak{g}}^*$ with respect to the brackets $\{\ ,\ \}$ that can be used as an algebra of the integrals of some Hamiltonian system on $\widetilde{\mathfrak{g}}^*$. For the Hamiltonian function one may chose one of the functions $I_k^{R_\pm}(L)$ or their linear combination. Let us consider the corresponding Hamiltonian equation:

$$\frac{dL_i}{dt_k^\pm} = \{L_i, I_k^{R_\pm}(L)\}. \tag{4}$$

The following proposition is true [13]:

Proposition 1. *The Hamiltonian equations of motion* (4) *can be written in the Euler–Arnold (generalized Lax) form:*

$$\frac{dL}{dt_k^\pm} = \mathrm{ad}^*_{V_k^\pm} L, \tag{5}$$

where $V_k^\pm \equiv \nabla I_k^{R_\pm}(L)$.

Remark 2. In the case when $\widetilde{\mathfrak{g}}^*$ can be identified with $\widetilde{\mathfrak{g}}$ and a coadjoint representation can be identified with an adjoint one, equation (5) may be written in the usual Lax (commutator) form.

In the present paper we will not consider finite-dimensional Hamiltonian systems that could be obtained in the framework of our construction but will use equation (5) in order to generate hierarchies of soliton equations in $1+1$ dimensions.

3 Integrable hierarchies and classical R-operators

Let us remind one of the main Lie algebraic approaches to the theory of soliton equations [10]. It is based on the zero-curvature condition and its interpretation as a consistency condition of two commuting Lax flows. The following Proposition is true:

Proposition 2. *Let* $H_1, H_2 \in C^\infty(\widetilde{\mathfrak{g}}^*)$ *be two Poisson commuting functions:* $\{H_1, H_2\} = 0$, *where* $\{\ ,\ \}$ *is a standard Lie–Poisson brackets on* $\widetilde{\mathfrak{g}}^*$. *Then their* $\widetilde{\mathfrak{g}}$*-valued gradients satisfy the "modified" zero-curvature equation:*

$$\frac{\partial \nabla H_1}{\partial t_2} - \frac{\partial \nabla H_2}{\partial t_1} + [\nabla H_1, \nabla H_2] = k \nabla I, \tag{6}$$

where I *is a Casimir function and* t_i *are parameters along the trajectories of Hamiltonian equations corresponding to the Hamiltonians* H_i *and* k *is an arbitrary constant.*

For the theory of soliton equations one needs the usual zero-curvature condition (case $k = 0$ in the above proposition) and the infinite set of the commuting Hamiltonians generating the corresponding U-V pairs. This can be achieved requiring that $\widetilde{\mathfrak{g}}$ is infinite-dimensional of a special type. In more detail, the following theorem is true:

Theorem 2. *Let $\widetilde{\mathfrak{g}}$ be an infinite-dimensional Lie algebra of $\mathfrak{g}$-valued function of the one complex variable u and a Lie algebra $\mathfrak{g}$ be semisimple. Let R be a classical R operator on $\widetilde{\mathfrak{g}}$, $L(u)$ be the generic element of the dual space $\widetilde{\mathfrak{g}}^*$, $I_k(L(u))$ be Casimir functions on $\widetilde{\mathfrak{g}}^*$ such that functions $I_k^{R_\pm}(L(u))$ are finite polynomials on $\widetilde{\mathfrak{g}}^*$. Then the $\widetilde{\mathfrak{g}}$-valued functions $\nabla I_k^{R_\pm}(L(u))$ satisfy the zero-curvature equations:*

$$\frac{\partial \nabla I_k^{R_\pm}(L(u))}{\partial t_l^{\pm}} - \frac{\partial \nabla I_l^{R_\pm}(L(u))}{\partial t_k^{\pm}} + [\nabla I_k^{R_\pm}(L(u)), \nabla I_l^{R_\pm}(L(u))] = 0, \tag{7}$$

$$\frac{\partial \nabla I_k^{R_\pm}(L(u))}{\partial t_l^{\mp}} - \frac{\partial \nabla I_l^{R_\mp}(L(u))}{\partial t_k^{\pm}} + [\nabla I_k^{R_\pm}(L(u)), \nabla I_l^{R_\mp}(L(u))] = 0. \tag{8}$$

Proof. As it follows from Proposition 2 and Theorem 1 the algebra-valued gradients $\nabla I_k^{R_\pm}(L(u))$, $\nabla I_l^{R_\pm}(L(u))$ and $\nabla I_k^{R_\pm}(L(u))$, $\nabla I_l^{R_\mp}(L(u))$ satisfy the "modified" zero-curvature equations (6). On the other hand, as it is not difficult to show, for the case of the Lie algebras $\widetilde{\mathfrak{g}}$ described in the theorem the algebra-valued gradients of the Casimir functions are proportional to powers of the generic element of the dual space $L(u)$, i.e. to the formal power series. On the other hand, due to the condition that all $I_k^{R_\pm}(L(u))$ are finite polynomials, their algebra-valued gradients are *finite* linear combinations of the basic elements of the Lie algebra $\widetilde{\mathfrak{g}}$. Hence the corresponding modified zero-curvature equations are satisfied if and only if the corresponding coefficients k in these equations are equal to zero, i.e. when they are reduced to the usual zero-curvature conditions. This proves the theorem. ■

Remark 3. Note that equations (7), (8) define three types of integrable hierarchies: two "small" hierarchies associated with Lie subalgebras $\widetilde{\mathfrak{g}}_{R_\pm}$ defined by equations (7) and one "large" hierarchy associated with the whole Lie algebra $\widetilde{\mathfrak{g}}$ that include both types of equations (7) and (8). Equations (8) have an interpretation of the "negative flows" of the integrable hierarchy associated with $\widetilde{\mathfrak{g}}_{R_\pm}$. In the case $\widetilde{\mathfrak{g}}_{R_+} \simeq \widetilde{\mathfrak{g}}_{R_-}$ the corresponding "small" hierarchies are equivalent and the large hierarchy associated with $\widetilde{\mathfrak{g}}$ may be called a "double" of the hierarchy associated with $\widetilde{\mathfrak{g}}_{R_\pm}$.

In the next subsection we will consider in detail a simple example of the above theorem when $\widetilde{\mathfrak{g}}$ is a loop algebra and R-operator is not of Kostant–Adler–Symes type but a triangular one. We will be interested in the "large" hierarchy associated with $\widetilde{\mathfrak{g}}$ and, in more detail, in the simplest equation of this hierarchy which will coincide with the generalization of the Thirring equation.

4 Loop algebras and generalized Thirring equations

4.1 Loop algebras and classical *R*-operators

In this subsection we remind several important facts from the theory of loop algebras [8].

Let $\mathfrak{g}$ be semisimple (reductive) Lie algebra. Let $\mathfrak{g} = \sum\limits_{j=0}^{p-1} \mathfrak{g}_{\overline{j}}$ be $Z_p = \mathbb{Z}/p\mathbb{Z}$ grading of $\mathfrak{g}$, i.e.: $[\mathfrak{g}_{\overline{i}}, \mathfrak{g}_{\overline{j}}] \subset \mathfrak{g}_{\overline{i+j}}$ where $\overline{j}$ denotes the class of equivalence of the elements $j \in \mathbb{Z}$ mod$p$ $\mathbb{Z}$. It is known that the $\mathbb{Z}_p$-grading of $\mathfrak{g}$ may be defined with the help of some automorphism σ of the order p, such that $\sigma(\mathfrak{g}_{\overline{i}}) = e^{2\pi i k/p}\mathfrak{g}_{\overline{i}}$ and $\mathfrak{g}_{\overline{0}}$ is the algebra of σ-invariants: $\sigma(\mathfrak{g}_{\overline{0}}) = \mathfrak{g}_{\overline{0}}$.

Let $\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathrm{Pol}(u, u^{-1})$ be a loop algebra. Let us consider the following subspace in $\widetilde{\mathfrak{g}}$:

$$\widetilde{\mathfrak{g}}^{\sigma} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_{\overline{j}} \otimes u^j.$$

It is known [8] that this subspace is a closed Lie subalgebra and if we extend the automorphism σ to the map $\tilde{\sigma}$ of the whole algebra $\widetilde{\mathfrak{g}}$, defining its action on the space $\mathfrak{g} \otimes \mathrm{Pol}(u, u^{-1})$ in the standard way [8]: $\tilde{\sigma}(X \otimes u^k) = \sigma(X) \otimes e^{-2\pi i k/p} u^k$, then the subalgebra $\widetilde{\mathfrak{g}}^{\sigma}$ can be defined as the subalgebra of $\tilde{\sigma}$-invariants in $\widetilde{\mathfrak{g}}$:

$$\widetilde{\mathfrak{g}}^{\sigma} = \{X \otimes p(u) \in \widetilde{\mathfrak{g}} |\ \tilde{\sigma}(X \otimes p(u)) = X \otimes p(u)\}.$$

We will call the algebra $\widetilde{\mathfrak{g}}^{\sigma}$ the loop subalgebra "twisted" with the help of σ. The basis in $\widetilde{\mathfrak{g}}^{\sigma}$ consists of algebra-valued functions $\{X^j_\alpha \equiv X^{\overline{j}}_\alpha u^j\}$, where $X^{\overline{j}}_\alpha \in \mathfrak{g}_{\overline{j}}$. Let us define the pairing between $\widetilde{\mathfrak{g}}^{\sigma}$ and $(\widetilde{\mathfrak{g}}^{\sigma})^*$ in the standard way:

$$\langle X, Y\rangle = \operatorname*{res}_{u=0} u^{-1}(X(u), Y(u)),$$

where $X \in \widetilde{\mathfrak{g}}^{\sigma}$, $Y \in (\widetilde{\mathfrak{g}}^{\sigma})^*$ and $(\ ,\ )$ is a bilinear, invariant, nondegenerate form on $\mathfrak{g}$. It is easy to see that with respect to such a pairing the dual space $(\widetilde{\mathfrak{g}}^{\sigma})^*$ may be identified with the Lie algebra $\widetilde{\mathfrak{g}}^{\sigma}$ itself. The dual basis in $(\widetilde{\mathfrak{g}}^{\sigma})^*$ has the form: $\{Y^j_\alpha \equiv X^{\overline{-j},\alpha} u^{-j}\}$, where $X^{\overline{-j},\alpha}$ is a dual basis in the space $\mathfrak{g}_{\overline{-j}}$.

The Lie algebra $\widetilde{\mathfrak{g}}^{\sigma}$ possesses KAS decomposition $\widetilde{\mathfrak{g}}^{\sigma} = \widetilde{\mathfrak{g}}^{\sigma+} + \widetilde{\mathfrak{g}}^{\sigma-}$ [11], where

$$\widetilde{\mathfrak{g}}^{\sigma+} = \bigoplus_{j\geq 0} \mathfrak{g}_{\overline{j}} \otimes u^j, \qquad \widetilde{\mathfrak{g}}^{\sigma-} = \bigoplus_{j<0} \mathfrak{g}_{\overline{j}} \otimes u^j.$$

It defines in a natural way the Kostant–Adler–Symes R-operator:

$$R = P^+ - P^-,$$

where $P^{\pm}$ are projection operators onto Lie algebra $\widetilde{\mathfrak{g}}^{\sigma\pm}$.

The twisted loop algebra $\widetilde{\mathfrak{g}}^{\sigma}$ also possesses "triangular" decomposition:

$$\widetilde{\mathfrak{g}}^{\sigma} = \widetilde{\mathfrak{g}}^{\sigma}_{+} + \mathfrak{g}_0 + \widetilde{\mathfrak{g}}^{\sigma}_{-},$$

where $\widetilde{\mathfrak{g}}^{\sigma-} \equiv \widetilde{\mathfrak{g}}^{\sigma}_{-}$, $\widetilde{\mathfrak{g}}^{\sigma+} \equiv \widetilde{\mathfrak{g}}^{\sigma}_{+} + \mathfrak{g}_0$.

It defines in a natural way the triangular R-operator

$$R = P_+ + R_0 - P_-,$$

where $P_{\pm}$ are the projection operators onto Lie algebra $\widetilde{\mathfrak{g}}_{\sigma\pm}$ and R_0 is an R-operator on $\mathfrak{g}_0$.

The Lie algebra $\mathfrak{g}_0$ in this decomposition is a reductive Lie subalgebra of the Lie algebra $\mathfrak{g}$. Due to the fact that a lot of solutions of the modified classical Yang–Baxter equations on reductive Lie algebras are known, one can construct explicitly R-operators R_0 on $\mathfrak{g}_0$, the Lie subalgebras $\widetilde{\mathfrak{g}}_{R_\pm} = \widetilde{\mathfrak{g}}_{\pm} + \mathrm{Im}(R_0)_{\pm}$ and the Poisson-commutative functions constructed in the Theorem 1.

4.2 Generalized Thirring models

As we have seen in the previous subsection, each of the gradings of the loop algebras, corresponding to the different automorphisms σ yields its own triangular decomposition. Let us consider the simplest equations of integrable hierarchies, corresponding to different triangular decompositions in more detail. The generic elements of the dual space to the Lie algebras $\widetilde{\mathfrak{g}}_{R_\pm}$ are written as follows:

$$L_\pm(u) = r^*_\pm\left(\sum_{\alpha=1}^{\dim\mathfrak{g}_{\bar{0}}} L^0_\alpha X^{\overline{0},\alpha}\right) + \sum_{j=\pm1}^{\pm\infty}\sum_{\alpha=1}^{\dim\mathfrak{g}_{\bar{j}}} L^{(j)}_\alpha X^{\overline{-j},\alpha}u^{-j} = r^*_\pm(L^{(0)}) + \sum_{j=\pm1}^{\pm\infty} L^{(j)}u^{-j},$$

where $L^{(-j)} \in \mathfrak{g}_{\bar{j}}$ and in order to simplify the notations we put $r_\pm \equiv \frac{(1\pm R_0)}{2}$.

Let $(\ ,\ )$ be an invariant nondegenerated form on the underlying semisimple (reductive) Lie algebra $\mathfrak{g}$. Then $I_2(L) = \frac{1}{2}(L, L)$ is a second order Casimir function on $\mathfrak{g}^*$. The corresponding generating function of the second order integrals has the form:

$$\begin{aligned} I_2^\pm(L(u)) = I_2(L_\pm(u)) &= \frac{1}{2}\left(r^*_\pm(L^{(0)}) + \sum_{j=\pm1}^{\pm\infty} L^{(j)}u^{-j}, r^*_\pm(L^{(0)}) + \sum_{j=\pm1}^{\pm\infty} L^{(j)}u^{-j}\right) \\ &= \sum_{k=0}^{\pm\infty} I_2^{pk}u^{-pk}, \end{aligned}$$

where p is an order of σ. The simplest of these integrals are:

$$I_2^{\pm0} = \frac{1}{2}\big(r^*_\pm(L^{(0)}), r^*_\pm(L^{(0)})\big), \qquad I_2^{\pm p} = \big(r^*_\pm(L^{(0)}), L^{(\pm p)}\big) + \frac{1}{2}\sum_{j=1}^{p-1}\big(L^{(\pm j)}, L^{(\pm(p-j))}\big).$$

The algebra-valued gradients of functions $I_2^{\pm p}$ read as follows:

$$\nabla I_2^{\pm p}(u) = u^{\pm p} r^*_\pm(L^{(0)}) + \sum_{j=1}^{p-1} u^{\pm(p-j)}L^{(\pm j)} + r_\pm(L^{(\pm p)}). \tag{9}$$

They coincide with the U-V pair of the generalized Thirring model. The corresponding zero-curvature condition:

$$\frac{\partial\nabla I_2^{+p}(L(u))}{\partial x_-} - \frac{\partial\nabla I_2^{-p}(L(u))}{\partial x_+} + [\nabla I_2^{+p}(L(u)), \nabla I_2^{-p}(L(u))] = 0 \tag{10}$$

yields the following system of differential equations in partial derivatives:

$$-\partial_{x_-} r^*_+(L^{(0)}) = [r^*_+(L^{(0)}), r_-(L^{(-p)})], \tag{11a}$$

$$\partial_{x_+} r_-(L^{(0)}) = [r_+(L^{(p)}), r^*_-(L^{(0)})], \tag{11b}$$

$$-\partial_{x_-} L^{(p-j)} = [L^{(p-j)}, r_-(L^{(-p)})] + \sum_{i=j+1}^{p-1}[L^{(p-i)}, L^{(i-j-p)}] + [r^*_+(L^{(0)}), L^{(-j)}], \tag{11c}$$

$$\begin{aligned} \partial_{x_+} L^{(j-p)} &= [r_+(L^{(p)}), L^{(j-p)})] + \sum_{i=j+1}^{p-1}[L^{(p+j-i)}, L^{(j-p)}] \\ &\quad + [L^{(j)}, r^*_-(L^{(0)})], j \in 1, p-1, \end{aligned} \tag{11d}$$

$$\begin{aligned} \partial_{x_+} r_-(L^{(-p)}) - \partial_{x_-} r_+(L^{(p)}) &= [r^*_+(L^{(0)}), r^*_-(L^{(0)})] + \sum_{i=1}^{p-1}[L^{(p-i)}, L^{(i-p)}] \\ &\quad + [r_+(L^{(p)}), r_-(L^{(-p)})]. \end{aligned} \tag{11e}$$

These equations are *integrable generalization of Thirring equations* corresponding to Lie algebra $\mathfrak{g}$, its Z_p-grading defined with the help of the automorphism σ of the order p and the classical R-operator R_0 on $\mathfrak{g}_{\bar{0}}$. We will call this system of equations the *non-Abelian generalized Thirring system*. In order to recognize in this complicated system of hyperbolic equations generalization of Thirring equations we will consider several examples.

4.2.1 Case of the second-order automorphism

Let us consider the case when automorphism σ is involutive, i.e. $p = 2$. Then $\mathfrak{g}_{\bar{1}} = \mathfrak{g}_{\overline{-1}}$ and Lax matrices $L_{\pm}(u)$ are the following:

$$L_{\pm}(u) = r^*_{\pm}(L^{(0)}) + u^{\mp 1}L^{(\pm 1)} + u^{\mp 2}L^{(\pm 2)} + \cdots ,$$

and the U-V pair (9) acquire more simple form:

$$\nabla I_2^{\pm 2}(u) = u^{\pm 2}r^*_{\pm}(L^{(0)}) + u^{\pm 1}L^{(\pm 1)} + r_{\pm}(L^{(\pm 2)}). \tag{12}$$

The corresponding zero-curvature condition yields the following system of differential equations in partial derivatives:

$$\partial_{x_-} r^*_+(L^{(0)}) = [r_-(L^{(-2)}), r^*_+(L^{(0)})], \tag{13a}$$
$$\partial_{x_+} r^*_-(L^{(0)}) = [r_+(L^{(2)}), r^*_-(L^{(0)})], \tag{13b}$$
$$\partial_{x_-} L^{(1)} = [r_-(L^{(-2)}), L^{(1)}] + [L^{(-1)}, r^*_+(L^{(0)})], \tag{13c}$$
$$\partial_{x_+} L^{(-1)} = [r_+(L^{(2)}), L^{(-1)}] + [L^{(1)}, r^*_-(L^{(0)})], \tag{13d}$$
$$\begin{aligned}\partial_{x_+} r_-(L^{(-2)}) - \partial_{x_-} r_+(L^{(2)}) &= [r^*_+(L^{(0)}), r^*_-(L^{(0)})] + [L^{(1)}, L^{(-1)}] \\ &\quad + [r_+(L^{(2)}), r_-(L^{(-2)})].\end{aligned} \tag{13e}$$

The system of equations (13) is still sufficiently complicated. In order to recognize in this system the usual Thirring equations we have to consider the case $\mathfrak{g} = sl(2)$.

Example 3. Let $\mathfrak{g} = sl(2)$ and σ be the Cartan involution, i.e. $sl(2)_{\bar{0}} = \mathfrak{h} = \mathrm{diag}\,(\alpha, -\alpha)$ and $sl(2)_{\bar{1}}$ consists of the matrices with zeros on the diagonal. The Lax matrices $L_{\pm}(u)$ have the following form:

$$L_{\pm}(u) = r^*_{\pm}\begin{pmatrix} \alpha^{(0)} & 0 \\ 0 & -\alpha^{(0)} \end{pmatrix} + u^{\mp 1}\begin{pmatrix} 0 & \beta^{(\pm 1)} \\ \gamma^{(\pm 1)} & 0 \end{pmatrix} + u^{\mp 2}\begin{pmatrix} \alpha^{(\pm 2)} & 0 \\ 0 & -\alpha^{(\pm 2)} \end{pmatrix} + \cdots .$$

Due to the fact that $sl(2)_{\bar{0}}$ is Abelian the one-dimensional linear maps $r_{\pm}$ are written as follows: $r^*_{\pm}(L^{(0)}) = k_{\pm}L^{(0)}$ where $k_{\pm}$ are some constants, such that $k_+ + k_- = 1$.

The simplest Hamiltonians obtained in the framework of our scheme are:

$$I_2^{\pm 0} = k_{\pm}(\alpha^{(0)})^2, \qquad I_2^{\pm 2} = 2k_{\pm}\alpha^{(0)}\alpha^{(\pm 2)} + \beta^{(\pm 1)}\gamma^{(\pm 1)}.$$

As it follows from the Theorem 1 and, as it is also easy to verify by the direct calculations, these Hamiltonians commute with respect to the standard Lie–Poisson brackets on $\widetilde{sl(2)}^{\sigma}$:

$$\{\alpha^{(2k)}, \beta^{(2l+1)}\} = \beta^{(2(k+l)+1)}, \qquad \{\alpha^{(2k)}, \gamma^{(2l+1)}\} = -\gamma^{(2(k+l)+1)},$$
$$\{\beta^{(2k+1)}, \gamma^{(2l+1)}\} = 2\alpha^{(2(k+l)+2)},$$
$$\{\alpha^{(2k)}, \alpha^{(2l)}\} = \{\beta^{(2k+1)}, \beta^{(2l+1)}\} = \{\gamma^{(2k+1)}, \gamma^{(2l+1)}\} = 0.$$

As a consequence, they produce the correct U-V pair for zero-curvature equations:

$$\nabla I_2^{\pm 2} = k_{\pm}\begin{pmatrix} \alpha^{(\pm 2)} & 0 \\ 0 & -\alpha^{(\pm 2)} \end{pmatrix} + u^{\pm 1}\begin{pmatrix} 0 & \beta^{(\pm 1)} \\ \gamma^{(\pm 1)} & 0 \end{pmatrix} + k_{\pm}u^{\pm 2}\begin{pmatrix} \alpha^{(0)} & 0 \\ 0 & -\alpha^{(0)} \end{pmatrix},$$

where $U \equiv \nabla I_2^{+2}$, $V \equiv \nabla I_2^{-2}$.

Due to the fact that $I_2^{\pm 0}$ are constants of motion we have that $\alpha^{(0)}$ is also a constant of motion and equations (13a), (13b) are satisfied automatically. The integrals $I_2^{\pm 2}$ are constants of motion too, that is why $\alpha^{(\pm 2)}$ are expressed via $\beta^{(\pm 1)}$, $\gamma^{(\pm 1)}$. Hence, equation (13e) is not independent and follows from equations (13c), (13d). These equations are independent. The last summand in the equations (13c), (13d) gives the linear "massive term" in the Thirring equation. The first summand gives the cubic non-linearity.

Let us write the equations (13c), (13d) in the case at hand in more detail taking into account the explicit form of the matrices $L^{(k)}$ and linear operators $r_\pm$:

$$\begin{aligned}
&\partial_{x_-}\gamma^{(1)} = 2k_+\alpha^{(0)}\gamma^{(-1)} - 2k_-\alpha^{(-2)}\gamma^{(1)},\\
&\partial_{x_-}\beta^{(1)} = -2k_+\alpha^{(0)}\beta^{(-1)} + 2k_-\alpha^{(-2)}\beta^{(1)},\\
&\partial_{x_+}\gamma^{(-1)} = 2k_-\alpha^{(0)}\gamma^{(1)} - 2k_+\alpha^{(2)}\gamma^{(-1)},\\
&\partial_{x_+}\beta^{(-1)} = -2k_-\alpha^{(0)}\beta^{(1)} + 2k_+\alpha^{(2)}\beta^{(-1)},
\end{aligned} \tag{14}$$

where we have put $I_2^{\pm 2} = 0$ and, hence, $\alpha^{(\pm 2)} = -\frac{1}{2k_\pm \alpha^{(0)}}(\beta^{(\pm 1)}\gamma^{(\pm 1)})$.

The system of equations (14) admits a reduction $\alpha^{(0)} = ic$, $\gamma^{(\pm 1)} = -\bar{\beta}^{(\pm 1)} \equiv -\bar{\psi}_{\pm 1}$, which corresponds to the restriction onto the real subalgebra $su(2)$ of the complex Lie algebra $sl(2)$.

After such a reduction the system of equations (14) is simplified and acquires the form:

$$ic\partial_{x_-}\psi_1 = 2k_+c^2\psi_{-1} + |\psi_{-1}|^2\psi_1, \qquad ic\partial_{x_+}\psi_{-1} = 2k_-c^2\psi_1 + |\psi_1|^2\psi_{-1}.$$

In the case $k_+ = k_- = \frac{1}{2}$ these equations are usual Thirring equations with a mass $m = c^2$.

4.2.2 Matrix generalization of Thirring equation

Let us return to the generalized Thirring model in the case of the higher rank Lie algebra $\mathfrak{g}$, its automorphism of the second order and corresponding Z_2-grading of $\mathfrak{g}$: $\mathfrak{g} = \mathfrak{g}_{\bar 0} + \mathfrak{g}_{\bar 1}$. We are interested in the cases that will be maximally close to the case of the ordinary Thirring equation. In particular, we wish to have $r^*_\pm(L^{(0)})$ that enter into our U-V pair (12) near the second order of spectral parameter to be constant along all time flows. As it follows from the equations (13a), (13b), this is not true for the general r-matrix R_0 on $\mathfrak{g}_{\bar 0}$. Fortunately there are very special cases when it is indeed so. The following proposition holds true:

Proposition 3. *Let $\mathfrak{g}_{\bar 0}$ admit the decomposition into direct sum of two reductive subalgebras: $\mathfrak{g}_{\bar 0} = \mathfrak{g}^+_{\bar 0} \oplus \mathfrak{g}^-_{\bar 0}$. Let $L^{(0)} = L^{(0)}_+ + L^{(0)}_-$ be the corresponding decomposition of the element of the dual space. Let $\zeta(\mathfrak{g}^\pm_{\bar 0})$ be a center of the subalgebra $\mathfrak{g}^\pm_{\bar 0}$, $K(L^{(0)}_\pm)$ be a part of $L^{(0)}_\pm$ dual to the center of the subalgebra $\mathfrak{g}^\pm_{\bar 0}$. Then $R_0 = P^+_0 - P^-_0$ is the R operator on $\mathfrak{g}_{\bar 0}$, $r_+(L^{(0)}) = L^{(0)}_+$, $r_-(L^{(0)}) = L^{(0)}_-$ are constant along all time flows and the reduction $L^{(0)} = K_+ + K_-$, where $K_\pm$ is a constant element of $\zeta(\mathfrak{g}^\pm_{\bar 0})^*$, is consistent with all equations of the hierarchy* (7), (8) *corresponding to the loop algebra $\widetilde{\mathfrak{g}}^\sigma$ and triangular R-operator on $\widetilde{\mathfrak{g}}^\sigma$ with the described above KAS R-operator R_0 on $\mathfrak{g}_{\bar 0}$.*

Proof. Let us at first note that in the case under consideration $r^*_\pm = r_\pm$. Let us show that $r^*_\pm(L^{(0)}) = r_\pm(L^{(0)})$ are constant along all time flows generated by the second order Hamiltonians $I_2^{\pm 2k}$. It is easy to show that the corresponding algebra-valued gradients have in this case the following form:

$$\nabla I_2^{\pm 2k} = P_\pm(u^{2k}L_\pm(u)) + r_\pm(L^{(\pm 2k)}) = u^{2k}L_\pm(u) - P_\mp(u^{2k}L_\pm(u)) - r_\mp(L^{(\pm 2k)}),$$

where $P_\pm$ are the projection operators onto the subalgebras $\widetilde{\mathfrak{g}}^\sigma_\pm$ in the triangular decomposition of the Lie algebra $\widetilde{\mathfrak{g}}^\sigma$ and we took into account that $r_+ + r_- = 1$. Let us substitute this expression into the Lax equation:

$$\frac{dL(u)}{dt^\pm_{2k}} = [\nabla I_2^{\pm 2k}, L(u)], \tag{15}$$

and take into account that the case under consideration corresponds to the Kostant–Adler–Symes decomposition and, hence, we have correctly defined the decomposition $L(u) = L_+(u) + L_-(u)$, and each of two infinite-component Lax equation (15) is correctly restricted to each of the subspaces $L_\pm(u)$:

$$\frac{dL_\pm(u)}{dt^\pm_{2k}} = [-P_\mp(u^{2k}L_\pm(u)) - r_\mp(L^{(\pm 2k)}), L_\pm(u)],$$
$$\frac{dL_\pm(u)}{dt^\mp_{2k}} = [P_\mp(u^{2k}L_\mp(u)) + r_\mp(L^{(\mp 2k)}), L_\pm(u)].$$

Making projection onto the subalgebra $\mathfrak{g}_{\bar{0}}$ in these equations we obtain the equations:

$$\frac{dr_\pm(L^{(0)})}{dt^\pm_{2k}} = -[r_\mp(L^{(\pm 2k)}), r_\pm(L^{(0)})], \qquad \frac{dr_\pm(L^{(0)})}{dt^\mp_{2k}} = [r_\mp(L^{(\mp 2k)}), r_\pm(L^{(0)})].$$

Due to the fact that our r-matrix is of Kostant–Adler–Symes type, we have $[r_\mp(X), r_\pm(Y)] = 0$, $\forall\, X, Y \in \mathfrak{g}_{\bar{0}}$ and, hence, $r_+(L^{(0)}) \equiv L^{(0)}_+$, $r_-(L^{(0)}) \equiv L^{(0)}_-$ are constant along all time flows generated by $I^{\pm 2k}_2$. In analogous way it is shown that $L^{(0)} = L^{(0)}_+ + L^{(0)}_-$ are constant with respect to the time flows generated by the higher order integrals $I^{\pm 2k}_m$. Hence, in this case components of $L^{(0)}$ belong to the algebra of the integrals of motion of our infinite-component Hamiltonian or Lax system. This algebra of integrals is, generally speaking, non-commutative i.e. $\{l^{(0)}_\alpha, l^{(0)}_\beta\} = c^\gamma_{\alpha\beta} l^{(0)}_\gamma$, where $c^\gamma_{\alpha\beta}$ are structure constants of the Lie algebra $\mathfrak{g}_{\bar{0}}$. That is why in order to have a correct and consistent reduction with respect to these integrals we have to restrict the dynamics to the surface of zero level of the integrals belonging to $([\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}])^*$. Other part of $(\mathfrak{g}_{\bar{0}})^*$, namely the one belonging to $(\mathfrak{g}_{\bar{0}}/[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}])^* = (\zeta(\mathfrak{g}_{\bar{0}}))^* = (\zeta(\mathfrak{g}^+_{\bar{0}}))^* + (\zeta(\mathfrak{g}^-_{\bar{0}}))^*$, may be put to be equal to a constant, i.e. correct reduction is: $L^{(0)} = K_+ + K_-$, where $K_\pm$ is a constant element of $\zeta(\mathfrak{g}^\pm_{\bar{0}})^*$. ■

Hence, in this case we have the following form of simplest U-V pair (12) for the zero-curvature condition:

$$\nabla I^{\pm 2}_2(u) = u^{\pm 2} K_\pm + u^{\pm 1} L^{(\pm 1)} + L^{(\pm 2)}_\pm, \tag{16}$$

where $L^{(\pm 2)}_\pm = P^\pm_0(L^{(\pm 2)}) \in \mathfrak{g}^\pm_{\bar{0}}$. The corresponding equations (13a), (13b) are satisfied automatically and the rest of equations of the system (13) are:

$$\partial_{x_-} L^{(1)} = [L^{(-2)}_-, L^{(1)}] + [L^{(-1)}, K_+], \tag{17a}$$
$$\partial_{x_+} L^{(-1)} = [L^{(2)}_+, L^{(-1)}] + [L^{(1)}, K_-], \tag{17b}$$
$$\partial_{x_+} L^{(-2)}_- - \partial_{x_-} L^{(2)}_+ = [L^{(1)}, L^{(-1)}], \tag{17c}$$

where we have again used that in our case $[r_+(X), r_-(Y)] = 0$, $\forall\, X, Y \in \mathfrak{g}_{\bar{0}}$.

The equations (17a), (17b) are *the simplest possible generalizations of the Thirring equations*. The first term in the right-hand-side of this equation is an analog of the cubic non-linearity of Thirring equation. The second term is an analog of linear "massive" term of the Thirring equations. In all the cases one can express $L^{(\pm 2)}_\pm$ via $L^{(\pm 1)}$, and equation (17c) will follow from equations (17a), (17b). In order to show this we will consider the following example.

Example 4. Let us consider the case $\mathfrak{g} = gl(n)$, with the following Z_2-grading: $\mathfrak{g} = gl(n)_{\bar{0}} + gl(n)_{\bar{1}}$, where $gl(n)_{\bar{0}} = gl(p) + gl(q)$ and $gl(n)_{\bar{1}} = \mathbb{C}^{2pq}$, i.e.

$$gl(n)_{\bar{0}} = \begin{pmatrix} \hat{\alpha} & 0 \\ 0 & \hat{\delta} \end{pmatrix}, \qquad gl(n)_{\bar{1}} = \begin{pmatrix} 0 & \hat{\beta} \\ \hat{\gamma} & 0 \end{pmatrix},$$

where $\hat{\alpha} \in gl(p)$, $\hat{\delta} \in gl(q)$, $\hat{\beta} \in \mathrm{Mat}(p, q)$, $\hat{\gamma} \in \mathrm{Mat}(q, p)$.

In this case $\mathfrak{g}_{\bar{0}}^{+} = gl(p)$, $\mathfrak{g}_{\bar{0}}^{-} = gl(q)$, $\zeta(\mathfrak{g}_{\bar{0}}^{+}) = k_{+}1_p$, $\zeta(\mathfrak{g}_{\bar{0}}^{-}) = k_{-}1_q$. The corresponding U-V pair (16) has the form:

$$\nabla I_2^{+2}(u) = k_+ u^2 \begin{pmatrix} 1_p & 0 \\ 0 & 0 \end{pmatrix} + u \begin{pmatrix} 0 & \hat{\beta}_+ \\ \hat{\gamma}_+ & 0 \end{pmatrix} + \begin{pmatrix} \hat{\alpha}_+ & 0 \\ 0 & 0 \end{pmatrix},$$

$$\nabla I_2^{-2}(u) = k_- u^{-2} \begin{pmatrix} 0 & 0 \\ 0 & 1_q \end{pmatrix} + u^{-1} \begin{pmatrix} 0 & \hat{\beta}_- \\ \hat{\gamma}_- & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \hat{\delta}_- \end{pmatrix}.$$

The corresponding zero-curvature condition yields in this case the following equations:

$$\partial_{x_-}\hat{\beta}_+ = -(\hat{\beta}_+\hat{\delta}_- + k_+\hat{\beta}_-), \qquad \partial_{x_-}\hat{\gamma}_+ = (\hat{\delta}_-\hat{\gamma}_+ + k_+\hat{\gamma}_-), \tag{18a}$$

$$\partial_{x_+}\hat{\beta}_- = (\hat{\alpha}_+\hat{\beta}_- + k_-\hat{\beta}_+), \qquad \partial_{x_+}\hat{\gamma}_- = -(\hat{\gamma}_-\hat{\alpha}_+ + k_-\hat{\gamma}_+), \tag{18b}$$

$$\partial_{x_+}\hat{\delta}_- = (\hat{\gamma}_+\hat{\beta}_- - \hat{\gamma}_-\hat{\beta}_+), \qquad \partial_{x_-}\hat{\alpha}_+ = (\hat{\beta}_-\hat{\gamma}_+ - \hat{\gamma}_-\hat{\beta}_+). \tag{18c}$$

By direct verification it is easy to show that the substitution of variables:

$$\hat{\delta}_- = -\frac{1}{k_-}\hat{\gamma}_-\hat{\beta}_-, \qquad \hat{\alpha}_+ = -\frac{1}{k_+}\hat{\beta}_+\hat{\gamma}_+$$

solves equation (18c) and yields the following non-linear differential equations:

$$\partial_{x_-}\hat{\beta}_+ = \frac{1}{k_-}\hat{\beta}_+(\hat{\gamma}_-\hat{\beta}_-) - k_+\hat{\beta}_-, \qquad \partial_{x_-}\hat{\gamma}_+ = -\frac{1}{k_-}(\hat{\gamma}_-\hat{\beta}_-)\gamma_+ + k_+\hat{\gamma}_-,$$

$$\partial_{x_+}\hat{\beta}_- = -\frac{1}{k_+}(\hat{\beta}_+\hat{\gamma}_+)\hat{\beta}_- + k_-\hat{\beta}_+, \qquad \partial_{x_+}\hat{\gamma}_- = \frac{1}{k_+}\hat{\gamma}_-(\hat{\beta}_+\hat{\gamma}_+) - k_-\hat{\gamma}_+.$$

These equations are *matrix generalization of the complex Thirring system* [15]. They admit several reductions (restrictions to the different real form of the algebra $gl(n)$). For the case of an $u(n)$ reduction we have

$$\hat{\gamma}_\pm = -\hat{\beta}_\pm^\dagger = -\Psi_\pm^\dagger, \qquad k_\pm = i\kappa_\pm, \qquad \kappa_\pm \in \mathbb{R}$$

and we obtain the following non-linear matrix equations in partial derivatives:

$$i\partial_{x_-}\Psi_+ = \frac{1}{\kappa_-}\Psi_+(\Psi_-^\dagger\Psi_-) + \kappa_+\Psi_-, \qquad i\partial_{x_+}\Psi_- = -\frac{1}{\kappa_+}(\Psi_+\Psi_+^\dagger)\Psi_- + \kappa_-\Psi_+.$$

These equations are *a matrix generalization of the massive Thirring equations* ($\Psi_\pm \in \mathrm{Mat}(p,q)$). In the case of $p = n-1$, $q = 1$ or $p = 1$, $q = n-1$ we obtain *a vector generalization of Thirring equations*. In the special case $n = 2$, $p = q = 1$ and $\kappa_- = \kappa_+$ we recover the usual scalar massive Thirring equations with mass $m = (\kappa_+)^2$.

4.2.3 Case of Coxeter automorphism

Let σ be a Coxeter automorphism. In this case $p = h$, where h is a Coxeter number of $\mathfrak{g}$ and algebra $\mathfrak{g}_{\bar{0}}$ is Abelian. That is why all tensors R_0 are solutions of the mYBE on $\mathfrak{g}_{\bar{0}}$ and maps $r_\pm$ are arbitrary (modulo the constraint $r_+ + r_- = 1$). Moreover, in this case the following proposition holds:

Proposition 4. *Let σ be a Coxeter automorphism of $\mathfrak{g}$ and maps $r_\pm$ on $\mathfrak{g}$ be nondegenerated. In this case $L^{(0)}$ is constant along all time flows and components of $L^{(\pm h)}$ are expressed as polynomials of components of $L^{(\pm 1)}, \ldots, L^{(\pm(h-1))}$.*

Proof. In the considered case we have that $\dim \mathfrak{g}_{\bar{0}} = \operatorname{rank} \mathfrak{g}$. On the other hand there exist $r = \operatorname{rank} \mathfrak{g}$ independent Casimir functions $I_k(L)$, $k \in \overline{1,h}$, on $\mathfrak{g}^*$. Hence there exist $2 \operatorname{rank} \mathfrak{g}$ integrals of the following form:

$$I_k^{\pm 0} = I_k(r_\pm^*(L^{(0)})).$$

They are constant along all time flows generated by all other integrals $I_k^{\pm l}$. That is why we may put them to be equal to constants. On the other hand, it is not difficult to see that all r independent components of $L^{(0)}$ can be functionally expressed via $I_k^{\pm 0}$. Hence, they are constants too. At last, we have $2 \operatorname{rank} \mathfrak{g}$ integrals $I_k^{\pm h}$, which are constant along all time flows and we may put them to be equal to constants $I_k^{\pm h} = \mathrm{const}_k^{\pm h}$. It easy to see that $I_k^{\pm h}$ are linear in $L^{(\pm h)}$ and, on the surface of level of $I_k^{\pm h}$, all components of $L^{(\pm h)}$ are expressed polynomially via components of $L^{(\pm 1)}, \ldots, L^{(\pm(h-1))}$ if the maps $r_\pm$ are non-degenerate. ■

This proposition has the following important corollary:

Corollary 1. *The number of independent fields in the generalized Thirring equation* (10)*, corresponding to Coxeter automorphism, is equal to* $2 \sum\limits_{j=1}^{h-1} \dim \mathfrak{g}_j = 2(\dim \mathfrak{g} - \operatorname{rank} \mathfrak{g})$.

Let us explicitly consider Thirring-type equations (11) in the case of the Coxeter automorphisms. In this case $\mathfrak{g}_{\bar{0}} \simeq \mathfrak{g}_{\bar{h}}$ is Abelian and equations (11a), (11b) become trivial. Moreover, due to the fact that $L^{(\pm h)}$ is expressed via $L^{(\pm j)}$ where $j < h$, equation (11d) becomes a consequence of equations (11c), (11d). In the resulting system of equations (11) is simplified to the following system:

$$\partial_{x_-} L^{(h-j)} = [r_-(L^{(-h)}), L^{(h-j)})] + \sum_{i=j+1}^{h-1} [L^{(i-j-h)}, L^{(h-i)}] + [L^{(-j)}, r_+^*(L^{(0)})],$$
$$\partial_{x_+} L^{(j-h)} = [r_+(L^{(h)}), L^{(j-h)})] + \sum_{i=j+1}^{h-1} [L^{(h+j-i)}, L^{(j-h)}] + [L^{(j)}, r_-^*(L^{(0)})], \tag{19}$$

where $j \in \{1, h-1\}$, $L^{(0)}$ is a constant matrix and $L^{(\pm h)}$ is polynomial in $L^{(\pm j)}$. We call this system of equations *the Abelian generalized Thirring equations*. The last summand in equations (19) is an analog of the "massive term" in the Thirring equation. The first summand is an analog of the cubic non-linearity in the Thirring equation, but with the growth of the rank of $\mathfrak{g}$ the degree of this term is also growing. The other terms are of the second order in dynamical variables. They are absent in the ordinary Thirring system corresponding to the case $\widetilde{\mathfrak{g}} = \widetilde{sl(2)}$. Let us explicitly consider the simplest example, which already possesses all features of the generalized Thirring equation:

Example 5. Let $\mathfrak{g} = gl(3)$. Its Z_3-grading corresponding to the Coxeter automorphism has the following form:

$$gl(3)_{\bar{0}} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \quad gl(3)_{\bar{1}} = \begin{pmatrix} 0 & \beta_1 & 0 \\ 0 & 0 & \beta_2 \\ \beta_3 & 0 & 0 \end{pmatrix}, \quad gl(3)_{\bar{2}} = \begin{pmatrix} 0 & 0 & \gamma_3 \\ \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \end{pmatrix},$$

$gl(3)_{\bar{3}} = gl(3)_{\bar{0}}$, $gl(3)_{\overline{-1}} = gl(3)_{\bar{2}}$, $gl(3)_{\overline{-2}} = gl(3)_{\bar{1}}$.

The Lax operators belonging to the dual spaces to $\widetilde{gl(3)}_{R_\pm}$ are:

$$L^\pm(u) = r_\pm^*(L^{(0)}) + L^{(\pm 1)} u^{\mp 1} + L^{(\pm 2)} u^{\mp 2} + L^{(\pm 3)} u^{\mp 3} + \cdots,$$

where $L^{(0)}, L^{(\pm 3)} \in gl(3)_{\bar{0}}$, $L^{(1)}, L^{(-2)} \in gl(3)_{\bar{2}}$, $L^{(2)}, L^{(-1)} \in gl(3)_{\bar{1}}$.

In order to simplify the form of the resulting soliton equations we will use the following notations:

$$L^{(0)} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \quad L^{(2)} = \begin{pmatrix} 0 & \gamma_1^+ & 0 \\ 0 & 0 & \gamma_2^+ \\ \gamma_3^+ & 0 & 0 \end{pmatrix}, \quad L^{(1)} = \begin{pmatrix} 0 & 0 & \beta_3^+ \\ \beta_1^+ & 0 & 0 \\ 0 & \beta_2^+ & 0 \end{pmatrix},$$
$$L^{(\pm 3)} = \begin{pmatrix} \delta_1^\pm & 0 & 0 \\ 0 & \delta_2^\pm & 0 \\ 0 & 0 & \delta_3^\pm \end{pmatrix}, \quad L^{(-1)} = \begin{pmatrix} 0 & \beta_1^- & 0 \\ 0 & 0 & \beta_2^- \\ \beta_3^- & 0 & 0 \end{pmatrix}, \quad L^{(-2)} = \begin{pmatrix} 0 & 0 & \gamma_3^- \\ \gamma_1^- & 0 & 0 \\ 0 & \gamma_2^- & 0 \end{pmatrix}.$$

The generating functions of the Poisson-commuting integrals of the corresponding integrable system are:

$$I^k(L^\pm(u)) = \frac{1}{k} \operatorname{tr} (L^\pm(u))^k = \sum_{m=0}^{\infty} I_k^{\pm m} u^{\mp m}.$$

In particular, we have the following integrals:

$$I_k^{\pm 0} = \frac{1}{k} \operatorname{tr} (r_\pm^*(L^{(0)}))^k, \qquad k \in \overline{1,3}.$$

From the fact that they are fixed along all the time flows we obtain that all α_i are constants of motion. We also have the following integrals:

$$I_1^{\pm 3} = \operatorname{tr} L^{(\pm 3)}, \qquad I_2^{\pm 3} = \operatorname{tr} (r_\pm^*(L^{(0)}) L^{(\pm 3)}) + \operatorname{tr} (L^{(\pm 1)} L^{(\pm 2)}),$$
$$I_3^{\pm 3} = \frac{1}{3} \operatorname{tr} (L^{(\pm 1)})^3 + \operatorname{tr} \big(r_\pm^*(L^{(0)})(L^{(\pm 1)} L^{(\pm 2)} + L^{(\pm 2)} L^{(\pm 1)})\big) + \operatorname{tr} \big(r_\pm^*(L^{(0)})^2 L^{(\pm 3)}\big).$$

These integrals permit us to express components of $L^{(\pm 3)}$ via components of $L^{(\pm 1)}$ and $L^{(\pm 2)}$ in the case of the non-degenerated maps $r_\pm$. Let us do this explicitly. For the sake of simplicity we will put that $r_\pm^*(L^{(0)}) = k_\pm L^{(0)}$, where $k_\pm$ are some constants and $k_+ + k_- = 1$. In this case we will have the following explicit form of the above integrals:

$$I_1^{\pm 3} = \sum_{i=1}^{3} \delta_i^\pm, \qquad I_2^{\pm 3} = k_\pm \sum_{i=1}^{3} \alpha_i \delta_i^\pm + \sum_{i=1}^{3} \beta_i^\pm \gamma_i^\pm,$$
$$I_3^{\pm 3} = k_\pm^2 \sum_{i=1}^{3} \alpha_i^2 \delta_i^\pm + k_\pm \sum_{i=1}^{3} \alpha_i (\beta_i^\pm \gamma_i^\pm + \beta_{i-1}^\pm \gamma_{i-1}^\pm) + \beta_1^\pm \beta_2^\pm \beta_3^\pm,$$

where we have implied in the last summation that $\beta_0^\pm \equiv \beta_3^\pm, \gamma_0^\pm \equiv \gamma_3^\pm$.

The U-V pair corresponding to the Hamiltonians $I_2^{\pm 3}$ have the form:

$$\nabla I_2^{\pm 3}(u) = u^{\pm 3} r_\pm^*(L^{(0)}) + u^{\pm 2} L^{(\pm 1)} + u^{\pm 1} L^{(\pm 2)} + r_\pm(L^{(\pm 3)}).$$

The corresponding zero-curvature equation reads as follows:

$$\partial_{x_-} L^{(1)} = [r_-(L^{(-3)}), L^{(1)}] + [L^{(-2)}, r_+^*(L^{(0)})],$$
$$\partial_{x_+} L^{(-1)} = [r_+(L^{(3)}), L^{(-1)}] + [L^{(2)}, r_-^*(L^{(0)})],$$
$$\partial_{x_-} L^{(2)} = [r_-(L^{(-3)}), L^{(2)}] + [L^{(-1)}, r_+^*(L^{(0)})] + [L^{(1)}, L^{(-2)}],$$
$$\partial_{x_+} L^{(-2)} = [r_+(L^{(3)}), L^{(-2)}] + [L^{(1)}, r_-^*(L^{(0)})] + [L^{(2)}, L^{(-1)}].$$

In the component form we have the following equations:

$$\partial_{x_-} \beta_i^+ = k_-(\delta_i^- - \delta_{i+1}^-)\beta_i^+ + k_+(c_{i+1} - c_i)\gamma_i^-,$$

$$\partial_{x_+}\beta_i^- = k_+(\delta_i^+ - \delta_{i+1}^+)\beta_i^- + k_-(c_{i+1} - c_i)\gamma_i^+,$$
$$\partial_{x_-}\gamma_i^+ = k_-(\delta_{i+1}^- - \delta_i^-)\gamma_i^+ + k_+(c_i - c_{i+1})\beta_i^- + (\beta_k^+\gamma_j^- - \beta_j^+\gamma_k^-),$$
$$\partial_{x_+}\gamma_i^+ = k_+(\delta_{i+1}^- - \delta_i^-)\gamma_i^+ + k_-(c_i - c_{i+1})\beta_i^- + (\gamma_j^+\beta_k^- - \gamma_k^+\beta_j^-), \qquad (20)$$

where $c_i \equiv \alpha_i$, indices i, j, k constitute a cyclic permutation of the indices 1, 2, 3, and it is implied that $\delta_{3+1}^\pm \equiv \delta_1^\pm$, $c_{3+1} \equiv c_1$.

Taking into account that all the integrals $I_{\pm 3}^k$ are constants of motion and putting their values to be equal to zero we can explicitly express variables $\delta_i^\pm$ via $\alpha_i = c_i$, $\beta_i^\pm$, $\gamma_i^\pm$:

$$\delta_i^\pm = \frac{1}{k_\pm^2(c_i - c_j)(c_i - c_k)} \times \left(k_\pm(c_j + c_k)\sum_{l=1}^{3}\beta_l^\pm\gamma_l^\pm - k_\pm\sum_{l=1}^{3} c_l(\beta_l^\pm\gamma_l^\pm + \beta_{l-1}^\pm\gamma_{l-1}^\pm) - \beta_1^\pm\beta_2^\pm\beta_3^\pm \right), \qquad (21)$$

where indices j and k are complementary to the index i in the set $\{1, 2, 3\}$. Due to the fact that c_i are constants of motion this expression is polynomial of the third order in dynamical variables.

At last, substituting (21) into the equations (20) we obtain a system of nonlinear differential equations for the dynamical variables $\beta_i^\pm$ and $\gamma_i^\pm$. These equations are a $gl(3)$ generalization of "complex" Thirring equations. Unfortunately, neither these equations nor more complicated $gl(n)$ complex Thirring equations do not admit reductions to the real forms $u(3)$ or $u(n)$ respectively.

4.3 The case of Kostant–Adler–Symes R-operators

Let us now consider integrable hierarchies corresponding to the case $r_+ = 1$, $r_- = 0$ (or $r_- = 0$, $r_- = 1$), i.e. corresponding to the Kostant–Adler–Symes decomposition of t loop algebras.

As it was reminded in Subsection 4.1, each of the gradings of the loop algebras corresponding to the different automorphisms σ of order p provides its own Kostant–Adler–Symes decomposition: $\widetilde{\mathfrak{g}}^\sigma = \widetilde{\mathfrak{g}}^{\sigma+} + \widetilde{\mathfrak{g}}^{\sigma-}$, and, hence, provides commuting Hamiltonian flows and hierarchies of integrable equations.

Let us consider the simplest equations of integrable hierarchies corresponding to them. We have the following generic elements of the dual spaces $(\widetilde{\mathfrak{g}}^{\sigma\pm})^*$:

$$L_+(u) = L^{(0)} + \sum_{j=1}^{\infty} L^{(j)}u^{-j}, \qquad L_-(u) = \sum_{j=1}^{\infty} L^{(-j)}u^j,$$

where $L^{(-k)} \in \mathfrak{g}_{\bar{k}}$, and the following generating functions of the commutative integrals:

$$I_k(L^+(u)) = \sum_{m=0}^{\infty} I_k^m u^{-m}, \qquad I_l(L^-(u)) = \sum_{n=1}^{\infty} I_l^{-n}u^n,$$

where $I_k(L)$ are Casimir functions of $\mathfrak{g}$.

We will be interested in the following Hamiltonians and their Hamiltonian flows:

$$I_2^p = \left(L^{(0)}, L^{(p)}\right) + \frac{1}{2}\sum_{j=1}^{p-1}\left(L^{(j)}, L^{(p-j)}\right), \qquad I_p^{-p} = I_p(L^{(-1)}).$$

The algebra-valued gradients of functions I_2^p, I_p^{-p} read as follows:

$$\nabla I_2^p(u) = u^p L^{(0)} + \sum_{j=1}^{p-1} u^{(p-j)} L^{(j)} + L^{(p)}, \qquad \nabla I_p^{-p}(u) = u^{-1}\tilde{L}^{(1)},$$

where $\tilde{L}^{(1)} \equiv \sum_{i=1}^{\dim \mathfrak{g}_{\bar{1}}} \frac{\partial I_p^{-p}}{\partial L_\alpha^{(1)}} X_\alpha^{\overline{-1}} \in \mathfrak{g}_{\overline{-1}}$.

The corresponding zero-curvature condition yields the following system of equations:

$$\partial_{x_-} L^{(0)} = 0, \tag{22}$$

$$\partial_{x_-} L^{(k)} = [\tilde{L}^{(1)}, L^{(k-1)}], \qquad k \in \{1, p\}, \tag{23}$$

$$\partial_{x_+} \tilde{L}^{(1)} = [L^{(p)}, \tilde{L}^{(1)}]. \tag{24}$$

Remark 4. The gradient $\nabla I_2^p(u)$ is an analog of the U operator of the generalized Thirring hierarchy. The other gradient $\nabla I_p^{-p}(u)$ is an analog of the V operator of the non-Abelian Toda equation. Hence the corresponding integrable equations may be considered as an intermediate case between generalized Thirring and (non-Abelian) Toda equation.

4.3.1 Case of second order automorphism

Let us consider the case of the automorphism of the second order $\sigma^2 = 1$ ($p = 2$), the corresponding Hamiltonians, their matrix gradients and zero-curvature conditions. We will use commuting second order Hamiltonians of the following form:

$$I_2^2 = \big(L^{(0)}, L^{(2)}\big) + \tfrac{1}{2}\big(L^{(1)}, L^{(1)}\big), \qquad I_2^{-2} = \tfrac{1}{2}(L^{(-1)}, L^{(-1)}).$$

The algebra-valued gradients of functions I_2^2, I_2^{-2} read as follows:

$$\nabla I_2^2(u) = u^2 L^{(0)} + u L^{(1)} + L^{(2)}, \qquad \nabla I_p^{-p}(u) = u^{-1} L^{(-1)}.$$

The corresponding zero-curvature condition yields the following simple system of differential equations of hyperbolic type:

$$\begin{aligned} &\partial_{x_-} L^{(0)} = 0, \qquad \partial_{x_-} L^{(1)} = [L^{(-1)}, L^{(0)}], \qquad \partial_{x_-} L^{(2)} = [L^{(-1)}, L^{(1)}], \\ &\partial_{x_+} L^{(-1)} = [L^{(2)}, L^{(-1)}], \end{aligned} \tag{25}$$

where $L^{(0)}, L^{(2)} \in \mathfrak{g}_{\bar{0}}$, $L^{(\pm 1)} \in \mathfrak{g}_{\bar{1}}$.

Let us consider the following example of these equations:

Example 6. Let $\mathfrak{g} = sl(2)$ and σ be a Cartan involution, i.e. $sl(2)_{\bar{0}} = \operatorname{diag}(\alpha, -\alpha)$ and $sl(2)_{\bar{1}}$ consists of the matrices with zeros on the diagonal. The simplest Hamiltonians obtained in the framework of Kostant–Adler–Symes scheme are:

$$I_2^0 = (\alpha^{(0)})^2, \qquad I_2^2 = 2\alpha^{(0)}\alpha^{(2)} + \beta^{(1)}\gamma^{(1)}, \qquad I_2^2 = \beta^{(-1)}\gamma^{(-1)}.$$

They produce the following U-V pair for zero-curvature equations:

$$\begin{aligned} \nabla I_2^2 &= \begin{pmatrix} \alpha^{(2)} & 0 \\ 0 & -\alpha^{(2)} \end{pmatrix} + u \begin{pmatrix} 0 & \beta^{(1)} \\ \gamma^{(1)} & 0 \end{pmatrix} + u^2 \begin{pmatrix} \alpha^{(0)} & 0 \\ 0 & -\alpha^{(0)} \end{pmatrix}, \\ \nabla I_2^{-2} &= u^{-1} \begin{pmatrix} 0 & \beta^{(-1)} \\ \gamma^{(-1)} & 0 \end{pmatrix}. \end{aligned}$$

Due to the fact that I_2^0 are constants of motion we have that $\alpha^{(0)}$ is also a constant of motion. Moreover, using the fact that I_2^2 is a constant of motion one can express $\alpha^{(2)}$ via $\beta^{(1)}$, $\gamma^{(1)}$. Hence we obtain from the equations (25) the following independent equations for the variables $\beta^{(\pm 1)}$, $\gamma^{(\pm 1)}$:

$$\partial_{x_-}\gamma^{(1)} = 2\alpha^{(0)}\gamma^{(-1)}, \tag{26a}$$
$$\partial_{x_-}\beta^{(1)} = -2\alpha^{(0)}\beta^{(-1)}, \tag{26b}$$
$$\partial_{x_+}\gamma^{(-1)} = -2\alpha^{(2)}\gamma^{(-1)}, \tag{26c}$$
$$\partial_{x_+}\beta^{(-1)} = 2\alpha^{(2)}\beta^{(-1)}. \tag{26d}$$

From the equation (26b) it follows that $\beta^{(-1)} = -\frac{\partial_{x_-}\beta^{(1)}}{2\alpha^{(0)}}$. Substituting this into equation (26d) and taking into account that $\alpha^{(0)}$ is a constant along all time flows we obtain the following equation:

$$\partial^2_{x_+x_-}\beta^{(1)} = 2\alpha^{(2)}\partial_{x_-}\beta^{(1)}. \tag{27}$$

Taking into account that $\alpha^{(2)} = -\frac{1}{2\alpha^{(0)}}(\beta^{(1)}\gamma^{(1)})$, where we have put that $I_2^2 = 0$, and making the reduction to the Lie algebra $su(2)$: $\alpha^{(0)} = ic$, $\gamma^{(1)} = -\bar{\beta}^{(1)} \equiv -\bar{\psi}$ we obtain the following integrable equation in partial derivatives:

$$\partial^2_{x_+x_-}\psi + \frac{i}{c}|\psi|^2\partial_{x_-}\psi = 0.$$

This equation is (in a some sense) intermediate between Thirring and sine-Gordon equations.

4.3.2 Case of Coxeter automorphism (principal gradation)

Let us consider again the case of the principal gradation. In this case $p = h$ and a subalgebra $\mathfrak{g}_{\bar{0}}$ is Abelian. In the same way as it was done in the case of the "generalized Abelian Thirring models" it is possible to show that $L^{(0)}$ is constant along all time flows and components of $L^{(h)}$ are expressed polynomially via the components of $L^{(k)}$, $k < h$. Let us assume that constants of motion $L^{(0)}$ are such that the operator $\mathrm{ad}_{L^{(0)}}$ is nondegenerate. In such a case we may solve the first of the equations (23) in the following way:

$$\tilde{L}^{(1)} = -\mathrm{ad}^{-1}_{L^{(0)}}(\partial_{x_-}L^{(1)}).$$

Substituting this expression into equation (24) and taking into account commutativity of $\mathfrak{g}_{\bar{0}}$ and, hence, operators $\mathrm{ad}^{-1}_{L^{(0)}}$ and $\mathrm{ad}_{L^{(h)}}$, we finally obtain the following matrix differential equation in partial derivatives:

$$\partial_{x_+x_-}L^{(1)} = \big[L^{(h)}(L^{(1)},\dots,L^{(h-1)}), \partial_{x_-}L^{(1)}\big],$$

where $L^{(k)}$, $k \in \{2, h-1\}$ satisfy the following set of ordinary differential equations:

$$\partial_{x_-}L^{(k)} = [L^{(k-1)}, \mathrm{ad}^{-1}_{L^{(0)}}(\partial_{x_-}L^{(1)})].$$

Let us consider the following example.

Example 7. Let $\mathfrak{g} = gl(3)$, $h = 3$. In this case we have the following differential equations:

$$\partial_{x_+x_-}L^{(1)} = [L^{(3)}(L^{(1)}, L^{(2)}), \partial_{x_-}L^{(1)}], \tag{28}$$
$$\partial_{x_-}L^{(2)} = [L^{(1)}, ad^{-1}_{L^{(0)}}(\partial_{x_-}L^{(1)})].$$

Let $L^{(0)}$, $L^{(1)}$, $L^{(2)}$, $L^{(3)}$ are parametrized as in the Example 5, i.e.

$$L^{(0)} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \qquad L^{(1)} = \begin{pmatrix} 0 & 0 & \beta_3 \\ \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \end{pmatrix},$$

$$L^{(2)} = \begin{pmatrix} 0 & \gamma_1 & 0 \\ 0 & 0 & \gamma_2 \\ \gamma_3 & 0 & 0 \end{pmatrix}, \qquad L^{(3)} = \begin{pmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{pmatrix}.$$

In such coordinates equation (28) acquires the following form:

$$\partial^2_{x_- x_+} \beta_i = (\delta_{i+1}(\beta,\gamma) - \delta_i(\beta,\gamma))\partial_{x_-}\beta_i, \qquad i \in \{1,3\}. \tag{29}$$

where, like in the Example 5, $\delta_i(\beta,\gamma)$ are expressed via β_j, γ_k and constants $\alpha_i \equiv c_i$:

$$\delta_i(\beta,\gamma) = \frac{1}{(c_i - c_j)(c_i - c_k)} \left((c_j + c_k) \sum_{l=1}^{3} \beta_l \gamma_l - \sum_{l=1}^{3} c_l (\beta_l \gamma_l + \beta_{l-1}\gamma_{l-1}) - \beta_1\beta_2\beta_3 \right). \tag{30}$$

This equation is an exact analog of the equation (27). Unfortunately in this case there is no $su(3)$ reduction and variables γ_i are not conjugated to β_i but satisfy the following differential equations:

$$\partial_{x_-}\gamma_i = \epsilon_{ijk}\beta_j (c_k - c_{k+1})^{-1} \partial_{x_-}\beta_k, \qquad i \in \{1,3\}. \tag{31}$$

Equations (29)–(31) are intermediate between the Abelian $gl(3)$-Toda equations and the generalized Abelian $gl(3)$-Thirring equations.

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