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Conformal Powers of the Laplacian via Stereographic Projection*

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Abstract. A new derivation is given of Branson's factorization formula for the conformally invariant operator on the sphere whose principal part is the k-th power of the scalar Laplacian. The derivation deduces Branson's formula from knowledge of the corresponding conformally invariant operator on Euclidean space (the k-th power of the Euclidean Laplacian) via conjugation by the stereographic projection mapping.

Key words: conformal Laplacian; stereographic projection

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Dedicated to the memory of Tom Branson

1 Introduction

The powers of the Laplacian on $\mathbb{R}^n$ satisfy an invariance property with respect to conformal motions. If C is a conformal transformation satisfying $C^*g_E = \Omega^2 g_E$, where g_E denotes the Euclidean metric and Ω is the conformal factor, then

$$\Delta^k = \left(C^{-1}\right)^* \Omega^{-n/2-k} \Delta^k \, \Omega^{n/2-k} C^*, \qquad k \in \mathbb{N}, \tag{1}$$

where the powers of Ω act by multiplication. This observation is the motivation for consideration of the "conformally invariant powers of the Laplacian" on a general curved conformal manifold (see [5]). In [1], Tom Branson derived the explicit form of such operators on the sphere S^n. He showed that any operator on S^n which satisfies the transformation law analogous to (1), where now C is a conformal transformation of S^n with conformal factor Ω, necessarily is a multiple of

$$\prod_{j=1}^{k} (\Delta_S - c_j), \qquad c_j = \left(\tfrac{n}{2} + j - 1\right)\left(\tfrac{n}{2} - j\right). \tag{2}$$

Here Δ_S denotes the Laplacian on the sphere, and our sign convention is $\Delta = \sum \partial_i^2$ on $\mathbb{R}^n$. To prove this, he introduced what are now called spectrum generating functions, by showing how to use infinitesimal conformal invariance to derive the full spectral decomposition of such an invariant operator from knowledge of its eigenvalue on a single spherical harmonic. Branson also used this argument to give the form of the pseudodifferential intertwining operators satisfying a transformation law analogous to (1) but involving more general, possibly complex, powers of Ω.

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There are now (at least) two other derivations of the factorization (2). In [3], (2) is derived via a construction using tractors, and in [2] by explicit solution of the algorithm of [5] in terms of the dual Hahn polynomials, a family of discrete orthogonal polynomials. Both of these derivations show that the same formula gives a conformally invariant operator for any Einstein metric whose scalar curvature agrees with that of S^n. This can also be deduced directly from Branson's result for S^n and the form of the GJMS algorithm; see the discussion in [2]. A rescaling gives the corresponding formula for general Einstein metrics.

In this note we give a direct argument relating the operator Δ^k on $\mathbb{R}^n$ and the operator (2) on S^n under stereographic projection. Thus the conformal invariance of the operator (2) is a consequence of (1). The case $k = 1$ is the Yamabe operator, whose conformal invariance, and therefore whose behavior under stereographic projection, is well-known. The argument here deduces the relation for $k > 1$ from the case $k = 1$ together with a calculation of pullback under stereographic projection. From this perspective, the constants c_j for $j > 1$ are manufactured from c_1 by the stereographic projection mapping.

The derivation presented here is the analogue in the conformal case of an argument in [4] relating CR invariant operators on odd-dimensional spheres to corresponding operators on the Heisenberg group via the Cayley transform. The CR case is more complicated: there is a 1-parameter family of invariant operators for each k, and the operators on the Heisenberg group are not powers of a fixed operator, but rather are products of various of the Folland–Stein operators.

2 Derivation

Let $\Phi : S^n \setminus \{p\} \to \mathbb{R}^n$ be stereographic projection:

$$\Phi(x', x_{n+1}) = x'(1 + x_{n+1})^{-1} = y$$

for $x' \in \mathbb{R}^n$ and $|x'|^2 + x_{n+1}^2 = 1$, where $p = (0, -1)$ is the south pole. One has

$$\Phi^*\left(\frac{2}{1+|y|^2}\right) = 1 + x_{n+1}.$$

The map Φ is conformal:

$$\Phi^* g_E = (1 + x_{n+1})^{-2} g_S.$$

Define $M^w : C^\infty(S^n \setminus \{p\}) \to C^\infty(S^n \setminus \{p\})$ by

$$M^w f = (1 + x_{n+1})^w f$$

and $M_w : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ by $M^w\Phi^* = \Phi^* M_w$, so that

$$M_w f = 2^w (1 + |y|^2)^{-w} f.$$

The Yamabe operator on the sphere is $Y = \Delta_S - c_1$, and its conformal invariance implies

$$Y M^{1-n/2}\Phi^* = M^{-1-n/2}\Phi^*\Delta \tag{3}$$

acting on functions on $\mathbb{R}^n$.

Proposition 1. *For $k \in \mathbb{N}$,*

$$\left(\prod_{j=1}^{k}(\Delta_S - c_j)\right) M^{k-n/2}\Phi^* = M^{-k-n/2}\Phi^*\Delta^k. \tag{4}$$

The analogue of (1) for the operator (2) under conformal transformations of S^n follows from (1) and (4), since conjugation by Φ maps conformal transformations of $\mathbb{R}^n$ to conformal transformations of S^n.

The proof begins by noting that $c_1 - c_j = j(j-1)$, so that the left hand side of (4) may be written as

$$\left(\prod_{j=1}^{k} (Y + j(j-1)) \right) M^{k-n/2} \Phi^*.$$

Now pass Φ^* through each term using (3) and then cancel the Φ^* to obtain that (4) is equivalent to the following identity on $\mathbb{R}^n$:

$$\begin{aligned}[t] &[\Delta + k(k-1)M_2]M_{-2}[\Delta + (k-1)(k-2)M_2]M_{-2} \cdots [\Delta + 2M_2]M_{-2}\Delta \\ &\quad = M_{1-k}\Delta^k M_{1-k}. \end{aligned} \tag{5}$$

The identity (5) can be proved by induction on k. The induction uses some commutator identities. Denote by $X = \sum y_i \partial_{y_i}$ the Euler vector field on $\mathbb{R}^n$. The commutator identities are:

$$[\Delta, X] = 2\Delta, \tag{6}$$
$$[X, M_w] = -w|y|^2 M_{w+1}, \tag{7}$$
$$[\Delta, M_w] = -wM_w \left(2X + n - (w-1)M_1|y|^2\right) M_1, \tag{8}$$
$$[\Delta^k, M_{-1}] = k\left(2X + n + 2(k-1)\right)\Delta^{k-1}. \tag{9}$$

The first three are just direct calculations. The last is an easy induction on k. Equation (8) has been written in the form above because this is advantageous below, but it is easily seen using (7) that this may also be written perhaps a little more naturally as

$$[\Delta, M_w] = -wM_{w+1}\left(2X + n - (w+1)M_1|y|^2\right).$$

In this form it is clear that the $k = 1$ case of (9) is the $w = -1$ case of (8).

Now prove (5) by induction. The $k = 1$ case is a tautology. Assuming the result for k and substituting this in the left hand side for $k + 1$ gives

$$[\Delta + k(k+1)M_2]M_{-2}M_{1-k}\Delta^k M_{1-k},$$

which equals

$$\begin{aligned} &\Delta M_{-k-1}\Delta^k M_{1-k} + k(k+1)M_{1-k}\Delta^k M_{1-k} \\ &\quad = M_{-k}\Delta M_{-1}\Delta^k M_{1-k} + [\Delta, M_{-k}]M_{-1}\Delta^k M_{1-k} + k(k+1)M_{1-k}\Delta^k M_{1-k} \\ &\quad = M_{-k}\Delta^{k+1}M_{-k} - M_{-k}\Delta[\Delta^k, M_{-1}]M_{1-k} + [\Delta, M_{-k}]M_{-1}\Delta^k M_{1-k} \\ &\qquad + k(k+1)M_{1-k}\Delta^k M_{1-k}. \end{aligned}$$

Upon substituting (9) and (8) and then using (6) to commute the Δ through the X which arises in the second term and finally simplifying, one finds that the last three terms add up to 0, thus completing the induction step.

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