E-Orbit Functions

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Abstract. We review and further develop the theory of E-orbit functions. They are functions on the Euclidean space E_n obtained from the multivariate exponential function by symmetrization by means of an even part W_e of a Weyl group W, corresponding to a Coxeter–Dynkin diagram. Properties of such functions are described. They are closely related to symmetric and antisymmetric orbit functions which are received from exponential functions by symmetrization and antisymmetrization procedure by means of a Weyl group W. The E-orbit functions, determined by integral parameters, are invariant with respect to even part $W_e^{\rm aff}$ of the affine Weyl group corresponding to W. The E-orbit functions determine a symmetrized Fourier transform, where these functions serve as a kernel of the transform. They also determine a transform on a finite set of points of the fundamental domain F^e of the group $W_e^{\rm aff}$ (the discrete E-orbit function transform).

Key words: E-orbit functions; orbits; products of orbits; symmetric orbit functions; E-orbit function transform; finite E-orbit function transform; finite Fourier transforms

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1 Introduction

In [1] and [2] it was initiated a study of orbit functions which are closely related to finite groups W of geometric symmetries generated by reflection transformations r_i (that is, such that $r_i^2 = 1$), $i = 1, 2, \ldots, n$, of the n-dimensional Euclidean space E_n with respect to $(n-1)$-dimensional subspaces containing the origin. In fact, orbit functions are multivariate exponential functions symmetrized or antisymmetrized by means of a Weyl group W of a semisimple Lie algebra or symmetrized by means of its subgroup W_e consisting of even elements of W. Orbit functions on the 2-dimensional Euclidean space E_2, invariant or anti-invariant with respect to W, were considered in detail in [3, 4, 5, 6]. A detailed description of symmetric and antisymmetric orbit functions on any Euclidean space E_n is given in [7] and in [8]. Orbit functions on E_2, invariant with respect to W_e are studied in [9].

The important peculiarity of orbit functions is a possibility of their discretization [10], which is made by using the results of paper [11]. This possibility makes orbit functions useful for applications. In particular, on this way multivariate discrete Fourier transform and multivariate discrete sine and cosine transforms are received (see [8, 12, 13]).

In order to obtain a symmetric orbit function we take a point $\lambda \in E_n$ and act upon λ by all elements of the group W. If $O(\lambda)$ is the W-orbit of the point λ, that is, the set of all different points of the form $w\lambda$, $w \in W$, then the symmetric orbit function, determined by λ, coincides with

$$\phi_\lambda(x) = \sum_{\mu \in O(\lambda)} e^{2\pi i \langle \mu, x\rangle},$$

where $\langle \mu, x\rangle$ is the scalar product on E_n. These functions are invariant with respect to the action by elements of the group W: $\phi_\lambda(wx) = \phi_\lambda(x)$, $w \in W$. If λ is an integral point of E_n, then $\phi_\lambda(x)$ is invariant with respect to the affine Weyl group $W^{\rm aff}$ corresponding to W.

Symmetry is the main property of symmetric orbit functions which make them useful in applications. Being a modification of monomial symmetric functions, they are directly related to the theory of symmetric (Laurent) polynomials [14, 15, 16, 17] (see Section 11 in [7]).

Symmetric orbit functions $\phi_\lambda(x)$ for integral λ are closely related to the representation theory of compact groups G. In particular, they were effectively used for different calculations in representation theory [18, 19, 20, 21, 22]. They are constituents of traces (characters) of irreducible unitary representations of G.

Antisymmetric orbit functions are given by

$$\varphi_\lambda(x) = \sum_{w\in W} (\det w) e^{2\pi {\rm i}\langle w\lambda, x\rangle}, \qquad x \in E_n,$$

where λ is an element, which does not lie on a wall of a Weyl chamber, and $\det w$ is a determinant of the transformation w (it is equal to 1 or -1, depending on either w is a product of even or odd number of reflections). The orbit functions φ_λ have many properties that the symmetric orbit functions ϕ_λ do. However, antisymmetry leads to some new properties which are useful for applications [6]. For integral λ, antisymmetric orbit functions are closely related to characters of irreducible representations of the corresponding compact Lie group G. Namely, the character χ_λ of the irreducible representation T_λ, $\lambda \in P_+$, coincides with $\varphi_{\lambda+\rho}/\varphi_\rho$, where ρ is the half-sum of positive roots related to the Weyl group W.

A symmetric (antisymmetric) orbit function is the exponential function $e^{2\pi{\rm i}\langle\lambda,x\rangle}$ on E_n symmetrized (antisymmetrized) by means of the group W. For each transformation group W, the symmetric (antisymmetric) orbit functions, characterized by integral λ, form a complete basis in the space of symmetric (antisymmetric) with respect to W polynomials in $e^{2\pi{\rm i}x_j}$, $j = 1, 2, \ldots, n$, or an orthogonal basis in the Hilbert space obtained by closing this space of polynomials with respect to an appropriate scalar product.

Orbit functions $\phi_\lambda(x)$ (or $\varphi_\lambda(x)$), when λ runs over integral elements, determine the so-called symmetric (antisymmetric) orbit function transform, which is a symmerization (antisymmetrization) of the usual Fourier series expansion on E_n. If λ runs over the dominant Weyl chamber in the space E_n, then $\phi_\lambda(x)$ (or $\varphi_\lambda(x)$) determine a symmetric (antisymmetric) orbit function transform, which is a symmetrization (antisymmetrization) of the usual continuous Fourier expansion in E_n (that is, of the Fourier integral).

The Fourier transform on $\mathbb{R}$ leads to the discrete Fourier transform on grids. In the same way the symmetric and antisymmetric orbit function transforms lead to discrete analogues of these transforms (which are generalizations of the discrete cosine and sine transforms, respectively, [23]). These discrete transforms are useful in many things related to discretization (see [3, 4, 5]). Construction of the discrete orbit function transforms are fulfilled by means of the results of paper [11].

Symmetric orbit functions are a generalization of the cosine function, whereas antisymmetric orbit functions are a generalization of the sine function. There appears a natural question: What is a generalization of the exponential function of one variable? This generalization is given by orbit functions symmetric with respect to the subgroup W_e of even elements in W.

Our goal in this paper is to give in a full generality the theory of orbit functions symmetric with respect to the group W_e. We shall call these functions E-orbit functions, since they are an analogue of the well-known exponential function (since symmetric and antisymmetric orbit functions ϕ_λ and φ_λ are generalizations of the cosine and sine functions, they are often called as C-functions and S-functions, respectively). This paper is a natural continuation of our papers [7] and [8]. Under our exposition we use the results of papers [2] and [10], where E-orbit functions are defined.

Roughly speaking, E-orbit functions are related to symmetric and antisymmetric orbit functions in the same way as the exponential function in one variable is related to the sine and cosine functions.

E-orbit functions are symmetric with respect to the subgroup W_e of the Weyl group W, that is, $E_\lambda(wx) = E_\lambda(x)$ for any $w \in W_e$. The subgroup W_e is of index 2 in W, that is $|W/W_e| = 2$, where $|X|$ denote a number of elements in the corresponding set X. This means that E-orbit functions are determined not only for λ from dominant Weyl chamber D_+ (as in the case of symmetric orbit functions), but also for elements from the set r_iD_+, where r_i is a fixed reflection from W.

If λ is an *integral* element, then the corresponding E-orbit function $E_\lambda(x)$ is symmetric also with respect to elements of the affine Weyl group $W_e^{\rm aff}$, corresponding to the group W_e (in fact, the group $W_e^{\rm aff}$ consists of even elements of the whole affine Weyl group $W^{\rm aff}$, corresponding to the Weyl group W).

Symmetry with respect to $W_e^{\rm aff}$ is a main property of E-orbit functions with integral λ. Because of this symmetry, it is enough to determine E-orbit functions only on the fundamental domain $F(W_e^{\rm aff})$ of the group $W_e^{\rm aff}$ (if λ is integral). This fundamental domain consists of two fundamental domains of the whole affine Weyl group $W^{\rm aff}$.

When the group W is a direct product of its subgroups, say $W = W_1 \times W_2$, then $W_e = (W_1)_e \times (W_2)_e$. In this case E-orbit functions of W_e are products of E-orbit functions of $(W_1)_e$ and $(W_2)_e$. Hence it suffices to carry out our considerations for groups W_e which cannot be represented as a product of its subgroups (that is, for such W for which a corresponding Coxeter–Dynkin diagram is connected).

E-orbit functions with integral λ determine the so-called E-orbit function transform on the fundamental domain $F(W_e^{\rm aff})$ of the group $W_e^{\rm aff}$. It is an expansion of functions on $F(W_e^{\rm aff})$ in these E-orbit functions. E-orbit functions $E_\lambda(x)$ with λ from E_n determine E-orbit function transform on the fundamental domain $F(W_e)$ of the group W_e. It is an analogue of the usual integral Fourier transform.

E-orbit functions determine also the discrete E-orbit function transforms. They are transforms on grids of the domain $F(W_e^{\rm aff})$. These transforms are an analogue of the usual discrete Fourier transforms.

We need for our exposition a general information on Weyl groups, affine Weyl groups and root systems. We have given this information in [7] and [8]. In order to make this paper self-contained we repeat shortly a part of that information in Section 2. In this section we also describe even Weyl groups and affine even Weyl groups.

In Section 3 we define and study W_e-orbits. It is shown how W_e-orbits are related to W-orbits. Each W-orbit is a W_e-orbit or consists of two W_e-orbit. To each W_e-orbit there corresponds an E-orbit function. W_e-orbits are parametrized by elements of even dominant Weyl chamber D_+^e.

We describe in Section 3 all W_e-orbits for A_2 and C_2. A big class of W_e-orbits for G_2 is also given. All W_e-orbits of A_3, B_3 and C_3 are derived. It is proposed to describe points of W_e-orbits of A_n, B_n, C_n and D_n by means of orthogonal coordinates. Then elements of the group W_e and W_e-orbits are described in a simple way. Section 3 contains also a description of fundamental domains for the groups $W_e^{\rm aff}(A_n)$, $W_e^{\rm aff}(B_n)$, $W_e^{\rm aff}(C_n)$, and $W_e^{\rm aff}(D_n)$.

Section 4 is devoted to description of E-orbit functions. E-orbit functions, corresponding to Coxeter–Dynkin diagrams, containing only two nodes, are given in an explicit form. In this section we also give explicit formulas for E-orbit functions, corresponding to the cases A_n, B_n, C_n and D_n, in the corresponding orthogonal coordinate systems.

In Section 5, properties of E-orbit functions are derived. If λ is integral, then a main property of the E-orbit function $E_\lambda(x)$ is an invariance with respect to the affine even Weyl group $W_e^{\rm aff}$.

Relation of E-orbit functions to symmetric and antisymmetric orbit functions (that is, to W-orbit functions) is described.

E-orbit functions $E_\lambda(x)$ with integral λ are orthogonal on the closure of the fundamental domain of the group $W_e^{\rm aff}$. This orthogonality is given in Section 5.

E-orbit functions are solutions of the corresponding Laplace equation. This description is exposed in Section 5. E-orbit functions also are solutions of some other differential equations.

In Section 6 we consider expansions of products of E-orbit functions into a sum of E-orbit functions. These expansions are closely related to properties of W_e-orbits, namely, the expansions are reduced to decomposition of products of W_e-orbits into separate W_e-orbits. In general, it is a complicated problem (especially, when multiple W_e-orbits appear in the decomposition). Several propositions, describing decomposition of products of W_e-orbits, are given. Many examples for expansions in the case of Coxeter–Dynkin diagrams A_2, C_2, and G_2 are considered.

Section 7 is devoted to expansion of W_e-orbit functions into a sum of W'_e-orbit functions, where W'_e is an even Weyl subgroup of the even Weyl group W_e. The cases of restriction of A_n to A_{n-1}, of B_n to B_{n-1}, of C_n to C_{n-1}, and of D_n to D_{n-1} are described in detail.

In Section 8 we expose E-orbit function transforms. There are two types of such transforms. The first one is an analogue of the expansion into Fourier series (it is an expansion on the fundamental domain of the group $W_e^{\rm aff}$) and the second one is an analogue of the Fourier integral transform (it is an expansion on the even dominant Weyl chamber).

In Section 9 a description of a W_e-generalization of the multi-dimensional finite Fourier transforms is given. This generalization is connected with grids on the corresponding fundamental domains for the affine even Weyl groups $W_e^{\rm aff}$. These grids are determined by a positive integer M. To each such an integer there corresponds a grid on the fundamental domain. Examples of such grids for A_2, C_2 and G_2 are given.

Section 10 is devoted to exposition of W_e-symmetric functions, which are symmetric analogues of special functions of mathematical physics or orthogonal polynomials. In particular, we find eigenfunctions of the W_e-orbit function transforms. These eigenfunctions are connected with classical Hermite polynomials.

2 Root systems and Weyl groups

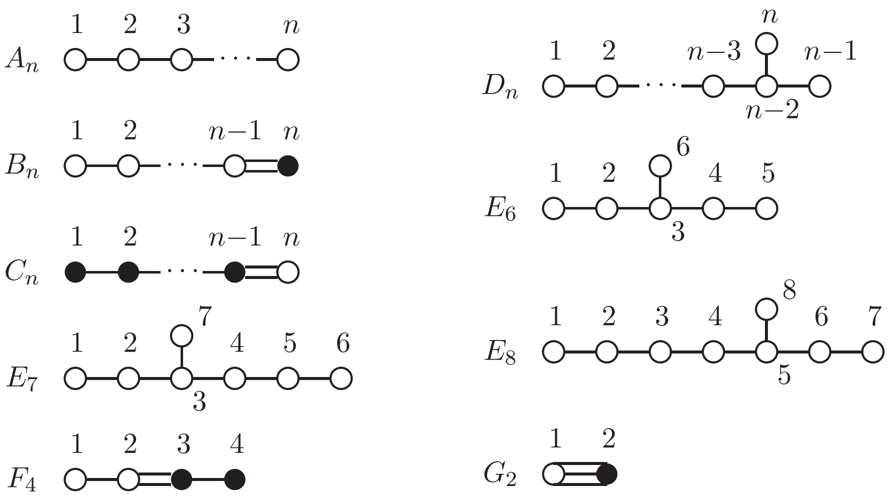
2.1 Coxeter–Dynkin diagrams and simple roots

We need finite transformation groups W, acting on the n-dimensional Euclidean space E_n, which are generated by reflections r_i, $i = 1, 2, \dots, n$ (that is, $r_i^2 = 1$); the theory of such groups see, for example, in [24] and [25]. We are interested in those groups W which are Weyl groups of semisimple Lie groups (semisimple Lie algebras). It is well-known that such Weyl groups together with the corresponding systems of reflections r_i, $i = 1, 2, \dots, n$, are determined by Coxeter–Dynkin diagrams. There are 4 series of simple Lie algebras and 5 separate simple Lie algebras, which uniquely determine their Weyl groups W. These algebras are denoted as

$$A_n\ (n \geq 1),\ B_n\ (n \geq 3),\ C_n\ (n \geq 2),\ D_n\ (n \geq 4),\ E_6,\ E_7,\ E_8,\ F_4,\ G_2.$$

To these simple Lie algebras there correspond connected Coxeter–Dynkin diagrams.

To semisimple Lie algebras (they are direct sums of simple Lie subalgebras) there correspond Coxeter–Dynkin diagrams, which consist of connected parts, corresponding to simple Lie subalgebras; these parts are not connected with each other (a description of the correspondence between simple Lie algebras and Coxeter–Dynkin diagrams see, for example, in [26]). Thus, we describe only Coxeter–Dynkin diagrams, corresponding to simple Lie algebras. They are of the form



A diagram determines a certain non-orthogonal basis $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ in the Euclidean space E_n. Each node of a diagram is associated with a basis vector α_k, called a *simple root*. A direct link between two nodes indicates that the corresponding basis vectors are not orthogonal. Conversely, an absence of a direct link between nodes implies orthogonality of the corresponding vectors. Single, double, and triple links indicate that the relative angles between the two simple roots are $2\pi/3$, $3\pi/4$, $5\pi/6$, respectively. There can be only two cases: all simple roots are of the same length or there are only two different lengths of simple roots. In the first case all simple roots are denoted by white nodes. In the case of two lengths, shorter roots are denoted by black nodes and longer ones by white nodes. Lengths of roots are determined uniquely up to a common constant. For the cases B_n, C_n, and F_4, the squared longer root length is double the squared shorter root length. For G_2, the squared longer root length is triple the squared shorter root length. Simple roots of the same length are orthogonal to each other or an angle between them is $2\pi/3$. A number of simple roots is called a *rank* of the corresponding Lie algebra.

To each Coxeter–Dynkin diagram there corresponds a Cartan matrix M, consisting of the entries

$$M_{jk} = \frac{2\langle \alpha_j, \alpha_k \rangle}{\langle \alpha_k, \alpha_k \rangle}, \qquad j,k \in \{1,2,\dots,n\}, \tag{2.1}$$

where $\langle x, y \rangle$ denotes the scalar product of $x, y \in E_n$. Cartan matrices of simple Lie algebras are given in many places (see, for example, [27]). For ranks 2 and 3 they are of the form:

$$A_2: \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad C_2: \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \qquad G_2: \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix},$$

$$A_3: \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \qquad B_3: \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}, \qquad C_3: \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}.$$

Lengths of the basis vectors α_i are fixed by the corresponding Coxeter–Dynkin diagram up to a constant. We adopt the standard choice in the Lie theory, namely

$$\langle \alpha, \alpha \rangle = 2$$

for all simple roots of A_n, D_n, E_6, E_7, E_8 and for the longer simple roots of B_n, C_n, F_4, G_2.

2.2 Weyl group and even Weyl group

A Coxeter–Dynkin diagram determines uniquely the corresponding transformation group W of the Euclidean space E_n, generated by reflections r_i, $i = 1, 2, \dots, n$. These reflections correspond to simple roots α_i, $i = 1, 2, \dots, n$. Namely, the transformation r_i corresponds to the simple root α_i and is the reflection with respect to $(n-1)$-dimensional linear subspace (hyperplane) of E_n (containing the origin), orthogonal to α_i. Such reflections are given by the formula

$$r_i x = x - \frac{2\langle x, \alpha_i\rangle}{\langle \alpha_i, \alpha_i\rangle}\alpha_i, \qquad i = 1, 2, \dots, n, \qquad x \in E_n. \tag{2.2}$$

Each reflection r_i can be thought as attached to the i-th node of the corresponding diagram.

A finite group W, generated by the reflections r_i, $i = 1, 2, \dots, n$, is called a *Weyl group*, corresponding to a given Coxeter–Dynkin diagram. If a Weyl group W corresponds to a Coxeter–Dynkin diagram of a simple Lie algebra L, then this Weyl group is often denoted by $W(L)$. Properties of Weyl groups are well known (see [24] and [25]). The orders (numbers of elements) of Weyl groups are given by the formulas

$$\begin{aligned} &|W(A_n)| = (n+1)!, \quad &&|W(B_n)| = |W(C_n)| = 2^n n!, \quad &&|W(D_n)| = 2^{n-1} n!,\\ &|W(E_6)| = 51\,840, &&|W(E_7)| = 2\ 903\,040, &&|W(E_8)| = 696\,729\ 600,\\ &|W(F_4)| = 1\,152, &&|W(G_2)| = 12. \end{aligned} \tag{2.3}$$

In particular,

$$|W(A_2)| = 6, \qquad |W(C_2)| = 8, \qquad |W(A_3)| = 24, \qquad |W(B_3)| = |W(C_3)| = 48.$$

Elements of the Weyl groups are linear transformations of the Euclidean space E_n. To these transformations there correspond in an orthonormal basis of E_n the corresponding $n \times n$ matrices. Since these transformations are orthogonal, then determinants of these matrices are $+1$ or -1. We say that a transformation $w \in W$ is even if $\det w = 1$ and odd if $\det w = -1$. Clearly, for reflections r_α corresponding to roots α we have $\det r_\alpha = -1$. If $w \in W$ is a product of even (odd) number of reflections, then $\det w = 1$ ($\det w = -1$).

The set of all elements $w \in W$ with $\det w = 1$ constitute a subgroup of W which will be denoted by W_e. One says that it is a subgroup of even elements of W. Moreover, W_e is a normal subgroup of W, that is, $wW_ew^{-1} = W_e$ for any $w \in W$. The group W_e is a basic group for definition of E-orbit functions.

Elements of W, which do not belong to W_e, are called odd. The number of even elements in W is equal to the number of odd elements, that is, $|W/W_e| = 2$. In particular, we have

$$\begin{aligned} &|W_e(A_2)| = 3, \quad |W_e(C_2)| = 4, \quad |W_e(G_2)| = 6, \quad |W_e(A_3)| = 12,\\ &|W_e(B_3)| = |W_e(C_3)| = 24. \end{aligned}$$

The Weyl groups $W(B_3)$ and $W(C_3)$ are isomorphic. For this reason, the even Weyl groups $W_e(B_3)$ and $W_e(C_3)$ are isomorphic.

Elements of W_e are orthogonal transformations of E_n with a unit determinant. Therefore, W_e is a finite subgroup of the rotation group $SO(n)$ of E_n. That is, the group W_e consists of rotations of the space E_n. In particular, for rank 2 case the even Weyl groups consist of rotations of a plane:

$$\begin{aligned} &W_e(A_2) = \{1, \mathrm{rot}(2\pi/3), \mathrm{rot}(4\pi/3)\},\\ &W_e(C_2) = \{1, \mathrm{rot}(\pi/2), \mathrm{rot}(\pi), \mathrm{rot}(3\pi/2)\},\\ &W_e(G_2) = \{1, \mathrm{rot}(k\pi/3),\ k = 1, 2, 3, 4, 5\}. \end{aligned}$$

2.3 Root and weight lattices

A Coxeter–Dynkin diagram determines a system of simple roots in the Euclidean space E_n. Acting by elements of the Weyl group W upon simple roots we obtain a finite system of vectors, which is invariant with respect to W. A set of all these vectors is called a *system of roots* associated with a given Coxeter–Dynkin diagram. It is denoted by R.

It is proved (see, for example, [26]) that roots of R are linear combinations of simple roots with integral coefficients. Moreover, there exist no roots, which are linear combinations of α_i, $i = 1, 2, \dots, n$, both with positive and negative coefficients. Therefore, the set of roots R can be represented as a union $R = R_+ \cup R_-$, where R_+ (respectively R_-) is the set of roots which are linear combinations of simple roots with positive (negative) coefficients. The set R_+ (the set R_-) is called a *set of positive (negative) roots*.

As mentioned above, a set of roots R is invariant under the action of elements of the Weyl group $W(R)$. However, $wR_+ \neq R_+$ if w is not a trivial element of W.

Let X_α be the $(n-1)$-dimensional linear subspace (hyperplane) of E_n (containing the origin) which is orthogonal to the root α. The hyperplane X_α consists of all points $x \in E_n$ such that $\langle x, \alpha\rangle = 0$. Clearly, $X_\alpha = X_{-\alpha}$. The set of reflections with respect to X_α, $\alpha \in R_+$, coincides with the set of all reflections of the corresponding Weyl group W.

The subspaces X_α, $\alpha \in R_+$, split the Euclidean space E_n into connected parts which are called *Weyl chambers*. (We assume that boundaries of Weyl chambers belong to the corresponding chambers. Therefore, Weyl chambers can have common points; they belong to boundaries of the corresponding chambers.) A number of Weyl chambers coincides with the number of elements of the Weyl group W. Elements of the Weyl group permute Weyl chambers. A part of a Weyl chamber, which belongs to some hyperplane X_α is called a *wall* of this Weyl chamber. If for some element x of a Weyl chamber we have $\langle x, \alpha\rangle = 0$ for some root α, then this point belongs to a wall. The Weyl chamber consisting of points x such that

$$\langle x, \alpha_i\rangle \geq 0, \qquad i = 1, 2, \dots, n,$$

is called the *dominant Weyl chamber*. It is denoted by D_+. Elements of D_+ are called *dominant*. If $\langle x, \alpha_i\rangle > 0$, $i = 1, 2, \dots, n$, then x is called *strictly dominant element*.

If we act by elements of the even subgroup W_e of W upon a fixed Weyl chamber, then we do not obtain all Weyl chambers. In order to have transitive action of W_e on parts of the Euclidean space E_n, we have to split E_n into a parts larger than Weyl chambers. In order to obtain such parts, we take the dominant Weyl chamber D_+ and act upon it by one of the reflections r_α, where α is a root. Denote the union $D_+ \cup r_\alpha D_+$ (where each point is taken only once) by D^e_+. Then acting upon D^e_+ by elements of W_e we cover the whole Euclidean space E_n. The domains wD^e_+, $w \in W_e$, are called *even Weyl chambers*. The procedure of splitting of E_n into even Weyl chambers is not unique. It depends on the reflection r_α taken for obtaining the first even Weyl chambers. For different roots α sets of even Weyl chambers are different. However, for each fixed root α the corresponding set of even Weyl chambers is transitive for the group W_e. The set $D^e_+ \equiv D^e_+(\alpha)$ is called an *even dominant Weyl chamber*.

The set Q of all linear combinations

$$Q = \left\{ \sum_{i=1}^{n} a_i\alpha_i \mid a_i \in \mathbb{Z} \right\} \equiv \bigoplus_i \mathbb{Z}\alpha_i$$

is called a *root lattice* corresponding to a given Coxeter–Dynkin diagram. Its subset

$$Q_+ = \left\{ \sum_{i=1}^{n} a_i\alpha_i \mid a_i = 0, 1, 2, \dots \right\}$$

is called a *positive root lattice*.

To each root $\alpha \in R$ there corresponds a coroot $\alpha^\vee$ defined by the formula

$$\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}.$$

It is easy to see that $\alpha^{\vee\vee} = \alpha$. The set $Q^\vee$ of all linear combinations

$$Q^\vee = \left\{ \sum_{i=1}^n a_i \alpha_i^\vee \mid a_i \in \mathbb{Z} \right\} \equiv \bigoplus_i \mathbb{Z}\alpha_i^\vee$$

is called a *coroot lattice* corresponding to a given Coxeter–Dynkin diagram. The subset

$$Q_+^\vee = \left\{ \sum_{i=1}^n a_i \alpha_i^\vee \mid a_i = 0, 1, 2, \dots \right\}$$

is called a *positive coroot lattice*.

As noted above, the set of simple roots α_i, $i = 1, 2, \dots, n$, form a basis of the space E_n. In addition to the α-basis, it is convenient to introduce the so-called ω-basis, $\omega_1, \omega_2, \dots, \omega_n$ (also called the *basis of fundamental weights*). The two bases are dual to each other in the following sense:

$$\frac{2\langle \alpha_j, \omega_k \rangle}{\langle \alpha_j, \alpha_j \rangle} \equiv \langle \alpha_j^\vee, \omega_k \rangle = \delta_{jk}\,, \qquad j, k \in \{1, 2, \dots, n\}\,. \tag{2.4}$$

The ω-basis (as well as the α-basis) is not orthogonal.

Note that the factor $2/\langle \alpha_j, \alpha_j \rangle$ can take only three values. Indeed, with the standard normalization of root lengths (see Subsection 2.1), we have

$$\begin{aligned}
&\frac{2}{\langle \alpha_k, \alpha_k \rangle} = 1 \quad \text{for roots of} \quad A_n,\ D_n,\ E_6,\ E_7,\ E_8,\\
&\frac{2}{\langle \alpha_k, \alpha_k \rangle} = 1 \quad \text{for long roots of} \quad B_n,\ C_n,\ F_4,\ G_2,\\
&\frac{2}{\langle \alpha_k, \alpha_k \rangle} = 2 \quad \text{for short roots of} \quad B_n,\ C_n,\ F_4,\\
&\frac{2}{\langle \alpha_k, \alpha_k \rangle} = 3 \quad \text{for short roots of} \quad G_2.
\end{aligned}$$

For this reason, we get

$$\begin{aligned}
&\alpha_k^\vee = \alpha_k \quad \text{for roots of} \quad A_n,\ D_n,\ E_6,\ E_7,\ E_8,\\
&\alpha_k^\vee = \alpha_k \quad \text{for long roots of} \quad B_n,\ C_n,\ F_4,\ G_2,\\
&\alpha_k^\vee = 2\alpha_k \quad \text{for short roots of} \quad B_n,\ C_n,\ F_4,\\
&\alpha_k^\vee = 3\alpha_k \quad \text{for short roots of} \quad G_2.
\end{aligned}$$

The α- and ω-bases are related by the Cartan matrix (2.1) and by its inverse:

$$\alpha_j = \sum_{k=1}^n M_{jk}\, \omega_k\,, \qquad \omega_j = \sum_{k=1}^n (M^{-1})_{jk}\, \alpha_k \tag{2.5}$$

For ranks 2 and 3 the inverse Cartan matrices are of the form

$$A_2 : \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad C_2 : \begin{pmatrix} 1 & 1/2 \\ 1 & 1 \end{pmatrix}, \qquad G_2 : \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix},$$

$$A_3: \ \frac{1}{4}\begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \qquad B_3: \ \frac{1}{2}\begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 4 \\ 1 & 2 & 3 \end{pmatrix}, \qquad C_3: \ \frac{1}{2}\begin{pmatrix} 2 & 2 & 1 \\ 2 & 4 & 2 \\ 2 & 4 & 3 \end{pmatrix}.$$

Later on we need to calculate the scalar product $\langle x, y\rangle$ when x and y are given by coordinates x_i and y_i in ω-basis. It is given by the formula

$$\langle x, y\rangle = \frac{1}{2}\sum_{j,k=1}^{n} x_j y_k (M^{-1})_{jk}\langle \alpha_k \mid \alpha_k\rangle = xM^{-1}Dy^T = xSy^T, \tag{2.6}$$

where D is the diagonal matrix $\operatorname{diag}\left(\frac{1}{2}\langle\alpha_1,\alpha_1\rangle,\dots,\frac{1}{2}\langle\alpha_n,\alpha_n\rangle\right)$. Matrices S, called 'quadratic form matrices', are shown in [27] for all connected Coxeter–Dynkin diagrams.

The sets P and P_+, defined as

$$P = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n \ \supset \ P_+ = \mathbb{Z}^{\geq 0}\,\omega_1 + \cdots + \mathbb{Z}^{\geq 0}\,\omega_n,$$

are called respectively the *weight lattice* and the *cone of dominant weights*. The set P can be characterized as a set of all $\lambda \in E_n$ such that

$$\frac{2\langle\alpha_j,\lambda\rangle}{\langle\alpha_j,\alpha_j\rangle} = \langle\alpha_j^\vee,\lambda\rangle \in \mathbb{Z}$$

for all simple roots α_j. Clearly, $Q \subset P$. Below we shall need also the set P_+^+ of dominant weights of P_+, which do not belong to any Weyl chamber (the set of *integral strictly dominant weights*). Then $\lambda \in P_+^+$ means that $\langle\lambda,\alpha_i\rangle > 0$ for all simple roots α_i. We have

$$P_+^+ = \mathbb{Z}^{>0}\omega_1 + \mathbb{Z}^{>0}\omega_2 + \cdots + \mathbb{Z}^{>0}\omega_n.$$

The smallest dominant weights of P_+, different from zero, coincide with the elements $\omega_1,\omega_2,\dots,\omega_n$ of the ω-basis. They are called *fundamental weights*. They are highest weights of fundamental irreducible representations of the corresponding simple Lie algebra L.

Through the paper we often use the following notation for weights in ω-basis:

$$z = \sum_{j=1}^{n} a_j\omega_j = (a_1 \ a_2 \ \dots \ a_n), \qquad a_1,\dots,a_n \in \mathbb{Z}.$$

If $x = \sum_{j=1}^{n} b_j\alpha_j^\vee$, then

$$\langle z, x\rangle = \sum_{j=1}^{n} a_j b_j. \tag{2.7}$$

2.4 Highest root and affine root system

There exists a unique highest (long) root ξ and a unique highest short root ξ_s. The highest (long) root can be written as

$$\xi = \sum_{i=1}^{n} m_i\alpha_i = \sum_{i=1}^{n} m_i\frac{\langle\alpha_i,\alpha_i\rangle}{2}\alpha_i^\vee \equiv \sum_{i=1}^{n} q_i\alpha_i^\vee. \tag{2.8}$$

The coefficients m_i and q_i can be viewed as attached to the i-th node of the diagram. They are called *marks* and *comarks* (see, for example, [27]). In root systems with two lengths of roots, that is, in B_n, C_n, F_4 and G_2, the highest (long) root ξ is of the form

$$B_n: \quad \xi = (0\,1\,0\,\dots\,0) = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_n, \tag{2.9}$$

$$C_n: \quad \xi = \quad (2\,0\,\dots\,0) = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n, \tag{2.10}$$
$$F_4: \quad \xi = \quad (1\,0\,0\,0) = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, \tag{2.11}$$
$$G_2: \quad \xi = \quad (1\,0) = 2\alpha_1 + 3\alpha_2. \tag{2.12}$$

For A_n, D_n, and E_n, all roots are of the same length, hence $\xi_s = \xi$. We have

$$A_n: \quad \xi = (1\,0\,\dots\,0\,1) = \alpha_1 + \alpha_2 + \cdots + \alpha_n, \tag{2.13}$$
$$D_n: \quad \xi = (0\,1\,0\,\dots\,0) = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \tag{2.14}$$
$$E_6: \quad \xi = (0\,1\,0\,\dots\,0) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6, \tag{2.15}$$
$$E_7: \quad \xi = (1\,0\,0\,\dots\,0) = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7, \tag{2.16}$$
$$E_8: \quad \xi = (0\,0\,\dots\,0\,1) = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8. \tag{2.17}$$

For highest root ξ we have

$$\xi^\vee = \xi. \tag{2.18}$$

Moreover, if all simple roots are of the same length, then

$$\alpha_i^\vee = \alpha_i.$$

For this reason,

$$(q_1, q_2, \dots, q_n) = (m_1, m_2, \dots, m_n).$$

for A_n, D_n and E_n. Formulas (2.13)–(2.18) determine these numbers. For short roots α_i of B_n, C_n and F_4 we have $\alpha_i^\vee = 2\alpha_i$. For short root α_2 of G_2 we have $\alpha_2^\vee = 3\alpha_2$. For this reason,

$$(q_1, q_2, \dots, q_n) = (1, 2, \dots, 2, 1) \qquad \text{for} \quad B_n,$$
$$(q_1, q_2, \dots, q_n) = (1, 1, \dots, 1, 1) \qquad \text{for} \quad C_n,$$
$$(q_1, q_2, q_3, q_4) = (2, 3, 2, 1) \qquad \text{for} \quad F_4,$$
$$(q_1, q_2) = (2, 1) \qquad \text{for} \quad G_2.$$

To each root system R there corresponds an *extended root system* (which is also called an *affine root system*). It is constructed with the help of the highest root ξ of R. Namely, if $\alpha_1, \alpha_2, \dots, \alpha_n$ is a set of all simple roots, then *the roots*

$$\alpha_0 := -\xi, \alpha_1, \alpha_2, \dots, \alpha_n$$

constitute a set of simple roots of the corresponding extended root system. Taking into account the orthogonality (non-orthogonality) of the root α_0 to other simple roots, a diagram of an extended root system can be constructed (which is an extension of the corresponding Coxeter–Dynkin diagram; see, for example, [28]). Note that for all simple Lie algebras (except for A_n) only one simple root is orthogonal to the root α_0. In the case of A_n, the two simple roots α_1 and α_n are not orthogonal to α_0.

2.5 Affine Weyl group and even affine Weyl group

We are interested in E-orbit functions which are given on the Euclidean space E_n. These functions are invariant with respect to action by elements of an even Weyl group W_e, which is a transformation group of E_n. However, W_e does not describe all symmetries of E-orbit functions corresponding to weights $\lambda \in P_+^e \equiv P_+ \cup r_\alpha P_+$. A whole group of symmetries of these

E-orbit functions is isomorphic to the even affine Weyl group $W_e^{\rm aff}$ which is an extension of the even Weyl group W. To describe the group $W_e^{\rm aff}$ we first define the affine Weyl group $W^{\rm aff}$.

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be simple roots in the Euclidean space E_n and let W be the corresponding Weyl group. The group W is generated by reflections r_{α_i}, $i = 1, 2, \dots, n$. In order to construct the affine Weyl group $W^{\rm aff}$, corresponding to W, we have to add an additional reflection. This reflection is constructed as follows.

We consider the reflection r_ξ with respect to the $(n-1)$-dimensional subspace (hyperplane) X_{n-1} containing the origin and orthogonal to the highest (long) root ξ, given in (2.8):

$$r_\xi x = x - \frac{2\langle x, \xi\rangle}{\langle \xi, \xi\rangle}\xi. \tag{2.19}$$

Clearly, $r_\xi \in W$. We shift the hyperplane X_{n-1} by the vector $\xi^\vee/2$, where $\xi^\vee = 2\xi/\langle\xi,\xi\rangle$. (Note that by (2.18) we have $\xi^\vee = \xi$. However, it is convenient here to use $\xi^\vee$.) The reflection with respect to the hyperplane $X_{n-1} + \xi^\vee/2$ will be denoted by r_0. In order to fulfill the transformation r_0 we have to fulfill the transformation r_ξ and then to shift the result by $\xi^\vee$, that is,

$$r_0 x = r_\xi x + \xi^\vee.$$

We have $r_0 0 = \xi^\vee$ and it follows from (2.19) that r_0 maps $x + \xi^\vee/2$ to

$$r_\xi(x + \xi^\vee/2) + \xi^\vee = x + \xi^\vee/2 - \langle x, \xi^\vee\rangle\xi$$

Therefore,

$$\begin{aligned} r_0(x + \xi^\vee/2) &= x + \xi^\vee/2 - \frac{2\langle x, \xi\rangle}{\langle \xi, \xi\rangle}\xi = x + \xi^\vee/2 - \frac{2\langle x, \xi^\vee\rangle}{\langle \xi^\vee, \xi^\vee\rangle}\xi^\vee \\ &= x + \xi^\vee/2 - \frac{2\langle x + \xi^\vee/2, \xi^\vee\rangle}{\langle \xi^\vee, \xi^\vee\rangle}\xi^\vee + \frac{2\langle \xi^\vee/2, \xi^\vee\rangle}{\langle \xi^\vee, \xi^\vee\rangle}\xi^\vee. \end{aligned}$$

Denoting $x + \xi^\vee/2$ by y we obtain that r_0 is given also by the formula

$$r_0 y = y + \left(1 - \frac{2\langle y, \xi^\vee\rangle}{\langle \xi^\vee, \xi^\vee\rangle}\right)\xi^\vee = \xi^\vee + r_\xi y. \tag{2.20}$$

The element r_0 does not belongs to W since elements of W do not move the point $0 \in E_n$.

The hyperplane $X_{n-1} + \xi^\vee/2$ coincides with the set of points y such that $r_0 y = y$. It follows from (2.20) that this hyperplane is given by the equation

$$1 = \frac{2\langle y, \xi^\vee\rangle}{\langle \xi^\vee, \xi^\vee\rangle} = \langle y, \xi\rangle = \sum_{k=1}^n a_k q_k, \tag{2.21}$$

where

$$y = \sum_{k=1}^n a_k \omega_k, \qquad \xi = \sum_{k=1}^n q_k \alpha_k^\vee$$

(see (2.7)).

A group of transformations of the Euclidean space E_n generated by reflections $r_0, r_{\alpha_1}, \dots, r_{\alpha_n}$ is called the *affine Weyl group* of the root system R and is denoted by $W^{\rm aff}$ or by $W_R^{\rm aff}$ (if is necessary to indicate the initial root system), see [28]. Adjoining the reflection r_0 to the Weyl group W completely changes properties of the group $W^{\rm aff}$.

Due to (2.19) and (2.20) for any $x \in E_n$ we have

$$r_0 r_\xi x = r_0(r_\xi x) = \xi^\vee + r_\xi r_\xi x = x + \xi^\vee.$$

Clearly, $(r_0 r_\xi)^k x = x + k\xi^\vee$, $k = 0, \pm1, \pm2, \dots$, that is, the set of elements $(r_0 r_\xi)^k$, $k = 0, \pm1, \pm2, \dots$, is an infinite commutative subgroup of $W^{\rm aff}$. This means that (unlike to the Weyl group W) *$W^{\rm aff}$ is an infinite group.*

Since $r_0 0 = \xi^\vee$, for any $w \in W$ we have

$$w r_0 0 = w\xi^\vee = \xi_w^\vee,$$

where $\xi_w^\vee$ is a coroot of the same length as the coroot $\xi^\vee$. For this reason, wr_0 is the reflection with respect to the $(n-1)$-hyperplane perpendicular to the root $\xi_w^\vee$ and containing the point $\xi_w^\vee/2$. Moreover,

$$(wr_0) r_{\xi_w^\vee} x = x + \xi_w^\vee$$

We also have $((wr_0) r_{\xi_w^\vee})^k x = x + k\xi_w^\vee$, $k = 0, \pm1, \pm2, \dots$. Since w is any element of W, then the set $w\xi^\vee$, $w \in W$, coincides with the set of coroots of R, corresponding to all long roots of the root system R. Thus, *the set $W^{\rm aff} \cdot 0$ coincides with the lattice $Q_l^\vee$ generated by coroots $\alpha^\vee$ taken for all long roots α from R.*

It is checked for each type of root systems that each coroot $\xi_s^\vee$ for a short root ξ_s of R is a linear combination of coroots $w\xi^\vee \equiv \xi_w$, $w \in W$, with integral coefficients, that is, $Q^\vee = Q_l^\vee$. Therefore, *The set $W^{\rm aff} \cdot 0$ coincides with the coroot lattice $Q^\vee$ of R.*

Let $\hat{Q}^\vee$ be the subgroup of $W^{\rm aff}$ generated by the elements

$$(wr_0) r_w, \qquad w \in W, \tag{2.22}$$

where $r_w \equiv r_{\xi_w^\vee}$ for $w \in W$. Since elements (2.22) pairwise commute with each other (since they are shifts), $\hat{Q}^\vee$ is a commutative group. The subgroup $\hat{Q}^\vee$ can be identified with the coroot lattice $Q^\vee$. Namely, if for $g \in \hat{Q}^\vee$ we have $g \cdot 0 = \gamma \in Q^\vee$, then g is identified with γ. This correspondence is one-to-one.

The subgroups W and $\hat{Q}^\vee$ generate $W^{\rm aff}$ since a subgroup of $W^{\rm aff}$, generated by W and $\hat{Q}^\vee$, contains the element r_0. *The group $W^{\rm aff}$ is a semidirect product of its subgroups W and $\hat{Q}^\vee$, where $\hat{Q}^\vee$ is an invariant subgroup* (see Section 5.2 in [7] for details).

We shall need not the whole affine Weyl group $W^{\rm aff}$ but only its subgroup, constructed on the base of the even Weyl group W_e. The subgroup $W_e^{\rm aff}$, coinciding with the semidirect product of the even Weyl group W_e and $\hat{Q}^\vee$, will be called an *even affine Weyl group*. This subgroup does not contain the reflection $r_0 \in W^{\rm aff}$. One says that $W_e^{\rm aff}$ consists of even elements of $W^{\rm aff}$.

2.6 $W^{\rm aff}$-fundamental domain and $W_e^{\rm aff}$-fundamental domain

An open connected simply connected set $F(G) \subset E_n$ is called a *fundamental domain* for the group G ($G = W, W^{\rm aff}, W_e, W_e^{\rm aff}$) if it does not contains equivalent points (that is, points x and x' such that $x = wx$, $w \in G$) and if its closure contains at least one point from each G-orbit. It is evident that *the dominant Weyl chamber D_+ without walls of this chamber is a fundamental domain for the Weyl group W*. Recall that this domain consists of all points $x = a_1\omega_1 + \cdots + a_n\omega_n \in E_n$ for which

$$a_i = \langle x, \alpha_i^\vee \rangle > 0, \qquad i = 1, 2, \dots, n.$$

For any fixed root α, the domain $D_+^e = D_+ \cup r_\alpha D_+$ (where each point is taken only once) can be taken as a closure of the fundamental domain of the even Weyl group W_e. The fundamental domain of W_e is the set D_+^e without its boundary.

Let us describe a fundamental domain for the group $W^{\rm aff}$. Since $W\subset W^{\rm aff}$, it can be chosen as a subset of the dominant Weyl chamber for W.

We have seen that the element $r_0\in W^{\rm aff}$ is a reflection with respect to the hyperplane $X_{n-1}+\xi^\vee/2$, orthogonal to the root ξ and containing the point $\xi^\vee/2$. This hyperplane is given by the equation (2.21). This equation shows that the hyperplane $X_{n-1}+\xi^\vee/2$ intersects the axes, determined by the vectors ω_i, in the points ω_i/q_i, $i=1,2,\dots,n$, where q_i are such as in (2.21). We create the simplex with $n+1$ vertices in the points

$$0,\ \frac{\omega_1}{q_1},\dots,\frac{\omega_n}{q_n}. \tag{2.23}$$

By (2.21), this simplex consists of all points y of the dominant Weyl chamber for which $\langle y,\xi\rangle\le 1$. Clearly, the interior F of this simplex belongs to the dominant Weyl chamber. The following theorem is true (see, for example, [7]):

Theorem 1. *The set F is a fundamental domain for the affine Weyl group $W^{\rm aff}$.*

For the rank 2 cases the fundamental domain is the interior of the simplex with the following vertices:

$$\begin{aligned} &A_2: && \{0,\ \omega_1,\ \omega_2\},\\ &C_2: && \{0,\ \omega_1,\ \omega_2\},\\ &G_2: && \{0,\ \tfrac{\omega_1}{2},\ \omega_2\}. \end{aligned}$$

A fundamental domain of the even affine group $W_e^{\rm aff}$ can be taken in such a way that it is contained in the fundamental domain D_+^e of W_e. Namely, the set $\bar F\cup r_\alpha\bar F$ (where each point is taken only once) without its boundary satisfies this condition and is a $W_e^{\rm aff}$-fundamental domain.

3 W_e-orbits

3.1 Definition

As we have seen, the $(n-1)$-dimensional linear subspaces X_α of E_n, orthogonal to positive roots α and containing the origin, divide the space E_n into connected parts, which are called Weyl chambers. A number of such chambers is equal to an order of the corresponding Weyl group W. Elements of the Weyl group permute these chambers. A single chamber D_+ such that $\langle\alpha_i,x\rangle\ge 0$, $x\in D_+$, $i=1,2,\dots,n$, is the dominant Weyl chamber. We fix a root α and create a set $D_+^e=D_+\cup r_\alpha D_+$, where r_α is the reflection corresponding to the root α. The sets, received from D_+^e by action by elements of W_e are called *even Weyl chambers*. Clearly, they depend on choosing of the root α. However, different choices of α (and therefore, of even Weyl chambers) does not change a set of W_e-orbits, which are considered below.

The cone of dominant integral weights P_+ belongs to the dominant Weyl chamber D_+. By P_+^e we denote the set $P_+^e\cup r_\alpha P_+^e$ (where each point is taken only once). Then $P_+^e\subset D_+^e$.

Let y be an arbitrary dominant element of the Euclidean space E_n. We act upon y by all elements of the Weyl group W. As a result, we obtain the set wy, $w\in W/W_y$ (where W_y is the subgroup of elements of W leaving y invariant), which is called a *Weyl group orbit* or a W-orbit of the point y. A W-orbit of a point $y\in D_+$ is denoted by $O(y)$ or $O_W(y)$. A size of an orbit $O(y)$ is a number $|O(y)|$ of its elements. Each Weyl chamber contains only one point of a fixed orbit $Q(y)$.

Now we act upon element $y\in E_n$ by all elements of the even Weyl group W_e. As a result, we obtain a set of elements wy, $w\in W_e$ (each point is taken only once), which is called an *even*

Weyl group orbit or a W_e-orbit of the point y. A W_e-orbit of a point $y \in D_+^e$ is denoted by $O_e(y)$ or $O_{W_e}(y)$. Each even Weyl chamber contains only one point of a fixed orbit $Q_e(y)$.

W_e-orbits $O_e(y)$ do not depend on a choice of a root α with respect to which we construct the even dominant Weyl chamber $D_+^e = D_+ \cup r_\alpha D_+$.

There are two types of W_e-orbits: orbits $O_e(y)$ which contain a point of the dominant Weyl chamber D_+ (we call them *W_e-orbits of the fist type*) and orbits which do not contain such a point (we call them *W_e-orbits of the second type*). It is easy to see that each W-orbit $O(y)$ with strictly dominant y consists of two W_e-orbits, one of them is of the first type and the second of the second type. Namely,

$$O(y) = O_e(y) \bigcup O_e(r_\alpha y),$$

where the W_e-orbit $O_e(r_\alpha y)$ does not depend on a choice of the root α, that is, $O_e(r_\alpha y)$ are the same for all positive roots α of a given root system. Moreover, the orbit $O_e(r_\alpha y)$ is obtained from the orbit $O_e(y)$ by acting upon each point of $O_e(y)$ by reflection r_α, $O_e(r_\alpha y) = r_\alpha O_e(y)$.

If y belongs to some wall of the dominant Weyl chamber, then the W_e-orbit $O_e(y)$ coincides with the W-orbit $O(y)$.

Example. *The case* A_1. The Weyl group W of A_1 consists of two elements 1 and r_α, where α is a unique positive root of A_1. The element r_α is a reflection and, therefore, $\det r_\alpha = -1$. Thus, the subgroup W_e in this case contains only one element 1. This means that each point of the real line is a W_e-orbit for A_1. In particular, $O_e(y)$, $y > 0$, belongs to orbits, corresponding to dominant elements. The other orbits (except for the orbit $O_e(0)$) are obtained by acting by the reflection r_α.

3.2 W_e-orbits of A_2, C_2, G_2

In this subsection we give W_e-orbits for the rank two cases. Orbits will be given by coordinates in the ω-basis. Points of the orbits will be denoted as $(a\ b)$, where a and b are ω-coordinates.

If $a > 0$ and $b > 0$, then the W_e-orbits $O_e(a\ b)$ and $r_\alpha O(a\ b) \equiv O_e(-a\ a+b)$ of A_2 contain points

$$A_2: \quad O_e(a\ b) \ni (a\ b),\ (-a{-}b\ a),\ (b\ {-a{-}b}), \tag{3.1}$$
$$O_e(-a\ a{+}b) \ni (-a\ a{+}b),\ (a{+}b\ {-b}),\ (-b\ {-a}). \tag{3.2}$$

The other W_e-orbits of A_2 are

$$A_2: \quad O_e(a\ 0) \ni (a\ 0),\ (-a\ a),\ (0\ {-a}), \tag{3.3}$$
$$O_e(0\ b) \ni (0\ b),\ (b\ {-b}),\ (-b\ 0). \tag{3.4}$$

In the cases of C_2 and G_2 (where the second simple root is the longer one for C_2 and the shorter one for G_2) for $a > 0$ and $b > 0$ we have

$$C_2: \quad O_e(a\ b) \ni (a\ b),\ (a{+}2b\ {-a{-}b}),\ (-a\ {-b}),\ (-a{-}2b\ a{+}b), \tag{3.5}$$
$$O_e(-a\ a{+}b) \ni (-a\ a{+}b),\ (a{+}2b\ {-b}),\ (a\ {-a{-}b}),\ (-a{-}2b\ b), \tag{3.6}$$
$$G_2: \quad O_e(a\ b) \ni \pm(a\ b),\ \pm(2a{+}b\ {-3a{-}b}),\ \pm(-a{-}b\ 3a{+}2b), \tag{3.7}$$
$$O_e(-a\ 3a{+}b) \ni \pm(-a\ 3a{+}b),\ \pm(a{+}b\ {-b}),\ \pm(-2a{-}b\ 3a{+}2b), \tag{3.8}$$

where $\pm(c\ d)$ means two points $(c\ d)$ and $(-c\ \ {-d})$.

We also give W_e-orbits for which one of the numbers a, b vanish:

$$C_2: \quad O_e(a\ 0) \ni \pm(a\ 0),\ \pm(-a\ a), \tag{3.9}$$

$$O_e(0\ b) \ni \pm(0\ b),\ \pm(2b\ -b), \tag{3.10}$$

$$G_2 : \quad O_e(a\ 0) \ni \pm(a\ 0),\ \pm(-a\ 3a),\ \pm(2a\ -3a), \tag{3.11}$$

$$O_e(0\ b) \ni \pm(0\ b),\ \pm(b\ -b),\ \pm(-b\ 2b). \tag{3.12}$$

As we see, for each point $(c\ d)$ of an orbit O_e of C_2 or G_2 there exists in the orbit the point $(-c\ -d)$.

3.3 The case of A_n

In the cases of Coxeter–Dynkin diagrams A_{n-1}, B_n, C_n, D_n, root and weight lattices, even Weyl groups and orbits are described in a simple way if to use the orthogonal coordinate system in E_n. In particular, this coordinate system is useful under practical work with orbits.

In the case A_n it is convenient to describe root and weight lattices, even Weyl group and orbit functions in the subspace of the Euclidean space E_{n+1}, given by the equation

$$x_1 + x_2 + \cdots + x_{n+1} = 0,$$

where $x_1, x_2, \dots, x_{n+1}$ are orthogonal coordinates of a point $x \in E_{n+1}$. The unit vectors in directions of these coordinates are denoted by $\mathbf{e}_j$, respectively. Clearly, $\mathbf{e}_i \perp \mathbf{e}_j$, $i \neq j$. The set of roots of A_n is given by the vectors

$$\alpha_{ij} = \mathbf{e}_i - \mathbf{e}_j, \qquad i \neq j.$$

The roots $\alpha_{ij} = \mathbf{e}_i - \mathbf{e}_j$, $i < j$, are positive and the roots

$$\alpha_i \equiv \alpha_{i,i+1} = \mathbf{e}_i - \mathbf{e}_{i+1}, \qquad i = 1, 2, \dots, n,$$

constitute the system of simple roots.

If $x = \sum_{i=1}^{n+1} x_i \mathbf{e}_i$, $x_1 + x_2 + \cdots + x_{n+1} = 0$, is a point of E_{n+1}, then this point belongs to the dominant Weyl chamber D_+ if and only if

$$x_1 \geq x_2 \geq \cdots \geq x_{n+1}.$$

Indeed, if this condition is fulfilled, then $\langle x, \alpha_i \rangle = x_i - x_{i+1} \geq 0$, $i = 1, 2, \dots, n$, and x is dominant. Conversely, if x is dominant, then $\langle x, \alpha_i \rangle \geq 0$ and this condition is fulfilled. *The point x is strictly dominant if and only if*

$$x_1 > x_2 > \cdots > x_{n+1}.$$

If $\alpha = \alpha_{12}$, the even dominant chamber $D_+^e = D_+ \cup r_\alpha D_+$ consists of points x such that

$$x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_{n+1} \qquad \text{or} \qquad x_2 \geq x_1 \geq x_3 \geq \cdots \geq x_{n+1},$$

that is, such that

$$x_2, x_2 \geq x_3 \geq \cdots \geq x_{n+1}.$$

If $\lambda = \sum_{i=1}^{n} \lambda_i \omega_i$, then the ω-coordinates λ_i are connected with the orthogonal coordinates x_j of $\lambda = \sum_{i=1}^{n+1} x_i \mathbf{e}_i$ by the formulas

$$x_1 = \frac{n}{n+1}\lambda_1 + \frac{n-1}{n+1}\lambda_2 + \frac{n-2}{n+1}\lambda_3 + \cdots + \frac{2}{n+1}\lambda_{n-1} + \frac{1}{n+1}\lambda_n,$$

$$x_2 = -\frac{1}{n+1}\lambda_1 + \frac{n-1}{n+1}\lambda_2 + \frac{n-2}{n+1}\lambda_3 + \cdots + \frac{2}{n+1}\lambda_{n-1} + \frac{1}{n+1}\lambda_n,$$
$$x_3 = -\frac{1}{n+1}\lambda_1 - \frac{2}{n+1}\lambda_2 + \frac{n-2}{n+1}\lambda_3 + \cdots + \frac{2}{n+1}\lambda_{n-1} + \frac{1}{n+1}\lambda_n,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x_n = -\frac{1}{n+1}\lambda_1 - \frac{2}{n+1}\lambda_2 - \frac{3}{n+1}\lambda_3 - \cdots - \frac{n-1}{n+1}\lambda_{n-1} + \frac{1}{n+1}\lambda_n,$$
$$x_{n+1} = -\frac{1}{n+1}\lambda_1 - \frac{2}{n+1}\lambda_2 - \frac{3}{n+1}\lambda_3 - \cdots - \frac{n-1}{n+1}\lambda_{n-1} - \frac{n}{n+1}\lambda_n.$$

The inverse formulas are

$$\lambda_i = x_i - x_{i+1}, \qquad i = 1, 2, \dots, n. \tag{3.13}$$

By means of the formula

$$r_\alpha\lambda = \lambda - \frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha \tag{3.14}$$

for the reflection with respect to the hyperplane, orthogonal to a root α, we can find how elements of the Weyl group $W(A_n)$ act upon points $\lambda \in E_{n+1}$. We conclude that the Weyl group $W(A_n)$ consists of all permutations of the orthogonal coordinates $x_1, x_2, \dots, x_{n+1}$ of a point λ, that is, $W(A_n)$ coincides with the symmetric group S_{n+1}. Even permutations of $W(A_n)$ constitute the even Weyl group $W_e(A_n)$. It is the alternating subgroup S^e_{n+1} of the group S_{n+1}. This subgroup is simple. Transformations of S^e_{n+1} are elements of the rotation group $SO(n+1)$.

Sometimes (for example, if we wish for coordinates to be integers or non-negative integers), it is convenient to introduce orthogonal coordinates $y_1, y_2, \dots, y_{n+1}$ for A_n in such a way that

$$y_1 + y_2 + \cdots + y_{n+1} = m,$$

where m is some fixed real number. They are obtained from the previous orthogonal coordinates by adding the same number $m/(n+1)$ to each coordinate. Then, as one can see from (3.13), ω-coordinates $\lambda_i = y_i - y_{i+1}$ and the group W and W_e do not change. Sometimes, it is natural to use orthogonal coordinates $y_1, y_2, \dots, y_{n+1}$ for which all y_i are non-negative.

W_e-orbits $O_e(\lambda)$ for strictly dominant λ can be constructed by means of signed W-orbits. Signed W-orbit $O^\pm(\lambda)$ were introduced in [8] . The signed orbit $O^\pm(\lambda)$, $\lambda = (x_1, x_2, \dots, x_{n+1})$, $x_1 > x_2 > \cdots > x_{n+1}$, consists of all points

$$(x_{i_1}, x_{i_2}, \dots, x_{i_{n+1}})^{\mathrm{sgn}\,(\det w)}$$

obtained from $(x_1, x_2, \dots, x_{n+1})$ by permutations $w \in W \equiv S_{n+1}$ (instead of $\mathrm{sgn}\,(\det w)$ we sometimes write simply $\det w$).

The signed W-orbit $O^\pm(\lambda)$ splits into two W_e-orbits: one of them coincides with $O_e(\lambda)$ and the second with $O_e(r_\alpha\lambda)$, where α is a positive root of A_n. The W_e-orbit $O_e(\lambda)$ contains the points of the W-orbit $O^\pm(\lambda)$, which have the sign plus and $O_e(r_\alpha\lambda)$ contains the points with the sign minus.

If λ is dominant but not strictly dominant, then the W_e-orbit $O_e(\lambda)$ coincides with the W-orbit $O(\lambda)$. Description of W-orbits of A_n in orthogonal coordinates see in [7, Subsection 3.1].

3.4 The case of B_n

Orthogonal coordinates of a point $x \in E_n$ are denoted by $x_1, x_2, \dots, x_n$. We denote by $\mathbf{e}_i$ the corresponding unit vectors. Then the set of roots of B_n is given by the vectors

$$\alpha_{\pm i,\pm j} = \pm\mathbf{e}_i \pm \mathbf{e}_j, \qquad i \neq j, \qquad \alpha_{\pm i} = \pm\mathbf{e}_i, \qquad i = 1, 2, \dots, n$$

(all combinations of signs must be taken). The roots $\alpha_{i,\pm j} = \mathbf{e}_i \pm \mathbf{e}_j$, $i < j$, $\alpha_i = \mathbf{e}_i$, $i = 1, 2, \ldots, n$, are positive and n roots

$$\alpha_i := \mathbf{e}_i - \mathbf{e}_{i+1}, \qquad i = 1, 2, \ldots, n-1, \qquad \alpha_n = \mathbf{e}_n$$

constitute the system of simple roots.

A point $\lambda = \sum_{i=1}^{n} x_i \mathbf{e}_i \in E_n$ belongs to the dominant Weyl chamber D_+ if and only if

$$x_1 \geq x_2 \geq \cdots \geq x_n \geq 0.$$

Moreover, *this point is strictly dominant if and only if*

$$x_1 > x_2 > \cdots > x_n > 0.$$

The even dominant chamber can be taken consisting of points x such that

$$x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \geq 0 \qquad \text{or} \qquad x_2 \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq 0,$$

that is, such that

$$x_1, x_2 \geq x_3 \geq \cdots \geq x_n \geq 0.$$

If $\lambda = \sum_{i=1}^{n} \lambda_i \omega_i$, then the ω-coordinates λ_i are connected with the orthogonal coordinates x_j of $\lambda = \sum_{i=1}^{n} x_i \mathbf{e}_i$ by the formulas

$$\begin{array}{lrr}
x_1 = \lambda_1 + & \lambda_2 + \cdots + \lambda_{n-1} + & \frac{1}{2}\lambda_n, \\
x_2 = & \lambda_2 + \cdots + \lambda_{n-1} + & \frac{1}{2}\lambda_n, \\
\multicolumn{3}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
x_n = & & \frac{1}{2}\lambda_n,
\end{array}$$

The inverse formulas are

$$\lambda_i = x_i - x_{i+1}, \qquad i = 1, 2, \ldots, n-1, \qquad \lambda_n = 2x_n.$$

It is easy to see that if $\lambda \in P_+$, then the coordinates $x_1, x_1, \ldots, x_n$ are all integers or all half-integers.

The Weyl group $W(B_n)$ of B_n consists of all permutations of the orthogonal coordinates $x_1, x_2, \ldots, x_n$ of a point λ with possible sign alternations of any number of them. Moreover, $\det w$ is equal to ± 1 depending on whether w is a product of even or odd number of reflections and alternations of signs. A sign of $\det w$ can be determined as follows. We represent w as a product $w = \epsilon s$, where s is a permutation of $(x_1, x_2, \ldots, x_n)$ and ϵ is an alternation of signs of coordinates. Then $\det w = (\det s)\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_n}$, where $\det s$ is defined as in the previous subsection and $\epsilon_{i_j} = -1$ in the case of change of a sign of i_j-th coordinate and $\epsilon_{i_j} = 1$ otherwise. This show haw to determine the even Weyl group $W_e(B_n)$ as a subgroup of $W(B_n)$.

$W_e(B_n)$-orbits $O_e(\lambda)$ with strictly dominant λ can be constructed by means of signed orbits $O^{\pm}(\lambda)$. The signed orbit $O^{\pm}(\lambda)$, $\lambda = (x_1, x_2, \ldots, x_n)$, $x_1 > x_2 > \cdots > x_n > 0$, consists of all points

$$(\pm x_{i_1}, \pm x_{i_2}, \ldots, \pm x_{i_n})^{\det w} \tag{3.15}$$

(each combination of signs is possible) obtained from $(x_1, x_2, \ldots, x_n)$ by permutations and alternations of signs which constitute an element w of the Weyl group $W(B_n)$. The signed orbit $O^{\pm}(\lambda)$ splits into two W_e-orbits: one of them $O_e(\lambda)$ consists of all points of $O^{\pm}(\lambda)$, which have the sign plus, and the second one consists of all points with the sign minus.

If a dominant element λ is not strictly dominant, then the W_e-orbit $O_e(\lambda)$ coincides with the W-orbit $O(\lambda)$.

3.5 The case of C_n

In the orthogonal coordinate system of the Euclidean space E_n the set of roots of C_n is given by the vectors

$$\alpha_{\pm i,\pm j} = \pm\mathbf{e}_i \pm \mathbf{e}_j, \qquad i \neq j, \qquad \alpha_{\pm i} = \pm 2\mathbf{e}_i, \qquad i = 1, 2, \ldots, n,$$

where $\mathbf{e}_i$ is the unit vector in the direction of i-th coordinate x_i (all combinations of signs must be taken). The roots $\alpha_{i,\pm j} = \mathbf{e}_i \pm \mathbf{e}_j$, $i < j$, and $\alpha_i = 2\mathbf{e}_i$, $i = 1, 2, \ldots, n$, are positive and n roots

$$\alpha_i := \mathbf{e}_i - \mathbf{e}_{i+1}, \qquad i = 1, 2, \ldots, n-1, \qquad \alpha_n = 2\mathbf{e}_n$$

constitute the system of simple roots.

A point $\lambda = \sum\limits_{i=1}^{n} x_i\mathbf{e}_i \in E_n$ belongs to the dominant Weyl chamber D_+ if and only if

$$x_1 \geq x_2 \geq \cdots \geq x_n \geq 0.$$

This point is strictly dominant if and only if

$$x_1 > x_2 > \cdots > x_n > 0.$$

The even dominant chamber can be taken consisting of points x such that

$$x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \geq 0 \qquad \text{or} \qquad x_2 \geq x_1 \geq x_3 \geq \cdots \geq x_n \geq 0,$$

that is, such that

$$x_1, x_2 \geq x_3 \geq \cdots \geq x_n \geq 0.$$

If $\lambda = \sum\limits_{i=1}^{n} \lambda_i\omega_i$, then the ω-coordinates λ_i are connected with the orthogonal coordinates x_j of $\lambda = \sum\limits_{i=1}^{n} x_i\mathbf{e}_i$ by the formulas

$$\begin{aligned}
x_1 &= \lambda_1 + \lambda_2 + \cdots + \lambda_{n-1} + \lambda_n,\\
x_2 &= \phantom{\lambda_1 + {}} \lambda_2 + \cdots + \lambda_{n-1} + \lambda_n,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
x_n &= \phantom{\lambda_1 + \lambda_2 + \cdots + \lambda_{n-1} + {}} \lambda_n.
\end{aligned}$$

The inverse formulas are

$$\lambda_i = x_i - x_{i+1}, \qquad i = 1, 2, \ldots, n-1, \qquad \lambda_n = x_n.$$

If $\lambda \in P_+$, then all coordinates x_i are integers.

The Weyl group $W(C_n)$ of C_n consists of all permutations of the orthogonal coordinates $x_1, x_2, \ldots, x_n$ of a point λ with sign alternations of some of them, that is, this Weyl group acts on orthogonal coordinates exactly in the same way as the Weyl group $W(B_n)$ does. Moreover, $\det w$ is equal to ± 1 depending on whether w consists of even or odd numbers of reflections and alternations of signs. Since $W(C_n) = W(B_n)$, then a sign of $\det w$ is determined as in the case B_n.

The signed orbit $O^{\pm}(\lambda)$, $\lambda = (x_1, x_2, \ldots, x_n)$, $x_1 > x_2 > \cdots > x_n > 0$, consists of all points

$$(\pm x_{i_1}, \pm x_{i_2}, \ldots, \pm x_{i_{n+1}})^{\det w}$$

(each combination of signs is possible) obtained from $(x_1, x_2, \dots, x_n)$ by permutations and alternations of signs which constitute an element w of the Weyl group $W(C_n)$, that is, in the orthogonal coordinates signed orbits for C_n coincide with signed orbits of B_n. This determine how to separate the subgroup $W_e(C_n)$ in the group $W(C_n)$.

The signed orbit $O^{\pm}(\lambda)$, $\lambda = (x_1, x_2, \dots, x_n)$, $x_1 > x_2 > \cdots > x_n > 0$, splits into two W_e-orbits: one of them $O_e(\lambda)$ consists of all points of $O^{\pm}(\lambda)$, which have the sign plus, and the second one consists of all points with the sign minus.

If a dominant element λ is not strictly dominant, then the W_e-orbit $O_e(\lambda)$ coincides with the W-orbit $O(\lambda)$.

3.6 The case of D_n

In the orthogonal coordinate system of the Euclidean space E_n the set of roots of D_n is given by the vectors

$$\alpha_{\pm i,\pm j} = \pm \mathbf{e}_i \pm \mathbf{e}_j, \qquad i \neq j,$$

where $\mathbf{e}_i$ is the unit vector in the direction of i-th coordinate (all combinations of signs must be taken). The roots $\alpha_{i,\pm j} = \mathbf{e}_i \pm \mathbf{e}_j$, $i < j$, are positive and n roots

$$\alpha_i := \mathbf{e}_i - \mathbf{e}_{i+1}, \qquad i = 1, 2, \dots, n-1, \qquad \alpha_n = \mathbf{e}_{n-1} + \mathbf{e}_n$$

constitute the system of simple roots.

If $\lambda = \sum\limits_{i=1}^{n} x_i \mathbf{e}_i$, then this point belongs to the dominant Weyl chamber D_+ if and only if

$$x_1 \geq x_2 \geq \cdots \geq x_{n-1} \geq |x_n|.$$

This point is strictly dominant if and only if

$$x_1 > x_2 > \cdots > x_{n-1} > |x_n|$$

(in particular, x_n can take the value 0).

The even dominant chamber can be taken consisting of points x such that

$$x_1 \geq x_2 \geq x_3 \geq \cdots \geq |x_n| \qquad \text{or} \qquad x_2 \geq x_1 \geq x_3 \geq \cdots \geq |x_n|,$$

that is, such that

$$x_1, x_2 \geq x_3 \geq \cdots \geq x_{n-1} \geq |x_n|.$$

If $\lambda = \sum\limits_{i=1}^{n} \lambda_i \omega_i$, then the ω-coordinates λ_i are connected with the orthogonal coordinates x_j of $\lambda = \sum\limits_{i=1}^{n} x_i \mathbf{e}_i$ by the formulas

$$\begin{aligned}
x_1 &= \lambda_1 + \lambda_2 + \cdots + \lambda_{n-2} + \tfrac{1}{2}(\lambda_{n-1} + \lambda_n), \\
x_2 &= \phantom{\lambda_1 + {}} \lambda_2 + \cdots + \lambda_{n-2} + \tfrac{1}{2}(\lambda_{n-1} + \lambda_n), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
x_{n-1} &= \tfrac{1}{2}(\lambda_{n-1} + \lambda_n), \\
x_n &= \tfrac{1}{2}(\lambda_{n-1} - \lambda_n),
\end{aligned}$$

The inverse formulas are

$$\lambda_i = x_i - x_{i+1}, \qquad i = 1, 2, \ldots, n-2, \qquad \lambda_{n-1} = x_{n-1} + x_n, \qquad \lambda_n = x_{n-1} - x_n.$$

If $\lambda \in P_+$, then the coordinates $x_1, x_2, \ldots, x_n$ are all integers or all half-integers.

The Weyl group $W(D_n)$ of D_n consists of all permutations of the orthogonal coordinates $x_1, x_2, \ldots, x_n$ of a point λ with sign alternations of even number of them. Moreover, $\det w$ is equal to ± 1 and a sign of $\det w$ is determined as follows. The element $w \in W(D_n)$ can be represented as a product $w = \tau s$, where s is a permutation from S_n and τ is an alternation of even number of coordinates. Then $\det w = \det s$. Indeed, since an alternation of signs of two coordinates x_i and x_j is a product of two reflections r_α with $\alpha = \mathbf{e}_i + \mathbf{e}_j$ and with $\alpha = \mathbf{e}_i - \mathbf{e}_j$, a sign of the determinant of this alternation is plus. (Note that $|W(D_n)| = \frac{1}{2}|W(B_n)|$.)

Now we may state that the even Weyl group $W_e(D_n)$ consists of products τs, where s runs over even permutations of S_n and τ runs over alternations of even numbers of coordinates.

The signed orbit $O^{\pm}(\lambda)$, $\lambda = (x_1, x_2, \ldots, x_n)$, $x_1 > x_2 > \cdots > x_n > 0$, for D_n consists of all points

$$(\pm x_{i_1}, \pm x_{i_2}, \ldots, \pm x_{i_n})^{\det w}$$

obtained from $(x_1, x_2, \ldots, x_n)$ by permutations and alternations of even number of signs which constitute an element w of the Weyl group $W(D_n)$. This signed orbit splits into two W_e-orbits: one of them $O_e(\lambda)$ consists of all points of $O^{\pm}(\lambda)$, which have the sign plus, and the second one consists of all points with the sign minus.

If a dominant element λ is not strictly dominant, then the $W_e(D_n)$-orbit $O_e(\lambda)$ coincides with the $W(D_n)$-orbit $O(\lambda)$.

3.7 Fundamental domains of W_e^{aff}

Using the explicit formula for the antisymmetric orbit function $\varphi_\rho(x)$, where ρ is a half of positive roots, we have derived in [8] explicit forms of the fundamental domains of W^{aff} for the cases A_n, B_n, C_n, D_n. They easily determine fundamental domains F_e for the corresponding even affine Weyl groups W_e^{aff}.

(a) The fundamental domain $F_e(A_n)$ of the even affine Weyl group $W_e^{\mathrm{aff}}(A_n)$ is contained in the domain of real points $x = (x_1, x_2, \ldots, x_{n+1})$ such that

$$x_1, x_2 > x_3 > \cdots > x_{n+1}, \qquad x_1 + x_2 + \cdots + x_{n+1} = 0.$$

Moreover, a point x of this domain belongs to $F_e(A_n)$ if and only if $x_1 + |x_{n+1}| < 1$ and $x_1 > x_2$, or $x_2 + |x_{n+1}| < 1$ and $x_2 > x_1$.

(b) The fundamental domain $F_e(B_n)$ of $W_e^{\mathrm{aff}}(B_n)$ is contained in the domain of points $x = (x_1, x_2, \ldots, x_n)$ such that

$$1 > x_1, x_2 > x_3 > \cdots > x_n > 0.$$

Moreover, a point x of this domain belongs to $F_e(B_n)$ if and only if $x_1 + x_2 < 1$.

(c) The fundamental domain $F_e(C_n)$ of $W_e^{\mathrm{aff}}(C_n)$ consists of all points $x = (x_1, x_2, \ldots, x_n)$ such that

$$\frac{1}{2} > x_1, x_2 > x_3 > \cdots > x_n > 0.$$

(d) The fundamental domain $F_e(D_n)$ of $W_e^{\mathrm{aff}}(D_n)$ is contained in the domain of points $x = (x_1, x_2, \ldots, x_n)$ such that

$$1 > x_1, x_2 > x_3 > \cdots > x_{n-1} > |x_n|.$$

Moreover, a point x of this domain belongs to $F_e(D_n)$ if and only if $x_1 + x_2 < 1$.

3.8 W_e-orbits of A_3

W_e-orbits for A_3, B_3 and C_3 can be calculated by using the orthogonal coordinates on the corresponding Euclidean space, described above, and the description of action of the Weyl groups $W_e(A_3)$, $W_e(B_3)$ and $W_e(C_3)$ in the orthogonal coordinate systems. Below we give results of such calculations. Points λ of W_e-orbits are given in the ω-coordinates in the form $(a\,b\,c)$, where $\lambda = a\omega_1 + b\omega_2 + c\omega_3$.

If $a > 0$, $b > 0$, $c > 0$, then W_e-orbits $O_e(a\ b\ c)$ and $O_e(a{+}b\ {-b}\ b{+}c) \equiv r_\alpha O_e(a\ b\ c)$ of A_3 contain the points

$$\begin{aligned} O_e(a\ b\ c) \ni{} & (a\ b\ c),\ (a{+}b\ c\ {-b}{-}c),\ (a{+}b{+}c\ {-b}{-}c\ b),\\ & ({-a}\ a{+}b{+}c\ {-c}),\ (b\ {-a}{-}b\ a{+}b{+}c),\ ({-a}{-}b\ a\ b{+}c),\\ O_e(a{+}b\ {-b}\ b{+}c) \ni{} & (a{+}b\ {-b}\ b{+}c),\ (a\ b{+}c\ {-c}),\ (a{+}b{+}c\ {-c}\ {-b}),\\ & ({-a}\ a{+}b\ c),\ (b{+}c\ {-a}{-}b{-}c\ a{+}b),\ ({-b}\ {-a}\ a{+}b{+}c) \end{aligned}$$

and the points, contragredient to these points, where the contragredient of the point $(a'\ b'\ c')$ is $(-c'\ {-b'}\ {-a'})$.

There exist also the W_e-orbits

$$\begin{aligned} O_e(a\ b\ 0) \ni{} & (a\ b\ 0), (a{+}b\ {-b}\ b), (a{+}b\ 0\ {-b}), ({-a}\ a{+}b\ 0), ({-a}{-}b\ a\ b),\\ & (b\ {-a}{-}b\ a{+}b) \text{ and contragredient points;}\\ O_e(a\ 0\ c) \ni{} & (a\ 0\ c), (a\ c\ {-c}), (a{+}c\ {-c}\ 0), ({-a}\ a\ c), (0\ {-a}\ a{+}c),\\ & ({-a}\ a{+}c\ {-c}) \text{ and contragredient points;}\\ O_e(0\ b\ c) \ni{} & (0\ b\ c), (b\ {-b}\ b{+}c), (0\ b{+}c\ {-c}), (b{+}c\ {-b}{-}c\ b), ({-b}\ 0\ b{+}c),\\ & (b\ c\ {-b}{-}c) \text{ and contragredient points;}\\ O_e(a\ 0\ 0) \ni{} & (a\ 0\ 0), ({-a}\ a\ 0), (0\ 0\ {-a}), (0\ {-a}\ a);\\ O_e(0\ b\ 0) \ni{} & (0\ b\ 0), (b\ {-b}\ b), (b\ 0\ {-b}), ({-b}\ b\ {-b}), (0\ {-b}\ 0), ({-b}\ 0\ b);\\ O_e(0\ 0\ c) \ni{} & (0\ 0\ c), (0\ c\ {-c}), (c\ {-c}\ 0), ({-c}\ 0\ 0). \end{aligned}$$

3.9 W_e-orbits of B_3

As in the previous case, points λ of W_e-orbits are given by the ω-coordinates $(a\ b\ c)$, where $\lambda = a\omega_1 + b\omega_2 + c\omega_3$. The W_e-orbits $O_e(a\ b\ c)$ and $O_e(a{+}b\ {-b}\ 2b{+}c) \equiv r_\alpha O_e(a\ b\ c)$, $a > 0$, $b > 0$, $c > 0$, of B_3 contain the points

$$\begin{aligned} O_e(a\ b\ c) \ni{} & (a\ b\ c),\ (b\ {-a}{-}b\ 2a{+}2b{+}c),\ ({-a}{-}b\ a\ 2b{+}c),\ (a{+}b{+}c\ {-b}{-}c\ 2b{+}c),\\ & ({-a}\ a{+}b{+}c\ {-c}),\ ({-b}{-}c\ {-a}\ 2a{+}2b{+}c),\ ({-a}{-}b{-}c\ {-b}\ 2b{+}c),\\ & (a{+}2b{+}c\ {-a}{-}b{-}c\ c),\ ({-b}\ a{+}2b{+}c\ {-2a}{-}2b{-}c),\ ({-a}{-}2b{-}c\ b{+}c\ {-c}),\\ & (b{+}c\ a{+}b\ {-2a}{-}2b{-}c),\ (a{+}b\ {-a}{-}2b{-}c\ 2b{+}c),\\ O_e(a{+}b\ {-b}\ 2b{+}c) \ni{} & (a{+}b\ {-b}\ 2b{+}c),\ ({-a}\ a{+}b\ c),\ ({-b}\ {-a}\ 2a{+}2b{+}c),\ (a\ b{+}c\ {-c}),\\ & (b{+}c\ {-a}{-}b{-}c\ 2a{+}2b{+}c),\ ({-a}{-}b{-}c\ a\ 2b{+}c),\ ({-a}{-}2b{-}c\ b\ c),\\ & (b\ a{+}b{+}c\ {-2a}{-}2b{-}c),\ (a{+}b{+}c\ {-a}{-}2b{-}c\ 2b{+}c),\ ({-a}{-}b\ {-b}{-}c\ 2b{+}c),\\ & (a{+}2b{+}c\ {-a}{-}b\ {-c}),\ ({-b}{-}c\ a{+}2b{+}c\ {-2a}{-}2b{-}c) \end{aligned}$$

and also all these points taken with opposite signs of coordinates.

In the case B_3 there exist also the W_e-orbits $O_e(a\ b\ 0)$, $O_e(a\ 0\ c)$, $O_e(0\ b\ c)$, which are of the form

$$O_e(a\ b\ 0) \ni \pm(a\ b\ 0), \pm(a{+}b\ {-b}\ 2b), \pm({-a}\ a{+}b\ 0), \pm(b\ {-a}{-}b\ 2a{+}2b),$$

$$\pm(-a{-}b\ a\ 2b), \pm(-b\ {-a}\ 2a{+}2b), \pm(-a{-}2b\ b\ 0), \pm(-a{-}b\ {-b}\ 2b),$$
$$\pm(a{+}2b\ {-a}{-}b\ 0), \pm(b\ a{+}b\ {-2a}{-}2b), \pm(a{+}b\ {-a}{-}2b\ 2b), \pm(-b\ a{+}2b\ {-2a}{-}2b);$$
$$O_e(a\ 0\ c) \ni \pm(a\ 0\ c), \pm(-a\ a\ c), \pm(0\ {-a}\ 2a{+}c), \pm(a\ c\ {-c}),$$
$$\pm(a{+}c\ {-c}\ c), \pm(-a\ a{+}c\ {-c}), \pm(c\ {-a}{-}c\ 2a{+}c), \pm(-a{-}c\ a\ c),$$
$$\pm(-c\ {-a}\ 2a{+}c), \pm(-a{-}c\ 0\ c), \pm(a{+}c\ {-a}{-}c\ c), \pm(0\ a{+}c\ {-2a}{-}c);$$
$$O_e(0\ b\ c) \ni \pm(0\ b\ c), \pm(b\ {-b}\ 2b{+}c), \pm(-b\ 0\ 2b{+}c), \pm(0\ b{+}c\ {-c}),$$
$$\pm(b{+}c\ {-b}{-}c\ 2b{+}c), \pm(-b{-}c\ 0\ 2b{+}c), \pm(-2b{-}c\ b\ c), \pm(-b{-}c\ {-b}\ 2b{+}c),$$
$$\pm(2b{+}c\ {-b}{-}c\ c), \pm(b\ b{+}c\ {-2b}{-}c), \pm(b{+}c\ {-2b}{-}c\ 2b{+}c), \pm(-b\ 2b{+}c\ {-2b}{-}c).$$

The W_e-orbits $O_e(a\ 0\ 0)$, $O_e(0\ b\ 0)$ and $O_e(0\ 0\ c)$ consist of the points

$$O_e(a\ 0\ 0) \ni \pm(a\ 0\ 0), \pm(a\ {-a}\ 0), \pm(0\ a\ {-2a});$$
$$O_e(0\ b\ 0) \ni \pm(0\ b\ 0), \pm(b\ {-b}\ 2b), \pm(-b\ 0\ 2b), \pm(-2b\ b\ 0), \pm(-b\ {-b}\ 2b), \pm(b\ {-2b}\ 2b);$$
$$O_e(0\ 0\ c) \ni \pm(0\ 0\ c), \pm(c\ {-c}\ c), \pm(0\ c\ {-c}), \pm(-c\ 0\ c).$$

3.10 W_e-orbits of C_3

As in the previous cases, points λ of W_e-orbits are given by the ω-coordinates $(a\ b\ c)$, where $\lambda = a\omega_1 + b\omega_2 + c\omega_3$. The W_e-orbits $O_e(a\ b\ c)$ and $O_e(a{+}b\ {-b}\ b{+}c) \equiv r_\alpha O_e(a\ b\ c)$, $a > 0$, $b > 0$, $c > 0$, of C_3 contain the points

$$O_e(a\ b\ c) \ni (a\ b\ c),\ (b\ {-a}{-}b\ a{+}b{+}c),\ (-a{-}b\ a\ b{+}c),\ (a{+}b{+}2c\ {-b}{-}2c\ b{+}c),$$
$$(-a\ a{+}b{+}2c\ {-c}),\ (-b{-}2c\ {-a}\ a{+}b{+}c),\ (-a{-}b{-}2c\ {-b}\ b{+}c),$$
$$(a{+}2b{+}2c\ {-a}{-}b{-}2c\ c),\ (-b\ a{+}2b{+}2c\ {-a}{-}b{-}c),\ (-a{-}2b{-}2c\ b{+}2c\ {-c}),$$
$$(b{+}2c\ a{+}b\ {-a}{-}b{-}c),\ (a{+}b\ {-a}{-}2b{-}2c\ b{+}c),$$
$$O_e(a{+}b\ {-b}\ b{+}c) \ni (a{+}b\ {-b}\ b{+}c),\ (-a\ a{+}b\ c),\ (-b\ {-a}\ a{+}b{+}c),\ (a\ b{+}2c\ {-c}),$$
$$(b{+}2c\ {-a}{-}b{-}2c\ a{+}b{+}c),\ (-a{-}b{-}2c\ a\ b{+}c),\ (-a{-}2b{-}2c\ b\ c),$$
$$(b\ a{+}b{+}2c\ {-a}{-}b{-}c),\ (a{+}b{+}2c\ {-a}{-}2b{-}2c\ b{+}c),\ (-a{-}b\ {-b}{-}2c\ b{+}c),$$
$$(a{+}2b{+}2c\ {-a}{-}b\ {-c}),\ (-b{-}2c\ a{+}2b{+}2c\ {-a}{-}b{-}c)$$

and also all these points taken with opposite signs of coordinates.

For the W_e-orbits $O_e(a\ b\ 0)$, $O_e(a\ 0\ c)$ and $O_e(0\ b\ c)$ we have

$$O_e(a\ b\ 0) \ni \pm(a\ b\ 0), \pm(a{+}b\ {-b}\ b), \pm(-a\ a{+}b\ 0), \pm(b\ {-a}{-}b\ a{+}b),$$
$$\pm(-a{-}b\ a\ b), \pm(-b\ {-a}\ a{+}b), \pm(-a{-}2b\ b\ 0), \pm(-a{-}b\ {-b}\ b),$$
$$\pm(a{+}2b\ {-a}{-}b\ 0), \pm(b\ a{+}b\ {-a}{-}b), \pm(a{+}b\ {-a}{-}2b\ b), \pm(-b\ a{+}2b\ {-a}{-}b);$$
$$O_e(a\ 0\ c) \ni \pm(a\ 0\ c), \pm(-a\ a\ c), \pm(0\ {-a}\ a{+}c), \pm(a\ 2c\ {-c}),$$
$$\pm(a{+}2c\ {-2c}\ c), \pm(a{+}2c\ {-a}{-}2c\ c), \pm(0\ a{+}2c\ {-a}{-}c), \pm(-a\ a{+}2c\ {-c}),$$
$$\pm(2c\ {-a}{-}2c\ a{+}c), \pm(-a{-}2c\ a\ c), \pm(-2c\ {-a}\ a{+}c), \pm(-a{-}2c\ 0\ c);$$
$$O_e(0\ b\ c) \ni \pm(0\ b\ c), \pm(b\ {-b}\ b{+}c), \pm(-b\ 0\ b{+}c), \pm(0\ b{+}2c\ {-c}),$$
$$\pm(b{+}2c\ {-b}{-}2c\ b{+}c), \pm(-b{-}2c\ 0\ b{+}c), \pm(-2b{-}2c\ b\ c), \pm(-b{-}2c\ \ {-b}\ b{+}c),$$
$$\pm(2b{+}2c\ {-b}{-}2c\ c), \pm(b\ b{+}2c\ {-b}{-}c), \pm(b{+}2c\ {-2b}{-}2c\ b{+}c), \pm(-b\ 2b{+}2c\ {-b}{-}c).$$

The W_e-orbits $O_e(a\ 0\ 0)$, $O_e(0\ b\ 0)$ and $O_e(0\ 0\ c)$ consist of the points

$$O_e(a\ 0\ 0) \ni \pm(a\ 0\ 0), \pm(a\ {-a}\ 0), \pm(0\ a\ {-a});$$
$$O_e(0\ b\ 0) \ni \pm(0\ b\ 0), \pm(b\ {-b}\ b), \pm(b\ 0\ {-b}), \pm(2b\ {-b}\ 0), \pm(-b\ {-b}\ b), \pm(b\ {-2b}\ b);$$
$$O_e(0\ 0\ c) \ni \pm(0\ 0\ c), \pm(0\ 2c\ {-c}), \pm(2c\ {-2c}\ c), \pm(2c\ 0\ {-c}).$$

4 E-orbit functions

4.1 Definition

E-orbit functions are obtained from the exponential functions $e^{2\pi i\langle\lambda,x\rangle}$, $x \in E_n$, with fixed $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ by the procedure of symmetrization by means of the even Weyl group W_e. E-orbit functions are closely related to symmetric and antisymmetric W-orbit functions. For this reason, we first define the last functions.

Let W be a Weyl group of transformations of the Euclidean space E_n. To each element $\lambda \in E_n$ from the dominant Weyl chamber (that is, $\langle\lambda, \alpha_i\rangle \geq 0$ for all simple roots α_i) there corresponds a *symmetric orbit function* ϕ_λ on E_n, which is given by the formula

$$\phi_\lambda(x) = \sum_{\mu\in O(\lambda)} e^{2\pi i\langle\mu,x\rangle}, \qquad x \in E_n, \tag{4.1}$$

where $O(\lambda)$ is the W-orbit of the element λ. The number of summands is equal to the size $|O(\lambda)|$ of the orbit $O(\lambda)$ and we have $\phi_\lambda(0) = |O(\lambda)|$. Sometimes (see, for example, [4] and [5]), it is convenient to use a modified definition of orbit functions:

$$\hat{\phi}_\lambda(x) = |W_\lambda|\phi_\lambda(x), \tag{4.2}$$

where W_λ is a subgroup in W whose elements leave λ fixed. Then for all orbit functions $\hat{\phi}_\lambda$ we have $\hat{\phi}_\lambda(0) = |W|$. The functions $\hat{\phi}_\lambda(x)$ can be defined as

$$\hat{\phi}_\lambda(x) = \sum_{w\in W} e^{2\pi i\langle\mu,x\rangle}.$$

Antisymmetric orbit functions are defined (see [2] and [6]) for dominant elements λ, which do not belong to a wall of the dominant Weyl chamber (that is, for strictly dominant elements λ). The *antisymmetric orbit function*, corresponding to such an element, is defined as

$$\varphi_\lambda(x) = \sum_{w\in W} (\det w) e^{2\pi i\langle w\lambda,x\rangle}, \qquad x \in E_n. \tag{4.3}$$

A number of summands in (4.3) is equal to the size $|W|$ of the Weyl group W. We have $\varphi_\lambda(0) = 0$.

E-orbit functions are defined for each element λ of the domain $D_+^e = D_+ \cup r_\alpha D_+$, where D_+ is the set of dominant elements of E_n and α is a root of the root system (we assume that each point is taken only once). The E-orbit function $E_\lambda(x)$, $\lambda \in D_+^e$, is given by the formula

$$E_\lambda(x) = \sum_{\mu\in O_e(\lambda)} e^{2\pi i\langle\mu,x\rangle}, \qquad x \in E_n, \tag{4.4}$$

where $O_e(\lambda)$ is the W_e-orbit of the point λ.

Sometimes, it is convenient to use normalized E-orbit functions defined as

$$\hat{E}_\lambda(x) = \sum_{w\in W_e} e^{2\pi i\langle w\lambda,x\rangle}. \tag{4.5}$$

We have $\hat{E}_\lambda(x) = |W_e^\lambda| E_\lambda(x)$, where W_e^λ is a subgroup of W_e whose elements leave λ invariant.

Example. *E-orbit functions for A_1.* In this case, there exists only one simple (positive) root α. We have $\langle\alpha, \alpha\rangle = 2$. Then the relation $2\langle\omega, \alpha\rangle/\langle\alpha, \alpha\rangle = 1$ means that $\langle\omega, \alpha\rangle = 1$. Therefore, $\omega = \alpha/2$ and $\langle\omega, \omega\rangle = 1/2$. We identify points x of $E_1 \equiv \mathbb{R}$ with $\theta\omega$. The Weyl group $W(A_1)$

consists of two elements 1 and r_α and $\det r_\alpha = -1$. The even Weyl group $W_e(A_1)$ consists of one element 1. For this reason, W_e-orbit functions $\varphi_\lambda(x)$, $\lambda = m\omega$, $m \in \mathbb{R}$, in this case coincide with exponential functions:

$$E_\lambda(x) = e^{2\pi \mathrm{i}\langle m\omega,\theta\omega\rangle} = e^{\pi \mathrm{i} m\theta}.$$

Note that for the symmetric and antisymmetric orbit functions $\phi_\lambda(x)$ and $\varphi_\lambda(x)$ we have

$$\phi_\lambda(x) = 2\cos(\pi m\theta), \qquad \varphi_\lambda(x) = 2\mathrm{i}\sin(\pi m\theta).$$

Therefore, $\phi_\lambda(x) = E_\lambda(x) + E_{-\lambda}(x)$ and $\varphi_\lambda(x) = E_\lambda(x) - E_{-\lambda}(x)$.

4.2 *E*-orbit functions of A_2

Put $\lambda = a\omega_1 + b\omega_2 \equiv (a\ b)$ with $a > 0$, $b > 0$. Then for $\varphi_\lambda(x) \equiv \varphi_{(a\ b)}(x)$ we receive from (4.3) that

$$E_{(a\ b)}(x) = e^{2\pi i\langle (a\ b),x\rangle} + e^{2\pi i\langle (b\ -a-b),x\rangle} + e^{2\pi i\langle (-a-b\ a),x\rangle},$$
$$E_{(-a\ a+b)}(x) = e^{2\pi i\langle (-a\ a+b),x\rangle} + e^{2\pi i\langle (a+b\ -b),x\rangle} + e^{2\pi i\langle (-b\ -a),x\rangle}.$$

Using the representation $x = \psi_1\alpha_1 + \psi_2\alpha_2$, one obtains

$$E_{(a\ b)}(x) = e^{2\pi i(a\psi_1+b\psi_2)} + e^{2\pi i(b\psi_1-(a+b)\psi_2)} + e^{2\pi i((-a-b)\psi_1+a\psi_2)}, \tag{4.6}$$
$$E_{(-a\ a+b)}(x) = e^{2\pi i(-a\psi_1+(a+b)\psi_2)} + e^{2\pi i((a+b)\psi_1-b\psi_2)} + e^{2\pi i(-b\psi_1-a\psi_2)}. \tag{4.7}$$

An expression for $E_{(a\ b)}(x)$ depends on a choice of coordinate systems for λ and x. Setting $x = \theta_1\omega_1 + \theta_2\omega_2$ and λ as before, we get

$$E_{(a\ b)}(x) = e^{\frac{2\pi i}{3}((2a+b)\theta_1+(a+2b)\theta_2)} + e^{-\frac{2\pi i}{3}((a-b)\theta_1+(2a+b)\theta_2)} + e^{-\frac{2\pi i}{3}((a+2b)\theta_1+(-a+b)\theta_2)},$$
$$E_{(-a\ a+b)}(x) = e^{\frac{2\pi i}{3}((-a+b)\theta_1+(a+2b)\theta_2)} + e^{\frac{2\pi i}{3}((2a+b)\theta_1+(a-b)\theta_2)} + e^{-\frac{2\pi i}{3}((a+2b)\theta_1+(2a+b)\theta_2)}.$$

Similarly one finds that $E_{(a\ 0)}(x)$ and $E_{(0\ b)}(x)$ are of the form

$$E_{(a\ 0)}(x) = e^{\frac{2\pi i}{3}a(2\theta_1+\theta_2)} + e^{\frac{2\pi i}{3}a(-\theta_1+\theta_2)} + e^{\frac{2\pi i}{3}a(-\theta_1-2\theta_2)}, \tag{4.8}$$
$$E_{(0\ b)}(x) = e^{\frac{2\pi i}{3}b(\theta_1+2\theta_2)} + e^{\frac{2\pi i}{3}b(\theta_1-\theta_2)} + e^{\frac{2\pi i}{3}b(-2\theta_1-\theta_2)}. \tag{4.9}$$

Note that the pairs $E_{(a\ b)}(x) + E_{(b\ a)}(x)$ are always real functions.

4.3 *E*-orbit functions of C_2 and G_2

Putting again $\lambda = a\omega_1 + b\omega_2 = (a\,b)$, $x = \theta_1\omega_1 + \theta_2\omega_2$ and using the matrices S from (2.6), which are of the form

$$S(C_2) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \qquad S(G_2) = \frac{1}{6}\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix},$$

we find the orbit functions $E_{(a\ b)}(x)$ for C_2 and G_2 with $a > 0$ and $b > 0$:

$$C_2: \quad E_{(a\ b)}(x) = 2\cos\pi((a+b)\theta_1 + (a+2b)\theta_2) + 2\cos\pi(b\theta_1 - a\theta_2), \tag{4.10}$$
$$E_{(-a\ a+b)}(x) = 2\cos\pi(b\theta_1 + (a+2b)\theta_2) + 2\cos\pi((a+b)\theta_1 + a\theta_2), \tag{4.11}$$
$$G_2: \quad E_{(a\ b)}(x) = 2\cos\pi(2a+b)\theta_1 + (a + \tfrac{2}{3}b)\theta_2)$$

$$+ 2\cos\pi((a+b)\theta_1 + \tfrac{1}{3}b\theta_2) + 2\cos\pi(a\theta_1 + (a+\tfrac{1}{3}b)\theta_2), \tag{4.12}$$

$$E_{(-a\ 3a+b)}(x) = 2\cos\pi((a+b)\theta_1 + (a+\tfrac{2}{3}b)\theta_2)$$
$$+ 2\cos\pi((2a+b)\theta_1 + (a+\tfrac{1}{3}b)\theta_2) + 2\cos\pi(a\theta_1 - \tfrac{1}{3}b\theta_2). \tag{4.13}$$

When one of the numbers a and b vanishes, then we have

$$C_2: \qquad E_{(a\ 0)}(x) = 2\cos\pi a(\theta_1+\theta_2) + 2\cos\pi a\theta_2, \tag{4.14}$$
$$E_{(0\ b)}(x) = 2\cos\pi b(\theta_1+2\theta_2) + 2\cos\pi b\theta_1, \tag{4.15}$$
$$G_2: \qquad E_{(a\ 0)}(x) = 2\cos\pi a(2\theta_1+\theta_2) + 2\cos\pi a(\theta_1+\theta_2) + 2\cos\pi a\theta_1, \tag{4.16}$$
$$E_{(0\ b)}(x) = 2\cos\pi b(\theta_1+\tfrac{2}{3}\theta_2) + 2\cos\pi b(\theta_1+\tfrac{1}{3}\theta_2) + 2\cos\pi\tfrac{1}{3}b\theta_2. \tag{4.17}$$

As we see, E-functions for C_2 and G_2 are real.

4.4 E-orbit functions of A_n

It is difficult to write down an explicit form of E-orbit functions for A_n, B_n, C_n and D_n in coordinates with respect to the ω- or α-bases. For this reason, for these cases we use the orthogonal coordinate systems, described in Section 3.

Let $\lambda = (m_1, m_2, \ldots, m_{n+1})$ be a strictly dominant element for A_n in orthogonal coordinates described in Subsection 3.3. Then $m_1 > m_2 > \cdots > m_{n+1}$. The Weyl group in this case coincides with the symmetric group S_{n+1} and the even Weyl group coincides with the alternating subgroup S^e_{n+1} of S_{n+1}. The W_e-orbit $O_e(\lambda)$ consists of points $(w\lambda)$, $w \in W_e$. Representing points $x \in E_{n+1}$ in the orthogonal coordinate system, $x = (x_1, x_2, \ldots, x_{n+1})$, and using formula (4.3) we find that

$$E_\lambda(x) = \sum_{w\in S^e_{n+1}} e^{2\pi\mathrm{i}\langle w(m_1,\ldots,m_{n+1}),(x_1,\ldots,x_{n+1})\rangle} = \sum_{w\in S^e_{n+1}} e^{2\pi\mathrm{i}((w\lambda)_1x_1+\cdots+(w\lambda)_{n+1}x_{n+1})}, \tag{4.18}$$

where $(w\lambda)_1, (w\lambda)_2, \ldots, (w\lambda)_{n+1}$ are the orthogonal coordinates of the point $w\lambda$.

The second type of E-orbit functions correspond to elements $\lambda = (m_1, m_2, \ldots, m_{n+1})$, for which $m_2 > m_1 > \cdots > m_{n+1}$. For this case the E-orbit functions are given by the same formula (4.18).

If $\lambda = (m_1, m_2, \ldots, m_{n+1})$ is dominant but not strictly dominant (that is, some of m_i are coinciding), then the corresponding E-orbit function is equal to the symmetric orbit function $\phi_{(m_1,m_2,\ldots,m_{n+1})}(x)$ and, thus, we have

$$E_\lambda(x) = \sum_{w\in S_{n+1}/S_\lambda} e^{2\pi\mathrm{i}\langle w(m_1,\ldots,m_{n+1}),(x_1,\ldots,x_{n+1})\rangle}$$
$$= \sum_{w\in S_{n+1}/S_\lambda} e^{2\pi\mathrm{i}((w\lambda)_1x_1+\cdots+(w\lambda)_{n+1}x_{n+1})}, \tag{4.19}$$

where S_λ is a subgroup of element of S_{n+1} leaving λ invariant.

Note that the element $-(m_{n+1}, m_n, \ldots, m_1)$ is strictly dominant if the element $(m_1, m_2, \ldots, m_{n+1})$ is strictly dominant. In the Weyl group $W(A_n)$ there exists an element w_0 such that

$$w_0(m_1, m_2, \ldots, m_{n+1}) = (m_{n+1}, m_n, \ldots, m_1).$$

Moreover, we have

$$\det w_0 = 1 \quad \text{for} \quad A_{4k-1} \quad \text{and} \quad A_{4k},$$
$$\det w_0 = -1 \quad \text{for} \quad A_{4k+1} \quad \text{and} \quad A_{4k+2}.$$

This means that $w_0 \in W_e$ for A_{4k-1} and A_{4k} and $w_0 \notin W_e$ for A_{4k+1} and A_{4k+2}.

It follows from here that for A_{4k-1} and A_{4k} in the expressions for the orbit functions $E_{(m_1,m_2,\dots,m_{n+1})}(x)$ and $E_{-(m_{n+1},m_n,\dots,m_1)}(x)$ there are summands

$$e^{2\pi \mathrm{i}\langle w_0\lambda,x\rangle}=e^{2\pi \mathrm{i}(m_{n+1}x_1+\cdots+m_1x_{n+1})}\qquad\text{and}\qquad e^{-2\pi \mathrm{i}(m_{n+1}x_1+\cdots+m_1x_{n+1})}, \tag{4.20}$$

respectively, which are complex conjugate to each other.

Similarly, for A_{4k-1} and A_{4k}, in the expressions (4.18) for the functions $E_{(m_1,m_2,\dots,m_{n+1})}(x)$ and $E_{-(m_{n+1},m_n,\dots,m_1)}(x)$ all other summands are pairwise complex conjugate. Therefore,

$$E_{(m_1,m_2,\dots,m_{n+1})}(x)=\overline{E_{-(m_{n+1},m_n,\dots,m_1)}(x)} \tag{4.21}$$

for $n=4k-1,4k$. If to use for λ the coordinates $\lambda_i=\langle\lambda,\alpha_i^\vee\rangle$ in the ω-basis instead of the orthogonal coordinates m_j, then this equation can be written as

$$E_{(\lambda_1,\dots,\lambda_n)}(x)=\overline{E_{(\lambda_n,\dots,\lambda_1)}(x)}.$$

If $n=4k+1$ or $n=4k+2$, then the E-orbit function $E_{-(m_{n+1},m_n,\dots,m_1)}(x)$ belongs to the second type of E-orbit functions. In this case the orbit functions $E_{(m_1,m_2,\dots,m_{n+1})}(x)$ and $E_{-(m_{n+1},m_n,\dots,m_1)}(x)$ have no common summands. In this case we have

$$E_{-r_\alpha(m_{n+1},m_n,\dots,m_1)}(x)=\overline{E_{(m_1,m_2,\dots,m_{n+1})}(x)},$$

where α is a positive root of our root system.

According to (4.21), if

$$(m_1,m_2,\dots,m_{n+1})=-(m_{n+1},m_n,\dots,m_1) \tag{4.22}$$

(that is, the element λ has in the ω-basis the coordinates $(\lambda_1,\lambda_2,\dots,\lambda_2,\lambda_1)$), then *the E-orbit function E_λ is real for $n=4k-1,4k$.* This orbit function can be represented as a sum of cosines of angles.

It is know from Proposition 2 in [7] that in the orthogonal coordinates antisymmetric orbit functions $\varphi_{(m_1,m_2,\dots,m_{n+1})}(x)$, $m_1>m_2>\cdots>m_{n+1}$, of A_n can be represented as determinants of certain matrices:

$$\begin{aligned}\varphi_{(m_1,m_2,\dots,m_{n+1})}(x)&=\det\left(e^{2\pi \mathrm{i}m_ix_j}\right)_{i,j=1}^{n+1}\\ &\equiv\det\begin{pmatrix} e^{2\pi \mathrm{i}m_1x_1} & e^{2\pi \mathrm{i}m_1x_2} & \dots & e^{2\pi \mathrm{i}m_1x_{n+1}}\\ e^{2\pi \mathrm{i}m_2x_1} & e^{2\pi \mathrm{i}m_2x_2} & \dots & e^{2\pi \mathrm{i}m_2x_{n+1}}\\ \dots & \dots & \dots & \dots\\ e^{2\pi \mathrm{i}m_{n+1}x_1} & e^{2\pi \mathrm{i}m_{n+1}x_2} & \dots & e^{2\pi \mathrm{i}m_{n+1}x_{n+1}}\end{pmatrix}.\end{aligned} \tag{4.23}$$

It follows from this formula that the corresponding E-orbit functions $E_{(m_1,m_2,\dots,m_{n+1})}(x)$ and $E_{(m_2,m_1,m_3,\dots,m_{n+1})}(x)$ can be represented as

$$E_{(m_1,m_2,m_3,\dots,m_{n+1})}(x)=\left[\det\left(e^{2\pi \mathrm{i}m_ix_j}\right)_{i,j=1}^{n+1}\right]^+, \tag{4.24}$$

$$E_{(m_2,m_1,m_3,\dots,m_{n+1})}(x)=\left[\det\left(e^{2\pi \mathrm{i}m_ix_j}\right)_{i,j=1}^{n+1}\right]^-, \tag{4.25}$$

where $[\det C]^+$ and $[\det C]^-$ mean a parts of the expression for the determinant of C containing all terms with sigh plus and with sign minus, respectively.

4.5 E-orbit functions of B_n

Let $\lambda = (m_1, m_2, \dots, m_n)$ be a strictly dominant element for B_n in orthogonal coordinates described in Subsection 3.4. Then $m_1 > m_2 > \cdots > m_n > 0$. The Weyl group $W(B_n)$ consists of permutations of the coordinates m_i with sign alternations of some of them. The even Weyl group $W_e(B_n)$ consists of those elements of $w \in W(B_n)$ for which $\det w = 1$. Representing points $x \in E_n$ also in the orthogonal coordinate system, $x = (x_1, x_2, \dots, x_n)$, and using formula (4.3) we find that the antisymmetric orbit function of $W(B_n)$, corresponding to element λ, coincides with

$$\begin{aligned}\varphi_\lambda(x) &= \sum_{\varepsilon_i=\pm1}\sum_{w\in S_n}(\det w)\varepsilon_1\varepsilon_2\cdots\varepsilon_n e^{2\pi\mathrm{i}\langle w(\varepsilon_1 m_1,\dots,\varepsilon_n m_n),(x_1,\dots,x_n)\rangle}\\ &= \sum_{\varepsilon_i=\pm1}\sum_{w\in S_n}(\det w)\varepsilon_1\varepsilon_2\cdots\varepsilon_n e^{2\pi\mathrm{i}((w(\varepsilon\lambda))_1x_1+\cdots+(w(\varepsilon\lambda))_nx_n)},\end{aligned}\tag{4.26}$$

where $(w(\varepsilon\lambda))_1, \dots, (w(\varepsilon\lambda))_n$ are the orthogonal coordinates of the points $w(\varepsilon\lambda)$ if $\varepsilon\lambda = (\varepsilon_1 m_1, \dots, \varepsilon_n m_n)$.

In order to obtain the corresponding E-orbit function $E_\lambda(x)$ we have to take in the expression (4.26) only those terms, for which $(\det w)\varepsilon_1\varepsilon_2\cdots\varepsilon_n = 1$. It is easy to see that

$$\begin{aligned}E_\lambda(x) = &\sum_{\varepsilon_i=\pm1}{}' \sum_{w\in S_n^e} e^{2\pi\mathrm{i}((w(\varepsilon\lambda))_1x_1+\cdots+(w(\varepsilon\lambda))_nx_n)}\\ &+\sum_{\varepsilon_i=\pm1}{}'' \sum_{w\in S_n\backslash S_n^e} e^{2\pi\mathrm{i}((w(\varepsilon\lambda))_1x_1+\cdots+(w(\varepsilon\lambda))_nx_n)},\end{aligned}\tag{4.27}$$

where $\sum\limits_{\varepsilon_i=\pm1}{}'$ means the sum over ε_i such that $\varepsilon_1\varepsilon_2\cdots\varepsilon_n = 1$ and $\sum\limits_{\varepsilon_i=\pm1}{}''$ means the sum over ε_i such that $\varepsilon_1\varepsilon_2\cdots\varepsilon_n = -1$. The notation $S_n\backslash S_n^e$ means a complement of S_n to S_n^e, where, as before, S_n^e is the alternating subgroup of S_n.

For the E-orbit function $E_{r_\alpha\lambda}(x)$ we respectively have

$$\begin{aligned}E_{r_\alpha\lambda}(x) = &\sum_{\varepsilon_i=\pm1}{}'' \sum_{w\in S_n^e} e^{2\pi\mathrm{i}((w(\varepsilon\lambda))_1x_1+\cdots+(w(\varepsilon\lambda))_nx_n)}\\ &+\sum_{\varepsilon_i=\pm1}{}' \sum_{w\in S_n\backslash S_n^e} e^{2\pi\mathrm{i}((w(\varepsilon\lambda))_1x_1+\cdots+(w(\varepsilon\lambda))_nx_n)},\end{aligned}\tag{4.28}$$

where $\sum\limits_{\varepsilon_i=\pm1}{}'$ and $\sum\limits_{\varepsilon_i=\pm1}{}''$ are such as in (4.27).

In $W(B_n)$ there exists an element w_- which change signs of all coordinates m_i. Then $\det w_- = 1$ if $n = 2k$ and $\det w_- = -1$ if $n = 2k+1$. Therefore, for each summand $e^{2\pi\mathrm{i}((w(\varepsilon\lambda))_1x_1+\cdots+(w(\varepsilon\lambda))_nx_n)}$ in the expressions (4.27) for the E-orbit function $E_{(m_1,m_2,\dots,m_n)}(x)$ there exists exactly one summand complex conjugate to it, $e^{-2\pi\mathrm{i}(((w(\varepsilon\lambda))_1x_1+\cdots+(w(\varepsilon\lambda))_nx_n)}$, if $n = 2k$. This means that *E-orbit functions of B_n are real if $n = 2k$.* These orbit functions can be represented as sums of cosines of the corresponding angles. If $n = 2k+1$, then the expression $e^{-2\pi\mathrm{i}(((w(\varepsilon\lambda))_1x_1+\cdots+(w(\varepsilon\lambda))_nx_n)}$ is not contained in the E-orbit function $E_{(m_1,m_2,\dots,m_n)}(x)$. Therefore, in this case this expression belongs to the E-orbit function of the second type, that is, to the E-orbit function $E_{(m_2,m_1,m_3,\dots,m_n)}(x)$. We conclude that when $n = 2k+1$, then the E-orbit functions $E_\lambda(x)$ and $E_{r_\alpha\lambda}(x)$ are pairwise complex conjugate to each other.

If λ is dominant but not strictly dominant, then the E-orbit function $E_\lambda(x)$ coincides with the symmetric orbit function $\phi_\lambda(x)$ and we have

$$E_\lambda(x) = \sum_{\varepsilon_i=\pm1}\sum_{w\in S_n/S_\lambda} e^{2\pi\mathrm{i}\langle w(\varepsilon_1 m_1,\dots,\varepsilon_n m_n),(x_1,\dots,x_n)\rangle}$$

$$= \sum_{\varepsilon_i=\pm1} \sum_{w\in S_n/S_\lambda} e^{2\pi \mathrm{i}((w(\varepsilon\lambda))_1 x_1+\cdots+(w(\varepsilon\lambda))_n x_n)}. \tag{4.29}$$

The summation here is over those $\varepsilon_i = \pm1$ for which $m_i \neq 0$.

4.6 E-orbit functions of C_n

Let $\lambda = (m_1, m_2, \dots, m_n)$ be a strictly dominant element for C_n in the orthogonal coordinates described in Subsection 3.5. Then $m_1 > m_2 > \cdots > m_n > 0$. Representing points $x \in E_n$ also in the orthogonal coordinate system, $x = (x_1, x_2, \dots, x_n)$, we find that the antisymmetric orbit function of $W(C_n)$, corresponding to λ, coincides with

$$\varphi_\lambda(x) = \sum_{\varepsilon_i=\pm1} \sum_{w\in S_n} (\det w)\varepsilon_1\varepsilon_2\ldots\varepsilon_n e^{2\pi\mathrm{i}\langle w(\varepsilon_1 m_1,\dots,\varepsilon_n m_n),(x_1,\dots,x_n)\rangle}$$

$$= \sum_{\varepsilon_i=\pm1} \sum_{w\in S_n} (\det w)\varepsilon_1\varepsilon_2\cdots\varepsilon_n e^{2\pi \mathrm{i}((w(\varepsilon\lambda))_1 x_1+\cdots+(w(\varepsilon\lambda))_n x_n)}, \tag{4.30}$$

where, as above, $(w(\varepsilon\lambda))_1, \dots, (w(\varepsilon\lambda))_n$ are the orthogonal coordinates of the points $w(\varepsilon\lambda)$, where $\varepsilon\lambda = (\varepsilon_1 m_1, \dots, \varepsilon_n m_n)$.

In order to obtain the corresponding E-orbit function $E_\lambda(x)$ we have to take in the expression (4.30) only those terms, for which $(\det w)\varepsilon_1\varepsilon_2\cdots\varepsilon_n = 1$. It is easy to see that

$$E_\lambda(x) = {\sum_{\varepsilon_i=\pm1}}' \sum_{w\in S_n^e} e^{2\pi \mathrm{i}((w(\varepsilon\lambda))_1 x_1+\cdots+(w(\varepsilon\lambda))_n x_n)}$$

$$+ {\sum_{\varepsilon_i=\pm1}}'' \sum_{w\in S_n\backslash S_n^e} e^{2\pi \mathrm{i}((w(\varepsilon\lambda))_1 x_1+\cdots+(w(\varepsilon\lambda))_n x_n)}, \tag{4.31}$$

where ${\sum_{\varepsilon_i=\pm1}}'$ means the sum over ε_i such that $\varepsilon_1\varepsilon_2\cdots\varepsilon_n = 1$ and ${\sum_{\varepsilon_i=\pm1}}''$ means the sum over ε_i such that $\varepsilon_1\varepsilon_2\cdots\varepsilon_n = -1$. The notation $S_n\backslash S_n^e$ means a complement of S_n to S_n^e. For the E-orbit functions $E_{r_\alpha\lambda}(x)$ we have the expression

$$E_\lambda(x) = {\sum_{\varepsilon_i=\pm1}}'' \sum_{w\in S_n^e} e^{2\pi \mathrm{i}((w(\varepsilon\lambda))_1 x_1+\cdots+(w(\varepsilon\lambda))_n x_n)}$$

$$+ {\sum_{\varepsilon_i=\pm1}}' \sum_{w\in S_n\backslash S_n^e} e^{2\pi \mathrm{i}((w(\varepsilon\lambda))_1 x_1+\cdots+(w(\varepsilon\lambda))_n x_n)}, \tag{4.32}$$

where ${\sum_{\varepsilon_i=\pm1}}'$ and ${\sum_{\varepsilon_i=\pm1}}''$ are such as in (4.31).

In $W(C_n)$ there exists an element w_- which change signs of all coordinates m_i. Then $\det w_- = 1$ if $n = 2k$ and $\det w_- = -1$ if $n = 2k+1$. Therefore, for each summand $e^{2\pi \mathrm{i}((w(\varepsilon\lambda))_1 x_1+\cdots+(w(\varepsilon\lambda))_n x_n)}$ in the expressions (4.31) for the E-orbit function $E_{(m_1,m_2,\dots,m_n)}(x)$ there exists exactly one summand complex conjugate to it, $e^{-2\pi \mathrm{i}(((w(\varepsilon\lambda))_1 x_1+\cdots+(w(\varepsilon\lambda))_n x_n)}$, if $n = 2k$. This means that *E-orbit functions of C_n are real if $n = 2k$*. This orbit function of C_n can be represented as a sum of cosines of the corresponding angles. If $n = 2k+1$, then the expression $e^{-2\pi \mathrm{i}(((w(\varepsilon\lambda))_1 x_1+\cdots+(w(\varepsilon\lambda))_n x_n)}$ is contained in the E-orbit function $E_{r_\alpha(m_1,m_2,\dots,m_n)}(x)$. We conclude that when $n = 2k+1$, then the E-orbit functions $E_\lambda(x)$ and $E_{r_\alpha\lambda}(x)$ are pairwise complex conjugate to each other.

If λ is dominant but not strictly dominant, then the E-orbit function $E_\lambda(x)$ coincides with the symmetric orbit function $\phi_\lambda(x)$ and we have

$$E_\lambda(x) = \sum_{\varepsilon_i=\pm1} \sum_{w\in S_n/S_\lambda} e^{2\pi\mathrm{i}\langle w(\varepsilon_1 m_1,\dots,\varepsilon_n m_n),(x_1,\dots,x_n)\rangle}$$

$$= \sum_{\varepsilon_i=\pm 1} \sum_{w\in S_n/S_\lambda} e^{2\pi \mathrm{i}((w(\varepsilon\lambda))_1 x_1+\cdots+(w(\varepsilon\lambda))_n x_n)}, \tag{4.33}$$

where the summation is over those $\varepsilon_i = \pm 1$ for which $m_i \neq 0$.

Note that in the orthogonal coordinates expressions for the E-orbit functions $E_{(m_1,m_2,\dots,m_n)}(x)$ of C_n coincide with the expressions for the corresponding E-orbit functions $E_{(m_1,m_2,\dots,m_n)}(x)$ of B_n. However, α-coordinates of the element $(m_1, m_2, \dots, m_n)$ for C_n do not coincide with α-coordinates of the element $(m_1, m_2, \dots, m_n)$ for B_n, that is, in α-coordinates expressions for the corresponding E-orbit functions of B_n and C_n are different.

4.7 *E*-orbit functions of D_n

Let $\lambda = (m_1, m_2, \dots, m_n)$ be a strictly dominant element for D_n in the orthogonal coordinates described in Subsection 3.6. Then $m_1 > m_2 > \cdots > m_{n-1} > |m_n|$. The Weyl group $W(D_n)$ consists of permutations of the coordinates with sign alternations of even number of them. The even Weyl group $W_e(D_n)$ consists of even permutations of the coordinates with sign alternations of even number of them.

Representing points $x \in E_n$ also in the orthogonal coordinate system, $x = (x_1, x_2, \dots, x_n)$, and using formula (4.3) we find that the antisymmetric orbit function $\varphi_\lambda(x)$ of D_n is given by the formula

$$\varphi_\lambda(x) = {\sum_{\varepsilon_i=\pm 1}}' \sum_{w\in S_n} (\det w) e^{2\pi \mathrm{i}\langle w(\varepsilon_1 m_1,\dots,\varepsilon_n m_n),(x_1,\dots,x_n)\rangle}$$
$$= {\sum_{\varepsilon_i=\pm 1}}' \sum_{w\in S_n} (\det w) e^{2\pi \mathrm{i}((w(\varepsilon\lambda))_1 x_1+\cdots+(w(\varepsilon\lambda))_n x_n)}, \tag{4.34}$$

where $(w(\varepsilon\lambda))_1, \dots, (w(\varepsilon\lambda))_n$ are the orthogonal coordinates of the points $w(\varepsilon\lambda)$ and the prime at the sum symbol means that the summation is over values of ε_i with even number of sign minus. We have taken into account that an alternation of coordinates without any permutation does not change a determinant.

Therefore, if λ is strictly dominant, then the E-orbit function $E_\lambda(x)$ is of the form

$$E_\lambda(x) = {\sum_{\varepsilon_i=\pm 1}}' \sum_{w\in S_n^e} e^{2\pi \mathrm{i}((w(\varepsilon\lambda))_1 x_1+\cdots+(w(\varepsilon\lambda))_n x_n)}, \tag{4.35}$$

where S_n^e is the even part of the symmetric group S_n and the first sum is such as in (4.34). For the corresponding E-orbit functions $E_{r_\alpha\lambda}(x)$ we have

$$E_{r_\alpha\lambda}(x) = {\sum_{\varepsilon_i=\pm 1}}' \sum_{w\in S_n\backslash S_n^e} e^{2\pi \mathrm{i}((w(\varepsilon\lambda))_1 x_1+\cdots+(w(\varepsilon\lambda))_n x_n)}, \tag{4.36}$$

where the first sum is such as in (4.35).

Let $m_n \neq 0$. Then in the expressions (4.35) for the E-orbit function $E_{(m_1,m_2,\dots,m_n)}(x)$ of $D_{n=2k}$ for each summand $e^{2\pi \mathrm{i}((w(\varepsilon\lambda))_1 x_1+\cdots+(w(\varepsilon\lambda))_n x_n)}$ there exists exactly one summand complex conjugate to it. This means that these *E-orbit functions of D_{2k} are real.* Each orbit function of D_{2k} can be represented as a sum of cosines of the corresponding angles.

It is also proved by using (4.35) that for $m_n \neq 0$ *the E-orbit functions $E_{(m_1,\dots,m_{2k},m_{2k+1})}(x)$ and $E_{(m_1,\dots,m_{2k},-m_{2k+1})}(x)$ of D_{2k+1} are complex conjugate.*

If $m_n = 0$, then it follows from (4.35) that E-orbit functions $E_\lambda(x)$ of D_n are real and can be represented as a sum of cosines of certain angles.

For dominant, but not strictly dominant, elements λ the E-orbit functions coincide with the corresponding symmetric orbit functions and we have

$$\begin{aligned} E_\lambda(x) &= \sum_{\varepsilon_i=\pm1}{}' \sum_{w\in S_n/S_\lambda} e^{2\pi \mathrm{i}\langle w(\varepsilon_1 m_1,\dots,\varepsilon_n m_n),(x_1,\dots,x_n)\rangle} \\ &= \sum_{\varepsilon_i=\pm1}{}' \sum_{w\in S_n/S_\lambda} e^{2\pi \mathrm{i}((w(\varepsilon\lambda))_1 x_1+\dots+(w(\varepsilon\lambda))_n x_n)}, \end{aligned} \tag{4.37}$$

where S_λ is the subgroup of S_n consisting of elements leaving λ invariant.

Note that in the expressions (4.37) for the orbit function $E_{(m_1,m_2,\dots,m_n)}(x)$ of D_{2k} for each summand $e^{2\pi \mathrm{i}((w(\varepsilon\lambda))_1 x_1+\dots+(w(\varepsilon\lambda))_n x_n)}$ there exists exactly one summand complex conjugate to it. This means that *all orbit functions of* D_{2k} *are real.* Each orbit function of D_{2k} can be represented as a sum of cosines of the corresponding angles.

It also follows from (4.37) that *for* D_{2k+1} *the* E*-orbit function* $E_{(m_1,m_2,\dots,m_n)}(x)$ *is real if and only if the condition* $m_{2k+1}=0$ *is fulfilled. The* E*-orbit functions* $E_{(m_1,\dots,m_{2k},m_{2k+1})}(x)$ *and* $E_{(m_1,\dots,m_{2k},-m_{2k+1})}(x)$ *of* D_{2k+1} *are complex conjugate.*

5 Properties of E-orbit functions

5.1 Invariance with respect to the even Weyl group

Since the scalar product $\langle\cdot,\cdot\rangle$ in E_n is invariant with respect to the Weyl group W, that is,

$$\langle wx, wy\rangle = \langle x,y\rangle, \qquad w\in W, \qquad x,y\in E_n,$$

it is invariant with respect to the even Weyl group W_e, which is a subgroup of W. It follows from here that E-orbit functions E_λ for strictly dominant elements λ are invariant with respect to W_e, that is,

$$E_\lambda(w'x) = E_\lambda(x), \qquad w'\in W_e.$$

Indeed, this relation is equivalent to the relation $\hat{E}_\lambda(w'x) = \hat{E}_\lambda(x)$ for the functions (4.5). For $\hat{E}_\lambda(x)$ we have

$$\hat{E}_\lambda(w'x) = \sum_{w\in W_e} e^{2\pi \mathrm{i}\langle w\lambda, w'x\rangle} = \sum_{w\in W_e} e^{2\pi \mathrm{i}\langle {w'}^{-1}w\lambda, x\rangle} = \sum_{w\in W_e} (\det w) e^{2\pi \mathrm{i}\langle w\lambda, x\rangle} = \hat{E}_\lambda(x)$$

since ${w'}^{-1}w$ runs over the whole group W_e when w runs over W_e.

In the same way it is shown that the corresponding E-orbit functions $E_{r_\alpha\lambda}(x)$, where λ is as above and α is a root, is invariant with respect to W_e.

If λ is dominant, but not strictly dominant, then $E_\lambda(x)$ coincides with the corresponding symmetric orbit function, which is invariant with respect to the Weyl group W, and therefore with respect to W_e.

5.2 Invariance with respect to the even affine Weyl group

If λ belongs to the set of dominant integral elements P_+ or to $r_\alpha P_+$, then $E_\lambda(x)$ is invariant with respect to the even affine group W_e^{aff}. Recall that W_e^{aff} is a semidirect product of the even Weyl group W_e and the group $\hat{Q}$ of translations by elements of the dual root system $Q^\vee$. In order to prove this invariance of E-orbit functions $E_\lambda(x)$ we need only to prove their invariance with respect to the subgroup $\hat{Q}$. For this we note that for $\mu\in P$ and for $\nu\in Q^\vee$ we have

$$\langle \mu, x+\nu\rangle = \langle\mu,x\rangle + \langle\mu,\nu\rangle = \langle\mu,x\rangle + \text{integer}.$$

Hence,

$$E_\lambda(x+\nu) = |W_\lambda|^{-1} \sum_{w\in W_e} e^{2\pi \mathrm{i}\langle w\lambda, x+\nu\rangle} = |W_\lambda|^{-1} \sum_{w\in W_e} e^{2\pi \mathrm{i}\langle \lambda, x\rangle} = E_\lambda(x), \tag{5.1}$$

where W_λ is the subgroup of W_e consisting of elements w such that $w\lambda = \lambda$. Thus, $E_\lambda(x)$ is invariant with respect to the even affine Weyl group W_e^{aff}.

If $\lambda \notin P$, then E_λ is not invariant with respect to W_e^{aff}. It is invariant only under action by elements of the even Weyl group W_e.

Due to the invariance of E-orbit functions E_λ, $\lambda \in P_+$, with respect to the group W_e^{aff}, it is enough to consider them only on the even fundamental domain F_e of W_e^{aff} (see Subsection 2.6). Values of E_λ on other points of E_n are determined by using the action of W_e^{aff} on F_e or taking a limit.

5.3 Relation to symmetric and antisymmetric orbit functions

E-orbit functions are closely related to symmetric and antisymmetric orbit functions $\phi_\lambda(x)$ and $\varphi_\lambda(x)$. We have seen that if λ is lying on a wall of the dominant Weyl chamber, then

$$E_\lambda(x) = \phi_\lambda(x). \tag{5.2}$$

Since the Weyl group W can be represented as a union of the even Weyl group W_e and of the set $r_\alpha W_e$,

$$W = W_e \bigcup r_\alpha W_e,$$

then it follows from the definitions of symmetric and antisymmetric orbit functions that for strictly dominant λ we have

$$\phi_\lambda(x) = E_\lambda(x) + E_{r_\alpha\lambda}(x), \tag{5.3}$$

$$\varphi_\lambda(x) = E_\lambda(x) - E_{r_\alpha\lambda}(x). \tag{5.4}$$

It follows from here that for such α one gets

$$E_\lambda(x) = \frac{1}{2}\left(\phi_\lambda(x) + \varphi_\lambda(x)\right), \tag{5.5}$$

$$E_{r_\alpha\lambda}(x) = \frac{1}{2}\left(\phi_\lambda(x) - \varphi_\lambda(x)\right). \tag{5.6}$$

It is directly derived from (5.3)–(5.6) that the relations

$$\phi_\lambda^2(x) - \varphi_\lambda^2(x) = 4E_\lambda(x)E_{r_\alpha\lambda}(x), \tag{5.7}$$

$$\phi_\lambda^2(x) + \varphi_\lambda^2(x) = 2(E_\lambda^2(x) + E_{r_\alpha\lambda}^2(x)) \tag{5.8}$$

are true, where λ is strictly dominant.

5.4 Continuity

An E-orbit function $E_\lambda(x)$ is a finite sum of exponential functions. Therefore, it is continuous and has continuous derivatives of all orders on the Euclidean space E_n.

Antisymmetric orbit functions φ_λ vanish on the boundary of the fundamental domain $F(W)$ of the Weyl group W. The normal derivative $\partial\mathbf{n}$ of symmetric orbit functions ϕ_λ to the boundary $\partial F(W)$ of the fundamental domain $F(W)$ equals zero. The reason of these properties is (anti)symmetry with respect to the reflections with respect to walls of the dominant Weyl chamber. These reflections do not belong to W_e and, therefore, E-orbit functions do not have these properties.

5.5 Scaling symmetry

Let $O_e(\lambda)$ be an E-orbit of λ, $\lambda \in D_+$. Since $w(c\lambda) = cw(\lambda)$ for any $c \in \mathbb{R}$ and for any $w \in W_e$, then the orbit $O_e(c\lambda)$ is an orbit consisting of the points $cw\lambda$, $w \in W_e$. Let $E_\lambda = \sum_{w\in W_e} e^{2\pi i w\lambda}$ be the E-orbit function for $\lambda \in D_+$. Then

$$E_{c\lambda}(x) = |W_\lambda|^{-1} \sum_{w\in W_e} e^{2\pi i\langle cw\lambda,x\rangle} = |W_\lambda|^{-1} \sum_{w\in W_e} e^{2\pi i\langle w\lambda,cx\rangle} = E_\lambda(cx).$$

The equality $E_{c\lambda}(x) = E_\lambda(cx)$ expresses the *scaling symmetry of E-orbit functions.*

5.6 Duality

Due to the invariance of the scalar product $\langle\cdot,\cdot\rangle$ with respect to the even Weyl group W_e, $\langle w\mu, wy\rangle = \langle\mu, y\rangle$, for E-orbit functions $\hat{E}_\lambda(x)$ (see Subsection 4.1) we have

$$\hat{E}_\lambda(x) = \sum_{w\in W_e} e^{2\pi i\langle\lambda,w^{-1}x\rangle} = \sum_{w\in W_e} e^{2\pi i\langle\lambda,wx\rangle} = \hat{E}_x(\lambda).$$

It is easy to see that this relation is also true for E-orbit functions $E_\lambda(x)$. The relation $E_\lambda(x) = E_x(\lambda)$ expresses the *duality* of E-orbit functions.

E-orbit functions have also the following property

$$E_{r_\alpha\lambda}(x) = E_\lambda(r_\alpha x), \tag{5.9}$$

where α is a root of the corresponding root system. This relation follows from the fact that $r_\alpha W_e = W_e r_\alpha$ and from the equalities

$$\hat{E}_{r_\alpha\lambda}(x) = \sum_{w\in W_e} e^{2\pi i\langle wr_\alpha\lambda,x\rangle} = \sum_{w\in W_e} e^{2\pi i\langle r_\alpha w\lambda,x\rangle} = \sum_{w\in W_e} e^{2\pi i\langle w\lambda,r_\alpha x\rangle} = \hat{E}_\lambda(r_\alpha x),$$

where λ is strictly dominant.

5.7 Orthogonality

If values of λ are integral points lying inside of the fundamental domain F_e of the even affine Weyl group $W_e^{\rm aff}$, then the corresponding E-orbit functions are orthogonal on the closure $\overline{F_e}$ of the fundamental domain F_e with respect to the Euclidean measure:

$$|\overline{F_e}|^{-1} \int_{\overline{F_e}} E_\lambda(x)\overline{E_{\lambda'}(x)}dx = |W_e|\delta_{\lambda\lambda'}, \tag{5.10}$$

where the overbar over $E_{\lambda'}(x)$ means complex conjugation. This relation directly follows from the orthogonality of the exponential functions $e^{2\pi i\langle\mu,x\rangle}$ (entering into the definition of E-orbit functions) for different weights μ and from the fact that a given element $\nu \in P$ belongs to precisely one E-orbit function. In (5.10), $|\overline{F_e}|$ means an area of the domain $\overline{F_e}$.

Sometimes, it is difficult to find the area $|\overline{F_e}|$. In this case it is useful the following form of the formula (5.10):

$$\int_{\mathsf{T}} E_\lambda(x)\overline{E_{\lambda'}(x)}dx = |W_e|\delta_{\lambda\lambda'},$$

where T is the torus in E_n described in Subsection 9.1 of [8]. If to assume that an area of T is equal to 1, $|\mathsf{T}| = 1$, then $|\overline{F_e}| = |W_e|^{-1}$ and formula (5.10) takes the form

$$\int_{\overline{F_e}} E_\lambda(x)\overline{E_{\lambda'}(x)}dx = \delta_{\lambda\lambda'}. \tag{5.11}$$

If λ is an integral point which lies on a wall of the even dominant Weyl chamber D_+^e, then instead of (5.11) we have the relation

$$\int_{F_e} E_\lambda(x)\overline{E_{\lambda'}(x)}dx = |W_\lambda|^{-1}\delta_{\lambda\lambda'}, \tag{5.12}$$

where W_λ is the subgroup of W_e consisting of elements $w \in W_e$ such that $w\lambda = \lambda$.

5.8 Solutions of the Laplace equation: the cases A_n, B_n, C_n and D_n

We use orthogonal coordinates $x_1, x_2, \dots, x_{n+1}$ in the case of A_n and the orthogonal coordinates $x_1, x_2, \dots, x_n$ in the cases B_n, C_n and D_n (see section 3). The Laplace operator on E_r in the orthogonal coordinates has the form

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_r^2},$$

where $r = n+1$ for A_n and $r = n$ for B_n, C_n and D_n. Let us consider the case B_n. We take a summand from the expression (4.26) for the E-orbit function $E_\lambda(x)$ of B_n and act upon it by the operator Δ. We get

$$\begin{aligned}
&\Delta e^{2\pi \mathrm{i}((w(\varepsilon\lambda))_1 x_1 + \cdots + (w(\varepsilon\lambda))_n x_n)} \\
&\qquad = -4\pi^2[(\varepsilon_1 m_1)^2 + \cdots + (\varepsilon_n m_n)^2] e^{2\pi \mathrm{i}((w(\varepsilon\lambda))_1 x_1 + \cdots + (w(\varepsilon\lambda))_n x_n)} \\
&\qquad = -4\pi^2(m_1^2 + \cdots + m_n^2) e^{2\pi \mathrm{i}((w(\varepsilon\lambda))_1 x_1 + \cdots + (w(\varepsilon\lambda))_n x_n)} \\
&\qquad = -4\pi^2\langle\lambda,\lambda\rangle\, e^{2\pi \mathrm{i}((w(\varepsilon\lambda))_1 x_1 + \cdots + (w(\varepsilon\lambda))_n x_n)},
\end{aligned}$$

where $\lambda = (m_1, m_2, \dots, m_n)$ is the element of $D_+^e(B_n)$, determining $E_\lambda(x)$, in the orthogonal coordinates and $w \in S_n^e \equiv S_n/S_2$. Since this action of Δ does not depend on a summand from (4.26), we have

$$\Delta E_\lambda(x) = -4\pi^2\langle\lambda,\lambda\rangle E_\lambda(x). \tag{5.13}$$

For A_n, C_n and D_n this formula also holds and the corresponding proofs are the same. Remark that in the case A_n the scalar product $\langle\lambda,\lambda\rangle$ is equal to

$$\langle\lambda,\lambda\rangle = m_1^2 + m_2^2 + \cdots + m_{n+1}^2.$$

Thus, E-orbit functions are eigenfunctions of the Laplace operator on the Euclidean space E_r.

5.9 The Laplace operator in ω-basis

We may parametrize points of $E - n$ by coordinates in the ω-basis: $x = \theta_1\omega_1 + \cdots + \theta_n\omega_n$. Denoting by ∂_k partial derivative with respect to θ_k, we have the Laplace operator Δ in the form

$$\Delta = \frac{1}{2}\sum_{i,j=1}^{n} \langle\alpha_j,\alpha_j\rangle^{-1} M_{ij}\partial_i\partial_j, \tag{5.14}$$

where (M_{ij}) is the corresponding Cartan matrix. One can see that it is indeed the Laplace operator as follows. The matrix $(S_{ij}) = (\langle\alpha_j,\alpha_j\rangle^{-1}M_{ij})$ is symmetric with respect to transposition and its determinant is positive. Hence it can be diagonalized, so that Δ becomes a sum of second derivatives with no mixed derivative terms.

We write down explicit form of the Laplace operators in coordinates in the ω-basis for ranks 2 and 3. For rank two the operator Δ is of the form

$$A_2: \quad (\partial_1^2 - \partial_1\partial_2 + \partial_2^2)E_\lambda(x) = -\tfrac{4\pi^2}{3}(a^2 + ab + b^2)E_\lambda(x), \tag{5.15}$$

$$C_2: \quad (2\partial_1^2 - 2\partial_1\partial_2 + \partial_2^2)E_\lambda(x) = -2\pi^2(a^2 + 4ab + 4b^2)E_\lambda(x), \tag{5.16}$$

$$G_2: \quad (\partial_1^2 - 3\partial_1\partial_2 + 3\partial_2^2)E_\lambda(x) = -\tfrac{4\pi^2}{3}(3a^2 + 3ab + b^2)E_\lambda(x). \tag{5.17}$$

Here, $\lambda = (a\ b)$ and $x = (\theta_1\ \theta_2)$.

In the semisimple case $A_1 \times A_1$ one has $M = 2\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$, therefore $\Delta = 2\partial_1^2 + 2\partial_2^2$, and $E_\lambda(x)$ is the product of two E-orbit functions, one from each A_1.

There are three 3-dimensional cases, namely A_3, B_3, and C_3. For A_3, B_3, and C_3 the result can be represented by the formulas

$$\begin{aligned} A_3: &\quad \Delta = \partial_1^2 + \partial_2^2 + \partial_3^2 - \partial_1\partial_2 - \partial_2\partial_3,\\ B_3: &\quad \Delta = \partial_1^2 + \partial_2^2 + 2\partial_3^2 - \partial_1\partial_2 - 2\partial_2\partial_3,\\ C_3: &\quad \Delta = 2\partial_1^2 + 2\partial_2^2 + 2\partial_3^2 - 2\partial_1\partial_2 - 2\partial_2\partial_3. \end{aligned} \tag{5.18}$$

In all these case we have

$$\Delta E_\lambda(x) = -4\pi^2\langle\lambda,\lambda\rangle E_\lambda(x). \tag{5.19}$$

5.10 *E*-orbit functions as eigenfunctions of other operators

E-orbit functions are eigenfunctions of many other operators. Let us consider examples of such operators.

With each $y \in E_n$ we associate the shift operator T_y which acts on the exponential functions $e^{2\pi \mathrm{i}\langle\lambda,x\rangle}$ as

$$T_y e^{2\pi \mathrm{i}\langle\lambda,x\rangle} = e^{2\pi \mathrm{i}\langle\lambda,x+y\rangle} = e^{2\pi \mathrm{i}\langle\lambda,y\rangle}e^{2\pi \mathrm{i}\langle\lambda,x\rangle}.$$

An action of elements of the even Weyl group W_e on functions, given on E_n, is given as $wf(x) = f(wx)$. For each $y \in E_n$ we define an operator acting on orbit functions by the formula

$$D_y = \sum_{w\in W_e} (\det w) w T_y.$$

Then

$$\begin{aligned} D_y\hat{E}_\lambda(x) &= D_y \sum_{w\in W} e^{2\pi \mathrm{i}\langle w\lambda,x\rangle} = \sum_{w'\in W_e}\sum_{w\in W_e} e^{2\pi \mathrm{i}\langle w\lambda,y\rangle}e^{2\pi \mathrm{i}\langle w\lambda,w'x\rangle}\\ &= \sum_{w\in W_e} e^{2\pi \mathrm{i}\langle w\lambda,y\rangle}\sum_{w'\in W_e} e^{2\pi \mathrm{i}\langle w\lambda,w'x\rangle} = \sum_{w\in W_e} e^{2\pi \mathrm{i}\langle w\lambda,y\rangle}\sum_{w'\in W_e} e^{2\pi \mathrm{i}\langle w'^{-1}w\lambda,x\rangle}\\ &= \sum_{w\in W_e} e^{2\pi \mathrm{i}\langle w\lambda,y\rangle}\hat{E}_\lambda(x) = \hat{E}_\lambda(y)\hat{E}_\lambda(x), \end{aligned}$$

that is, $\hat{E}_\lambda(x)$ (and therefore $E_\lambda(x)$) is an eigenfunction of the operator D_y with eigenvalue $\hat{E}_\lambda(y)$.

It is shown similarly that in the cases of A_n, B_n, C_n, D_n the functions $E_\lambda(x)$ are eigenfunctions of the operators

$$\sum_{w\in W_e} w\frac{\partial^2}{\partial x_i^2}, \qquad i = 1, 2, \ldots, r,$$

where $x_1, x_2, \ldots, x_r$ are orthogonal coordinates of the point x, $r = n + 1$ for A_n and $r = n$ for other cases. In fact, these operators are multiple to the Laplace operator Δ.

6 Decomposition of products of E-orbit functions

In this section we show how to decompose products of E-orbit functions into sums of E-orbit functions. Such operations are fulfilled by means of the corresponding decompositions of W_e-orbits.

6.1 Products of orbit functions

Invariance of E-orbit functions E_λ with respect to the even Weyl group W_e leads to the following statement:

Proposition 1. *A product of E-orbit functions expands into a sum of W_e-orbit functions:*

$$E_\lambda E_\mu = \sum_\nu n_\nu E_\nu, \tag{6.1}$$

where n_ν are non-negative integers, which shows how many times the orbit function E_ν is contained in the product $E_\lambda E_\mu$.

Proof. For $w \in W_e$ we have

$$E_\lambda(wx)E_\mu(wx) = E_\lambda(x)E_\mu(x).$$

Therefore, the product $E_\lambda(x)E_\mu(x)$ is a finite sum of exponential functions, which is invariant with respect to W_e. Hence, it can be expanded into E-orbit functions. Representing the product $E_\lambda(x)E_\mu(x)$ as a sum of exponential functions we see that these exponential functions enter into this sum with positive integral coefficients. This means that coefficients n_ν in (6.1) are non-negative integers. The proposition is proved. ■

Under termwise multiplication of E-orbit functions $E_\lambda(x)$ and $E_\mu(x)$ we multiply exponential functions,

$$e^{2\pi \mathrm{i}\langle\nu,x\rangle} e^{2\pi \mathrm{i}\langle\nu',x\rangle} = e^{2\pi \mathrm{i}\langle\nu+\nu',x\rangle}.$$

This reduces the procedure of multiplication of E-orbit functions $E_\lambda(x)$ and $E_\mu(x)$ to the procedure of multiplication of the corresponding W_e-orbits $O_e(\lambda)$ and $O_e(\mu)$. A product of these orbits is defined as follows.

A product $O_e(\lambda)\otimes O_e(\lambda')$ of two W_e-orbits $O_e(\lambda)$ and $O_e(\lambda')$ is the set of all points of the form $\lambda_1+\lambda_2$, where $\lambda_1 \in O_e(\lambda)$ and $\lambda_2 \in O_e(\lambda')$. Since a set of points $\lambda_1+\lambda_2$, $\lambda_1 \in O_e(\lambda)$, $\lambda_2 \in O_e(\lambda')$, is invariant with respect to action of the corresponding even Weyl group W_e, each product of W_e-orbits is decomposable into a sum of W_e-orbits. If $\lambda = 0$, then $O_e(\lambda) \otimes O_e(\lambda') = O_e(\lambda')$. If $\lambda' = 0$, then $O_e(\lambda) \otimes O_e(\lambda') = O_e(\lambda)$. In what follows we assume that $\lambda \neq 0$ and $\lambda' \neq 0$. Decomposition of products of two W_e-orbits into separate W_e-orbits is not a simple task.

6.2 Decomposition of products of W_e-orbits

If $O_e(\lambda)$ and $O_e(\lambda')$ are two W_e-orbits such that λ and λ' are dominant and lie on walls of the dominant Weyl chamber, then these orbits in fact coincide with the corresponding W-orbits $O(\lambda)$ and $O(\lambda')$, respectively. In this case we can apply to this product a procedure of decomposition of a product of W-orbits (see Section 4 in [7]). Decomposing this product into W-orbits, we make further the following. If a resulting W-orbit $O(\mu)$ is such that μ lies on a wall of the dominant Weyl chamber, then $O(\mu)$ is W_e-orbit, and $O_e(\mu)$ is contained in the product $O_e(\lambda) \otimes O_e(\lambda')$ of W_e-orbits with a multiplicity equal to a multiplicity of W-orbit $O(\mu)$ in the product $O(\mu)\otimes O(\mu')$. If a resulting W-orbit $O(\mu)$ is such that μ does not lie on a wall of the dominant Weyl chamber, then this W-orbit consists of two W_e-orbit $O_e(\mu)$ and $O(r_\alpha\mu)$. Moreover, multiplicities of these W_e-orbits are determined as in the previous case. Thus, in our case a decomposition of W_e-orbits is completely determined by a decomposition of the corresponding W-orbits.

Let now the W_e-orbits $O_e(\lambda)$ and $O_e(\lambda')$ be such that λ is on a wall of the dominant Weyl chamber and λ' is a strictly dominant element. Then $O_e(\lambda)$ can be considered as W-orbit $O(\lambda)$. Instead of the W_e-orbit $O_e(\lambda')$ we consider the signed W-orbit $O^\pm(\lambda')$. (This signed orbit consists of two W_e-orbits $O_e(\lambda)$ and $O_e(r_\alpha\lambda)$, and points of $O_e(r_\alpha\lambda)$ are taken with the sign minus.) Our problem of decomposition of the product $O_e(\lambda)\otimes O_e(\lambda')$ is reduced to decomposition of the product $O(\lambda)\otimes O^\pm(\lambda')$ of a W-orbit and a signed W-orbit. Decompositions of such products are studied in Section 7 in [8]. The product $O(\lambda)\otimes O^\pm(\lambda')$ decomposes into signed orbits, which are taken with sign plus or sign minus, that is,

$$O(\lambda)\otimes O^\pm(\lambda') = \bigcup_\mu n_\mu O^\pm(\mu),$$

where n_μ are positive or negative integers. Now a result for decomposition of the product $O_e(\lambda)\otimes O_e(\lambda')$ can be formulated as follows. If a sign orbit $O^\pm(\mu)$ is contained in the decomposition of $O(\lambda)\otimes O^\pm(\lambda')$ with positive coefficient n_μ, then the product $O_e(\lambda)\otimes O_e(\lambda')$ contains the W_e-orbit $O_e(\mu)$ with multiplicity n_μ. If an orbit $O^\pm(\mu)$ is contained in decomposition of $O(\lambda)\otimes O^\pm(\lambda')$ with negative coefficient n_μ, then the product $O_e(\lambda)\otimes O_e(\lambda')$ contains the W_e-orbit $O_e(r_\alpha\mu)$ with multiplicity $-n_\mu$. This procedure determines in the decomposition of $O_e(\lambda)\otimes O_e(\lambda')$ all W_e-orbits $O_e(\mu)$ and $O_e(r_\alpha\mu)$ for which μ is strictly positive and does not give W_e-orbits $O_e(\mu)$ with μ lying on a wall of the dominant Weyl chamber.

In order to find in the decomposition $O_e(\lambda)\otimes O_e(\lambda')$ the W_e-orbits $O_e(\mu)$ with μ lying on a wall of the dominant Weyl chamber, along with the product $O(\lambda)\otimes O^\pm(\lambda')$ we have to consider also the product $O(\lambda)\otimes O(\lambda')$ of W-orbit functions (which are not signed orbits).

Below we consider the case of products $O_e(\lambda)\otimes O_e(\lambda')$ of W_e-orbits, when λ and λ' do not lie on walls of even dominant Weyl chamber. For simplicity we consider the case when λ and λ' are strictly dominant.

Let $O_e(\lambda) = \{w\lambda | w\in W_e\}$ and $O_e(\mu) = \{w\mu | w\in W_e\}$ be two W_e-orbits with strictly dominant λ and μ. Then

$$\begin{aligned} O_e(\lambda)\otimes O_e(\mu) &= \{w\lambda + w'\mu | w, w'\in W_e\} \\ &= \{w\lambda+w_1\mu|w\in W_e\}\cup\{w\lambda+w_2\mu|w\in W_e\}\cup\cdots\cup\{w\lambda+w_s\mu|w\in W_e\}, \end{aligned} \tag{6.2}$$

where $w_1, w_2, \dots, w_s$ is the set of elements of W_e. Since a product of W_e-orbits is invariant with respect to W_e, for decomposition of the product $O_e(\lambda)\otimes O_e(\mu)$ into separate W_e-orbits it is necessary to separate from each term of the right hand side of (6.2) elements, which belong to the even Weyl chamber D_+^e. That is, $O_e(\lambda)\otimes O_e(\mu)$ is a union of the W_e-orbits, determined by points from

$$D_e(\{w\lambda + w_1\mu | w\in W_e\}),\ D(\{w\lambda + w_2\mu | w\in W_e\}),\dots, D_e(\{w\lambda + w_s\mu | w\in W_e\}), \tag{6.3}$$

where $D_e(\{w\lambda + w_i\mu | w\in W_e\})$ means the set of elements in the collection $\{w\lambda + w_i\mu | w\in W_e\}$ belonging to D_+^e.

Proposition 2. *Let λ and μ be elements of D_+^e, which do not lie on walls of D_+^e. The product $O_e(\lambda)\otimes O_e(\mu)$ consists only of W_e-orbits of the form $O_e(|w\lambda+\mu|)$, $w\in W_e$, where $|w\lambda+\mu|$ is an element of D_+^e in the W_e-orbit containing $w\lambda+\mu$. Moreover, each such W_e-orbit $O_e(|w\lambda+\mu|)$, $w\in W_e$, belongs to the product $O_e(\lambda)\otimes O_e(\mu)$.*

Proof. For each dominant element $w\lambda + w_i\mu$ from (6.3) there exists an element $w''\in W_e$ such that $w''(w\lambda + w_i\mu) = w'\lambda + \mu$. It means that $w\lambda + w_i\mu$ is of the form $|w'\lambda + \mu|$, $w'\in W_e$.

Conversely, take any element $w\lambda+\mu$, $w\in W_e$. It belongs to the product $O_e(\lambda)\otimes O_e(\mu)$. This means that $|w\lambda+\mu|$ also belongs to this product. Therefore, the orbit $O_e(|w\lambda+\mu|)$ is contained in $O_e(\lambda)\otimes O_e(\mu)$. Proposition is proved. ■

It follows from Proposition 2 that for decomposition of the product $O_e(\lambda)\otimes O_e(\mu)$ into separate orbits we have to take all elements $w\lambda+\mu$, $w\in W_e$, and to find the corresponding dominant elements $|w\lambda+\mu|$, $w\in W_e$.

For λ and μ from Proposition 2 the product $O_e(\lambda)\otimes O_e(\mu)$ contains W_e-orbits with multiplicities 1, that is,

$$O_e(\lambda)\otimes O_e(\mu)=\bigcup_{w\in W_e} O_e(|w\lambda+\mu|). \tag{6.4}$$

If λ and μ lie on walls of D_+^e, then some orbits can be contained in the decomposition of $O_e(\lambda)\otimes O_e(\mu)$ with a multiplicity. The most difficult problem under consideration of products of orbits is to find these multiplicities.

Formulas (6.2) and (6.3) are related to decompositions of products $O_e(\lambda)\otimes O_e(\mu)$, when λ and μ do not lie on walls of the domain D_+^e. Now we assume that λ or/and μ may lie on walls of D_+^e. Then formula (6.2) is replaced by

$$\begin{aligned} O_e(\lambda)\otimes O_e(\mu) &= \{w\lambda+w'\mu|w\in W_e/W_\lambda, w'\in W_e/W_\mu\}\\ &= \{w{+}w_1\mu|w{\in}W_e/W_\lambda\}\cup\{w\lambda{+}w_2\mu|w{\in}W_e/W_\lambda\}\cup\cdots\cup\{w\lambda{+}w_r\mu|w{\in}W_e/W_\lambda\}, \end{aligned} \tag{6.5}$$

where W_λ is the subgroup of W_e consisting of elements leaving λ invariant and $w_1,w_2,\dots,w_r$ is the set of elements of W_e/W_μ. In this case, $O_e(\lambda)\otimes O_e(\mu)$ is a union of the W_e-orbits, determined by points from

$$\begin{aligned} &D_e(\{w\lambda+w_1\mu|w\in W_e/W_\lambda\}),\ D(\{w\lambda+w_2\mu|w\in W_e/W_\lambda\}),\ \dots,\\ &\qquad D_e(\{w\lambda+w_r\mu|w\in W_e/W_\lambda\}), \end{aligned} \tag{6.6}$$

where $D_e(\{w\lambda+w_i\mu|w\in W_e\})$ has the same sense as in (6.3).

Proposition 3. *Let $O_e(\lambda)$ and $O_e(\mu)$, $\lambda,\mu\in D_+^e$, be W_e-orbits such that $\lambda\neq 0$ and $\mu\neq 0$, and let elements $w\lambda+\mu$, $w\in W_e/W_\lambda$, belong to D_e^+ and do not belong to walls of even Weyl dominant chamber D_+^e. Then*

$$O_e(\lambda)\otimes O_e(\mu)=\bigcup_{w\in W_e/W_\lambda} O_e(w\lambda+\mu). \tag{6.7}$$

Proof. Under the conditions of the proposition the set of elements $w\lambda+\mu$, $w\in W_e/W_\lambda$, is contained in the first set of (6.6) if w_1 coincides with the identical transformation. Moreover, $W_{w\lambda+\mu}\equiv\{w'\in W_e; w'(w\lambda+\mu)=w\lambda+\mu\}=\{1\}$ for all elements $w\in W_e$. Then μ does not lies on the boundary of D_e^+, that is, $W_\mu=\{1\}$. Let us show that the collection (6.6) contains only one non-empty set $D(\{w\lambda+\mu|w\in W_e/W_\lambda\})$. Indeed, let $w\lambda+w_i\mu$, $w_i\neq 1$, belongs to D_e^+. Then

$$w_i^{-1}(w\lambda+w_i\mu)=w_i^{-1}w\lambda+\mu\in D_+^e.$$

Since $W_{w_i^{-1}w\lambda+\mu}=\{1\}$, then $w_i^{-1}(w\lambda+w_i\mu)$ is an intrinsic point of D_+^e. Therefore, $w\lambda+w_i\mu\notin D_+^e$. This contradicts to the condition that $w\lambda+w_i\mu\in D_+^e$ and, therefore, the collection (6.6) contains only one non-empty set. The set of orbits, corresponding to the points of $D(\{w\lambda+\mu|w\in W_e/W_\lambda\})$, coincides with the right hand side of (6.7). Proposition is proved. ■

Proposition 4. *If $\lambda, \mu \in D_+^e$, then $O_e(\lambda) \otimes O_e(\mu)$ consists only of W_e-orbits of the form $O_e(|w\lambda + \mu|)$, $w \in W_e/W_\lambda$, where $|w\lambda + \mu|$ is an element of D_+^e in the W_e-orbit containing $w\lambda + \mu$. Moreover, each such W_e-orbit $O_e(|w\lambda + \mu|)$ belongs to the product $O_e(\lambda) \otimes O_e(\mu)$.*

Proof is similar to that of Proposition 2 and we omit it.

Under conditions of Proposition 4 the relation (6.4) in general is not true. The simplest counterexample is when $\mu = 0$. Then according to this formula $O_e(\lambda) \otimes O_e(\mu) = O_e(\lambda) \cup O_e(\lambda) \cup \cdots \cup O_e(\lambda)$ ($|W_e/W_\lambda$ times). However, as we know, $O_e(\lambda) \otimes O_e(\mu) = O_e(\lambda)$.

Proposition 3 states that instead of (6.4) we have

$$O_e(\lambda) \otimes O_e(\mu) \subseteq \bigcup_{w \in W_e} O_e(|w\lambda + \mu|). \tag{6.8}$$

Note that some orbits on the right hand side can coincide.

Proposition 5. *Let $O_e(\mu)$ and $O_e(\mu)$ be W_e-orbits such that $\lambda \neq 0$ and $\mu \neq 0$, and let all elements $w\lambda + \mu$, $w \in W_e$, belong to D_+^e. Then*

$$O_e(\lambda) \otimes O_e(\mu) = \bigcup_{w \in W/W_\lambda} n_{w\lambda+\mu} O_e(w\lambda + \mu),$$

where $n_{w\lambda+\mu} = |W_{w\lambda+\mu}|$.

Proof. Since $\lambda \neq 0$, all elements $w\lambda + \mu$, $w \in W_e/W_\lambda$, belong to D_e^+ if and only if $W_\mu = \{1\}$. Then on the right hand side of (6.5) there are $|W_e|$ terms. If the element $w\lambda + \mu$ belongs to D_e^+ and does not lie on a wall, that is, $W_{w\lambda+\mu} = \{1\}$, then this element is met only in one term. This means that the multiplicity $n_{w\lambda+\mu}$ of $O_e(w\lambda + \mu)$ in the product $O_e(\lambda) \otimes O_e(\mu)$ is 1. If $w\lambda + \mu$ is placed on some wall of D_+^e, then it is met in $n_{w\lambda+\mu} = |W_{w\lambda+\mu}|$ terms (since the elements $w'w\lambda + w'\mu$, $w' \in W_{w\lambda+\mu}$, belong to pairwise different terms in (6.5)). Therefore, there are $n_{w\lambda+\mu}$ orbits $O_e(w\lambda + \mu)$ in the decomposition of $O_e(\lambda) \otimes O_e(\mu)$. Proposition is proved. ■

Proposition 6. *If $W_\mu = \{1\}$ and none of the points $w\lambda + \mu$, $w \in W_e/W_\lambda$, lies on a wall of some even Weyl chamber, then*

$$O_e(\lambda) \otimes O_e(\mu) = \bigcup_{w \in W_e/W_\lambda} O_e(|w\lambda + \mu|).$$

Proof. For the product $O_e(\lambda) \times O_e(\mu)$ the inclusion

$$O_e(\lambda) \otimes O_e(\mu) \subseteq \bigcup_{w \in W_e} O_e(|w\lambda + \mu|) \tag{6.9}$$

holds. Each orbit $O_e(|w\lambda + \mu|)$, $w \in W_e/W_\lambda$, has $|W_e|$ elements and is contained in $O_e(\lambda) \otimes O_e(\mu)$. Therefore, numbers of elements in both sides of (6.9) coincide. This means that the inclusion (6.9) is in fact an equality. Proposition is proved. ■

We formulate a conjecture concerning decomposition of products of W_e-orbits.

Conjecture. *Let $O_e(\lambda)$ and $O_e(\mu)$, $\lambda \neq 0$, $\mu \neq 0$, be orbits, and let μ belong to D_e^+ and do not lie on a wall of D_+^e. If for some $w \in W_e$ the element $w\lambda + \mu$ belongs to D_e^+ and does not lie on a wall, then a multiplicity of $O_e(w\lambda + \mu)$ in $O_e(\lambda) \otimes O_e(\mu)$ is 1.*

At the end of this subsection we formulate a method for decomposition of products $O_e(\lambda) \otimes O_e(\mu)$, which follows from the statement of Proposition 4. On the first step we shift all points of the orbit $O_e(\lambda)$ by μ. As a result, we obtain the set of points $w\lambda + \mu$, $w \in W_e$. On the second step, we map elements of this set, which do not belong to D_e^+, by elements of the even Weyl group W_e to the chamber D_e^+. On this step we obtain the set $|w\lambda + \mu|$, $w \in W_e$. Then by Proposition 4, $O_e(\lambda) \otimes O_e(\mu)$ consists of the orbits $O_e(|w\lambda + \mu|)$. On the third step, we determine multiplicities of these orbits, taking into account the above propositions or making direct calculations.

6.3 Decomposition of products for A_2

We give examples of decompositions of products of W_e-orbits for the cases A_2 and C_2. Orbits for these cases are placed on a plane. Therefore, decompositions can be done by geometrical calculations on this plane. These cases can be easily considered also by using for orbit points orthogonal coordinates from Section 3. The corresponding even Weyl groups have a simple description in these coordinates and this gives a possibility to make calculations in a simple manner.

For the case of A_2 at $a \neq b$, $a > 0$, $b > 0$, and at $c > 0$ we have

$$
\begin{aligned}
A_2:\quad & O_e(a\ b) \otimes O_e(c\ 0) = O_e(a{+}c\ b) \cup O_e(a{-}c\ b{+}c) \\
& \qquad\qquad \cup O_e(-a{-}b{+}c\ a) && (a > c > b),\\
& O_e(a\ b) \otimes O_e(c\ 0) = O_e(a{+}c\ b) \cup O_e(a{-}c\ b{+}c) \cup O_e(a\ b{-}c) && (a > c, b > c),\\
& O_e(a\ b) \otimes O_e(c\ 0) = O_e(a{+}c\ b) \cup O(-a{-}b{+}c\ a)) \cup O_e(0\ a+b) && (a = c > b),\\
& O_e(a\ b) \otimes O_e(c\ 0) = O_e(a{+}c\ b) \cup O_e(a\ b{-}c) \cup O_e(0\ a+b) && (b > a = c),\\
& O_e(a\ b) \otimes O_e(c\ 0) = O_e(a{+}c\ b) \cup O_e(a{-}c\ b{+}c) \cup O_e(a\ 0) && (a < b = c),\\
& O_e(a\ b) \otimes O_e(c\ 0) = O_e(a{+}c\ b) \cup O_e(a{-}c\ b{+}c) \\
& \qquad\qquad \cup O_e(c{-}a{-}b\ a) && (c > a+b),\\
& O_e(a\ b) \otimes O_e(c\ 0) = O_e(a{+}c\ b) \cup O^{\pm}(a{-}c\ b{+}c) \\
& \qquad\qquad \cup O^{\pm}(-a{-}b{+}c\ b) && (a+b > c > b),\\
& O_e(a\ b) \otimes O_e(c\ 0) = O_e(a{+}c\ b) \cup O_e(a{-}c\ b{+}c) \cup O_e(a\ b{-}c) && (a+b > b > c),
\end{aligned}
$$

$$
\begin{aligned}
A_2:\quad & O_e(-a\ a{+}b) \otimes O_e(c\ 0) = O_e(-a{-}c\ a{+}b{+}c) \cup O_e(c{-}a\ a{+}b) \\
& \qquad\qquad \cup O_e(a{+}b{-}c\ c{-}b) && (a > c > b),\\
& O_e(-a\ a{+}b) \otimes O_e(c\ 0) = O_e(-a{-}c\ a{+}b{+}c) \cup O_e(c{-}a\ a{+}b) \\
& \qquad\qquad \cup O_e(-a\ a{+}b{-}c) && (a > c, b > c),\\
& O_e(-a\ a{+}b) \otimes O_e(c\ 0) = O_e(-a{-}c\ a{+}b{+}c) \cup O_e(a{+}b{-}c\ c{-}b)) \\
& \qquad\qquad \cup O_e(0\ a+b) && (a = c > b),\\
& O_e(-a\ a{+}b) \otimes O_e(c\ 0) = O_e(-a{-}c\ a{+}b{+}c) \cup O_e(-a\ a{+}b{-}c) \\
& \qquad\qquad \cup O_e(0\ a+b) && (b > a = c),\\
& O_e(-a\ a{+}b) \otimes O_e(c\ 0) = O_e(-a{-}c\ a{+}b{+}c) \cup O_e(c{-}a\ a{+}b) \\
& \qquad\qquad \cup O_e(a\ 0) && (a < b = c),\\
& O_e(-a\ a{+}b) \otimes O_e(c\ 0) = O_e(-a{-}c\ a{+}b{+}c) \cup O_e(c{-}a\ a{+}b) \\
& \qquad\qquad \cup O_e(a{+}b{-}c\ c{-}b) && (c > a+b),\\
& O_e(-a\ a{+}b) \otimes O_e(c\ 0) = O_e(-a{-}c\ a{+}b{+}c) \cup O_e(c{-}a\ a{+}b) \\
& \qquad\qquad \cup O_e(a{+}b{-}c\ c{-}a) && (a+b > c > b),\\
& O_e(-a\ a{+}b) \otimes O(c\ 0) = O_e(-a{-}c\ a{+}b{+}c) \cup O_e(c{-}a\ a{+}b) \\
& \qquad\qquad \cup O_e(-a\ a{+}b{-}c) && (a+b > b > c).
\end{aligned}
$$

Note that $(-a\ a{+}b)$ belongs to D_+^e and does not belong to D_+. If $a = b$, then we get

$$
\begin{aligned}
&O_e(a\ a) \otimes O_e(c\ 0) = O_e(a{+}c\ a) \cup O_e(c{-}2a\ a) \cup O_e(a{-}c\ a{+}c) \quad (c > 2a),\\
&O_e(a\ a) \otimes O_e(c\ 0) = O_e(a{+}c\ a) \cup O_e(c{-}2a\ a) \cup O_e(a{-}c\ a + c) \quad (2a > c > a),\\
&O_e(a\ a) \otimes O_e(c\ 0) = O_e(a{+}c\ a) \cup O_e(a{-}c\ a{+}c) \cup O_e(a\ a{-}c) \quad (a > c).
\end{aligned}
$$

The W_e-orbit $O_e(r_\alpha(a\ a))$ has the form $O_e(-a\ 2a)$. The products of such orbits with the W_e-orbit $O_e(c\ 0)$ decompose into W_e-orbits as

$$
\begin{aligned}
O_e(-a\ 2a) \otimes O_e(c\ 0) &= O_e(-a{-}c\ 2a + c) \cup O_e(2a{-}c\ c - a)\\
&\quad \cup O_e(c{-}a\ 2a) \qquad (c > 2a),\\
O_e(-a\ 2a) \otimes O_e(c\ 0) &= O_e(-a{-}c\ 2a + c) \cup O_e(2a{-}c\ c{-}a)\\
&\quad \cup O_e(c{-}a\ 2a) \qquad (2a > c > a),\\
O_e(-a\ 2a) \otimes O_e(c\ 0) &= O_e(-a{-}c\ 2a + c) \cup O_e(c{-}a\ 2a)\\
&\quad \cup O_e(-a\ 2a{-}c) \qquad (a > c).
\end{aligned}
$$

We also give the following decompositions for W_e-orbits of A_2:

$$
\begin{aligned}
A_2 :\quad &O_e(a\ 0) \otimes O_e(b\ 0) = O_e(a{+}b\ 0) \cup O_e(-a{+}b\ a) \cup O_e(a{-}b\ b) \quad (a < b),\\
&O_e(a\ 0) \otimes O_e(a\ 0) = O_e(2a\ 0) \cup 2O_e(0\ a),\\
&O_e(a\ 0) \otimes O_e(0\ b) = O_e(a\ b) \cup O_e(-a\ a{+}b) \cup O_e(0\ -a{+}b) \quad (a < b),\\
&O_e(a\ 0) \otimes O_e(0\ a) = O_e(a\ a) \cup O_e(-a\ 2a) \cup 3O_e(0\ 0).
\end{aligned}
$$

6.4 Decomposition of products for C_2

For dominant elements $(a\ b)$ products W_e-orbits for C_2 are of the form

$$
\begin{aligned}
C_2 :\quad O_e(a\ b) \otimes O_e(c\ 0) &= O_e(a{+}c\ b) \cup O_e(a{+}2b{-}c\ -a{-}b{+}c) \cup O_e(a{-}c\ b{+}c)\\
&\quad \cup O_e(c{-}2b{-}a\ a{+}b) \qquad (a{+}b{-}c{<}b),\\
O_e(a\ b) \otimes O_e(c\ 0) &= O_e(a{+}c\ b) \cup O_e(a{+}2b{-}c\ c{-}a{-}b) \cup O_e(a{-}c\ b{+}c)\\
&\quad \cup O_e(c{-}a{-}2b\ a{+}b) \qquad (b{>}c{-}a{-}b, a{>}c),\\
O_e(a\ b) \otimes O_e(c\ 0) &= O_e(a{+}c\ b) \cup O_e(a{+}2b{-}c\ c{-}a{-}b) \cup O_e(a{-}c\ b{+}c)\\
&\quad \cup O_e(c{-}a{-}2b\ a{+}b) \qquad (b{>}c{-}a{-}b, c{>}a),\\
O_e(a\ b) \otimes O_e(c\ 0) &= O_e(a{+}c\ b) \cup O_e(a{-}c\ b) \cup O_e(a{-}c\ b{+}c)\\
&\quad \cup O_e(c{-}a{-}2b\ a{+}b) \qquad (a{+}b{>}b{+}c),\\
O_e(a\ b) \otimes O_e(c\ 0) &= O_e(a{+}c\ b) \cup O_e(a{-}c\ b) \cup O_e(a{-}c\ b{+}c)\\
&\quad \cup O_e(c{-}a{-}2b\ a{+}b) \qquad (b{+}c{>}a{+}b{>}c{-}b),\\
O_e(a\ b) \otimes O_e(c\ 0) &= O_e(a{+}c\ b) \cup O_e(a{-}c\ b) \cup O_e(a{-}c\ b{+}c)\\
&\quad \cup O_e(c{-}a{-}2b\ a{+}b) \qquad (a{+}b{<}c{-}b).
\end{aligned}
$$

The W_e-orbit $O_e(r_\alpha(a\ b))$ has the form $O_e(-a\ a + b)$. The products of such orbits with the W_e-orbit $O_e(c\ 0)$ decompose into W_e-orbits as

$$
\begin{aligned}
C_2 :\quad O_e(-a\ a{+}b) \otimes O_e(c\ 0) &= O_e(-a{-}c\ a{+}b{+}c) \cup O_e(-a{-}2b{+}c\ b) \cup O_e(c{-}a\ a{+}b)\\
&\quad \cup O_e(a{+}2b{-}c\ b{+}c) \qquad (a{+}b{-}c{<}b),\\
O_e(-a\ a{+}b) \otimes O_e(c\ 0) &= O_e(-a{-}c\ a{+}b{+}c) \cup O_e(c{-}a{-}2b\ b) \cup O_e(c{-}a\ a{+}b)\\
&\quad \cup O_e(a{+}2b{-}c\ c{-}b) \qquad (b{>}c{-}a{-}b, a{>}c),
\end{aligned}
$$

$$
\begin{aligned}
O_e(-a\ a{+}b)\otimes O_e(c\ 0) &= O_e(-a{-}c\ a{+}b{+}c)\cup O_e(c{-}a{-}2b\ b)\cup O_e(c{-}a\ a{+}b)\\
&\quad \cup O_e(a{+}2b{-}c\ c{-}b) \qquad\qquad (b{>}c{-}a{-}b, c{>}a),\\
O_e(-a\ a{+}b)\otimes O_e(c\ 0) &= O_e(-a{-}c\ a{+}b{+}c)\cup O_e(c{-}a\ a{+}b{-}c)\cup O_e(c{-}a\ a{+}b)\\
&\quad \cup O_e(a{+}2b{-}c\ c{-}b) \qquad\qquad (a{+}b{>}b{+}c),\\
O_e(-a\ a{+}b)\otimes O_e(c\ 0) &= O_e(-a{-}c\ a{+}b{+}c)\cup O_e(c{-}a\ a{+}b{-}c)\cup O_e(c{-}a\ a{+}b)\\
&\quad \cup O_e(a{+}2b{-}c\ c{-}b) \qquad\qquad (b{+}c{>}a{+}b{>}c{-}b),\\
O_e(-a\ a{+}b)\otimes O_e(c\ 0) &= O_e(-a{-}c\ a{+}b{+}c)\cup O_e(c{-}a\ a{+}b{-}c)\cup O_e(c{-}a\ a{+}b)\\
&\quad \cup O_e(a{+}2b{-}c\ c{-}b) \qquad\qquad (a{+}b{<}c{-}b).
\end{aligned}
$$

We also give the following decompositions of E-orbits of C_2:

$$
\begin{aligned}
C_2:\quad & O_e(a\ 0)\otimes O_e(b\ 0) = O_e(a{+}b\ 0)\cup O_e(a{-}b\ 0)\cup O_e(a{-}b\ b)\cup O_e(b{-}a\ a) \qquad (a>b),\\
& O_e(a\ 0)\otimes O_e(a\ 0) = O_e(2a\ 0)\cup 2O_e(0\ 2a)\cup 4O_e(0\ 0),\\
& O_e(0\ a)\otimes O_e(0\ b) = O_e(0\ a{+}b)\cup O_e(2b\ a{-}b)\cup O_e(-2b\ a{+}b)\cup O_e(0\ a{-}b) \qquad (a>b),\\
& O_e(0\ a)\otimes O_e(0\ a) = O_e(0\ 2a)\cup 2O_e(2a\ 0)\cup 4O_e(0\ 0),\\
& O_e(a\ 0)\otimes O_e(0\ b) = O_e(a\ b)\cup O_e(-a\ a{+}b)\cup O_e(a{-}2b\ b)\cup O_e(2b{-}a\ a{-}b) \qquad (a>2b),\\
& O_e(a\ 0)\otimes O_e(0\ b) = O_e(a\ b)\cup O_e(-a\ a{+}b)\cup O_e(2b{-}a\ a{-}b)\cup O_e(a{-}2b\ b) \qquad (2b{>}a{>}b),\\
& O_e(a\ 0)\otimes O_e(0\ b) = O_e(a\ b)\cup O_e(-a\ a{+}b)\cup O(a\ b{-}a)\cup O(-a\ b) \qquad (b>a),\\
& O_e(a\ 0)\otimes O_e(0\ a) = O_e(a\ a)\cup O_e(-a\ 2a)\cup 2O_e(a\ 0),\\
& O_e(a\ 0)\otimes O_e(0\ 2a) = O_e(a\ 2a)\cup O_e(-a\ 3a)\cup O_e(a\ a)\cup O_e(-a\ 2a),\\
& O_e(2a\ 0)\otimes O_e(0\ a) = O_e(2a\ a)\cup O_e(-2a\ 3a)\cup 2O_e(0\ a).
\end{aligned}
$$

6.5 Decomposition of products for G_2

We give some examples of decomposition of products of W_e-orbits of G_2 using ω-coordinates for elements of orbits:

$$
\begin{aligned}
G_2:\quad & O_e(a\ 0)\otimes O_e(b\ 0) = O_e(a{+}b\ 0)\cup O(b{-}a\ 3a)\cup O(2a{+}b\ {-3a})\cup O(2a{-}b\ 3b{-}3a)\\
&\qquad \cup O(b{-}a\ 3a{-}3b)\cup O_e(b{-}a\ 0) \qquad\qquad (a<b<2a),\\
& O_e(a\ 0)\otimes O_e(b\ 0) = O_e(a{+}b\ 0)\cup O_e(b{-}a\ 3a)\cup O_e(2a{+}b\ {-3a})\cup O(b{-}2a\ 3a)\\
&\qquad \cup O_e(a{+}a\ {-3a})\cup O_e(b{-}a\ 0) \qquad\qquad (b>2a),\\
& O_e(a\ 0)\otimes O_e(a\ 0) = O_e(2a\ 0)\cup 2O_e(0\ 3a)\cup 2O_e(a\ 0)\cup 6O_e(0\ 0),\\
& O_e(a\ 0)\otimes O_e(2a\ 0) = O_e(3a\ 0)\cup O_e(a\ 0)\cup O_e(a\ 3a)\cup O_e(4a\ {-3a})\cup 2O_e(0\ 3a),\\
& O_e(0\ a)\otimes O_e(0\ b) = O_e(0\ a{+}b)\cup O_e(a\ b{-}a)\cup O_e(b\ a{-}b)\cup O_e(b{-}a\ 2a{-}b)\\
&\qquad \cup O_e(a\ b{-}2a)\cup O_e(0\ b{-}a) \qquad\qquad (a<b<2a),\\
& O_e(0\ a)\otimes O_e(0\ b) = O_e(0\ a{+}b)\cup O_e(0\ b{-}a)\cup O_e(a\ b{-}a)\cup O_e(b\ a{-}b)\\
&\qquad \cup O_e(a\ b{-}2a)\cup O_e(b{-}a\ 2a{-}b) \qquad\qquad (b>2a),\\
& O_e(0\ a)\otimes O_e(0\ a) = O_e(0\ 2a)\cup 2O_e(a\ 0)\cup 2O_e(0\ a)\cup 6O_e(0\ 0),\\
& O_e(0\ a)\otimes O_e(0\ 2a) = O_e(0\ 3a)\cup 2O_e(a\ 0)\cup O_e(a\ a)\cup O_e(2a\ {-a})\cup O_e(0\ a).
\end{aligned}
$$

7 Decomposition of W_e-orbit functions into W'_e-orbit functions

For these decompositions it is enough to obtain the corresponding decompositions for signed W-orbits and then to make a corresponding separation of W_e-orbits. For this reason, we shall deal mainly with signed orbits. Our reasoning here is similar to that of Section 4 in [7].

7.1 Introduction

Let R be a root system with a Weyl group W and let R' be another root system which is a subset of the set R. Then the Weyl group W' for R' can be considered as a subgroup of W. Moreover, W'_e is a subgroup of the even Weyl group W_e.

Let $O_e^W(\lambda)$ be a W_e-orbit. The set of points of $O_e^W(\lambda)$ is invariant with respect to W'_e. This means that the orbit $O_e^W(\lambda)$ consists of W'_e-orbits. In this section we deal with representing $O_e^W(\lambda)$ as a union of W'_e-orbits. Properties of such a representation depend on root systems R and R' (or on Weyl groups W and W'). We distinguish two cases:

Case 1. *Root systems R and R' span vector spaces of the same dimension.* In this case Weyl chambers for W are smaller than Weyl chambers for W'. Moreover, each Weyl chamber for W' consists of $|W/W'|$ chambers for W. Therefore, an even Weyl chamber for W'_e consists of $|W/W'| = |W_e/W'_e|$ even Weyl chambers of W_e. Let D_e^+ be an even dominant Weyl chamber for the root system R. Then the even dominant Weyl chamber for W'_e consists of W_e-chambers $w_iD_e^+$, $i = 1, 2, \ldots, k$, $k = |W/W'|$, where w_i, $i = 1, 2, \ldots, k$, are representatives of cosets in W_e/W'_e. If λ does not lie on any wall of the even dominant Weyl chamber D_e^+, then

$$O_e^W(\lambda) = \bigcup_{i=1}^{k} O_e^{W'}(w_i\lambda), \tag{7.1}$$

where $O_e^{W'}$ are W'_e-orbits.

Representing λ by coordinates in ω-basis it is necessary to take into account that coordinates of the same point in ω-bases related to the root systems R and R' are different. There exist matrices connecting coordinates in these different ω-bases (see [29]).

To the decomposition (7.1) there corresponds the following expansion for E-orbit functions:

$$E_\lambda^{(W)}(x) = \sum_{i=1}^{k} E_{w_i\lambda}^{(W')}(x).$$

Case 2. *Root systems R and R' span vector spaces of different dimensions.* This case is more complicated. In order to represent $O_e^W(\lambda)$ as a union of W'_e-orbits, it is necessary to project points μ of $O_e^W(\lambda)$ to the vector subspace $E_{n'}$ spanned by R' and to select in the set of these projected points dominant points with respect to the root system R'. Note that under projection, different points of $O_e^W(\lambda)$ can give the same point in $E_{n'}$. This leads to appearing of coinciding W'_e-orbits in a representation of $O_e^W(\lambda)$ as a union of W'_e-orbits.

Under expansion of an E^W-orbit function $E_\lambda^{(W)}(x)$ into $E^{(W')}$-orbit functions we have to consider $E_\lambda^{(W)}(x)$ on the subspace $E_{n'} \subset E_n$ and to take into account the corresponding decomposition of the orbit $O_e^W(\lambda)$. For this reason, below in this section we consider decomposition of W_e-orbits into W'_e-orbits. They uniquely determine the corresponding expansions for E-orbit functions.

7.2 Decomposition of $W_e(A_n)$-orbits into $W_e(A_{n-1})$-orbits

Below we shall consider decompositions for $W_e(A_n)$-orbits $O_e(\lambda)$ and $O_e(r_\alpha\lambda)$ such that λ is strictly dominant.

If λ is not strictly dominant, then $W_e(A_n)$-orbit $O_e(\lambda) \equiv O(\lambda)$ coincides with the $W(A_n)$-orbit $O(\lambda)$. In this case, in order to decompose $W_e(A_n)$-orbit $O_e(\lambda)$ into $W_e(A_{n-1})$-orbits we have to decompose the $W(A_n)$-orbit $O(\lambda)$ into $W(A_{n-1})$-orbits and then to split $W(A_{n-1})$-orbits into $W_e(A_{n-1})$-orbits. Namely, if a $W(A_{n-1})$-orbit $O(\mu)$ is such that μ is not strictly dominant, then

$O(\mu)$ is in fact a $W_e(A_{n-1})$-orbit. If μ is strictly dominant, then $O(\mu)$ consists of two $W_e(A_{n-1})$-orbits $O_e(\mu)$ and $O(r_\alpha\mu)$, where α is a root of A_{n-1}. Thus, if λ is not strictly dominant, then decomposition of $W_e(A_n)$-orbits into $W_e(A_{n-1})$-orbits are reduced to decomposition of $W(A_n)$-orbits into $W(A_{n-1})$-orbits. The last decomposition are studied in Subsection 4.5 in [7].

So, let λ be a strictly dominant element for A_n. It is convenient to fulfil the decomposition of $W_e(A_n)$-orbits $O_e(\lambda)$ and $O_e(r_\alpha\lambda)$ by using a corresponding decomposition of a signed $W(A_n)$-orbit $O^\pm(\lambda)$ into signed $W(A_{n-1})$-orbits considered in Subsection 8.2 in [8]. For this we have to take into account that a signed $W(A_n)$-orbit $O^\pm(\lambda)$ consists of two $W_e(A_n)$-orbits. One of them consists of points with the sign $+$ and the second with the sign $-$.

For such decomposition it is convenient to represent orbit elements in orthogonal coordinates (see Section 3). Let $m_1, m_2, \ldots, m_{n+1}$ be orthogonal coordinates of a strictly dominant element λ, that is,

$$m_1 > m_2 > \cdots > m_n > m_{n+1}.$$

The orthogonal coordinates $m_1, m_2, \ldots, m_{n+1}$ satisfy the conditions $m_1 + m_2 + \cdots + m_{n+1} = 0$. However, we may add to all coordinates m_i the same real number, since under this procedure the ω-coordinates $\lambda_i = m_i - m_{i+1}$, $i = 1, 2, \ldots, n$ do not change (see Section 3).

Let $O^\pm(\lambda) \equiv O^\pm(m_1, m_2, \ldots, m_{n+1})$ be a signed $W(A_n)$-orbit with dominant element $\lambda = (m_1, m_2, \ldots, m_{n+1})$. This orbit consists of all points

$$w(m_1, m_2, \ldots, m_{n+1}) = (m_{i_1}, m_{i_2}, \ldots, m_{i_{n+1}}), \qquad w \in W(A_n), \tag{7.2}$$

where $(i_1, i_2, \ldots, i_{n+1})$ is a permutation of the numbers $1, 2, \ldots, n+1$, determined by w. The sign of $(\det w)$ is attached to such a point. Points of $O^\pm(\lambda)$ belong to the Euclidean space E_{n+1}. We restrict these points to the subspace E_n, spanned by the simple roots $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ of A_n, which form a set of simple roots of A_{n-1}. This restriction is reduced to removing the last coordinate $m_{i_{n+1}}$ in points $(m_{i_1}, m_{i_2}, \ldots, m_{i_{n+1}})$ of the signed orbit $O^\pm(\lambda)$ (see (7.2)). As a result, we obtain the set of points

$$(m_{i_1}, m_{i_2}, \ldots, m_{i_n}) \tag{7.3}$$

received from the points (7.2). The point (7.3) is dominant if and only if

$$m_{i_1} \geq m_{i_2} \geq \cdots \geq m_{i_n}.$$

It is easy to see that after restriction to E_n (that is, under removing the last coordinate) we obtain from the set of points (7.2) the following set of dominant elements:

$$(m_1, \ldots, m_{i-1}, \hat{m}_i, m_{i+1}, \ldots, m_{n+1}), \qquad i = 1, 2, \ldots, n+1,$$

where a hat over m_i means that the coordinate m_i must be omitted.

Thus, the signed $W(A_n)$-orbit $O^\pm(m_1, m_2, \ldots, m_{n+1})$ consists of the following signed $W(A_{n-1})$-orbits:

$$O^\pm(m_1, \ldots, m_{i-1}, \hat{m}_i, m_{i+1}, \ldots, m_{n+1}), \qquad i = 1, 2, \ldots, n+1.$$

Each of these signed orbits must be taken with a coefficient $+1$ or -1. Moreover, a coefficient at the orbit $O^\pm(m_1, \ldots, m_{i-1}, \hat{m}_i, m_{i+1}, \ldots, m_{n+1})$ is 1, if after $\hat{m}_i$ in the point

$$(m_1, \ldots, m_{i-1}, \hat{m}_i, m_{i+1}, \ldots, m_{n+1})$$

there exists an even number of coordinates, and -1 otherwise (see Section 8.2 in [8]). These statements can be written in the form

$$O^\pm_W(m_1, m_2, \ldots, m_{n+1}) = \bigcup_{i=1}^{n+1} (\det w(m_i)) O^\pm_{W'}(m_1, \ldots, m_{i-1}, \hat{m}_i, m_{i+1}, \ldots, m_{n+1}), \tag{7.4}$$

where $w(m_i)$ is the permutation which send the coordinate m_i to the end, not changing an order of other coordinates.

Now we have to split the left and the right hand sides of (7.4) into two parts: the first part has to consist of points with the sign plus and the second with the sign minus. This splitting for the left hand side leads to two $W_e(A_n)$-orbits $O_e(\lambda)$ and $O_e(r_\alpha\lambda)$. Each signed $W(A_{n-1})$-orbit on the right hand side splits into two $W(A_{n-1})$-orbits, one contains points with the sign plus and other with the sign minus; one of them is contained in $O_e(\lambda)$ and another in $O_e(r_\alpha\lambda)$. If $\det w(m_i) = 1$, then the $W_e(A_{n-1})$-orbit of $O^{\pm}_{W'}(m_1, \dots, m_{i-1}, \hat{m}_i, m_{i+1}, \dots, m_{n+1})$ with points having the sign plus belongs to $O_e(\lambda)$. The $W_e(A_{n-1})$-orbit with points having the sign minus belongs to $O_e(r_\alpha\lambda)$. If $\det w(m_i) = -1$, then the $W_e(A_{n-1})$-orbit of $O^{\pm}_{W'}(m_1, \dots, m_{i-1}, \hat{m}_i, m_{i+1}, \dots, m_{n+1})$ with points having the sign plus belongs to $O_e(r_\alpha\lambda)$ and the $W_e(A_{n-1})$-orbit with points having the sign minus belongs to $O_e(\lambda)$. Fulfilling this splitting we obtain lists of $W_e(A_{n-1})$-orbits, which are contained in $W_e(A_n)$-orbits $O_e(\lambda)$ and $O_e(r_\alpha\lambda)$.

7.3 Decomposition of $W_e(B_n)$-orbits into $W_e(B_{n-1})$-orbits and of $W_e(C_n)$-orbits into $W_e(C_{n-1})$-orbits

Decomposition of $W_e(B_n)$-orbits and decomposition of $W_e(C_n)$-orbits are fulfilled in the same way. For this reason, we give a proof only for the case of $W_e(C_n)$-orbits. As in the previous case for fulfilling the decompositions we use signed W-orbits.

A set of simple roots of C_n consists of roots $\alpha_1, \alpha_2, \dots, \alpha_n$. The roots $\alpha_2, \dots, \alpha_n$ constitute a set of simple roots of C_{n-1}. They span the subspace E_{n-1}.

We determine elements λ of E_n by using orthogonal coordinates $m_1, m_2, \dots, m_n$. Then λ is strictly dominant if and only if

$$m_1 > m_2 > \cdots > m_n > 0.$$

Then the signed $W(C_n)$-orbit $O^{\pm}(\lambda)$ consists of all points

$$w(m_1, m_2, \dots, m_n) = (\pm m_{i_1}, \pm m_{i_2}, \dots, \pm m_{i_n}), \qquad w \in W(C_n), \tag{7.5}$$

where $(i_1, i_2, \dots, i_n)$ is a permutation of the set $1, 2, \dots, n$, and all combinations of signs are possible.

Restriction of elements (7.5) to the vector subspace E_{n-1}, defined above, reduces to removing the first coordinate $\pm m_{i_1}$ in (7.5). As a result, we obtain from the set of points (7.5) the collection

$$(\pm m_{i_2}, \pm m_{i_3}, \dots, \pm m_{i_n}), \qquad w \in W(C_n).$$

Only the points $(m_{i_2}, m_{i_3}, \dots, m_{i_{n-1}}, m_{i_n})$ with positive coordinates may be dominant. Moreover, such a point is dominant if and only if

$$m_{i_2} > m_{i_3} > \cdots > m_{i_n}.$$

Therefore, under restriction of points (7.5) to E_{n-1} we obtain the following strictly $W(C_{n-1})$-dominant elements:

$$(m_1, m_2, \dots, m_{i-1}, \hat{m}_i, m_{i+1}, \dots, m_n), \qquad i = 1, 2, \dots, n, \tag{7.6}$$

where a hat over m_i means that the coordinate m_i must be omitted. Moreover, the element (7.6) with fixed i can be obtained from two elements in (7.5), namely, from $(m_1, m_2, \dots, m_{i-1}, \pm m_i, m_{i+1}, \dots, m_n)$. In the signed orbit $O^{\pm}(m_1, m_2, \dots, m_n)$ these two elements have opposite signs.

Thus, the signed $W(C_n)$-orbits $O^{\pm}(m_1, m_2, \dots, m_n)$ consists of the following signed $W(C_{n-1})$-orbits:

$$\begin{aligned} &O^{\pm}(m_1, m_2, \dots, m_{i-1}, \hat{m}_i, m_{i+1}, \dots, m_n) \\ &\qquad \equiv O^{\pm}(m_1, \dots, \hat{m}_i, \dots, m_n), \qquad i = 1, 2, \dots, n. \end{aligned} \tag{7.7}$$

Each such signed $W(C_{n-1})$-orbit is contained in $O^{\pm}(m_1, m_2, \dots, m_n)$ twice with opposite signs. Since the signed $W(C_n)$-orbit $O^{\pm}(m_1, m_2, \dots, m_n)$ consists of two $W_e(C_n)$-orbits (one consists of points with the sign $+$ and the second with the sign $-$), then it follows from these assertions that *the $W_e(C_n)$-orbits $O_e(m_1, m_2, \dots, m_n)$ and $O_e(r_\alpha(m_1, m_2, \dots, m_n))$ consist of the same set of $W_e(C_{n-1})$-orbits*, namely, of $W_e(C_{n-1})$-orbits which are contained in the signed $W(C_{n-1})$-orbits (7.7). This set consists of the $W_e(C_{n-1})$-orbits $O^{\pm}(m_1, , \dots, \hat{m}_i, \dots, m_n)$ and $O^{\pm}(r_\beta(m_1, , \dots, \hat{m}_i, \dots, m_n))$, $i = 1, 2, \dots, n$, where β is a root of C_{n-1}.

For $W_e(B_n)$-orbits we have similar assertions. A $W_e(B_n)$-orbit $O_e(m_1, m_2, \dots, m_n)$, $m_1 > m_2 > \cdots > m_n > 0$, consists of $W_e(B_{n-1})$-orbits which are contained in the signed $W^{\pm}(B_n)$-orbit

$$O^{\pm}(m_1, m_2, \dots, m_{i-1}, \hat{m}_i, m_{i+1}, \dots, m_n), \qquad i = 1, 2, \dots, n,$$

and each such $W_e(B_{n-1})$-orbit is contained in the decomposition only once.

7.4 Decomposition of $W_e(D_n)$-orbits into $W_e(D_{n-1})$-orbits

Assume that $\alpha_1, \alpha_2, \dots, \alpha_n$ is the set of simple roots of D_n, $n > 4$. Then $\alpha_2, \dots, \alpha_n$ are simple roots of D_{n-1}. The last roots span the subspace E_{n-1}.

For elements λ of E_n we use orthogonal coordinates $m_1, m_2, \dots, m_n$. Then λ is strictly dominant if and only if $m_1 > m_2 > \cdots > m_{n-1} > |m_n|$. We assume that λ satisfies the condition

$$m_1 > m_2 > \cdots > m_n > 0.$$

Then the signed $W(D_n)$-orbit $O^{\pm}(\lambda)$ consists of all points

$$w(m_1, m_2, \dots, m_n) = (\pm m_{i_1}, \pm m_{i_2}, \dots, \pm m_{i_n}), \qquad w \in W_{D_n}, \tag{7.8}$$

where $(i_1, i_2, \dots, i_n)$ is a permutation of the numbers $1, 2, \dots, n$ and there exists an even number of signs $-$. Restriction of elements (7.8) to the subspace E_{n-1} reduces to removing the first coordinate $\pm m_{i_1}$ in (7.8). As a result, we obtain from the set of points (7.8) the collection

$$(\pm m_{i_2}, \pm m_{i_3}, \dots, \pm m_{i_n}), \qquad w \in W(D_n),$$

where a number of signs $-$ may be either even or odd. Only points of the form $(m_{i_2}, m_{i_3}, \dots, m_{i_{n-1}}, \pm m_{i_n})$ may be dominant. Moreover, such a point is dominant if and only if

$$m_{i_2} > m_{i_3} > \cdots > m_{i_{n-1}} > |m_{i_n}|.$$

Therefore, under restriction of points (7.8) to E_{n-1} we obtain the following $W(D_{n-1})$-dominant elements:

$$(m_1, m_2, \dots, m_{i-1}, \hat{m}_i, m_{i+1}, \dots, m_{n-1}, \pm m_n), \qquad i = 1, 2, \dots, n, \tag{7.9}$$

where a hat over m_i means that the coordinate m_i must be omitted. Moreover, the element (7.9) with fixed i can be obtained only from one element in (7.8), namely, from element $(m_1, m_2, \dots, m_{i-1}, \pm m_i, m_{i+1}, \dots, \pm m_n)$, where at m_i and m_n signs are coinciding.

Thus, the signed $W(D_n)$-orbit $O^{\pm}(m_1, m_2, \ldots, m_n)$ with $m_1 > m_2 > \cdots > m_n > 0$ consists of the following signed $W(D_{n-1})$-orbits:

$$O^{\pm}(m_1, m_2, \ldots, m_{i-1}, \hat{m}_i, m_{i+1}, \ldots, \pm m_n), \qquad i = 1, 2, \ldots, n. \tag{7.10}$$

Each such signed $W(D_{n-1})$-orbit is contained in $O^{\pm}(m_1, m_2, \ldots, m_n)$ only once (with sign $+$ or sign $-$). A sign of such an orbit depends on a number i and does not depend on a sign at m_n. This sign is $+$ if after $\hat{m}_i$ in (7.10) there exists on even number of coordinates and -1 otherwise.

Now we split the signed $W(D_n)$-orbit $O^{\pm}(m_1, m_2, \ldots, m_n)$ into two $W_e(D_n)$-orbits $O_e(m_1, m_2, \ldots, m_n)$ and $O_e(r_\alpha(m_1, m_2, \ldots, m_n))$. Next, we split each of the signed $W(D_{n-1})$-orbits (7.10) also into two $W_e(D_{n-1})$-orbits

$$O_e(m_1, m_2, \ldots, m_{i-1}, \hat{m}_i, m_{i+1}, \ldots, \pm m_n), \qquad i = 1, 2, \ldots, n. \tag{7.11}$$

$$O^e(r_\beta(m_1, m_2, \ldots, m_{i-1}, \hat{m}_i, m_{i+1}, \ldots, \pm m_n)), \qquad i = 1, 2, \ldots, n. \tag{7.12}$$

It is necessary to split these $W_e(D_{n-1})$-orbits into two parts which constitute the $W_e(D_n)$-orbits $O_e(m_1, m_2, \ldots, m_n)$ and $O_e(r_\alpha(m_1, m_2, \ldots, m_n))$. This is done as follows. If a fixed signed orbit from (7.10) is contained in the signed $W(D_n)$-orbit $O^{\pm}(m_1, m_2, \ldots, m_n)$ with the sign $+$, then the corresponding $W_e(D_{n-1})$-orbit (7.11) is contained in the $W_e(D_n)$-orbit $O_e(m_1, m_2, \ldots, m_n)$ and the $W_e(D_{n-1})$-orbit (7.12) is contained in $O_e(r_\alpha(m_1, m_2, \ldots, m_n))$. If a fixed signed $W(D_{n-1})$-orbit (7.10) is contained in $O^{\pm}(m_1, m_2, \ldots, m_n)$ with the sign $-$, then the corresponding $W_e(D_{n-1})$-orbit (7.11) is contained in the $W_e(D_n)$-orbit $O_e(r_\alpha(m_1, m_2, \ldots, m_n))$ and the $W_e(D_{n-1})$-orbit (7.12) is contained in $O_e(m_1, m_2, \ldots, m_n)$. Therefore, the $W_e(D_n)$-orbit $O_e(m_1, m_2, \ldots, m_n)$ consists of $W_e(D_{n-1})$-orbit (7.11) such that $n - i$ is an even integer and of $W_e(D_{n-1})$-orbit (7.12) such that $n - i$ is odd. The $W_e(D_n)$-orbit $O_e(r_\alpha(m_1, m_2, \ldots, m_n))$ consists of $W_e(D_{n-1})$-orbit (7.11) such that $n - i$ is odd and of $W_e(D_{n-1})$-orbit (7.12) such that $n - i$ is even.

It is shown similarly that *the $W_e(D_n)$-orbits*

$$O_e(m_1, \ldots, m_{n-1}, -m_n), \qquad O_e(r_\alpha(m_1, \ldots, m_{n-1}, -m_n)),$$

with $m_1 > m_2 > \cdots > m_n > 0$ consists of the same $W_e(D_{n-1})$-orbits as the $W_e(D_n)$-orbits $O_e(m_1, \ldots, m_{n-1}, m_n)$ and $O_e(r_\alpha(m_1, \ldots, m_{n-1}, m_n))$ with the same numbers $m_1, \ldots, m_{n-1}, m_n$ do, respectively.

8 E-orbit function transforms

As in the case of symmetric and antisymmetric orbit functions, E-orbit functions determine certain orbit function transforms which generalize the Fourier transform (in the case of symmetric orbit functions these transforms generalize the cosine transform and in the case of antisymmetric orbit functions these transforms generalize the sine transform) [7, 8, 10].

As in the case of symmetric and antisymmetric orbit functions, E-orbit functions determine three types of orbit function transforms: the first one is related to the E-orbit functions $E_\lambda(x)$ with integral λ, the second one is related to $E_\lambda(x)$ with real values of coordinates of λ, and the third one is the related discrete transform.

8.1 Expansion in E-orbit functions on F_e

The aim of this subsection is to obtain formulas for expansions of functions on the closure of the fundamental domain F_e of the even affine Weyl group W_e^{aff} in E-orbit functions $E_\lambda(x)$ with integral λ.

Let us start with the usual Fourier expansion of functions on E_n,

$$f(x) = \sum_{\lambda\in\mathbb{Z}^n} c_\lambda e^{2\pi\langle\lambda,x\rangle}. \tag{8.1}$$

with coefficients

$$c_\lambda = \int_{x\in\mathsf{T}} f(x)e^{-2\pi\langle\lambda,x\rangle}dx, \tag{8.2}$$

where T is a torus in E_n.

Let the function $f(x)$ be invariant with respect to the even Weyl group W_e. It is easy to check that the coefficients c_λ are also W_e-invariant, $c_{w\lambda} = c_\lambda$, $w \in W_e$. Replace in (8.2) λ by $w\lambda$, $w \in W_e$, and sum up both side of (8.2) over $w \in W_e$. Then instead of (8.2) we obtain

$$c_\lambda = |W_e|^{-1}\int_{\mathsf{T}} f(x)\hat{E}_\lambda(x)dx,$$

where $\hat{E}_\lambda(x) = \sum\limits_{w\in W_e} e^{2\pi\mathrm{i}\langle w\lambda,x\rangle}$. (We have taken into account that both $f(x)$ and $E_\lambda(x)$ are W_e-invariant.) This formula can be written as

$$c_\lambda = \int_{\overline{F_e}} f(x)\hat{E}_\lambda(x)dx = |W_\lambda|\int_{\overline{F_e}} f(x)E_\lambda(x)dx, \tag{8.3}$$

where W_λ is the subgroup of W_e consisting of elements leaving λ invariant.

Similarly, starting from (8.1), we obtain an inverse formula:

$$f(x) = \sum_{\lambda\in P_e^+} c_\lambda\overline{E_\lambda(x)}, \tag{8.4}$$

where P_e^+ is the set of integral elements from D_e^+. For the transforms (8.3) and (8.4) the Plancherel formula

$$\int_{\overline{F_e}} |f(x)|^2dx = \sum_{\lambda\in P_e^+} |W_\lambda|^{-1}|c_\lambda|^2$$

holds, which means that the Hilbert spaces with the appropriate scalar products are isometric.

Formula (8.4) is the symmetrized (by means of the group W_e) Fourier transform of the function $f(x)$. Formula (8.3) gives an inverse transform. These formulas give the *E-orbit function transforms* corresponding to E-orbit functions E_λ, $\lambda \in P_e^+$.

Let $\mathcal{L}^2(F_e)$ denote the Hilbert space of functions on the closure of the fundamental domain F_e of the group $W_e^{\rm aff}$ with the scalar product

$$\langle f_1, f_2\rangle = \int_{\overline{F_e}} f_1(x)\overline{f_2(x)}dx.$$

The formulas (8.3) and (8.4) show that *the set of E-orbit functions E_λ, $\lambda \in P_e^+$, form an orthogonal basis of $\mathcal{L}^2(F_e)$.*

8.2 E-orbit function transform on even dominant Weyl chamber

The expansion (8.4) of functions on the domain $\overline{F_e}$ is an expansion in the E-orbit functions $E_\lambda(x)$ with integral elements λ. The E-orbit functions $E_\lambda(x)$ with λ lying in the even dominant Weyl chamber (and not obligatory integral) are not invariant with respect to the corresponding even affine Weyl group $W_e^{\rm aff}$. They are invariant only with respect to the even Weyl group W_e. A fundamental domain of W_e coincides with the even dominant Weyl chamber $D_e^+ = D_+ \cup r_\alpha D_+$. For this reason, the E-orbit functions $E_\lambda(x)$, $\lambda \in D_e^+$, determine another orbit function transform (a transform on D_e^+).

We began with the usual Fourier transforms on $\mathbb{R}^n$:

$$\tilde{f}(\lambda) = \int_{\mathbb{R}^n} f(x) e^{2\pi {\rm i}\langle\lambda,x\rangle} dx, \tag{8.5}$$

$$f(x) = \int_{\mathbb{R}^n} \tilde{f}(\lambda) e^{-2\pi {\rm i}\langle\lambda,x\rangle} d\lambda. \tag{8.6}$$

Let the function $f(x)$ be invariant with respect to the even Weyl group W_e, that is, $f(wx) = (\det w) f(x)$, $w \in W_e$. The function $\tilde{f}(\lambda)$ is also invariant with respect to the even Weyl group W_e. Replace in (8.5) λ by $w\lambda$, $w \in W_e$, and sum up these both side over $w \in W_e$. Then instead of (8.5) we obtain

$$\tilde{f}(\lambda) = |W_e|^{-1} \int_{\mathbb{R}^n} f(x) \hat{E}_\lambda(x) dx, \qquad \lambda \in D_e^+,$$

Therefore,

$$\tilde{f}(\lambda) = \int_{D_e^+} f(x) \hat{E}_\lambda(x) dx, \qquad \lambda \in D_e^+, \tag{8.7}$$

where we have taken into account that $f(x)$ is invariant with respect to W_e.

Similarly, starting from (8.6), we obtain the inverse formula:

$$f(x) = \int_{D_e^+} \tilde{f}(\lambda) \overline{\hat{E}_\lambda(x)} d\lambda. \tag{8.8}$$

For the transforms (8.7) and (8.8) the Plancherel formula

$$\int_{D_e^+} |f(x)|^2 dx = \int_{D_e^+} |\tilde{f}(\lambda)|^2 d\lambda$$

holds.

9 Finite *E*-orbit function transforms

9.1 Introduction

It is possible to introduce finite E-orbit function transforms, based on E-orbit functions (see [10]). It is done in the same way as in the case of symmetric orbit functions in [7] by using the results of paper [11]. Finite E-orbit function transforms are generalizations of the finite (discrete) Fourier transforms, which are defined as follows.

Let us fix a positive integer N and consider the numbers

$$e_{mn} := N^{-1/2} \exp(2\pi {\rm i} mn/N), \qquad m, n = 1, 2, \dots, N. \tag{9.1}$$

The matrix $(e_{mn})_{m,n=1}^N$ is unitary, that is,

$$\sum_k e_{mk} \overline{e_{nk}} = \delta_{mn}, \qquad \sum_k e_{km} \overline{e_{kn}} = \delta_{mn}. \tag{9.2}$$

Let $f(n)$ be a function of $n \in \{1, 2, \dots, N\}$. We may consider the transform

$$\sum_{n=1}^{N} f(n)e_{mn} \equiv N^{-1/2} \sum_{n=1}^{N} f(n) \exp(2\pi \mathrm{i} mn/N) = \tilde{f}(m). \tag{9.3}$$

Then due to unitarity of the matrix $(e_{mn})_{m,n=1}^{N}$, we express $f(n)$ as a linear combination of conjugates of the functions (9.1):

$$f(n) = N^{-1/2} \sum_{m=1}^{N} \tilde{f}(m) \exp(-2\pi \mathrm{i} mn/N). \tag{9.4}$$

The function $\tilde{f}(m)$ is a *finite Fourier transform* of $f(n)$. This transform is a linear map. The formula (9.4) gives an inverse transform. The Plancherel formula

$$\sum_{m=1}^{N} |\tilde{f}(m)|^2 = \sum_{n=1}^{N} |f(n)|^2$$

holds for transforms (9.3) and (9.4). This means that the finite Fourier transform conserves the norm introduced on the space of functions on $\{1, 2, \dots, N\}$.

The finite Fourier transform on the r-dimensional linear space E_r is defined similarly. We again fix a positive integer N. Let $\mathbf{m} = (m_1, m_2, \dots, m_r)$ be an r-tuple of integers such that each m_i runs over the integers $1, 2, \dots, N$. Then the finite Fourier transform on E_r is given by the kernel

$$e_{\mathbf{mn}} := e_{m_1 n_1} e_{m_2 n_2} \cdots e_{m_r n_r} = N^{-r/2} \exp(2\pi \mathrm{i} \mathbf{m} \cdot \mathbf{n}/N),$$

where $\mathbf{m} \cdot \mathbf{n} = m_1 n_1 + m_2 n_2 + \cdots + m_r n_r$. If $F(\mathbf{m})$ is a function of r-tuples $\mathbf{m}$, $m_i \in \{1, 2, \dots, N\}$, then the finite Fourier transform of F is given by

$$\tilde{F}(\mathbf{n}) = N^{-r/2} \sum_{\mathbf{m}} F(\mathbf{m}) \exp(2\pi \mathrm{i} \mathbf{m} \cdot \mathbf{n}/N).$$

The inverse transform is

$$F(\mathbf{m}) = N^{-r/2} \sum_{\mathbf{n}} \tilde{F}(\mathbf{n}) \exp(-2\pi \mathrm{i} \mathbf{m} \cdot \mathbf{n}/N).$$

The corresponding Plancherel formula is of the form $\sum_{\mathbf{m}} |F(\mathbf{m})|^2 = \sum_{\mathbf{n}} |\tilde{F}(\mathbf{n})|^2$.

9.2 Grids on the fundamental domain F_e

In order to determine an analogue of the finite Fourier transform, based on E-orbit functions, we need an analogue of the set

$$\{\mathbf{m} = \{m_1, m_2, \dots, m_n\} \mid m_i \in \{1, 2, \dots, N\}\},$$

used for multidimensional finite Fourier transform. Such a set has to be invariant with respect to the even Weyl group W_e (see [11]).

We know that the coroot lattice $Q^\vee$ is a discrete W-invariant subset of E_n. Clearly, the set $\frac{1}{m}Q^\vee$ is also W-invariant, where m is a fixed positive integer. Then the set

$$T_m = \tfrac{1}{m}Q^\vee/Q^\vee$$

is finite and W-invariant. If $\alpha_1, \alpha_2, \dots, \alpha_l$ is the set of simple root for the Weyl group W, then T_m can be identified with the set of elements

$$m^{-1} \sum_{i=1}^{l} d_i \alpha_i^\vee, \qquad d_i = 0, 1, 2, \dots, m-1. \tag{9.5}$$

We select from T_m the set of elements which belongs to the closure $\overline{F}_e$ of the fundamental domain F_e. These elements lie in the collection $\frac{1}{m}Q^\vee \cap \overline{F}_e$.

Let $\mu \in \frac{1}{m}Q^\vee \cap \overline{F}_e$ be an element determining an element of T_m and let M be the least positive integer such that $M\mu \in P^\vee$. Then there exists the least positive integer N such that $N\mu \in Q^\vee$. One has $M|N$ and $N|m$.

The collection of points of T_m belonging to $\overline{F}$ (we denote the set of these points by F_M), where F is the fundamental domain of the Weyl group W, coincides with the set of elements

$$s = \tfrac{s_1}{M}\omega_1^\vee + \dots + \tfrac{s_l}{M}\omega_l^\vee, \qquad \omega_i^\vee = \frac{2\omega_i}{\langle \alpha_i, \alpha_i \rangle}, \tag{9.6}$$

where $s_1, s_2, \dots, s_l$ runs over values from $\{0, 1, 2, \dots\}$ and satisfy the following condition: there exists a non-negative integer s_0 such that

$$s_0 + \sum_{i=1}^{l} s_i m_i = M, \tag{9.7}$$

where $m_1, m_2, \dots, m_l$ are non-negative integers from formula (2.8) (see [11]). (Values of m_i for all simple Lie algebras can be found in Subsection 2.4.)

To every positive integer M there corresponds the grid F_M of points (9.6) in $\overline{F}$ which corresponds to some set T_m such that $M|m$. The precise relation between M and m can be defined by the grid F_M (see [10]) . Acting upon the grid F_M by elements of the Weyl group W we obtain the whole set T_m. Below, we are interested in grids F_M and do not need the corresponding numbers m.

For studying finite E-orbit function transforms we need grids F_M^e such that T_m is obtained by action by elements of W_e^{aff}. In order to obtain F_M^e we fix a positive root α and construct the the set $F_M \cup r_\alpha F_M$. The set $F_M \cap r_\alpha F_M$ can be non-empty. Taking each point from $F_M \cup r_\alpha F_M$ only once we obtain the grid F_M^e. The set $\cup_{w \in W_e^{\mathrm{aff}}} w F_M^e$ coincides with T_m, where some points are taken several times. The set F_M^e depends on a choice of a root α.

9.3 Grids F_M^e for A_2, C_2 and G_2

In this section we give some examples of grids F_M^e for the rank two cases (see [9]). Since the long root ξ of A_2 is representable in the form $\xi = \alpha_1 + \alpha_2$, where α_1 and α_2 are simple roots, that is, $m_1 = m_2 = 1$ (see formula (9.7)), then

$$F_M(A_2) = \left\{ \tfrac{s_1}{M}\omega_1 + \tfrac{s_2}{M}\omega_2;\ s_0 + s_1 + s_2 = M,\ s_0, s_1, s_2 \in \mathbb{Z}^{\geq 0} \right\}.$$

A direct computation shows that in the ω-coordinates we have

$$F_2(A_2) = \left\{ (0,0), (1,0), (0,1), (\tfrac{1}{2}, 0), (0, \tfrac{1}{2}), (\tfrac{1}{2}, \tfrac{1}{2}) \right\}.$$

We take a root α coinciding with the first simple root. Then

$$r_\alpha F_2(A_2) = \left\{ (0,0), (-1,1), (0,1), (-\tfrac{1}{2}, \tfrac{1}{2}), (0, \tfrac{1}{2}), (-\tfrac{1}{2}, 1). \right\}.$$

Thus, the grid $F_2^e(A_2)$ consists of different points from $F_2(A_2) \cup r_\alpha F_2(A_2)$ (9 points).

For $F_3(A_2)$ we have

$$F_3(A_2) = \left\{(0,0),(1,0),(0,1),(\tfrac{1}{3},0),(0,\tfrac{1}{3}),(\tfrac{2}{3},0),(0,\tfrac{2}{3}),(\tfrac{2}{3},\tfrac{1}{3}),(\tfrac{1}{3},\tfrac{2}{3}),(\tfrac{1}{3},\tfrac{1}{3})\right\}.$$

Therefore,

$$r_\alpha F_3(A_2) = \left\{(0,0),(-1,1),(0,1),(-\tfrac{1}{3},\tfrac{1}{3}),(0,\tfrac{1}{3}),(-\tfrac{2}{3},\tfrac{2}{3}),(0,\tfrac{2}{3}),(-\tfrac{2}{3},1),(-\tfrac{1}{3},1),(-\tfrac{1}{3},\tfrac{2}{3})\right\}$$

and $F_3^e(A_2)$ consists of 16 points.

Since the long root ξ of C_2 is representable in the form $\xi = 2\alpha_1 + \alpha_2$, where α_1 and α_2 are simple roots, that is, $m_1 = 2, m_2 = 1$, then

$$F_M(C_2) = \left\{\tfrac{s_1}{M}\omega_1^\vee + \tfrac{s_2}{M}\omega_2^\vee;\ s_0 + 2s_1 + s_2 = M,\ s_0, s_1, s_2 \in \mathbb{Z}^{\geq 0}\right\}.$$

A direct computation shows that in the $\omega^\vee$-coordinates we have

$$F_2^e(C_2) = \left\{(0,0),(0,1),(\tfrac{1}{2},0),(0,\tfrac{1}{2}),(-\tfrac{1}{2},\tfrac{1}{2})\right\},$$
$$F_3^e(C_2) = \left\{(0,0),(0,1),(\tfrac{1}{3},0),(0,\tfrac{1}{3}),(0,\tfrac{2}{3}),(\tfrac{1}{3},\tfrac{1}{3}),(-\tfrac{1}{3},\tfrac{1}{3}),(-\tfrac{1}{3},\tfrac{2}{3})\right\}.$$

Since the long root ξ of G_2 is representable in the form $\xi = 2\alpha_1 + 3\alpha_2$, where α_1 and α_2 are simple roots, that is, $m_1 = 2$, $m_2 = 3$, then

$$F_M(G_2) = \left\{\tfrac{s_1}{M}\omega_1^\vee + \tfrac{s_2}{M}\omega_2^\vee;\ s_0 + 2s_1 + 3s_2 = M,\ s_0, s_1, s_2 \in \mathbb{Z}^{\geq 0}\right\}.$$

A computation shows that in the $\omega^\vee$-coordinates we have

$$F_2^e(G_2) = \{(0,0),(1,0),(-1,3)\},$$
$$F_3^e(G_2) = \left\{(0,0),(0,\tfrac{1}{3}),(\tfrac{1}{3},0),(-\tfrac{1}{3},1)\right\},$$
$$F_4^e(G_2) = F_2^e(G_2)\bigcup\left\{(\tfrac{1}{4},0),(0,\tfrac{1}{4}),(-\tfrac{1}{4},\tfrac{3}{4}\right\},$$
$$F_5^e(G_2) = \left\{(0,0),(0,\tfrac{1}{5}),(\tfrac{1}{5},0),(\tfrac{1}{5},\tfrac{1}{5}),(\tfrac{2}{5},0),(-\tfrac{1}{5},\tfrac{3}{5}),(-\tfrac{1}{5},\tfrac{4}{5}),(-\tfrac{2}{5},\tfrac{6}{5})\right\},$$
$$F_8^e(G_2) = F_4^e(G_2)\bigcup\left\{(\tfrac{1}{8},0),(0,\tfrac{1}{8}),(\tfrac{1}{8},\tfrac{1}{8}),(\tfrac{1}{4},\tfrac{1}{8}),(-\tfrac{1}{8},\tfrac{3}{8}),(-\tfrac{1}{8},\tfrac{1}{2}),(-\tfrac{1}{4},\tfrac{7}{8})\right\}.$$

9.4 Expansion in *E*-orbit functions through expansion on grids

Let us give an analogue of the finite Fourier transform when instead of exponential functions we use E-orbit functions. This analogue is not so simple as finite Fourier transform. It is called the finite E-orbit function transform. This transform is used in order to be able to recover (at least approximately) the expansion $f(x) = \sum_\lambda a_\lambda E_\lambda(x)$ for continuous values of x by values of $f(x)$ on a finite set of point.

Under considering the finite Fourier transform in Section 9.1, we have restricted the exponential function to a discrete set. Similarly, in order to determine finite transform, based on E-orbit functions, we have to restrict E-orbit functions $E_\lambda(x)$ to appropriate finite sets of values of x. Candidates for such finite sets are sets T_m. However, E-orbit functions $E_\lambda(x)$ with integral λ are invariant with respect to the affine even Weyl group $W_e^{\rm aff}$. For this reason, we consider E-orbit functions $E_\lambda(x)$ on grids F_M^e, which are parts of the sets T_m.

On the other side, we have also to choose a finite number of E-orbit functions, that is, a finite number of integral $\lambda \in P_+^e$. The best choose is when a number of E-orbit functions coincides with the number $|F_M^e|$ of elements in F_M^e. These E-orbit functions must be selected in such a way that the matrix

$$(E_{\lambda_i}(x_j))_{\lambda_i\in\Omega, x_j\in F_M} \qquad (9.8)$$

(where Ω is our finite set of integral elements λ) is not singular. In order to have non-singularity of this matrix some conditions must be satisfied. In general, they are not known. For this reason, we consider some, more weak, form of the transform (when $|\Omega| \geq |F^e_M|$) and then explain how the set $|\Omega|$ of $\lambda \in P^e_+$ can be chosen in such a way that $|\Omega| = |F^e_M|$.

Let $O_e(\lambda)$ and $O_e(\mu)$ be two different W_e-orbits for elements λ and μ of P^e_+. We say that the group T_m *separates* $O_e(\lambda)$ and $O_e(\mu)$ if for any two elements $\lambda_1 \in O_e(\lambda)$ and $\mu_1 \in O_e(\mu)$ there exists an element $x \in T_m$ such that $\exp(2\pi i\langle\lambda_1, x\rangle) \neq \exp(2\pi i\langle\mu_1, x\rangle)$. Note that λ may coincide with μ.

Let f_1 and f_2 be two functions on the Euclidean space E_n which are finite linear combinations of E-orbit functions. We introduce a T_m-scalar product for f_1 and f_2 by the formula

$$\langle f_1, f_2\rangle_{T_m} = \sum_{x\in T_m} f_1(x)\overline{f_2(x)}.$$

Then the following proposition is true (see [11] and [10]):

Proposition 7. *If T_m separates the orbits $O_e(\lambda)$ and $O_e(\mu)$, $\lambda, \mu \in P^e_+$, then*

$$\langle E_\lambda, E_\mu\rangle_{T_m} = m^n|O_e(\lambda)|\delta_{\lambda\mu}. \tag{9.9}$$

Proof. We have

$$\begin{aligned}\langle E_\lambda, E_\mu\rangle_{T_m} &= \sum_{x\in T_m}\sum_{\sigma\in O_e(\lambda)}\sum_{\tau\in O_e(\mu)} \exp(2\pi i\langle\sigma-\tau, x\rangle)\\ &= \sum_{\sigma\in O_e(\lambda)}\sum_{\tau\in O_e(\mu)}\left(\sum_{x\in T_m}\exp(2\pi i\langle\sigma-\tau, x\rangle)\right).\end{aligned}$$

Since T_m separates $O_e(\lambda)$ and $O_e(\mu)$, then none of the differences $\sigma-\tau$ in the last sum vanishes on T_m. Since T_m is a group and $|T_m| = m^n$, one has

$$\sum_{x\in T_m}\exp(2\pi i\langle\sigma-\tau, x\rangle) = m^n\delta_{\sigma,\tau}.$$

Therefore, $\langle E_\lambda, E_\mu\rangle_{T_m} = m^n|O_e(\lambda)|\delta_{\lambda\mu}$. The proposition is proved. ■

Let f be an invariant (with respect to $W^{\rm aff}_e$) function on the Euclidean space E_n which is a finite linear combination of E-orbit functions:

$$f(x) = \sum_{\lambda_j\in P^e_+} a_{\lambda_j}E_{\lambda_j}(x). \tag{9.10}$$

Our aim is to determine $f(x)$ by its values on a finite subset of E_n, namely, on T_m.

We suppose that T_m separate orbits $O_e(\lambda_j)$ with λ_j from the right hand side of (9.10). Then taking the T_m-scalar product of both sides of (9.10) with E_{λ_j} and using the relation (9.9) we obtain

$$a_{\lambda_j} = (m^n|O_e(\lambda_j)|)^{-1}\langle f, E_{\lambda_j}\rangle_{T_m}.$$

Let now $s^{(1)}, s^{(2)}, \ldots, s^{(h)}$ be all elements of $\overline{F}_e \cap \frac{1}{m}Q^\vee$. By $W_{s^{(i)}}$ we denote the subgroup of W_e whose elements leave $s^{(i)}$ invariant. Then

$$a_{\lambda_j} = m^{-n}|O_e(\lambda_j)|^{-1}\sum_{x\in T_m} f(x)\overline{E_{\lambda_j}(x)} = m^{-n}|W_{\lambda_j}|\sum_{i=1}^{h}|W_{s^{(i)}}|^{-1}f(s^{(i)})\overline{\varphi_{\lambda_j}(s^{(i)})}. \tag{9.11}$$

Thus, *the finite number of values* $f(s^{(i)})$, $i = 1, 2, \ldots, h$, *of the function* $f(x)$ *determines the coefficients* a_{λ_j} *and, therefore, determines the function* $f(x)$ *on the whole space* E_n.

This means that we can reconstruct a $W_e^{\rm aff}$-invariant function $f(x)$ on the whole Euclidean space E_n by its values on the finite set F_M under an appropriate value of M. Namely, we have to expand this function, taken on F_M, into the series (9.10) by means of the coefficients a_{λ_j}, determined by formula (9.11), and then to continue analytically the expansion (9.10) to the whole fundamental domain F_e (and, therefore, to the whole space E_n), that is, to consider the decomposition (9.10) for all $x \in E_n$.

We have assumed that the function $f(x)$ is a finite linear combination of E-orbit functions. If $f(x)$ expands into infinite sum of orbit functions, then for applying the above procedure we have to approximate the function $f(x)$ by taking a finite number of terms in this infinite sum and then to apply the procedure. That is, in this case we obtain an approximate expression of the function $f(x)$ by using a finite number of its values.

At last, we explain how to choose a set Ω in formula (9.8). The set F_M consists of the points (9.6). These points determines the set Ξ of points

$$\lambda = s_1\omega_1 + s_2\omega_2 + \cdots + s_l\omega_l,$$

where $s_1, s_2, \ldots, s_l$ run over the same values as for the set F_M. The set $\Xi \cup r_\alpha \Xi$, where each point is taken only once, can be taken as the set Ω (see [10]).

10 W_e-symmetric functions

E-orbit functions are symmetrized versions of the exponential function, when symmetrization is fulfilled by an even Weyl group W_e. Instead of the exponential function we can take any other set of functions, for example, a set of orthogonal polynomials or a countable set of functions. Then we obtain a corresponding set of orthogonal W_e-symmetric polynomials or a set of W_e-symmetric functions. Such sets of polynomials and functions are considered in this section.

10.1 Symmetrization by E-orbit functions

E-orbit functions can be used for obtaining W_e-symmetric sets of functions. Let $u_m(x)$, $m = 0, 1, 2, \ldots$, be a set of continuous functions of one variables. We create functions of n variables

$$u_{i_1,i_2,\ldots,i_n}(x_1, x_2, \ldots, x_n) \equiv u_{i_1}(x_1)u_{i_2}(x_2)\cdots u_{i_n}(x_n), \qquad i_k = 0, 1, 2, \ldots.$$

Then the functions

$$\tilde{u}_{i_1,i_2,\ldots,i_n}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \int_{F_e} u_{i_1,i_2,\ldots,i_n}(x_1, x_2, \ldots, x_n)E_\lambda(x_1, x_2, \ldots, x_n)dx, \tag{10.1}$$

where $\lambda \equiv (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $E_\lambda(x)$ is a E-orbit function, and dx is the Euclidean measure on E_n (that is, $dx = dx_1 \cdots dx_n$), is symmetric with respect to the action of the even Weyl group W_e. Indeed, for $w \in W_e$ we have

$$\begin{aligned}\tilde{u}_{i_1,i_2,\ldots,i_n}(w\lambda) &= \int_{F_e} u_{i_1,i_2,\ldots,i_n}(x_1, x_2, \ldots, x_n)E_{w\lambda}(x_1, x_2, \ldots, x_n)\, dx \\ &= \int_{F_e} u_{i_1,i_2,\ldots,i_n}(x_1, x_2, \ldots, x_n)E_\lambda(x_1, x_2, \ldots, x_n)\, dx = \tilde{u}_{i_1,i_2,\ldots,i_n}(\lambda).\end{aligned}$$

Formula (10.1) is used for obtaining W_e-symmetric functions or polynomials.

If $u_m(x)$, $m = 0, 1, 2, \ldots$, are orthogonal functions, then the functions (10.1), taken for $i_1 \geq i_2 \geq \cdots \geq i_n$, constitute a set of W_e-symmetric orthogonal functions on the domain D_+^e.

10.2 Eigenfunctions of E-orbit function transform for $W_e(A_n)$

Let $H_n(x)$, $n = 0, 1, 2, \dots$, be the well-known Hermite polynomials. They are defined by the formula

$$H_n(x) = n! \sum_{m=0}^{[n/2]} \frac{(-1)^m (2x)^{n-2m}}{m!(n-2m)!},$$

where $[n/2]$ is an integral part of the number $n/2$. They satisfy the relation

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathrm{i}px} e^{-p^2/2} H_m(p) dp = \mathrm{i}^{-m} e^{-x^2/2} H_m(x)$$

(see, for example, Subsection 12.2.4 in [30]), which can be written in the form

$$\int_{-\infty}^{\infty} e^{2\pi\mathrm{i}px} e^{-\pi p^2} H_m(\sqrt{2\pi}p) dp = \mathrm{i}^m e^{-\pi x^2} H_m(\sqrt{2\pi}x). \tag{10.2}$$

This relation shows that the function $e^{-\pi p^2} H_m(\sqrt{2\pi}p)$ is an eigenfunction of the Fourier transform of one variable with eigenvalue i^m.

Using the Hermite polynomials we create polynomials of many variables

$$H_{\mathbf{m}}(\mathbf{x}) \equiv H_{m_1,m_2,\dots,m_n}(x_1, x_2, \dots, x_n) := H_{m_1}(x_1) H_{m_2}(x_2) \cdots H_{m_n}(x_n). \tag{10.3}$$

The functions

$$e^{-|\mathbf{x}|^2/2} H_{\mathbf{m}}(\mathbf{x}), \qquad m_i = 0, 1, 2, \dots, \qquad i = 1, 2, \dots, n, \tag{10.4}$$

form an orthogonal basis of the Hilbert space $L^2(\mathbb{R}^n)$ with the scalar product

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^n} f_1(\mathbf{x}) \overline{f_2(\mathbf{x})} d\mathbf{x},$$

where $d\mathbf{x} = dx_1\, dx_2 \cdots dx_n$.

We make W_e-symmetrization of the functions

$$e^{-\pi|x|} H_{\mathbf{m}}(\sqrt{2\pi}\mathbf{x}), \qquad m_i = 0, 1, 2, \dots,$$

(obtained from (10.4) by replacing $\mathbf{x}$ by $\sqrt{2\pi}\mathbf{x}$) by means of E-orbit functions of A_{n-1}:

$$\int_{\mathbb{R}^n} E_\lambda(\mathbf{x}) e^{-\pi|\mathbf{x}|^2} H_{\mathbf{m}}(\sqrt{2\pi}\mathbf{x}) = \mathrm{i}^{|\mathbf{m}|} e^{-\pi|\lambda|^2} \mathcal{H}_{\mathbf{m}}(\sqrt{2\pi}\lambda), \tag{10.5}$$

where $E_\lambda(\mathbf{x})$ is an E-orbit function of A_{n-1} and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$.

The polynomials $\mathcal{H}_{\mathbf{m}}$ are symmetric with respect to the even Weyl group $W_e \equiv S_n/S_2 := S_n^e$ of A_{n-1}:

$$\mathcal{H}_{\mathbf{m}}(w\lambda) = \mathcal{H}_{\mathbf{m}}(\lambda), \qquad \mathcal{H}_{w\mathbf{m}}(\lambda) = \mathcal{H}_{\mathbf{m}}(\lambda), \qquad w \in S_n^e.$$

For this reason, $\mathcal{H}_{\mathbf{m}}(\lambda)$ can be considered for values of $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ such that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$.

The polynomials $\mathcal{H}_{\mathbf{m}}$ are of the form

$$\mathcal{H}_{\mathbf{m}}(\lambda) = \sum_{w \in S_n^e} H_{w\mathbf{m}}(\lambda), \tag{10.6}$$

where the polynomials $H_{w\mathbf{m}}(\lambda)$ are of the form (10.3).

Now we apply E-orbit function transform (8.7) (we denote this transform by $\mathfrak{F}$) to the W_e-symmetric function (10.6). Taking into account formula (10.5) we obtain

$$\mathfrak{F}\left(e^{-\pi|\mathbf{x}|^2}\mathcal{H}_{\mathbf{m}}(\sqrt{2\pi}\mathbf{x})\right) := \frac{2}{|S_n|}\int_{\mathbb{R}^n} E_\lambda(\mathbf{x})e^{-\pi|\mathbf{x}|^2}\mathcal{H}_{\mathbf{m}}(\sqrt{2\pi}\mathbf{x})d\mathbf{x}$$
$$= \mathrm{i}^{|\mathbf{m}|}e^{-\pi|\lambda|^2}\mathcal{H}_{\mathbf{m}}(\sqrt{2\pi}\lambda),$$

where $|S_n|$ is an order of the permutation group S_n, that is, functions (10.6) are eigenfunctions of the E-orbit function transform $\mathfrak{F}$. Since the functions (10.6) for $m_i = 0, 1, 2, \ldots$, $i = 1, 2, \ldots, n$, $m_1 \geq m_2 \geq \cdots \geq m_n$, form an orthogonal basis of the Hilbert space $L^2_{\mathrm{sym}}(\mathbb{R}^n)$ of functions from $L^2(\mathbb{R}^n)$ symmetric with respect to W_e, then they constitute a complete set of eigenfunctions of this transform. Thus, this transform has only four eigenvalues i, $-$i, 1, -1 in $L^2_{\mathrm{sym}}(\mathbb{R}^n)$. This means that, as in the case of the usual Fourier transform, we have

$$\mathfrak{F}^4 = 1.$$

10.3 $W_e(A_n)$-symmetric sets of polynomials

In the previous subsection we constructed W_e-symmetric sets of functions connected with Hermite polynomials. Other sets of orthogonal polynomials can be similarly constructed.

Let $p_m(x)$, $m = 0, 1, 2, \ldots$, be the set of orthogonal polynomials in one variable such that

$$\int_{\mathbb{R}} p_m(x)p_{m'}(x)d\sigma(x) = \delta_{mm'},$$

where $d\sigma(x)$ is some orthogonality measure, which may be continuous or discrete.

We create a set of symmetric polynomials of n variables as follows:

$$p^{\mathrm{sym}}_{\mathbf{m}}(\mathbf{x}) = \sum_{w\in S^e_n/S_{\mathbf{m}}} p_{m_{w(1)}}(x_1)p_{m_{w(2)}}(x_2)\cdots p_{m_{w(n)}}(x_n), \tag{10.7}$$
$$m_i = 0, 1, 2, \ldots, \qquad i = 1, 2, \ldots, n,$$

where $\mathbf{m} = (m_1, m_2, \ldots, m_n)$, $m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, and $w(1), w(2), \ldots, w(n)$ is a set of numbers $1, 2, \ldots, n$ transformed by the permutation $w \in S^e_n/S_{\mathbf{m}}$, where $S_{\mathbf{m}}$ is the subgroup of S_n consisting of elements leaving $\mathbf{m}$ invariant.

It is easy to check that the polynomials $p^{\mathrm{sym}}_{\mathbf{m}}(\mathbf{x})$ are symmetric with respect to transformations of S^e_n:

$$p^{\mathrm{sym}}_{\mathbf{m}}(w\mathbf{x}) = p^{\mathrm{sym}}_{\mathbf{m}}(\mathbf{x}), \qquad w \in S_n.$$

Thus, we may consider the polynomials (10.7) on the closure of the fundamental domain of the transformation group $W_e(A_{n-1}) \equiv S^e_n$. This closure (which is denoted as D^e_+) coincides with the set of points $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ for which

$$x_1, x_2 \geq \cdots \geq x_n.$$

The set of polynomials (10.7) is orthogonal with respect to the product measure $d\sigma(\mathbf{x}) \equiv d\sigma(x_1)\, d\sigma(x_2)\cdots d\sigma(x_n)$. Indeed, we have

$$\int_{D^e_+} p^{\mathrm{sym}}_{\mathbf{m}}(\mathbf{x})\overline{p^{\mathrm{sym}}_{\mathbf{m}'}(\mathbf{x})}d\sigma(\mathbf{x}) = \frac{|O_e(\mathbf{m})|}{|S^e_n|}\delta_{\mathbf{m}\mathbf{m}'} = \frac{1}{|S_{\mathbf{m}}|}\delta_{\mathbf{m}\mathbf{m}'},$$

where $O_e(\mathbf{m})$ is the S^e_n-orbit of the point $\mathbf{m}$.

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References

[1] Patera J., Orbit functions of compact semisimple Lie groups as special functions, in Proceedings of Fifth International Conference "Symmetry in Nonlinear Mathematical Physics" (June 23–29, 2003, Kyiv), Editors A.G. Nikitin, V.M. Boyko, R.O. Popovych and I.A. Yehorchenko, *Proceedings of Institute of Mathematics, Kyiv* **50** (2004), Part 3, 1152–1160.

[2] Patera J., Compact simple Lie groups and their C-, S-, and E-transforms, *SIGMA* **1** (2005), 025, 6 pages, math-ph/0512029.

[3] Atoyan A., Patera J., Properties of continuous Fourier extension of the discrete cosine transform and its multidimensional generalization, *J. Math. Phys.* **45** (2004), 2468–2491, math-ph/0309039.

[4] Patera J., Zaratsyan A., Discrete and continuous cosine transform generalized to Lie groups $SU(2)\times SU(2)$ and $O(5)$, *J. Math. Phys.* **46** (2005), 053514, 25 pages.

[5] Patera J., Zaratsyan A., Discrete and continuous cosine transform generalized to Lie groups $SU(2)$ and G_2, *J. Math. Phys.* **46** (2005), 113506, 17 pages.

[6] Patera J., Zaratsyan A., Discrete and continuous sine transforms generalized to compact semisimple Lie groups of rank two, *J. Math. Phys.* **47** (2006), 043512, 22 pages.

[7] Klimyk A.U., Patera J., Orbit functions, *SIGMA* **2** (2006), 006, 60 pages, math-ph/0601037.

[8] Klimyk A.U., Patera J., Antisymmetric orbit functions, *SIGMA* **3** (2007), 023, 83 pages, math-ph/0702040.

[9] Kashuba I., Patera J., Discrete and continuous exponential transforms of simple Lie groups of rank two, *J. Phys. A: Math. Theor.* **40** (2007), 1751–1774, math-ph/0702016.

[10] Moody R.V., Patera J., Orthogonality within the families of C-, S-, and E-functions of any compact semisimple Lie group, *SIGMA* **2** (2006), 076, 14 pages, math-ph/0611020.

[11] Moody R.V., Patera J., Computation of character decompositions of class functions on compact semisimple Lie groups, *Math. Comp.* **48** (1987), 799–827.

[12] Klimyk A.U., Patera J., (Anti)symmetric multivariate exponential functions and corresponding Fourier transforms, *J. Phys. A: Math. Theor.* **40** (2007), 10473–10489, arXiv:0705.3572.

[13] Klimyk A.U., Patera J., (Anti)symmetric multivariate trigonometric functions and corresponding Fourier transforms, *J. Math. Phys.* **48** (2007), 093504, 24 pages, arXiv:0705.4186.

[14] Macdonald I.G., Symmetric functions and hall polynomials, 2nd ed., Oxford Univ. Press, Oxford, 1995.

[15] Macdonald I.G., A new class of symmetric functions, *Publ. I.R.M.A. Strasbourg*, 372/S-20, Actes 20 Séminaire Lotharingien, 1988, 131–171.

[16] Macdonald I.G., Orthogonal polynomials associated with root systems, *Séminaire Lotharingien de Combinatoire*, Actes B45a, Stracbourg, 2000.

[17] Vilenkin N.Ja., Klimyk A.U., Representations of Lie groups and special functions: recent advances, Kluwer, Dordrecht, 1995.

[18] Moody R.V., Patera J., Elements of finite order in Lie groups and their applications, in Proceedings of XIII Int. Colloq. on Group Theoretical Methods in Physics, Editor W. Zachary, World Scientific Publishers, Singapore, 1984, 308–318.

[19] McKay W.G., Moody R.V., Patera J., Tables of E_8 characters and decomposition of plethysms, in Lie Algebras and Related Topics, Editors D.J. Britten, F.W. Lemire and R.V. Moody, Amer. Math. Society, Providence RI, 1985, 227–264.

[20] McKay W.G., Moody R.V., Patera J., Decomposition of tensor products of E_8 representations, *Algebras Groups Geom.* **3** (1986), 286–328.

[21] Patera J., Sharp R.T., Branching rules for representations of simple Lie algebras through Weyl group orbit reduction, *J. Phys. A: Math. Gen.* **22** (1989), 2329–2340.

[22] Grimm S., Patera J., Decomposition of tensor products of the fundamental representations of E_8, in Advances in Mathematical Sciences – CRM's 25 Years, Editor L. Vinet, *CRM Proc. Lecture Notes*, Vol. 11, Amer. Math. Soc., Providence, RI, 1997, 329–355.

[23] Rao K.R., Yip P., Discrete cosine transform – algorithms, advantages, applications, Academic Press, New York, 1990.

[24] Kane R., Reflection groups and invariants, Springer, New York, 2002.

[25] Humphreys J.E., Reflection groups and Coxeter groups, Cambridge Univ. Press, Cambridge, 1990.

[26] Humphreys J.E., Introduction to Lie algebras and representation theory, Springer, New York, 1972.

[27] Bremner M.R., Moody R.V., Patera J., Tables of dominant weight multiplicities for representations of simple Lie algebras, Marcel Dekker, New York, 1985.

[28] Kac V., Infinite dimensional Lie algebras, Birkhäuser, Basel, 1982.

[29] Mckay W.G., Patera J., Sannikoff D., The computation of branching rules for representations of semisimple Lie algebras, in Computers in Nonassociative Rings and Algebras, Editors R.E. Beck and B. Kolman, New York, Academic Press, 1977, 235–278.

[30] Vilenkin N.Ja., Klimyk A.U., Representations of Lie groups and special functions, Vol. 2, Kluwer, Dordrecht, 1993.