Generalized Symmetries of Massless Free Fields on Minkowski Space*

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Abstract. A complete and explicit classification of generalized, or local, symmetries of massless free fields of spin $s \geq 1/2$ is carried out. Up to equivalence, these are found to consists of the conformal symmetries and their duals, new chiral symmetries of order $2s$, and their higher-order extensions obtained by Lie differentiation with respect to conformal Killing vectors. In particular, the results yield a complete classification of generalized symmetries of the Dirac–Weyl neutrino equation, Maxwell's equations, and the linearized gravity equations.

Key words: generalized symmetries; massless free field; spinor field

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1 Introduction

Recent years have seen a growing interest in the study of the symmetry structure of the main field equations originating in mathematical physics. Generalized, or local, symmetries, which arise as vectors that are tangent to the solution jet space and preserve the contact ideal, are important for several reasons. Besides their original application to the construction of conservation laws, they play a central role in various methods, in particular in the classical symmetry reduction [16], Vessiot's method of group foliation [12], and separation of variables, for finding exact solutions to systems of partial differential equations. Generalized symmetries also arise in the study of infinite dimensional Hamiltonian systems [16] and are, moreover, connected with Bäcklund transformations and integrability [11]. In fact, the existence of an infinite number of independent generalized symmetries has been proposed as a test for complete integrability of a system of differential equations [13].

For the important examples of the Einstein gravitational field equations and the Yang–Mills field equations with semi-simple structure group, classifications of their symmetry structures [4, 19] have shown that, besides the obvious gauge symmetries, these equations essentially admit no generalized symmetries. In contrast, the linear graviton equations and the linear Abelian Yang–Mills equations possess a rich structure of generalized symmetries, which to-date has yet to be fully determined.

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In this paper we present a complete, explicit classification of the generalized symmetries for the massless field equations of any spin $s = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ on Minkowski spacetime, formulated in terms of spinor fields. These equations comprise as special cases Maxwell's equations, i.e., $U(1)$ Yang–Mills equations for $s = 1$, and graviton equations, i.e., linearized Einstein equations for $s = 2$. Other field equations of physical interest which are included are the Dirac–Weyl, or massless neutrino equation, and the gravitino equation, corresponding to the spin values $s = \frac{1}{2}$ and $s = \frac{3}{2}$, respectively.

There are two main results of the classification. First, we obtain spin s generalizations of constant coefficient linear second order symmetries found some years ago by Fushchich and Nikitin for Maxwell's equations [8] and subsequently generalized by Pohjanpelto [18]. The new second order symmetries for Maxwell's equations are especially interesting because, under duality rotations of the electromagnetic spinor, they possess odd parity as opposed to the even parity of the well-known conformal point symmetries. Consequently, we will refer to the new spin s generalizations as chiral symmetries. Furthermore, we show that the spacetime symmetries and chiral symmetries, together with scalings and duality rotations, generate the complete enveloping algebra of all generalized symmetries for the massless spin s field equations. In particular, these equations admit no other generalized symmetries apart from the elementary ones arising from the linearity of the field equations. It is also worth noting that, due to the conformal invariance of the massless field equations, our results provide as a by-product a complete classification of generalized symmetries of spin s fields on any locally conformally flat spacetime, extending earlier results for the electromagnetic field obtained by Kalnins et al. [10].

This classification is a counterpart to our results classifying all local conservation laws for the massless spin s field equations [1, 2, 3]. We emphasize, however, that there is no immediate Noether correspondence between conservation laws and symmetries in our situation, as the formulation of the massless spin s field equations in terms of spinor fields does not admit a local Lagrangian.

Our paper is organized as follows. First, in Section 2 we cover some background material on symmetries of differential equations and on spinorial formalism, including a factorization property of Killing spinors on Minkowski space that is pivotal in the symmetry analysis carried out in this paper. Then in Section 3 we state and prove our classification theorem for generalized symmetries of arbitrary order for the massless field equations. These, in particular, include novel chiral symmetries of order $2s$ for spin $s = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ fields. As applications, in Section 4 we transcribe our main result in the spin $s = 1$ case into tensorial form to derive a complete classification of generalized symmetries for the vacuum Maxwell's equations, and, finally, in Section 5, we employ the methods of Section 3 to carry out a full symmetry analysis of the Weyl system, or the massless Dirac equation, on Minkowski space. Our results for the Weyl system complement those found in [5, 7, 15], and, in particular, provide a classification of symmetries of arbitrarily high order for the massless neutrino equations.

2 Preliminaries

Let $\mathbf{M}$ be Minkowski space with coordinates x^i, $0 \le i \le 3$, and let E_s, $s = \frac{1}{2}$, 1, $\frac{3}{2}$, $\ldots$, stand for the coordinate bundle

$$\pi : E_s = \{(x^i, \phi_{A_1 A_2 \cdots A_{2s}})\} \to \{(x^i)\},$$

where $\phi_{A_1 A_2 \cdots A_{2s}}$ is a type $(2s, 0)$ spinor. We denote the kth order jet bundle of local sections of E_s by $J^k(E_s)$, $0 \le k \le \infty$. Recall that the infinite jet bundle $J^\infty(E_s)$ is the coordinate space

$$J^\infty(E_s) = \{(x^i, \phi_{A_1 A_2 \cdots A_{2s}}, \phi_{A_1 A_2 \cdots A_{2s}, j_1}, \ldots, \phi_{A_1 A_2 \cdots A_{2s}, j_1 j_2 \cdots j_p}, \ldots)\},$$

where $\phi_{A_1A_2\cdots A_{2s},j_1j_2\cdots j_p}$ stands for the pth order derivative variables. As is customary, we write

$$\phi_{A_1A_2\cdots A_{2s},B_1B_2\cdots B_p}{}^{B'_1B'_2\cdots B'_p} = \sigma^{j_1}{}_{B_1}{}^{B'_1}\sigma^{j_2}{}_{B_2}{}^{B'_2}\cdots\sigma^{j_p}{}_{B_p}{}^{B'_p}\phi_{A_1A_2\cdots A_{2s},j_1j_2\cdots j_p},$$

where $\sigma^j{}_{BB'}$, $0\le j\le 3$, are, up to a constant factor, the identity matrix and the Pauli spin matrices. We also write

$$\overline{\phi}_{A'_1A'_2\cdots A'_{2s},B'_1\cdots B'_p}{}^{B_1\cdots B_p} = \overline{\phi_{A_1A_2\cdots A_{2s},B_1\cdots B_p}{}^{B'_1\cdots B'_p}},$$

where the bar stands for complex conjugation. Here and in the sequel we employ the Einstein summation convention in both the space-time and spinorial indices, and we lower and raise spinorial indices using the spinor metric ϵ_{AB} and its inverse ϵ^{AB}; see [17] for further details.

In order to streamline our notation, we will employ boldface capital letters to designate spinorial multi-indices. Thus, for example, we will write

$$\phi_{\mathbf{A}_{2s},\mathbf{B}_p}{}^{\mathbf{B}'_p} = \phi_{A_1A_2\cdots A_{2s},B_1B_2\cdots B_p}{}^{B'_1B'_2\cdots B'_p},$$

and we will combine multi-indices by the rule $\mathbf{B}_p\mathbf{C}_q = (B_1B_2\cdots B_pC_1C_2\cdots C_q)$.

We let

$$\partial_{CC'} = \sigma^i_{CC'}\partial/\partial x^i$$

denote the spinor representative of the coordinate derivative $\partial/\partial x^i$. Moreover, we define partial derivative operators $\partial_\phi^{\mathbf{A}_{2s},\mathbf{B}_p}{}_{\mathbf{B}'_p}$ by

$$\partial_\phi^{\mathbf{A}_{2s},\mathbf{B}_p}{}_{\mathbf{B}'_p}\phi_{\mathbf{C}_{2s},\mathbf{D}_r}{}^{\mathbf{D}'_r} = \begin{cases}\epsilon_{(C_1}{}^{A1}\cdots\epsilon_{C_{2s})}{}^{A_{2s}}\epsilon_{(D_1}{}^{B_1}\cdots\epsilon_{D_p)}{}^{B_p}\epsilon_{(B'_1}{}^{D'_1}\cdots\epsilon_{B'_p)}{}^{D'_p}, & \text{if} \quad p=r,\\ 0, & \text{if} \quad p\neq r,\end{cases}$$

$$\partial_\phi^{\mathbf{A}_{2s},\mathbf{B}_p}{}_{\mathbf{B}'_p}\overline{\phi}_{\mathbf{C}'_{2s},\mathbf{D}'_r}{}^{\mathbf{D}_r} = 0,$$

and write

$$\overline{\partial}_\phi^{\mathbf{A}'_{2s},\mathbf{B}'_p}{}_{\mathbf{B}_p} = \overline{\partial_\phi^{\mathbf{A}_{2s},\mathbf{B}_p}{}_{\mathbf{B}'_p}}.$$

Here, in accordance with the standard spinorial notation, we have written $\epsilon_C{}^A$ for the Kronecker delta and we use round brackets to indicate symmetrization in the enclosed indices.

A generalized vector field X on E_s in spinor form is a vector field

$$X = P^{CC'}\partial_{CC'} + Q_{\mathbf{A}_{2s}}\partial_\phi^{\mathbf{A}_{2s}} + \overline{Q}_{\mathbf{A}'_{2s}}\overline{\partial}_\phi^{\mathbf{A}'_{2s}}, \tag{2.1}$$

where the coefficients $P^{CC'} = P^{CC'}(x^j, \phi^{[p]})$, $Q_{\mathbf{A}_{2s}} = Q_{\mathbf{A}_{2s}}(x^j,\phi^{[p]})$ are spinor valued functions in x^j and the derivative variables $\phi_{\mathbf{A}_{2s},\mathbf{B}_q}{}^{\mathbf{B}'_q}$ up to some finite order p. An evolutionary vector field Y, in turn, is a generalized vector field of the form

$$Y = Q_{\mathbf{A}_{2s}}\partial_\phi^{\mathbf{A}_{2s}} + \overline{Q}_{\mathbf{A}'_{2s}}\overline{\partial}_\phi^{\mathbf{A}'_{2s}},$$

where $Q_{\mathbf{A}_{2s}}$ is called the characteristic of Y.

Let

$$D_C^{C'} = \partial_C^{C'} + \sum_{p\ge 0}\big(\phi_{\mathbf{A}_{2s},\mathbf{B}_pC}{}^{\mathbf{B}'_pC'}\partial_\phi^{\mathbf{A}_{2s},\mathbf{B}_p}{}_{\mathbf{B}'_p} + \overline{\phi}_{\mathbf{A}'_{2s},\mathbf{B}'_pC}{}^{\mathbf{B}_pC'}\overline{\partial}_\phi^{\mathbf{A}'_{2s},\mathbf{B}'_p}{}_{\mathbf{B}_p}\big)$$

stand for the spinor representative of the standard total derivative operator, which, as is easily verified, satisfies the commutation formula

$$[\partial_\phi^{\mathbf{A}_{2s},\mathbf{B}_p}{}_{\mathbf{B}'_p}, D_C^{C'}] = \epsilon_{(B'_p|}{}^{C'}\epsilon_C{}^{(B_p|}\partial_\phi^{\mathbf{A}_{2s},|\mathbf{B}_{p-1})}{}_{|\mathbf{B}'_{p-1})}, \qquad p \geq 1. \tag{2.2}$$

The infinite prolongation $\operatorname{pr} X$ of X in (2.1) to a vector field on $J^\infty(E_s)$ is given by

$$\operatorname{pr} X = P^{CC'}D_{CC'} + \sum_{p\geq 0}\big((D_{B_1}^{B'_1}\cdots D_{B_p}^{B'_p}R_{\mathbf{A}_{2s}})\partial_\phi^{\mathbf{A}_{2s},\mathbf{B}_p}{}_{\mathbf{B}'_p} + (D_{B_1}^{B'_1}\cdots D_{B_p}^{B'_p}\overline{R}_{\mathbf{A}'_{2s}})\overline{\partial}_\phi^{\mathbf{A}'_{2s},\mathbf{B}_p}{}_{\mathbf{B}'_p}\big),$$

where $R_{\mathbf{A}_{2s}}$ is the characteristic of the evolutionary form

$$X_{\mathrm{ev}} = (Q_{\mathbf{A}_{2s}} - P^{CC'}\phi_{\mathbf{A}_{2s},CC'})\partial_\phi^{\mathbf{A}_{2s}} + (\overline{Q}_{\mathbf{A}'_{2s}} - P^{CC'}\overline{\phi}_{\mathbf{A}'_{2s},CC'})\overline{\partial}_\phi^{\mathbf{A}'_{2s}}$$

of X.

The massless free field equation of spin s and its differential consequences

$$\phi_{\mathbf{A}_{2s},A'}{}^{A_{2s}\mathbf{B}'_p}{}_{\mathbf{B}_p} = 0, \qquad p \geq 0, \tag{2.3}$$

determine the infinitely prolonged solution manifold $\mathcal{R}^\infty(E_s) \subset J^\infty(E_s)$ of the equations. According to [17], the symmetrized derivative variables

$$\phi_{\mathbf{A}_{2s}\mathbf{B}_p}{}^{\mathbf{B}'_p} = \phi_{(\mathbf{A}_{2s},\mathbf{B}_p)}{}^{\mathbf{B}'_p}, \qquad p \geq 0,$$

known as Penrose's exact sets of fields, together with the independent variables x^i provide coordinates for $\mathcal{R}^\infty(E_s)$. Moreover, as is easily verified, the unsymmetrized and symmetrized variables $\phi_{\mathbf{A}_{2s},\mathbf{B}_p}{}^{\mathbf{B}'_p}$, $\phi_{\mathbf{A}_{2s}\mathbf{B}_p}{}^{\mathbf{B}'_p}$ agree on $\mathcal{R}^\infty(E_s)$.

A generalized, or local, symmetry of massless free fields is a generalized vector field X satisfying

$$\operatorname{pr} X\phi_{\mathbf{A}_{2s},A'}{}^{A_{2s}} = 0 \qquad \text{on} \quad \mathcal{R}^\infty(E_s). \tag{2.4}$$

Note that any generalized vector field of the form

$$T_P = P^{CC'}(\partial_{CC'} + \phi_{\mathbf{A}_{2s},CC'}\partial_\phi^{\mathbf{A}_{2s}} + \overline{\phi}_{\mathbf{A}'_{2s},CC'}\overline{\partial}_\phi^{\mathbf{A}'_{2s}})$$

with the prolongation

$$\operatorname{pr} T_P = P^{CC'}D_{CC'}$$

automatically satisfies the determining equations (2.4) for a symmetry. Hence we will call a symmetry trivial if its prolongation agrees with a total vector field $\operatorname{pr} T_P = P^{CC'}D_{CC'}$ on $\mathcal{R}^\infty(E_s)$ and we call two symmetries equivalent if their difference is a trivial symmetry. See, e.g., [6, 16] for further details and background material on generalized symmetries.

In this paper we explicitly classify all equivalence classes of generalized symmetries of massless free fields of spin $s = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ on Minkowski space. By the above, in our classification we only need to consider symmetries in evolutionary form, and for such a vector field

$$Y = Q_{\mathbf{A}_{2s}}\partial_\phi^{\mathbf{A}_{2s}} + \overline{Q}_{\mathbf{A}'_{2s}}\overline{\partial}_\phi^{\mathbf{A}'_{2s}},$$

the determining equations (2.4) for the characteristic $Q_{\mathbf{A}_{2s}}$ become

$$D_{A'}^{A_{2s}}Q_{\mathbf{A}_{2s}} = 0 \qquad \text{on} \quad \mathcal{R}^\infty(E_s). \tag{2.5}$$

Moreover, after replacing X by an equivalent symmetry, we can always assume that the components $Q_{\mathbf{A}_{2s}}$ are functions of only the independent variables x^i and the symmetrized derivative variables $\phi_{\mathbf{B}_{2s+q}}^{\mathbf{B}'_q}$, $0 \le q \le p$, for some p.

Recall that a vector field $\xi = \xi^i(x^j)\partial_i$ on $\mathbf{M}$ is conformal Killing provided that

$$\partial_{(i}\xi_{j)} = k\eta_{ij} \tag{2.6}$$

for some function $k = k(x^i)$, where η_{ij} stands for the Minkowski metric. As can be verified by a direct computation, a conformal Killing vector field ξ gives rise to the symmetry

$$\mathcal{Z}[\xi] = \mathcal{Z}_{\mathbf{A}_{2s}}[\xi]\partial_\phi^{\mathbf{A}_{2s}} + \overline{\mathcal{Z}}_{\mathbf{A}'_{2s}}[\xi]\overline{\partial}_\phi^{\mathbf{A}'_{2s}} \tag{2.7}$$

of massless free fields of spin s, with the characteristic

$$\mathcal{Z}_{\mathbf{A}_{2s}}[\xi] = \xi^{CC'}\phi_{\mathbf{A}_{2s}CC'} + s\partial_{C'(A_{2s}}\xi^{CC'}\phi_{\mathbf{A}_{2s-1})C} + \frac{1-s}{4}(\partial_{CC'}\xi^{CC'})\phi_{\mathbf{A}_{2s}},$$

which agrees with the conformally weighted Lie derivative

$$\mathcal{L}_\xi^{(-1)}\phi_{\mathbf{A}_{2s}} = \mathcal{L}_\xi\phi_{\mathbf{A}_{2s}} + \frac{1}{4}(\partial_{CC'}\xi^{CC'})\phi_{\mathbf{A}_{2s}}$$

of the spinor field $\phi_{\mathbf{A}_{2s}}$; see [2, 17].

In the course of the present symmetry classification we will repeatedly use the fact that, on account of the linearity of the massless free field equations, the componentwise derivative

$$\operatorname{pr}\mathcal{Z}[\xi]Y$$

of an evolutionary symmetry Y with respect to the prolongation of the vector field $\mathcal{Z}[\xi]$ is again a symmetry of the equations; see, e.g., [18].

In spinor form the conformal Killing vector equation (2.6) becomes

$$\partial_{(B}^{(B'}\xi_{C)}^{C')} = 0. \tag{2.8}$$

An obvious generalization of equations (2.8) to spinor fields $\kappa_{\mathbf{A}_k}^{\mathbf{A}'_l} = \kappa_{\mathbf{A}_k}^{\mathbf{A}'_l}(x^{CC'})$ of type (k,l) is

$$\partial_{(A_{k+1}}^{(A'_{l+1}}\kappa_{\mathbf{A}_k)}^{\mathbf{A}'_l)} = 0, \tag{2.9}$$

and symmetric spinor fields $\kappa_{\mathbf{A}_k}^{\mathbf{A}'_l} = \kappa_{(\mathbf{A}_k)}^{(\mathbf{A}'_l)}(x^{CC'})$ satisfying these equations are called Killing spinors of type (k,l). Thus, in particular, a type $(1,1)$ Killing spinor $\kappa_A^{A'}$ corresponds to a complex conformal Killing vector. The following Lemma, which is a special case of the well-known factorization property of Killing spinors on Minkowski space, is pivotal in our classification of symmetries of massless free fields. For more details, see [17].

Lemma 1. *Let $\xi_{\mathbf{A}_k}^{\mathbf{A}'_k}$, $\kappa_{\mathbf{A}_k}^{\mathbf{A}'_{k+2s}}$ be Killing spinors of type (k,k) and $(k,k+2s)$. Then $\xi_{\mathbf{A}_k}^{\mathbf{A}'_k}$ can be expressed as a sum of symmetrized products of k Killing spinors of type $(1,1)$, and $\kappa_{\mathbf{A}_k}^{\mathbf{A}'_{k+2s}}$ can be expressed as a sum of symmetrized products of Killing spinors of type $(0,2s)$ and k Killing spinors of type $(1,1)$. The dimensions of the complex vector spaces of Killing spinors of type (k,k) and $(k,k+2s)$ are*

$$(k+1)^2(k+2)^2(2k+3)/12 \qquad \textit{and}$$
$$(k+1)(k+2)(k+2s+1)(k+2s+2)(2k+2s+3)/12,$$

respectively.

3 Main results

Let ξ, $\zeta_1, \ldots, \zeta_p$ be real conformal Killing vectors and let $\pi^{\mathbf{A}'_{4s}}$ be a type $(0, 4s)$ Killing spinor. Let $\mathcal{Z}[\xi]$ be the symmetry associated with ξ as in (2.7) and define $\mathcal{Z}[\mathrm{i}\xi]$ by

$$\mathcal{Z}[\mathrm{i}\xi] = \mathrm{i}\mathcal{Z}_{\mathbf{A}_{2s}}[\xi]\partial_\phi^{\mathbf{A}_{2s}} - \mathrm{i}\overline{\mathcal{Z}}_{\mathbf{A}'_{2s}}[\xi]\overline{\partial}_\phi^{\mathbf{A}'_{2s}}. \tag{3.1}$$

Furthermore, let

$$\mathcal{W}[\pi] = \mathcal{W}_{\mathbf{A}_{2s}}[\pi]\partial_\phi^{\mathbf{A}_{2s}} + \overline{\mathcal{W}}_{\mathbf{A}'_{2s}}[\pi]\overline{\partial}_\phi^{\mathbf{A}'_{2s}} \tag{3.2}$$

be an evolutionary vector field with components

$$\mathcal{W}_{\mathbf{A}_{2s}}[\pi] = \sum_{p=0}^{2s} \mathrm{c}_{2s,p}\partial_{B'_1(A_{2s-p+1}|}\partial_{B'_2|A_{2s-p+2}|}\cdots_|\,\partial_{B'_p|A_{2s}}\pi^{\mathbf{B}'_p\mathbf{C}'_{4s-p}}\overline{\phi}_{\mathbf{A}_{2s-p})\mathbf{C}'_{4s-p}}, \tag{3.3}$$

where the coefficients $\mathrm{c}_{2s,p}$ are given by

$$\mathrm{c}_{2s,p} = \frac{4s-p+1}{4s+1}\binom{2s}{p}, \qquad 0 \le p \le 2s. \tag{3.4}$$

Moreover, write

$$\mathcal{Z}[\xi;\zeta_1,\ldots,\zeta_p] = \mathrm{pr}\,\mathcal{Z}[\zeta_1]\cdots\mathrm{pr}\,\mathcal{Z}[\zeta_p]\mathcal{Z}[\xi], \tag{3.5}$$
$$\mathcal{Z}[\mathrm{i}\xi;\zeta_1,\ldots,\zeta_p] = \mathrm{pr}\,\mathcal{Z}[\zeta_1]\cdots\mathrm{pr}\,\mathcal{Z}[\zeta_p]\mathcal{Z}[\mathrm{i}\xi], \tag{3.6}$$
$$\mathcal{W}[\pi;\zeta_1,\ldots,\zeta_q] = \mathrm{pr}\,\mathcal{Z}[\zeta_1]\cdots\mathrm{pr}\,\mathcal{Z}[\zeta_q]\mathcal{W}[\pi], \tag{3.7}$$

for the repeated componentwise derivatives of the vector fields (2.7), (3.1), (3.2) with respect to conformal symmetries.

Proposition 1. *Let ξ, $\zeta_1,\ldots,\zeta_p$ be conformal Killing vectors and let $\pi_{\mathbf{A}_{4s}}$ be a Killing spinor of type $(0, 4s)$. Then the evolutionary vector fields*

$$\mathcal{Z}[\xi;\zeta_1,\ldots,\zeta_p], \qquad \mathcal{Z}[\mathrm{i}\xi;\zeta_1,\ldots,\zeta_p], \qquad \mathcal{W}[\pi;\zeta_1,\ldots,\zeta_q], \qquad p,q \ge 0, \tag{3.8}$$

are symmetries of the massless free field equations of spin s of order $p+1$ and $q+2s$, respectively. Moreover, when restricted to the solution manifold $\mathcal{R}^\infty(E_s)$, the leading order terms in the components $\mathcal{Z}_{\mathbf{A}_{2s}}[\xi;\zeta_1,\ldots,\zeta_p]$, $\mathcal{Z}_{\mathbf{A}_{2s}}[\mathrm{i}\xi;\zeta_1,\ldots,\zeta_p]$, $\mathcal{W}_{\mathbf{A}_{2s}}[\pi;\zeta_1,\ldots,\zeta_q]$ of the symmetries (3.8) reduce to

$$(-1)^{p+1}\xi^{(C_1}_{(C'_1}\zeta_1{}^{C_2}_{C'_2}\cdots\zeta_p{}^{C_{p+1})}_{C'_{p+1})}\phi_{\mathbf{A}_{2s}\mathbf{C}_{p+1}}{}^{\mathbf{C}'_{p+1}},$$
$$(-1)^{p+1}\mathrm{i}\xi^{(C_1}_{(C'_1}\zeta_1{}^{C_2}_{C'_2}\cdots\zeta_p{}^{C_{p+1})}_{C'_{p+1})}\phi_{\mathbf{A}_{2s}\mathbf{C}_{p+1}}{}^{\mathbf{C}'_{p+1}},$$
$$(-1)^{q}\zeta_1{}^{(C_1}_{(C'_1}\zeta_2{}^{C_2}_{C'_2}\cdots\zeta_q{}^{C_q)}_{C'_q}\pi_{\mathbf{B}'_{4s})}\overline{\phi}_{\mathbf{A}_{2s}\mathbf{C}_q}{}^{\mathbf{C}'_q\mathbf{B}'_{4s}},$$

respectively.

Proof. We only need to show that $\mathcal{W}[\pi]$ in (3.2), (3.3) satisfies the symmetry equations (2.4). First note that due to the Killing spinor equations (2.9) we have that

$$\partial_{CC'}\partial^{CD'}\pi^{\mathbf{B}'_{4s}} = 0, \tag{3.9}$$

and, consequently,

$$\partial_{CC'}\partial_{DD'}\pi^{\mathbf{B}'_{4s}} = \partial_{CD'}\partial_{DC'}\pi^{\mathbf{B}'_{4s}}. \tag{3.10}$$

Write

$$\Pi^1_{p,\mathbf{A}_{2s-1}A'} = \partial^{A_{2s}}_{A'}\partial_{B'_1(A_{2s-p+1}|}\partial_{B'_2|A_{2s-p+2}|}\cdots{}_|\,\partial_{B'_p|A_{2s}}\pi^{\mathbf{B}'_p\mathbf{C}'_{4s-p}}\overline{\phi}_{\mathbf{A}_{2s-p})\mathbf{C}'_{4s-p}},$$
$$\Pi^2_{p,\mathbf{A}_{2s-1}A'} = \partial_{B'_1(A_{2s-p+1}|}\partial_{B'_2|A_{2s-p+2}|}\cdots{}_|\,\partial_{B'_p|A_{2s}}\pi^{\mathbf{B}'_p\mathbf{C}'_{4s-p}}\overline{\phi}^{A_{2s}}_{\mathbf{A}_{2s-p})\mathbf{C}'_{4s-p}A'},$$

$0 \le p \le 2s$, so that $\mathcal{W}[\pi]$ is a symmetry of massless free field equations provided that

$$\sum_{p=0}^{2s} \mathrm{c}_{2s,p}(\Pi^1_{p,\mathbf{A}_{2s-1}A'} + \Pi^2_{p,\mathbf{A}_{2s-1}A'}) = 0. \tag{3.11}$$

We compute

$$\begin{aligned}
&\Pi^1_{p,\mathbf{A}_{2s-1}A'}\\
&= \frac{4s}{4s+1}\partial_{B'_1(A_{2s-p+1}|}\partial_{B'_2|A_{2s-p+2}|}\cdots{}_|\,\partial_{B'_p|A_{2s}}(\epsilon_{A'}{}^{(B'_p}\partial^{|A_{2s}|}_{|D'|}\pi^{\mathbf{B}'_{p-1}\mathbf{C}'_{4s-p})D'})\overline{\phi}_{\mathbf{A}_{2s-p})\mathbf{C}'_{4s-p}}\\
&= \frac{p}{4s+1}\partial_{A'(A_{2s-p+1}|}\partial_{B'_1|A_{2s-p+2}|}\cdots{}_|\,\partial_{B'_{p-1}|A_{2s}}\partial^{A_{2s}}_{|D'|}\pi^{\mathbf{B}'_{p-1}\mathbf{C}'_{4s-p}D'}\overline{\phi}_{\mathbf{A}_{2s-p})\mathbf{C}'_{4s-p}}\\
&\quad + \frac{4s-p}{4s+1}\partial_{B'_1(A_{2s-p+1}|}\partial_{B'_2|A_{2s-p+2}|}\cdots{}_|\,\partial_{B'_p|A_{2s}}\partial^{A_{2s}}_{|D'|}\pi^{\mathbf{B}'_p\mathbf{C}'_{4s-p-1}D'}\overline{\phi}_{\mathbf{A}_{2s-p})\mathbf{C}'_{4s-p-1}A'}\\
&= \frac{p}{4s+1}\Pi^1_{p,\mathbf{A}_{2s-1}A'} + \frac{4s-p}{4s+1}\frac{2s-p}{2s}\\
&\quad \times \partial_{B'_1(A_{2s-p}|}\partial_{B'_2|A_{2s-p+1}|}\cdots{}_|\,\partial_{B'_p|A_{2s-1}}\partial^{B}_{|D'|}\pi^{\mathbf{B}'_p\mathbf{C}'_{4s-p-1}D'}\overline{\phi}_{\mathbf{A}_{2s-p-1})\mathbf{C}'_{4s-p-1}A'B},
\end{aligned} \tag{3.12}$$

where we used (3.9) and (3.10). On the other hand,

$$\Pi^2_{p,\mathbf{A}_{2s-1}A'} = \frac{p}{2s}\partial_{D'B}\partial_{B'_1(A_{2s-p+1}|}\partial_{B'_2|A_{2s-p+2}|}\cdots{}_|\,\partial_{B'_p|A_{2s-1}}\pi^{\mathbf{B}'_p\mathbf{C}'_{4s-p}D'}\overline{\phi}^{B}_{\mathbf{A}_{2s-p})\mathbf{C}'_{4s-p}A'}. \tag{3.13}$$

Now it follows from (3.12), (3.13) that

$$\Pi^1_{p,\mathbf{A}_{2s-1}A'} = -\frac{(4s-p)(2s-p)}{(4s-p+1)(p+1)}\Pi^2_{p+1,\mathbf{A}_{2s-1}A'}.$$

Clearly

$$\Pi^1_{2s,\mathbf{A}_{2s-1}A'} = 0, \qquad \Pi^2_{0,\mathbf{A}_{2s-1}A'} = 0.$$

Consequently, by virtue of (3.4), equation (3.11) holds and hence $\mathcal{W}[\pi]$ is a symmetry of the massless free field equations. ■

The massless free field equations of spin s also admit the obvious scaling symmetry $\mathcal{S}$, its dual symmetry $\widetilde{\mathcal{S}}$, and the elementary symmetries $\mathcal{E}[\varphi]$ given by

$$\mathcal{S} = \phi_{\mathbf{A}_{2s}}\partial^{\mathbf{A}_{2s}} + \overline{\phi}_{\mathbf{A}'_{2s}}\overline{\partial}^{\mathbf{A}'_{2s}}, \qquad \widetilde{\mathcal{S}} = \mathrm{i}\phi_{\mathbf{A}_{2s}}\partial^{\mathbf{A}_{2s}} - \mathrm{i}\overline{\phi}_{\mathbf{A}'_{2s}}\overline{\partial}^{\mathbf{A}'_{2s}}, \tag{3.14}$$

and

$$\mathcal{E}[\varphi] = \varphi_{\mathbf{A}_{2s}}(x^i)\partial^{\mathbf{A}_{2s}} + \overline{\varphi}_{\mathbf{A}'_{2s}}(x^i)\overline{\partial}^{\mathbf{A}'_{2s}}, \tag{3.15}$$

where $\varphi_{\mathbf{A}_{2s}} = \varphi_{\mathbf{A}_{2s}}(x^i)$ is any solution of (2.3).

Theorem 1. *Let $\mathcal{Q}$ be a generalized symmetry of the massless free field equations of spin $s = \frac{1}{2}, 1, \frac{3}{2}, \dots$. If the evolutionary form of $\mathcal{Q}$ is of order r, then $\mathcal{Q}$ is equivalent to a symmetry $\widehat{\mathcal{Q}}$ of order at most r which can be written as*

$$\widehat{\mathcal{Q}} = \mathcal{V} + \mathcal{E}[\varphi], \tag{3.16}$$

where $\varphi_{\mathbf{A}_{2s}} = \varphi_{\mathbf{A}_{2s}}(x^i)$ is a solution of the massless free field equations of spin s, and where $\mathcal{V}$ is equivalent to a spinorial symmetry that is a linear combination of the symmetries

$$\mathcal{S}, \quad \widetilde{\mathcal{S}}, \quad \mathcal{Z}[\xi; \zeta_1, \dots, \zeta_p], \quad \mathcal{Z}[\mathrm{i}\xi; \zeta_1, \dots, \zeta_p], \quad \mathcal{W}[\pi; \zeta_1, \dots, \zeta_q] \tag{3.17}$$

with $p \le r - 1$, $q \le r - 2s$.

In particular, the dimension d_r of the real vector space of equivalence classes of spinorial symmetries of order at most r spanned by the symmetries (3.17) *is*

$$d_r = (r+1)^2 (r+2)^2 (r+3)^2/18, \qquad \text{if} \quad r < 2s,$$

and

$$\begin{aligned} d_r &= (r+1)^2 (r+2)^2 (r+3)^2/18 \\ &\quad + ((r+1)^2 - 4s^2)((r+2)^2 - 4s^2)((r+3)^2 - 4s^2)/18, \qquad \text{if} \quad r \ge 2s. \end{aligned}$$

The above result was originally announced without proof in [3].

Proof. Without loss of generality, we can assume that the components $\mathcal{Q}_{\mathcal{A}_{2s}}$ of $\mathcal{Q}$ are functions of the independent variables x^i and the symmetrized derivative variables $\phi_{\mathbf{A}_{2s+p}}^{\mathbf{A}'_p}$, $0 \le p \le r$. Consequently,

$$\partial_\phi{}^{\mathbf{B}_{2s}, \mathbf{C}_p}{}_{\mathbf{C}'_p} \mathcal{Q}_{\mathbf{A}_{2s}} = \partial_\phi{}^{(\mathbf{B}_{2s}, \mathbf{C}_p)}{}_{\mathbf{C}'_p} \mathcal{Q}_{\mathbf{A}_{2s}}, \quad \overline{\partial}_\phi{}^{\mathbf{B}'_{2s}, \mathbf{C}'_p}{}_{\mathbf{C}_p} \mathcal{Q}_{\mathbf{A}_{2s}} = \overline{\partial}_\phi{}^{(\mathbf{B}'_{2s}, \mathbf{C}'_p)}{}_{\mathbf{C}_p} \mathcal{Q}_{\mathbf{A}_{2s}}. \tag{3.18}$$

It follows from the determining equations for spinorial symmetries that

$$\partial_\phi{}^{(\mathbf{B}_{2s}, \mathbf{C}_p)}{}_{\mathbf{C}'_p} D^{A_{2s}}_{A'} \mathcal{Q}_{\mathbf{A}_{2s}} = 0, \qquad \overline{\partial}_\phi{}^{(\mathbf{B}'_{2s}, \mathbf{C}'_p)}{}_{\mathbf{C}_p} D^{A_{2s}}_{A'} \mathcal{Q}_{\mathbf{A}_{2s}} = 0 \tag{3.19}$$

on $\mathcal{R}^\infty(E_s)$. By virtue of the commutation formulas (2.2), the above equations with $p = r+1$ show that

$$\partial_\phi{}^{(\mathbf{B}_{2s}, \mathbf{C}_r}{}_{\mathbf{C}'_r} \mathcal{Q}^{A_{2s})}_{\mathbf{A}_{2s-1}} = 0, \qquad \overline{\partial}_\phi{}^{(\mathbf{B}'_{2s}, \mathbf{C}'_r)}{}_{(\mathbf{C}_r} \mathcal{Q}_{A_{2s})\mathbf{A}_{2s-1}} = 0 \tag{3.20}$$

identically on $J^\infty(E_s)$. Equations (3.20), in turn, combined with (3.18), imply that

$$\begin{aligned} &\partial_\phi{}^{\mathbf{B}_{2s}, \mathbf{C}_r}{}_{\mathbf{C}'_r} \mathcal{Q}_{\mathbf{A}_{2s}} = \epsilon_{A_1}{}^{(B_1} \epsilon_{A_2}{}^{B_2} \cdots \epsilon_{A_{2s}}{}^{B_{2s}} \mathcal{S}^{\mathbf{C}_r)}_{\mathbf{C}'_r}, \\ &\overline{\partial}_\phi{}^{\mathbf{B}'_{2s}, \mathbf{C}'_r}{}_{\mathbf{C}_r} \mathcal{Q}^{\mathbf{A}_{2s}} = \epsilon_{(C_{r-2s+1}}{}^{A_1} \epsilon_{C_{r-2s+2}}{}^{A_2} \cdots \epsilon_{C_r}{}^{A_{2s}} \mathcal{T}^{\mathbf{B}'_{2s} \mathbf{C}'_r}_{\mathbf{C}_{r-2s})}, \end{aligned} \tag{3.21}$$

for some spinor valued functions $\mathcal{S}^{\mathbf{C}_r}_{\mathbf{C}'_r}$, $\mathcal{T}^{\mathbf{C}_{r+2s'}}_{\mathbf{C}_{r-2s}}$ on $J^\infty(E_s)$ symmetric in their indices. Note in particular that $\mathcal{T}^{\mathbf{C}_{r+2s'}}_{\mathbf{C}_{r-2s}}$ vanishes if $r < 2s$.

Next use equations (3.19) with $p = r$ together with the commutation formulas (2.2) to conclude that

$$D^{A_{2s}}_{(A'} \partial_\phi{}^{\mathbf{B}_{2s}, \mathbf{C}_r}{}_{\mathbf{C}'_r)} \mathcal{Q}_{\mathbf{A}_{2s}} = 0, \qquad D^{A_{2s}(A'} \overline{\partial}_\phi{}^{\mathbf{B}'_{2s}, \mathbf{C}'_r)}{}_{\mathbf{C}_r} \mathcal{Q}_{\mathbf{A}_{2s}} = 0 \qquad \text{on} \quad \mathcal{R}^\infty(E_s). \tag{3.22}$$

Now substitute expressions (3.21) into (3.22) to deduce that

$$D^{(C_{r+1}}_{(C'_{r+1}}\mathcal{S}^{\mathbf{C}_r)}_{\mathbf{C}'_r)} = 0, \qquad D^{(C'_{r+2s+1}}_{(C_{r-2s+1}}\mathcal{T}^{\mathbf{C}'_{r+2s})}_{\mathbf{C}_{r-2s})} = 0 \qquad \text{on} \quad \mathcal{R}^\infty(E_s).$$

But it is easy to see that the above equations force $\mathcal{S}^{\mathbf{C}_r}_{\mathbf{C}'_r}$, $\mathcal{T}^{\mathbf{C}'_{r+2s}}_{\mathbf{C}_{r-2s}}$ to be independent of the symmetrized derivative variables $\phi^{\mathbf{B}'_p}_{\mathbf{B}_{p+2s}}$, $p \geq 0$, and, consequently, they must satisfy the Killing spinor equations

$$\partial^{(C_{r+1}}_{(C'_{r+1}}\mathcal{S}^{\mathbf{C}_r)}_{\mathbf{C}'_r)} = 0, \qquad \partial^{(C'_{r+2s+1}}_{(C_{r-2s+1}}\mathcal{T}^{\mathbf{C}'_{r+2s})}_{\mathbf{C}_{r-2s})} = 0.$$

Thus

$$\mathcal{Q}_{\mathbf{A}_{2s}} = \mathcal{S}^{\mathbf{B}_r}_{\mathbf{B}'_r}\phi_{\mathbf{A}_{2s}\mathbf{B}_r}{}^{\mathbf{B}'_r} + \mathcal{T}^{\mathbf{B}'_{r+2s}}_{\mathbf{B}_{r-2s}}\overline{\phi}_{\mathbf{A}_{2s}\mathbf{B}'_{r+2s}}{}^{\mathbf{B}_{r-2s}} + \mathcal{U}_{\mathbf{A}_{2s}},$$

where $\mathcal{U}_{\mathbf{A}_{2s}}$ only involves the derivative variables $\phi^{\mathbf{A}'_p}_{\mathbf{A}_{2s+p}}$ up to order $r-1$.

Now by the factorization property of Lemma 1, the Killing spinors $\mathcal{S}^{\mathbf{B}_r}_{\mathbf{B}'_r}$, $\mathcal{T}^{\mathbf{B}_{r+2s}}_{\mathbf{B}_{r-2s}}$ can be expressed as a sum of symmetrized products of r Killing spinors of type $(1,1)$, and as a sum of symmetrized products of a Killing spinor of type $(0,4s)$ and $r-2s$ Killing spinors of type $(1,1)$, respectively. Thus by Proposition 1 there is a linear combination $\mathcal{V}_r$ of the basic symmetries

$$Z[\xi;\zeta_1,\ldots,\zeta_{r-1}], \qquad Z[\mathrm{i}\xi;\zeta_1,\ldots,\zeta_{r-1}], \qquad W[\pi;\zeta_1,\ldots,\zeta_{r-2s}]$$

so that on $\mathcal{R}^\infty(E_s)$, the highest order terms in $\mathcal{V}_r$ agree with those in $\mathcal{Q}$, and, consequently, the symmetry $\mathcal{Q}$ is equivalent to a linear combination of the basic symmetries (3.5)–(3.7) with $p = r-1$, $q = r-2s$ and an evolutionary symmetry of order $r-1$.

Now proceed inductively in the order of the symmetry. In the last step the symmetry $\mathcal{Q}$ is equivalent to a linear combination of the symmetries (3.5)–(3.7) with $p \leq r-1$, $q \leq r-2s$, and an evolutionary symmetry $\mathcal{V}_o$ of order 0. But it is straightforward to solve the determining equations (2.5) for $\mathcal{V}_o$ to see that

$$\mathcal{V}_{o,\mathbf{A}_{2s}} = a\phi_{\mathbf{A}_{2s}} + \varphi_{\mathbf{A}_{2s}},$$

where $a \in \mathbf{C}$ is a constant and $\varphi_{\mathbf{A}_{2s}} = \varphi_{\mathbf{A}_{2s}}(x^i)$ is a solution of the massless free field equations of spin s. Thus (3.16) holds.

Finally, the above arguments show that the vector space of equivalence classes of symmetries of order $r \geq 1$ modulo symmetries of order $r-1$ is isomorphic with the real vector space of Killing spinors of type (r,r), if $r < 2s$ and with the direct sum of the real vector spaces of Killing spinors of type (r,r) and $(r-2s, r+2s)$ if $r \geq 2s$. The dimension of the space spanned by the spinorial symmetries (3.17) now can be computed by adding up the dimensions given in Lemma 1. This concludes the proof of the Theorem. ■

4 Symmetries of Maxwell's equations

In this section we transcribe the spinorial symmetries of Theorem 1 for $s=1$ to tensorial form in order to classify generalized symmetries of Maxwell's equations

$$F_{ij,}{}^{j} = 0, \qquad *F_{ij,}{}^{j} = 0 \tag{4.1}$$

on Minkowski space. Here $F_{ij} = -F_{ji}$ are the components of the electromagnetic field tensor F and $*$ stands for the Hodge dual.

We write $\Lambda^2(T^*\mathbf{M}) \to \mathbf{M}$ for the associated bundle with coordinates F_{ij}, $i < j$. Then $J^\infty\Lambda^2(T^*\mathbf{M})$ is the coordinate bundle

$$\{(x^i, F_{ij}, F_{ij,k_1}, F_{ij,k_1k_2}, \dots)\} \to \{(x^i)\}.$$

For notational convenience, we write $F_{ij,k_1\cdots k_p} = -F_{ji,k_1\cdots k_p}$ for $i \geq j$.

In spinor form the electromagnetic field tensor F becomes

$$\sigma^i_{AA'}\sigma^j_{BB'}F_{ij} = \epsilon_{A'B'}\phi_{AB} + \epsilon_{AB}\overline{\phi}_{A'B'},$$

while Maxwell's equations (4.1) correspond to the spin $s = 1$ massless free field equations (2.3) for the electromagnetic spinor $\phi_{AB} = \phi_{(AB)}$.

A generalized symmetry of Maxwell's equations in evolutionary form is a vector field $Y = Q_{ij}\partial^{ij}_F$ satisfying

$$D^jQ_{ij} = 0, \qquad D^j{*Q_{ij}} = 0 \tag{4.2}$$

on solutions of (4.1). If one defines $\mathcal{Q}_{AB}$ by

$$\sigma^i_{AA'}\sigma^j_{BB'}Q_{ij} = \epsilon_{A'B'}\mathcal{Q}_{AB} + \epsilon_{AB}\overline{\mathcal{Q}}_{A'B'}, \tag{4.3}$$

then it easily follows from (4.2), (4.3) that $\mathcal{Q}_{AB}$ are the components of a generalized symmetry of the massless field equations of spin $s = 1$. Thus, by employing the correspondence (4.3), we can obtain a complete classification of generalized symmetries of Maxwell's equations from the classification result in Theorem 1.

Symmetries (3.14), (3.15) clearly correspond to the symmetries

$$\mathcal{S} = F_{ij}\partial^{ij}_F, \qquad \widetilde{\mathcal{S}} = {*F_{ij}}\partial^{ij}_F, \qquad \mathcal{E}(\mathbf{F}) = \mathrm{F}_{ij}\partial^{ij}_F$$

of Maxwell's equations, where $\mathbf{F}$ is any solution of (4.1) with components $\mathrm{F}_{ij} = \mathrm{F}_{ij}(x^k)$. Conformal symmetries (2.7) and their duals (3.1) in turn give rise to the symmetries

$$\mathcal{Z}[F;\xi] = \mathcal{Z}_{ij}[F;\xi]\partial^{ij}_F, \qquad \mathcal{Z}[*F;\xi] = \mathcal{Z}_{ij}[*F;\xi]\partial^{ij}_F, \tag{4.4}$$

with components

$$\mathcal{Z}_{ij}[F;\xi] = \xi^kF_{ij,k} - 2\partial_{[i}\xi^kF_{j]k},$$

where ξ is a conformal Killing vector on $\mathbf{M}$ and where square brackets indicate skew-symmetrization in the enclosed indices.

In order to transcribe the second order chiral symmetries $\mathcal{W}[\pi]$ introduced in (3.2) to symmetries of Maxwell's equations in physical form, we first introduce the following polynomial tensors on $\mathbf{M}$. Let

$$p^0_{ijkl} = a^0_{ijkl}, \tag{4.5}$$

$$p^1_{ijkl} = x_{[i}a^1_{j]kl} + x_{[k}a^1_{l]ij} + (\eta_{[i|[k}a^1_{l]|j]n} + \eta_{[k|[i}a^1_{j]|l]n})x^n, \tag{4.6}$$

$$\begin{aligned} p^2_{ijkl} = {} & a^2_{[i|[k}x_{l]|}x_{j]} - \frac{1}{2}\eta_{[i|[k}a^2_{l]|j]}x_mx^m \\ & + \frac{1}{2}(\eta_{[i|[k}a^2_{l]n|}x_{j]} + \eta_{[k|[i}a^2_{j]n|}x_{l]})x^n - \frac{1}{6}\eta_{i[k}\eta_{l]j}a^2_{mn}x^mx^n, \end{aligned} \tag{4.7}$$

$$\begin{aligned} p^3_{ijkl} = {} & (x_{[i}a^3_{j]n[k}x_{l]} + x_{[k}a^3_{l]n[i}x_{j]})x^n + \frac{1}{4}(x_{[i}a^3_{j]kl} + x_{[k}a^3_{l]ij})x_nx^n \\ & + \frac{1}{2}(a^3_{mn[i}\eta_{j][k}x_{l]} + a^3_{mn[k}\eta_{l][i}x_{j]})x^mx^n + \frac{1}{4}(a^3_{[i|m[k}\eta_{l]|j]} + a^3_{[k|m[i}\eta_{j]|l]})x^mx_nx^n, \end{aligned} \tag{4.8}$$

$$p^4_{ijkl} = (a^4_{m[i|n[k}x_{l]|}x_{j]} - \frac{1}{2}a^4_{m[i|n[k}\eta_{l]|j]}x_px^p)x^mx^n - \frac{1}{16}a^4_{ijkl}x_mx^mx_nx^n, \tag{4.9}$$

where a^0_{ijkl}, a^1_{ijk}, a^2_{ij}, a^3_{ijk}, a^4_{ijkl} are real constants satisfying

$$a^{\mathtt{h}}_{ijkl} = a^{\mathtt{h}}_{[kl][ij]}, \qquad a^{\mathtt{h}}_{[ijkl]} = 0, \qquad a^{\mathtt{h}}_{ijk}{}^{j} = 0, \qquad \mathtt{h} = 0, 4, \tag{4.10}$$

$$a^{\mathtt{h}}_{ijk} = a^{\mathtt{h}}_{i[jk]}, \qquad a^{\mathtt{h}}_{[ijk]} = 0, \qquad a^{\mathtt{h}}_{ji}{}^{j} = 0, \qquad \mathtt{h} = 1, 3, \tag{4.11}$$

$$a^2_{ij} = a^2_{(ij)}, \qquad a^{2i}_{i} = 0, \tag{4.12}$$

and where we raise indices using the inverse η^{ij} of the Minkowski metric.

Note that $a^{\mathtt{h}}_{ijkl}$, $\mathtt{h} = 0, 4$, possesses the symmetries of Weyl's conformal curvature tensor. Hence their spinor representatives can be written in the form

$$a^{\mathtt{h}}_{II'JJ'KK'LL'} = \epsilon_{IJ}\epsilon_{KL}\alpha^{\mathtt{h}}_{I'J'K'L'} + \epsilon_{I'J'}\epsilon_{K'L'}\overline{\alpha}^{\mathtt{h}}_{IJKL}, \qquad \mathtt{h} = 0, 4,$$

where $\alpha^{\mathtt{h}}_{I'J'K'L'}$ is a constant, symmetric spinor. See, for example, [17]. Moreover, it is easy to verify that by (4.11), (4.12), the spinor representatives of a^2_{ij} and $a^{\mathtt{h}}_{ijk}$, $\mathtt{h} = 1, 3$, are given by

$$a^2_{II'JJ'} = \alpha^2_{IJI'J'} \qquad \text{and}$$
$$a^{\mathtt{h}}_{II'JJ'KK'} = \epsilon_{JK}\alpha^{\mathtt{h}}_{II'J'K'} + \epsilon_{J'K'}\overline{\alpha}^{\mathtt{h}}_{I'IJK}, \qquad \mathtt{h} = 1, 3,$$

where $\alpha^1_{II'J'K'}$, $\alpha^2_{IJI'J'} = \overline{\alpha}^2_{IJI'J'}$, $\alpha^3_{II'J'K'}$, are symmetric constant spinors.

For $0 \le \mathtt{h} \le 4$, let $\mathcal{W}[F; p^{\mathtt{h}}]$ denote the evolutionary vector field with components

$$\mathcal{W}_{ij}[F; p^{\mathtt{h}}] = p^{\mathtt{h}}_{klm[i}F^{kl}{}_{,}{}^{m}{}_{j]} + \partial_{[i}p^{\mathtt{h}}_{j]mkl}F^{kl}{}_{,}{}^{m} + \frac{3}{5}\partial^m p^{\mathtt{h}}_{klm[i}F^{kl}{}_{,j]} + \frac{3}{5}\partial^m\partial_{[i}p^{\mathtt{h}}_{j]mkl}F^{kl}, \tag{4.13}$$

where $p^{\mathtt{h}}_{ijkl}$ are the polynomials in (4.5)–(4.9), and write

$$\mathcal{Z}[F; \xi, \zeta_1, \dots, \zeta_q] = \operatorname{pr}\mathcal{Z}[F; \zeta_1] \cdots \operatorname{pr}\mathcal{Z}[F; \zeta_q] Z[F; \xi],$$
$$\mathcal{W}[F; p^{\mathtt{h}}; \zeta_1, \dots, \zeta_q] = \operatorname{pr}\mathcal{Z}[F; \zeta_1] \cdots \operatorname{pr}\mathcal{Z}[F; \zeta_q] W[F; p^{\mathtt{h}}]$$

for the componentwise Lie derivatives, where $\mathcal{Z}[F; \zeta_i]$ stands for the conformal symmetry (4.4) associated with the conformal Killing vector ζ_i.

Lemma 2. *Let $p^{\mathtt{h}}_{ijkl}$, $0 \le \mathtt{h} \le 4$, be the polynomials* (4.5)–(4.9)*. Then the spinor representative of $p^{\mathtt{h}}_{ijkl}$ is of the form*

$$p^{\mathtt{h}}_{II'JJ'KK'LL'} = \epsilon_{IJ}\epsilon_{KL}\pi^{\mathtt{h}}_{I'J'K'L'} + \epsilon_{I'J'}\epsilon_{K'L'}\overline{\pi}^{\mathtt{h}}_{IJKL}, \tag{4.14}$$

where

$$\begin{aligned}
&\pi^0_{I'J'K'L'} = \alpha^0_{I'J'K'L'}, && \pi^1_{I'J'K'L'} = \alpha^1_{L(I'J'K'}x^L_{L')},\\
&\pi^2_{I'J'K'L'} = \frac{1}{4}\alpha^2_{KL(I'J'}x^K_{K'}x^L_{L')}, && \pi^3_{I'J'K'L'} = -\frac{1}{2}\overline{\alpha}^3_{JKL(I'}x^J_{J'}x^K_{K'}x^L_{L')},\\
&\pi^4_{I'J'K'L'} = \frac{1}{4}\overline{\alpha}^4_{IJKL}x^I_{I'}x^J_{J'}x^K_{K'}x^L_{L'}.
\end{aligned} \tag{4.15}$$

Moreover, on the solution manifold $\mathcal{R}^\infty(E_1)$, the spinor representatives of the evolutionary vector fields $\mathcal{W}[F; p^{\mathtt{h}}]$ in (4.13) *reduce to*

$$\mathcal{W}_{II'JJ'}[F; p^{\mathtt{h}}] = \epsilon_{I'J'}\mathcal{W}_{IJ}[\pi^{\mathtt{h}}] + \epsilon_{IJ}\overline{\mathcal{W}}_{I'J'}[\pi^{\mathtt{h}}], \tag{4.16}$$

where $\mathcal{W}_{IJ}[\pi^{\mathtt{h}}]$ are the components (3.3) *of chiral symmetries for spin $s = 1$ fields. Thus $W_{ij}[F; p^{\mathtt{h}}]$, $0 \le \mathtt{h} \le 4$, is a symmetry of Maxwell's equations. Moreover,*

$$*W_{ij}[F; p^{\mathtt{h}}] = -W_{ij}[*F; p^{\mathtt{h}}]. \tag{4.17}$$

Proof. The proofs of the equations in the Lemma are based on straightforward albeit lengthy computations. We will therefore, as an example, derive equation (4.14) for $\mathtt{h}=2$ and equation (4.16), and omit the proofs of the remaining equations in the Lemma.

One can check by a direct computation that the polynomials $p^{\mathtt{h}}_{ijkl}$ possess the symmetries of Weyl's conformal curvature tensor, that is,

$$p^{\mathtt{h}}_{ijkl} = p^{\mathtt{h}}_{[kl][ij]}, \qquad p^{\mathtt{h}}_{[ijkl]} = 0, \qquad p^{\mathtt{h}}_{ijk}{}^{j} = 0, \qquad 0 \le \mathtt{h} \le 4.$$

Consequently, we can write the spinor representatives of $p^{\mathtt{h}}_{ijkl}$ in the form (4.14); see [17]. Then, in particular,

$$\pi^{\mathtt{h}}_{I'J'K'L'} = \frac{1}{4} p^{\mathtt{h}}_{P(I'}{}^{P}{}_{J'|Q|K'}{}^{Q}{}_{L')}. \tag{4.18}$$

Thus we can find $\pi^{\mathtt{h}}_{I'J'K'L'}$ by substituting the spinor representative of the expression on the right-hand side of equations (4.5)–(4.9) into (4.18) and, in each of the resulting terms, symmetrizing over primed indices. The computations can be further simplified by using the observation that given a tensor f_{ijkl} with the spinor representative $f_{II'JJ'KK'LL'}$, then the spinor representative $h_{II'JJ'KK'LL'}$ of $h_{ijkl} = f_{[ij][kl]}$ satisfies

$$h_{PI'}{}^{P}{}_{J'QK'}{}^{Q}{}_{L'} = f_{P(I'J')}{}^{P}{}_{Q(K'L')}{}^{Q}. \tag{4.19}$$

Now insert the spinor representative of the right-hand side of the equation (4.7) into (4.18). Then by virtue of (4.19), the first term yields the expression

$$\frac{1}{4}\alpha^{2}_{KL(I'J'}x^{K}_{K'}x^{L}_{L')}.$$

The spinor forms of each of the remaining terms on the right-hand side of (4.7) contain an instance of the conjugate of the epsilon tensor, and, consequently, upon symmetrization over primed indices these terms vanish. Hence we have that

$$\pi^{2}_{I'J'K'L'} = \frac{1}{4}\alpha_{KL(I'J'}x^{K}_{K'}x^{L}_{L')},$$

as required.

In order to derive (4.16), we first write

$$F^{+}{}_{kl} = \frac{1}{2}(F_{kl} - \mathrm{i}{*}F_{kl}), \qquad \mathcal{W}^{+}{}_{ij}[F, p^{\mathtt{h}}] = \frac{1}{2}(W_{ij}[F, p^{\mathtt{h}}] - \mathrm{i}W_{ij}[{*}F, p^{\mathtt{h}}]),$$

so that

$$\begin{aligned} \mathcal{W}^{+}{}_{ij}[F, p^{\mathtt{h}}] = {} & p^{\mathtt{h}}_{klm[i}F^{+kl}{}_{,}{}^{m}{}_{j]} + \partial_{[i}p^{\mathtt{h}}_{j]mkl}F^{+kl}{}_{,}{}^{m} \\ & + \frac{3}{5}\partial^{m}p^{\mathtt{h}}_{klm[i}F^{+kl}{}_{,j]} + \frac{3}{5}\partial^{m}\partial_{[i}p^{\mathtt{h}}_{j]mkl}F^{+kl}. \end{aligned} \tag{4.20}$$

We also have that $F^{+}{}_{KK'LL'} = \epsilon_{KL}\overline{\phi}_{K'L'}$.

Note that by virtue of the Killing spinor equations

$$\partial^{P}_{I'}\pi^{\mathtt{h}}_{J'K'L'M'} = -\frac{4}{5}\epsilon_{I'(J'}\partial^{PP'}\pi^{\mathtt{h}}_{K'L'M')P'}.$$

Hence

$$\partial_{P(I'}\pi^{\mathtt{h}}_{J')K'L'M'}\overline{\phi}^{K'L'M'P} = \frac{3}{5}\partial^{P'}_{P}\pi^{\mathtt{h}}_{K'L'P'(I'}\overline{\phi}_{J')}{}^{K'L'P}. \tag{4.21}$$

Consider, for example, the spinor representative of the second term on the right-hand side of equation (4.20). On the solution manifold $\mathcal{R}^\infty(E_1)$ we have

$$\begin{aligned}
&\sigma^i_{II'}\sigma^j_{JJ'}\partial_{[i}p^{\mathtt{h}}_{j]mkl}F^{+kl}{}_{,}{}^{m}\\
&\qquad= -\partial_{II'}\pi^{\mathtt{h}}_{J'K'L'M'}\overline{\phi}_J{}^{K'L'M'} + \partial_{JJ'}\pi^{\mathtt{h}}_{I'K'L'M'}\overline{\phi}_I{}^{K'L'M'}\\
&\qquad= \epsilon_{I'J'}\partial_{P'(I}\pi^{\mathtt{h}K'L'M'P'}\overline{\phi}_{J)K'L'M'} - \epsilon_{IJ}\partial_{P(I'}\pi^{\mathtt{h}}_{J')K'L'M'}\overline{\phi}^{K'L'M'P}\\
&\qquad= \epsilon_{I'J'}\partial_{P'(I}\pi^{\mathtt{h}K'L'M'P'}\overline{\phi}_{J)K'L'M'} - \frac{3}{5}\epsilon_{IJ}\partial^{P'}_{P}\,\pi^{\mathtt{h}}_{K'L'P'(I'}\overline{\phi}_{J')}{}^{K'L'P},
\end{aligned} \tag{4.22}$$

where we used (4.21). Similarly, on $\mathcal{R}^\infty(E_1)$, we see that

$$\sigma^i_{II'}\sigma^j_{JJ'}p^{\mathtt{h}}_{klm[i}F^{+kl}{}_{,}{}^{m}{}_{j]} = \epsilon_{I'J'}\pi^{\mathtt{h}}_{K'L'M'P'}\overline{\phi}_{IJ}{}^{K'L'M'P'}, \qquad \text{and} \tag{4.23}$$

$$\begin{aligned}
\sigma^i_{II'}\sigma^j_{JJ'}\partial^m p^{\mathtt{h}}_{klm[i}F^{+kl}{}_{,j]} &= \epsilon_{I'J'}\partial_{P'(I}\pi^{\mathtt{h}K'L'M'P'}\overline{\phi}_{J)K'L'M'}\\
&\quad - \epsilon_{IJ}\partial_{PP'}\pi^{\mathtt{h}K'L'P'}{}_{(I'}\overline{\phi}_{J')K'L'}{}^{P}.
\end{aligned} \tag{4.24}$$

Moreover, by virtue of (3.9),

$$\begin{aligned}
&\sigma^i_{II'}\sigma^j_{JJ'}\partial^m\partial_{[i}p^{\mathtt{h}}_{j]mkl}F^{+kl}\\
&\qquad= \epsilon_{I'J'}\partial_{M'I}\partial_{JP'}\pi^{\mathtt{h}K'L'M'P'}\overline{\phi}_{K'L'} - \epsilon_{IJ}\partial_{PM'}\partial^{P}_{(I'}\pi^{\mathtt{h}}_{J')}{}^{K'L'M'}\overline{\phi}_{K'L'}\\
&\qquad= \epsilon_{I'J'}\partial_{M'I}\partial_{JP'}\pi^{\mathtt{h}K'L'M'P'}\overline{\phi}_{K'L'}
\end{aligned} \tag{4.25}$$

on $\mathcal{R}^\infty(E_1)$.

Now equations (4.22)–(4.25) together show that the spinor form of $\mathcal{W}^+{}_{ij}[F,p^{\mathtt{h}}]$ is

$$\begin{aligned}
\mathcal{W}^+{}_{II'JJ'}[F,p^{\mathtt{h}}] &= \epsilon_{I'J'}\big(\pi^{\mathtt{h}K'L'M'P'}\overline{\phi}_{IJK'L'M'P'}\\
&\qquad + \frac{8}{5}\partial_{P'(I}\pi^{\mathtt{h}K'L'M'P'}\overline{\phi}_{J)K'L'M'} + \frac{3}{5}\partial_{M'I}\partial_{JP'}\pi^{\mathtt{h}K'L'M'P'}\overline{\phi}_{K'L'}\big)\\
&= \epsilon_{I'J'}\mathcal{W}_{IJ}[\pi^{\mathtt{h}}],
\end{aligned}$$

which immediately yields equation (4.16). ■

Theorem 2. *The space of equivalence classes of symmetries of Maxwell's equations of order r, $r \geq 2$, is spanned by the symmetries*

$$\begin{array}{llll}
\mathcal{E}(\mathbf{F}), \quad \mathcal{S}, \quad \widetilde{\mathcal{S}}, & & & \\
\mathcal{Z}[F;\xi,\zeta_1,\ldots,\zeta_q], & \mathcal{Z}[*F;\xi,\zeta_1,\ldots,\zeta_q], & q \leq r-1, & \\
\mathcal{W}[F;p^{\mathtt{h}};\zeta_1\ldots,\zeta_q], & \mathcal{W}[*F;p^2;\zeta_1,\ldots,\zeta_q], & 0 \leq \mathtt{h} \leq 4, & q \leq r-2,
\end{array}$$

where $\mathbf{F}$ is an arbitrary solution of Maxwell's equations, ξ, $\zeta_1,\ldots,\zeta_r$, are conformal Killing vectors and $p^{\mathtt{h}}_{ijkl}$, $0 \leq \mathtt{h} \leq 4$, are the polynomials given in (4.5)–(4.9). Apart from the trivial symmetries $\mathcal{E}(\mathbf{F})$, there are

$$d_r = (r+1)(r+3)(r^4 + 8r^3 + 17r^2 + 4r + 6)/9$$

independent symmetries of Maxwell's equations of order $r \geq 2$.

Proof. First note that if $Y = Q_{ij}\partial^{ij}_F$ is a symmetry of Maxwell's equations in evolutionary form with the spinor representative $\mathcal{Q}$ determined by (4.3) then the spinor representative of the symmetry $\operatorname{pr} Z[F;\zeta]Y$ is $\operatorname{pr} Z[\zeta]\mathcal{Q}$. Thus, in light of Theorem 1 and by linearity, it suffices to show that the spinorial symmetries $\mathcal{E}(\varphi)$, $\mathcal{S}$, $\widetilde{\mathcal{S}}$, $\mathcal{Z}[\xi]$, $\mathcal{Z}[\mathrm{i}\xi]$, $\mathcal{W}[\pi]$, where ξ is a real conformal Killing vector and π is a type $(0,4)$ Killing spinor, are contained in the span of the spinorial symmetries corresponding to

$$\mathcal{E}(\mathbf{F}), \quad \mathcal{S}, \quad \widetilde{\mathcal{S}}, \quad \mathcal{Z}[F;\xi], \quad *\mathcal{Z}[F;\xi], \quad *\mathcal{W}[F;p^2], \quad \mathcal{W}[F;p^{\mathtt{h}}], \qquad 0 \le \mathtt{h} \le 4,$$

under the identification (4.3).

Obviously the symmetries $\mathcal{E}(\mathbf{F})$, $\mathcal{S}$, $\widetilde{\mathcal{S}}$, $\mathcal{Z}[F;\xi]$, $*\mathcal{Z}[F;\xi]$ correspond via (4.3) to the spinorial symmetries $\mathcal{E}(\varphi)$, $\mathcal{S}$, $\widetilde{\mathcal{S}}$ $\mathcal{Z}[\xi]$, $\mathcal{Z}[\mathrm{i}\xi]$, where φ is the spinorial counterpart of the solution $\mathbf{F}$ of Maxwell's equations.

Next, by (4.15), the spinor fields $\pi^{\mathtt{h}}_{I'J'K'L'}$ corresponding to the polynomials $p^{\mathtt{h}}_{ijkl}$ span the space of Killing spinors of type $(0,4)$ provided that we allow $\alpha^2_{IJI'J'}$ also to take complex values. Thus, in light of (4.16), the first part of the Theorem now follows from the observation that if $\mathcal{W}[\pi]$ is the spinorial symmetry corresponding to $\mathcal{W}[F;p^{\mathtt{h}}]$, then $\mathcal{W}[\mathrm{i}\pi]$ corresponds to the symmetry $\widehat{\mathcal{W}}[F;p^{\mathtt{h}}] = -*\mathcal{W}[F;p^{\mathtt{h}}]$.

Finally, the dimension count follows from the dimension count in Theorem 1. This concludes the proof of the Theorem. ∎

Remark 1. The new chiral symmetries $\mathcal{W}[F;p^{\mathtt{h}}]$ are physically interesting since, as evidenced by equation (4.17), they possess odd parity, i.e., chirality, under the duality transformation interchanging the electric and magnetic fields. Hence, in a marked contrast to the conformal symmetries $\mathcal{Z}[F;\xi]$, which behave equivariantly under the duality transformation, the new second order symmetries, when regarded as differential operators acting on solutions of Maxwell's equations, map self-dual electromagnetic fields to anti-self-dual fields and vice versa.

5 Applications to the Dirac operator on Minkowski space

Recall that a Dirac spinor consists of a pair

$$\mathbf{\Psi} = \begin{pmatrix} \psi^{A'} \\ \varphi_A \end{pmatrix}$$

of spinor fields of type $(0,1)$ and $(1,0)$ (see, e.g., [20]), with the conjugate spinor $\mathbf{\Psi}^*$ given by

$$\mathbf{\Psi}^* = \begin{pmatrix} \overline{\varphi}^{A'} \\ \overline{\psi}_A \end{pmatrix}.$$

A vector $v^i \in \mathbf{M}$ determines a linear transformation $\not{v}$ on Dirac spinors given by

$$\not{v}\mathbf{\Psi} = \begin{pmatrix} v^i \sigma_i^{AA'} \varphi_A \\ v^i \sigma_{iAA'} \psi^{A'} \end{pmatrix}.$$

Then the Dirac γ-matrices are defined by the condition

$$v^i \gamma_i \mathbf{\Psi} = \not{v}\mathbf{\Psi},$$

so that

$$\gamma_i = \begin{pmatrix} 0 & \sigma_i^{A'B} \\ \sigma_{iAB'} & 0 \end{pmatrix}, \qquad 0 \le i \le 3.$$

Moreover, as is customary, we will write

$$\gamma_5 = -\mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}.$$

The Weyl system, or the massless Dirac equation, is given by

$$\partial\!\!\!/ \boldsymbol{\Psi} = 0, \tag{5.1}$$

where

$$\partial\!\!\!/ = \gamma^i \frac{\partial}{\partial x^i}$$

is the Dirac operator. As an application of the methods developed in Section 3 we present here a complete classification of generalized symmetries of the Weyl system on Minkowski space.

Evidently, the Weyl system is equivalent to the decoupled pair

$$\varphi_{A,}{}^{AA'} = 0, \qquad \psi^{A'}{}_{,AA'} = 0$$

of spin $s = \frac{1}{2}$ massless free field equation and its conjugate equation, and, consequently, we will be able to analyze symmetries of (5.1) by the methods employed in the proof of the classification result in Theorem 1.

Now an evolutionary vector field takes the form

$$X = Q^{A'}\partial_{\psi^{A'}} + \overline{Q}^{A}\overline{\partial}_{\psi^{A}} + R_A\partial_{\varphi_A} + \overline{R}_{A'}\overline{\partial}_{\varphi_{A'}},$$

where the characteristic

$$\mathcal{Q} = \begin{pmatrix} Q^{A'} \\ R_A \end{pmatrix}$$

can be assumed to depend on the spacetime variables x^i and the exact sets of fields, that is, the symmetrized derivatives

$$\psi^{A'\mathbf{A}'_p}_{\mathbf{A}_p} = \psi^{(A'\mathbf{A}'_p)}{}_{,\mathbf{A}_p}, \qquad \varphi_{A\mathbf{A}_p}^{\mathbf{A}'_p} = \varphi_{(A,\mathbf{A}_p)}^{\mathbf{A}'_p}$$

of the components of the Dirac spinor.

As in the case of massless free fields, the Weyl system obviously admits scaling, conformal and chiral symmetries and their duals. The scaling symmetry and its dual possess the characteristics

$$S[\boldsymbol{\Psi}] = \boldsymbol{\Psi}, \qquad S[\mathrm{i}\boldsymbol{\Psi}] = i\boldsymbol{\Psi}.$$

Let $\xi^{CC'}$ be a real conformal Killing vector, $\pi^{B'C'}$ a type $(0,2)$ Killing spinor. We write

$$\mathcal{Z}[\boldsymbol{\Psi};\xi] = \begin{pmatrix} \xi^{CC'}\psi^{A'}_{CC'} + \dfrac{1}{2}(\partial^{A'}_{C}\xi^{CC'})\psi_{C'} + \dfrac{1}{8}(\partial_{CC'}\xi^{CC'})\psi^{A'} \\ \xi^{CC'}\varphi_{ACC'} + \dfrac{1}{2}(\partial_{AC'}\xi^{CC'})\varphi_C + \dfrac{1}{8}(\partial_{CC'}\xi^{CC'})\varphi_A \end{pmatrix},$$

$$\mathcal{W}[\boldsymbol{\Psi};\pi] = \begin{pmatrix} \overline{\pi}^{BC}\varphi^{A'}_{BC} + \dfrac{2}{3}\,\partial^{A'}_{C}\overline{\pi}^{BC}\varphi_B \\ \pi^{B'C'}\psi_{AB'C'} + \dfrac{2}{3}\,\partial_{AC'}\pi^{B'C'}\psi_{B'} \end{pmatrix}$$

for the characteristics of the basic conformal and chiral symmetries of the Weyl system (5.1). Moreover, we define $\mathcal{Z}[\mathrm{i}\boldsymbol{\Psi};\xi]$ in the obvious fashion, and denote the componentwise Lie derivatives of $\mathcal{Z}[\boldsymbol{\Psi};\xi]$, $\mathcal{Z}[\mathrm{i}\boldsymbol{\Psi};\xi]$, $\mathcal{W}[\boldsymbol{\Psi};\pi]$ with respect to the conformal symmetries corresponding to $\zeta_1,\ldots,\zeta_p$ by

$$\mathcal{Z}[\boldsymbol{\Psi};\xi,\zeta_1,\ldots,\zeta_p], \qquad \mathcal{Z}[\mathrm{i}\boldsymbol{\Psi};\xi,\zeta_1,\ldots,\zeta_p], \qquad \mathcal{W}[\boldsymbol{\Psi};\pi;\zeta_1,\ldots,\zeta_p].$$

Theorem 3. *Let $\mathcal{Q}$ be a generalized symmetry of order r of the Weyl system* (5.1) *in evolutionary form. Then $\mathcal{Q}$ is equivalent to a symmetry $\widehat{\mathcal{Q}}$ of order at most r in evolutionary form which can be written as*

$$\widehat{\mathcal{Q}} = \mathcal{V} + \mathcal{E}[\Psi],$$

where $\mathcal{E}[\Psi]$ is an elementary symmetry corresponding to a solution $\Psi = \Psi(x^i)$ of the Weyl system, and where $\mathcal{V}$ is equivalent to a symmetry that is a linear combination of the symmetries

$$\begin{array}{llll}
\mathcal{S}[\boldsymbol{\Psi}], & \mathcal{S}[\mathrm{i}\boldsymbol{\Psi}], & \gamma_5\mathcal{S}[\boldsymbol{\Psi}], & \gamma_5\mathcal{S}[\mathrm{i}\boldsymbol{\Psi}], \\
\mathcal{S}[\boldsymbol{\Psi}^*], & \mathcal{S}[\mathrm{i}\boldsymbol{\Psi}^*], & \gamma_5\mathcal{S}[\boldsymbol{\Psi}^*], & \gamma_5\mathcal{S}[\mathrm{i}\boldsymbol{\Psi}^*], \\
\mathcal{Z}[\boldsymbol{\Psi};\xi,\zeta_1,\dots,\zeta_p], & & \gamma_5\mathcal{Z}[\boldsymbol{\Psi};\xi,\zeta_1,\dots,\zeta_p], & \\
\mathcal{Z}[\mathrm{i}\boldsymbol{\Psi};\xi,\zeta_1,\dots,\zeta_p], & & \gamma_5\mathcal{Z}[\mathrm{i}\boldsymbol{\Psi};\xi,\zeta_1,\dots,\zeta_p], & \\
\mathcal{Z}[\boldsymbol{\Psi}^*;\xi,\zeta_1,\dots,\zeta_p], & & \gamma_5\mathcal{Z}[\boldsymbol{\Psi}^*;\xi,\zeta_1,\dots,\zeta_p], & \\
\mathcal{Z}[\mathrm{i}\boldsymbol{\Psi}^*;\xi,\zeta_1,\dots,\zeta_p], & & \gamma_5\mathcal{Z}[\mathrm{i}\boldsymbol{\Psi}^*;\xi,\zeta_1,\dots,\zeta_p], & \\
\mathcal{W}[\boldsymbol{\Psi};\pi;\zeta_1,\dots,\zeta_p], & & \gamma_5\mathcal{W}[\boldsymbol{\Psi};\pi;\zeta_1,\dots,\zeta_p], & \\
\mathcal{W}[\boldsymbol{\Psi}^*;\pi;\zeta_1,\dots,\zeta_p], & & \gamma_5\mathcal{W}[\boldsymbol{\Psi}^*;\pi;\zeta_1,\dots,\zeta_p], &
\end{array} \tag{5.2}$$

where $p \le r-1$. In particular, the dimension d_r of the real vector space of equivalence classes of spinorial symmetries of order at most r spanned by the symmetries (5.2) *is*

$$d_r = \frac{2}{9}\big[(r+1)^2(r+2)^2(r+3)^2 + ((r+1)^2-1)((r+2)^2-1)((r+3)^2-1)\big], \quad \textit{for} \quad r \ge 1.$$

Proof. The proof of Theorem 3 follows very much the same lines as the proof of Theorem 1, and we can safely omit the details. ■

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