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Branson's Q-curvature in Riemannian and Spin Geometry*

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Abstract. On a closed 4-dimensional Riemannian manifold, we give a lower bound for the square of the first eigenvalue of the Yamabe operator in terms of the total Branson's Q-curvature. As a consequence, if the manifold is spin, we relate the first eigenvalue of the Dirac operator to the total Branson's Q-curvature. On a closed n-dimensional manifold, $n \geq 5$, we compare the three basic conformally covariant operators: the Branson–Paneitz, the Yamabe and the Dirac operator (if the manifold is spin) through their first eigenvalues. Equality cases are also characterized.

Key words: Branson's Q-curvature; eigenvalues; Yamabe operator; Branson–Paneitz operator; Dirac operator; σ_k-curvatures; Yamabe invariant; conformal geometry; Einstein metrics; Killing spinors

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To the memory of Tom Branson

Tom has deeply influenced my life. With him, I learned to push the limits of what would concretely mean to have a clear and deep thinking, to take a huge distance from things and events so that the essence could be touched.

Oussama Hijazi

1 Introduction

The scalar curvature function behaves nicely under a conformal change of the metric. The Yamabe operator relates the scalar curvatures associated with two metrics in the same conformal class. From the conformal point of view dimension 2 is special and the scalar curvature corresponds to (twice) the Gauss curvature. The problem of conformally deforming the scalar curvature into a constant is known as the Yamabe problem, it has been intensively studied in the seventies and solved in the beginning eighties.

On a spin compact manifold it is an important fact, through the Schrödinger–Lichnerowicz formula, that the scalar curvature is closely related to the eigenvalues of the Dirac operator and its sign influences the topology of the manifold.

The conformal behavior of the scalar curvature and that of the Dirac operator allowed to establish lower bounds for the square of the eigenvalues of the Dirac operator in terms of scalar conformal invariants, the Yamabe invariant in dimension at least 3 and the total scalar curvature in dimension 2 (see [12, 13, 4]).

Given this inequality for surfaces, one might ask whether such integral inequalities could be true in any dimension. It is shown in [1] that this could not be true with the total scalar curvature. However one could expect such integral lower bounds associated with other curvatures.

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The Branson Q-curvature is a scalar function which shares with the scalar curvature interesting conformal behavior. The operator which relates the Q-curvatures associated with two conformally related metrics is the Branson–Paneitz operator, a 4-th order differential operator. Another curvature function which is of special interest from the conformal aspect is the σ_k-curvature, which is the k-th symmetric function associated with the eigenvalues of the Schouten tensor (the tensor which measures the defect of the conformal invariance of the Riemann tensor).

Taking into account the analogy between the scalar curvature and the Q-curvature, some natural questions could be asked: what would be the role of the Q-curvature in Spin Geometry and is there any relation between the spectra of the three natural conformally covariant differential operators : Dirac, Yamabe and Branson–Paneitz?

In this paper, a first answer is given for n-dimensional closed Riemannian manifolds. For $n = 4$, we establish a new lower bound for the square of the first eigenvalue of the Yamabe operator in terms of the total Branson's Q-curvature (see Theorem 2). For $n \geq 5$, we show that, up to a constant, the square of the first eigenvalue of the Yamabe operator is at least equal to that of the Paneitz–Branson operator (see Theorem 3).

In case the manifold is spin, we make use of what is *called* the Hijazi inequality to relate the first eigenvalue of the Dirac operator to the total Branson's Q-curvature, if $n = 4$, and to the first eigenvalue of the Paneitz–Branson operator, if $n \geq 5$.

The key classical argument in Spin Geometry (see [12]) is to consider on a Riemannian manifold a special metric in the conformal class associated with an appropriate choice of the conformal factor, namely an eigenfunction associated with the first eigenvalue of the Yamabe operator.

For completeness, we sketch the proof of relevant classical results in Conformal Spin Geometry and we end by introducing the notion of σ_k-curvature and give the proof, on a 4-dimensional closed spin manifold, of a relation between the eigenvalues of the Dirac operator and the total σ_2-curvature established by G. Wang [23]. By the Chern–Gauss–Bonnet formula, it follows that the total σ_2-curvature is precisely a multiple of the total Branson's Q-curvature, hence another indirect proof of Corollary 1 may be obtained.

2 Natural geometric operators in conformal Riemannian geometry

Consider a compact Riemannian manifold (M^n, g) and let $[g] = \{g_u := e^{2u}g \,/\, u \in C^\infty(M)\}$ be the conformal class of the metric g. A class of differential operators of particular interest in Riemannian Geometry is that of conformally covariant operators. If $A := A_g$ is a formally self-adjoint geometric differential operator acting on functions (or on sections of vector bundles) over (M^n, g) and $A_u := A_{g_u}$ then A is said to be conformally covariant of bidegree $(a, b) \in \mathbb{R}^2$ if and only if

$$A_u(\cdot) = e^{-bu} A(e^{au} \cdot).$$

We now give some relevant examples of such operators.

2.1 The Yamabe operator

In dimension $n = 2$, the Laplacian $\Delta := \delta d$ acting on smooth functions is a conformally covariant differential operator of bidegree $(0, 2)$ since it satisfies $\Delta_u = e^{-2u}\Delta$. It is interesting to note that we have the *Gauss curvature equation*:

$$\Delta u + K = K_u e^{2u}, \tag{1}$$

where K_u is the Gauss curvature of (M^2, g_u). Using this formula, we can easily conclude that:

$$\int_M K dv = \int_M K_u dv_u,$$

is a conformal invariant of the surface M equipped with the conformal class of g. In fact, it is a topological invariant due to the Gauss–Bonnet formula:

$$2\pi\chi(M^2) = \int_M K dv, \tag{2}$$

where $\chi(M^2)$ is the Euler–Poincaré characteristic class of M.

In dimension $n \geq 3$, the Yamabe operator (or the conformal Laplacian):

$$L := 4\frac{n-1}{n-2}\Delta + R,$$

where R is the scalar curvature of (M^n, g), is conformally covariant of bidegree $(\frac{n-2}{2}, \frac{n+2}{2})$. Indeed, we have the following relation:

$$L_u f = e^{-\frac{n+2}{2}u} L(e^{\frac{n-2}{2}u} f),$$

for all $f \in C^\infty(M)$. It is important to note that this operator relates the scalar curvatures of the manifold M associated with two metrics in the same conformal class. Indeed, we have:

$$Lu = R_u\, u^{\frac{n+2}{n-2}}$$

for $g_u = u^{\frac{4}{n-2}} \in [g]$, where u is a smooth positive function.

2.2 The Branson–Paneitz operator

In dimension $n \geq 3$, the Branson–Paneitz operator, defined by

$$P := \Delta^2 + \delta(\alpha_n R\,\mathrm{Id} + \beta_n \mathrm{Ric})d + \frac{n-4}{2}Q$$

is a self-ajdoint elliptic fourth-order conformally covariant differential operator of bidegree $(\frac{n-4}{2}, \frac{n+4}{2})$, i.e.,

$$P_u f = e^{-\frac{n+4}{2}u}\, P(e^{\frac{n-4}{2}u} f),$$

for all $f \in C^\infty(M)$. Here $\alpha_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}$, $\beta_n = -\frac{4}{n-2}$, Ric is the Ricci tensor and Q is the Branson Q-curvature of (M^n, g) defined by:

$$Q = \frac{n}{8(n-1)^2}R^2 - 2|S|^2 + \frac{1}{2(n-1)}\Delta R, \tag{3}$$

where:

$$S_{ij} = \frac{1}{n-2}A_{ij} = \frac{1}{n-2}\left(\mathrm{Ric}_{ij} - \frac{1}{2(n-1)}R g_{ij}\right).$$

is the Schouten tensor. The Q-curvature together with the Branson–Paneitz operator share a similar conformal behavior as that of the scalar curvature and the Yamabe operator. Indeed, if $n \geq 5$ and $g_u = u^{\frac{4}{n-4}}$ we have:

$$Pu = \frac{n-4}{2}\, Q_u\, u^{\frac{n+4}{n-4}}, \tag{4}$$

where Q_u is the Q-curvature of (M^n, g_u). On a 4-dimensional manifold, the Branson–Paneitz operator P and the Q-curvature are analogues of the Laplacian and the Gauss curvature on 2-dimensional manifolds. In fact, we have the following *Q-curvature equation*:

$$Pu + Q = Q_u\, e^{4u}$$

for $g_u = e^{2u} g \in [g]$ (compare with (1)). Again, using this equation, we can easily deduce that:

$$\int_M Q\, dv = \int_M Q_u\, dv_u$$

is a conformal invariant. Another way to obtain the conformal covariance of the total Q-curvature functional comes from the Chern–Gauss–Bonnet formula for 4-dimensional manifolds:

$$16\pi^2 \chi(M^4) = \int_M \left(\frac{1}{2}|W|^2 + \frac{1}{12}R^2 - |E|^2 \right) dv, \tag{5}$$

where $E := \mathrm{Ric} - (1/n)Rg$ is the Einstein tensor of (M^4, g). Thus using (3) for $n = 4$, we obtain:

$$16\pi^2 \chi(M^4) = \frac{1}{2} \int_M |W|^2 dv + 2 \int_M Q dv. \tag{6}$$

Since the expression $|W|^2 dv$ is a pointwise conformal invariant, we conclude that the total Q-curvature is a conformal invariant.

3 Spin geometry, the Dirac operator and classical eigenvalue estimates

For convenience we briefly recall some standard facts about Riemannian Spin Geometry (see [16] or [9]) and establish with some details the fact that, as the Yamabe and Branson–Paneitz operators, the Dirac operator D acting on smooth sections of the spinor bundle is a conformal covariant operator of bidegree $(\frac{n-1}{2}, \frac{n+1}{2})$. We then give a proof of classical conformal eigenvalue lower bounds on the spectrum of the Dirac operator.

We consider a closed compact Riemannian manifold (M^n, g) equipped with a spin structure, which is a topological restriction corresponding to an orientability condition of order two. Thanks to this structure, one can construct a complex vector bundle $\Sigma_g M := \Sigma M$ (the bundle of complex spinors) of rank $2^{[n/2]}$ over M. A smooth section $\psi \in \Gamma(\Sigma M)$ of this vector bundle is called a spinor field. Note that this vector bundle depends on the Riemannian metric. The spinor bundle ΣM is endowed with

1) Clifford multiplication, that is an action of the tangent bundle on spinor fields:

$$\begin{array}{ccc} \mathrm{T}M \otimes \Sigma M & \longrightarrow & \Sigma M \\ X \otimes \psi & \longmapsto & X \cdot \psi, \end{array}$$

2) the natural spinorial Levi-Civita connection ∇ acting on smooth spinor fields corresponding to the Levi-Civita connection (also denoted by ∇) and satisfying:

$$\nabla_X (Y \cdot \varphi) = (\nabla_X Y) \cdot \varphi + Y \cdot \nabla_X \varphi,$$

3) a natural Hermitian scalar product $\langle\ ,\ \rangle$ such that:

$$\langle X \cdot \psi, X \cdot \varphi \rangle = g(X, X) \langle \psi, \varphi \rangle$$

and compatible with the spin connection, that is:

$$X\langle \varphi, \psi \rangle = \langle \nabla_X \varphi, \psi \rangle + \langle \varphi, \nabla_X \psi \rangle,$$

for all $X, Y \in \Gamma(\mathrm{T}M)$ and $\psi, \varphi \in \Gamma(\Sigma M)$. We can finally define a differential operator acting on smooth spinor fields, the Dirac operator, locally given by:

$$\begin{array}{rccl} D: & \Gamma(\Sigma M) & \longrightarrow & \Gamma(\Sigma M) \\ & \psi & \longmapsto & D\psi = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i} \psi, \end{array}$$

where $\{e_1, \ldots, e_n\}$ is a local g-orthonormal frame. This first order differential operator is elliptic and formally self adjoint.

3.1 Conformal covariance of the Dirac operator

We now focus on the conformal behavior of spinors on a Riemannian spin manifolds. We explain with some details the conformal covariance of the Dirac operator and give an application of this property. For more details, we refer to [15, 12] or [5]. So consider a smooth function u on the manifold M, and let $g_u = e^{2u} g$ be a conformal change of the metric. Then we have an obvious identification between the two SO_n-principal bundles of g and g_u-orthonormal oriented frames denoted respectively by $\mathrm{SO}M$ and $\mathrm{SO}_u M$. We can thus identify the corresponding Spin_n-principal bundles $\mathrm{Spin}M$ and $\mathrm{Spin}_u M$, leading to a bundle isometry

$$\begin{array}{ccc} \Sigma M & \longrightarrow & \Sigma_u M \\ \varphi & \longmapsto & \varphi_u. \end{array}$$

We can also relate the corresponding Levi-Civita connections, Clifford multiplications and Hermitian scalar products. Indeed, denoting by ∇^u, $\cdot_u$ and $\langle \ , \ \rangle_u$ the associated data which act on sections of the bundle $\Sigma_u M$, we can easily show that:

$$\begin{aligned} &\nabla^u_X \psi_u = \left(\nabla_X \psi - \frac{1}{2} X \cdot du \cdot \psi - \frac{1}{2} X(u) \psi \right)_u, \\ &X_u \cdot_u \psi_u = (X \cdot \psi)_u, \\ &\langle \psi_u, \varphi_u \rangle_u = \langle \psi, \varphi \rangle, \end{aligned}$$

for all $\psi, \varphi \in \Gamma(\Sigma M)$, $X \in \Gamma(\mathrm{T}M)$ and where $X_u := e^{-u} X$ denotes the vector field over (M^n, g_u) under the identification explained above. Using these identifications, one can deduce the relation between the Dirac operators D and D_u acting respectively on sections of ΣM and $\Sigma_u M$, that is:

$$D_u \psi_u = \left(e^{-\frac{n+1}{2} u} D(e^{\frac{n-1}{2} u} \psi) \right)_u. \tag{7}$$

This formula clearly shows that the Dirac operator is a conformally covariant differential operator of bidegree $(\frac{n-1}{2}, \frac{n+1}{2})$.

3.2 Eigenvalues of the Dirac operator

A powerfull tool in the study of the Dirac operator is the Schrödinger–Lichnerowicz formula which relates the square of the Dirac operator with the spinorial Laplacian. More precisely, we have:

$$D^2 = \nabla^* \nabla + \frac{1}{4} R,$$

where ∇^* is the L^2-formal adjoint of ∇. An integration by parts using this formula leads to the following integral identity:

$$\int_M |D\psi|^2 dv = \int_M |\nabla \psi|^2 dv + \frac{1}{4} \int_M R |\psi|^2 dv \tag{8}$$

for all $\psi \in \Gamma(\Sigma M)$. Combining this fundamental identity with the Atiyah–Singer Index Theorem [2] implies topological obstructions to the existence of metrics with positive scalar curvature (see [19]). This vanishing result can be seen as a non-optimal estimate on the spectrum of the Dirac operator. In fact, if the scalar curvature is positive and if ψ is an eigenspinor associated with the first eigenvalue $\lambda_1(D)$ of the Dirac operator, then by (8) one gets:

$$\lambda_1^2(D) > \frac{1}{4}\inf_M(R).$$

Optimal eigenvalues estimate could be obtained by introducing the twistor operator T, which is the projection of ∇ on the kernel of the Clifford multiplication. It is locally given by:

$$T_X\psi = \nabla_X\psi + \frac{1}{n}X\cdot D\psi$$

for all $\psi \in \Gamma(\Sigma M)$ and $X \in \Gamma(\mathrm{T}M)$. Thus using the relation:

$$|\nabla\psi|^2 = |T\psi|^2 + \frac{1}{n}|D\psi|^2, \tag{9}$$

T. Friedrich [7] proved that the first eigenvalue $\lambda_1(D)$ of the Dirac operator satisfies:

$$\lambda_1(D)^2 \geq \frac{n}{4(n-1)}\inf_M(R)$$

with equality if and only if the eigenspinor associated with the first eigenvalue is a Killing spinor, that is for all $X \in \Gamma(\mathrm{T}M)$:

$$\nabla_X\psi = -\frac{\lambda_1(D)}{n}X\cdot\psi. \tag{10}$$

3.3 Conformal lower bounds for the eigenvalues of the Dirac operator

We now prove the following result due to the first author:

Theorem 1 ([12, 13]). *Let (M^n,g) be a closed compact Riemannian spin manifold of dimension $n \geq 2$. Then the first eigenvalue of the Dirac operator satisfies:*

$$\lambda_1(D)^2 \geq \frac{n}{4(n-1)}\sup_u\inf_M\big(R_u e^{2u}\big). \tag{11}$$

Moreover, equality is achieved if and only if the eigenspinor associated with the eigenvalue $\lambda_1(D)$ is a Killing spinor. In particular, the manifold (M^n,g) is Einstein.

Proof. Consider an eigenspinor $\psi \in \Gamma(\Sigma M)$ of the Dirac operator associated with the eigenvalue λ. Now if we let $\varphi = e^{-\frac{n-1}{2}u}\psi \in \Gamma(\Sigma M)$ then the relation (7) gives $D_u\varphi_u = \lambda e^{-u}\varphi_u$. Thus combining formulae (8) and (9), we have:

$$\frac{n-1}{n}\int_M |D_u\varphi_u|^2 dv_u = \int_M |T^u\varphi_u|^2 dv_u + \frac{1}{4}\int_M R_u|\varphi_u|^2 dv_u,$$

which leads to:

$$\frac{n-1}{n}\lambda^2\int_M e^{-2u}|\varphi_u|^2 dv_u \geq \frac{1}{4}\int_M R_u|\varphi_u|^2 dv_u \geq \frac{1}{4}\inf_M(R_u e^{2u})\int_M e^{-2u}|\varphi_u|^2 dv_u$$

for all $u \in C^\infty(M)$. Inequality (11) follows directly. Suppose now that equality is achieved in (11), then we have:

$$T^u_{X_u}\varphi_u = \nabla^u_{X_u}\varphi_u + \big(\lambda_1(D)/n\big)e^{-u}X_u\cdot_u\varphi_u = 0,$$

for all $X \in \Gamma(\mathrm{T}M)$. However, one can compute that in this case, the function u has to be constant (see [14] for example) and thus $\psi \in \Gamma(\Sigma M)$ is a Killing spinor (that is it satisfies equation (10)). ■

We will now derive two results from the interpretation of the right-hand side of inequality (11). As we will see, these estimates will lead to some natural geometric invariants.

3.3.1 The 2-dimensional case

We focus here on the case of compact closed surfaces and show that

$$\lambda_1(D)^2 \geq \frac{2\pi\,\chi(M^2)}{\mathrm{Area}(M^2,g)}.$$

This has first been observed by Bär (see [4]). For $n=2$, inequality (11) reads:

$$\lambda_1(D)^2 \geq \frac{1}{2}\sup_u \inf_M \left(R_u e^{2u}\right).$$

First note that for all $u \in C^\infty(M)$:

$$\inf_M \left(R_u e^{2u}\right) \leq \frac{1}{\mathrm{Area}(M^2,g)}\int_M R_u dv_u = \frac{1}{\mathrm{Area}(M^2,g)}\int_M R dv = \frac{4\pi\,\chi(M^2)}{\mathrm{Area}(M^2,g)},$$

where we have used the fact that $R_u = 2K_u$ and the Gauss–Bonnet formula (2). On the other hand, one can easily show the existence of a unique (up to an additive constant) smooth function $u_0 \in C^\infty(M)$ such that:

$$\Delta u_0 = \frac{1}{\mathrm{Area}(M^2,g)}\int_M K dv - K,$$

which proves the relation:

$$\frac{1}{2}\sup_u \inf_M \left(R_u e^{2u}\right) = \frac{1}{2} R_{u_0} e^{2u_0} = \frac{2\pi\,\chi(M^2)}{\mathrm{Area}(M^2,g)}.$$

3.3.2 The case $n \geq 3$

Here we will show that the right-hand side of inequality (11) is given by the first eigenvalue $\lambda_1(L)$ of the conformal Laplacian L (see Section 2.1), that is we get:

$$\lambda_1(D)^2 \geq \frac{n}{4(n-1)}\lambda_1(L). \tag{12}$$

First note that since $n \geq 3$, one can consider the conformal change of metrics defined by $g_h = h^{\frac{4}{n-2}} g$ with h a smooth positive function on M and thus we have:

$$R_u e^{2u} = R_h h^{\frac{4}{n-2}} = h^{-1} L h.$$

Now we choose the conformal weight $h = h_1$ as being an eigenfunction of the conformal Laplacian associated with the first eigenvalue $\lambda_1(L)$. Such a function can be assumed to be positive and normalized (with unit L^2-norm) and satisfies:

$$\lambda_1(L) = 4\frac{n-1}{n-2}\int_M h_1 \Delta h_1 dv + \int_M R h_1^2 dv.$$

On the other hand, for any f smooth and positive function on M, we can write $h_1 = fF$ with F a smooth positive function on M. Using this expression in the preceding integral equality with an integration by parts leads to:

$$\lambda_1(L) = \int_M \left(4\frac{n-1}{n-2} f^{-1}\Delta f + R\right) h_1^2 dv + 4\frac{n-1}{n-2}\int_M f^2 |\nabla F|^2 dv,$$

then

$$\lambda_1(L) \geq \inf_M \left(4\frac{n-1}{n-2}f^{-1}\Delta f + R\right) = \inf_M \big(f^{-1}Lf\big)$$

for all f smooth and positive. It is clear that the above inequality is achieved if and only if the function f is an eigenfunction of L associated with its first eigenvalue. Using this inequality, we can compare the spectrum of the Dirac operator with a conformal invariant of the manifold (M^n, g), *the Yamabe invariant* $Y(M^n, [g])$. This invariant appears naturally in the context of the Yamabe problem (see [17] for example) and plays a fundamental role in its solution. It is defined by:

$$Y(M^n, [g]) = \inf_{f \in H_1^2 \setminus \{0\}} \frac{\int_M \big(4\frac{n-1}{n-2}|\nabla f|^2 + R f^2\big) dv}{\Big(\int_M |f|^{\frac{2n}{n-2}} dv\Big)^{\frac{n-2}{2}}},$$

where H_1^2 denotes the space of L^2-integrable functions as well as their first derivatives. Indeed, using the variational characterization of $\lambda_1(L)$ given by the Rayleigh quotient:

$$\lambda_1(L) = \inf_{f \in H_1^2 \setminus \{0\}} \frac{\int_M \big(4\frac{n-1}{n-2}|\nabla f|^2 + R f^2\big) dv}{\int_M f^2 dv},$$

and applying the Hölder inequality leads to $\lambda_1(L)\mathrm{Vol}(M^n, g)^{\frac{2}{n}} \geq Y(M^n, [g])$. Combining this estimate with Inequality (12) shows that:

$$\lambda_1(D)^2 \mathrm{Vol}(M^n, g)^{\frac{2}{n}} \geq \frac{n}{4(n-1)} Y(M^n, [g]).$$

4 First eigenvalues of conformally covariant differential operators

In this section, we show that in dimension 4 the total Q-curvature bounds from below the square of the Yamabe invariant. Using the conformal covariance of the Yamabe and the Branson–Paneitz operators together with a special choice of the conformal factor, we get a relation between their first eigenvalues. This combined with the inequality (12) relate the Dirac, Yamabe and Branson–Paneitz operators through appropriate powers of their first eigenvalues. We show:

Theorem 2. *Let (M^4, g) be a closed compact 4-dimensional Riemannian manifold. Then, the first eigenvalue of the Yamabe operator satisfies*

$$\lambda_1(L)^2 \geq \frac{24}{\mathrm{Vol}(M^4, g)} \int_M Q dv. \tag{13}$$

Moreover, equality occurs if and only if g is an Einstein metric.

Proof. Recall that if $n = 4$, the Q-curvature is defined by:

$$6Q = R^2 - 3|\mathrm{Ric}|^2 + \Delta R$$

and then one can easily check that:

$$24Q = R^2 - 12|E|^2 + 4\Delta R,$$

where $E := \mathrm{Ric} - (R/4)g$ is the Einstein tensor of (M^4, g). Now for $g_u = e^{2u}g$, we can write:

$$24 \int_M Q_u dv_u = \int_M R_u^2 dv_u - 12 \int_M |E_u|_u^2 dv_u \leq \int_M R_u^2 dv_u \tag{14}$$

since $|E_u|_u^2 \geq 0$. We now choose an adapted conformal weight, namely an eigenfunction of the Yamabe operator associated with its first eigenvalue that is a smooth positive function h_1 such that:

$$Lh_1 = \lambda_1(L)\, h_1.$$

Consider the conformal change of metrics $g_{h_1} = h_1^2 g \in [g]$ and inequality (14) written for the metric g_{h_1} reads:

$$24 \int_M Q_{h_1} dv_{h_1} \leq \int_M R_{h_1}^2 dv_{h_1} = \lambda_1(L)^2 \mathrm{Vol}(M^4, g)$$

where we used the fact that:

$$R_{h_1} = h_1^{-3} L h_1 = \lambda_1(L) h_1^{-2}$$

and $dv_{h_1} = h_1^4 dv$. Finally since $n = 4$, we use the conformal invariance of the left-hand side of the preceding inequality (see Section 2.2) to get inequality (13). ∎

If we now apply inequality (12), we obtain:

Corollary 1. *Under the assumptions of Theorem 2, if M is spin and $\lambda_1(L) > 0$, then:*

$$\lambda_1(D)^4 \geq \frac{1}{9}\lambda_1(L)^2 \geq \frac{8}{3} \frac{\int_M Q dv}{\mathrm{Vol}(M^4, g)}.$$

Equality in both inequalities is characterized by the existence of a Killing spinor, in particular, the manifold is the round sphere.

We now consider the general case:

Theorem 3. *Let (M^n, g) be a closed compact Riemannian manifold with $n \geq 5$. If the first eigenvalue $\lambda_1(P)$ of the Branson–Paneitz operator is positive, then we have:*

$$\lambda_1^2(L) \geq \frac{16n(n-1)^2}{(n^2-4)(n-4)} \lambda_1(P). \tag{15}$$

Moreover equality occurs if and only if g is an Einstein metric.

Proof. First note that the Q-curvature, defined in (3), can be written as:

$$Q = \frac{n^2-4}{8n(n-1)^2} R^2 - \frac{2}{(n-2)^2} |E|^2 + \frac{1}{2(n-1)} \Delta R,$$

where $E := \mathrm{Ric} - \frac{R}{n} g$ is the Einstein tensor of (M^n, g). We now consider the metric $g_u = u^{\frac{4}{n-4}} g$ where u is a smooth positive function. Stokes formula gives:

$$\begin{aligned} \int_M Q_u\, dv_u &= \int_M \left(\frac{n^2-4}{8n(n-1)^2} R_u^2 - \frac{2}{(n-2)^2} |E_u|_u^2 + \frac{1}{2(n-1)} \Delta_u R_u \right) dv_u \\ &= \int_M \left(\frac{n^2-4}{8n(n-1)^2} R_u^2 - \frac{2}{(n-2)^2} |E_u|_u^2 \right) dv_u \leq \frac{n^2-4}{8n(n-1)^2} \int_M R_u^2\, dv_u. \end{aligned}$$

On the other hand, with the help of the relation (4) and since $dv_u = u^{\frac{2n}{n-4}}dv$, one can check that:

$$\frac{n-4}{2}\int_M Q_u\,dv_u = \int_M uPu\,dv,$$

which gives:

$$\int_M uPu\,dv \leq \frac{(n^2-4)(n-4)}{16n(n-1)^2}\int_M R_u^2\,dv_u \tag{16}$$

for all u smooth and positive on M. We will now see that a suitable choice of the conformal weight will lead to the desired inequality. We choose $h_1 \in C^\infty(M)$ a smooth eigenfunction of the conformal Laplacian associated with its first eigenvalue, that is $Lh_1 = \lambda_1(L)h_1$. It is a standard fact that h_1 can be chosen to be positive on M. Let $g_{u_1} = u_1{}^{\frac{4}{n-4}}g \in [g]$ where $u_1 := h_1^{\frac{n-4}{n-2}}$ is a smooth positive function on M. Applying (16) in the metric g_{u_1} leads to:

$$\int_M u_1Pu_1\,dv \leq \frac{(n^2-4)(n-4)}{16n(n-1)^2}\int_M R_{u_1}^2\,dv_{u_1}.$$

The choice of u_1 allows to compute that the scalar curvature of the manifold (M^n, g_{u_1}) is given by:

$$R_{u_1} = h_1^{-\frac{n+2}{n-2}}L(h_1) = \lambda_1(L)h_1^{-\frac{4}{n-2}} \tag{17}$$

and thus (16) reads:

$$\int_M u_1Pu_1\,dv \leq \frac{(n^2-4)(n-4)}{16n(n-1)^2}\lambda_1(L)^2\int_M h_1^{2\frac{n-4}{n-2}}\,dv = \frac{(n^2-4)(n-4)}{16n(n-1)^2}\lambda_1(L)^2\int_M u_1^2\,dv.$$

Inequality (15) follows directly from the variational characterization of $\lambda_1(P)$. If equality is achieved, then it is clear that the manifold (M^n, g_{u_1}) is Einstein. However, with the help of (17) we easily conclude that u_1 has to be constant and thus (M^n, g) is also an Einstein manifold. ■

Inequality (12) and Theorem 3 then give:

Corollary 2. *Under the assumptions of Theorem* 3*, if M is spin and $\lambda_1(L) > 0$, then:*

$$\lambda_1(D)^4 \geq \frac{n^2}{16(n-1)^2}\lambda_1^2(L) \geq \frac{n^3}{(n^2-4)(n-4)}\lambda_1(P).$$

Equality in both inequalities is characterized by the existence of a Killing spinor, in particular the manifold is Einstein.

5 Relation with the σ_2-scalar curvatures

For completeness, we recall the notion of σ_k-scalar curvatures introduced by Viaclovsky [22] which is essential for Wang's result (see Theorem 6). We first briefly recall standard properties of these curvatures and then we will recall some key facts for the proof of inequality (20). For a complete introduction to the subject, the reader may consult [22, 11].

On a closed compact Riemannian n-dimensional manifold (M^n, g), the Riemann curvature tensor can be decomposed as:

$$\mathrm{Riem} = W + S \odot g$$

where $\odot$ denotes the Kulkarni–Nomizu product, W the Weyl tensor and S the Schouten tensor. To define the σ_k-scalar curvature, we need to introduce the k-th elementary functions associated with a symmetric $n \times n$ matrix A. Namely, if A is such a matrix, we let:

$$\sigma_k(A) = \sigma_k(\Lambda),$$

where $\Lambda := (\lambda_1, \ldots, \lambda_n)$ denotes the set of eigenvalues of A and σ_k is the classical k-th elementary function given by:

$$\sigma_k(\lambda_1, \ldots, \lambda_n) := \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.$$

Following Viaclovsky [22], the σ_k-scalar curvature is then defined by:

$$\sigma_k(g) := \sigma_k(g^{-1}S)$$

$g^{-1}S$ is the $(1-1)$-tensor locally defined by: $(g^{-1}S)_i^j = g^{jk}S_{ki}$. One can explicitly compute $\sigma_k(g)$ in terms of curvature invariants. Indeed, for $k = 1$, we get that:

$$\sigma_1(g) = \mathrm{Tr}(S) = \frac{1}{2(n-1)} R. \tag{18}$$

For $k = 2$, one has:

$$\sigma_2(g) := \sum_{1 \le i < j \le n} \lambda_i \lambda_j = \frac{1}{2}\big(\mathrm{Tr}(S)^2 - |S|^2\big). \tag{19}$$

We also define the Garding's cone by:

$$\Gamma_k^+ = \{\Lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \,/\, \forall j \le k,\ \sigma_j(\Lambda) > 0\}$$

and we say that a Riemannian metric g belongs to Γ_k^+ if and only if $\sigma_k(g_x) \in \Gamma_k^+$ for all $x \in M$. In his thesis, Jeff Viaclovsky studies the following σ_k-*Yamabe problem*: can one find a Riemannian metric g_u in the conformal class of g such that its σ_k-scalar curvature is constant? The classical Yamabe problem is a famous problem in Riemannian Geometry which consists of finding a metric in the conformal class of g with constant scalar curvature R. This problem has been solved by Yamabe [25], Trüdinger [21], Aubin [3] and finally Schoen [20] in the middle of the eighties (see also [17] for a complete review and a unified approach). Using (18), one can note that finding a metric $g_u \in [g]$ with $\sigma_1(g_u)$ constant is equivalent to solve the Yamabe problem. As shown in [22], the σ_k-Yamabe problem is closely related to the behavior of the functional:

$$\mathcal{F}_k : g_u \mapsto \int_M \sigma_k(g_u)\, dv_u$$

on the space of metrics in the conformal class of g with unit volume. Indeed, among other things he proves the following result:

Theorem 4 ([22]). *If $k \neq n/2$ and $(M^n, [g])$ is a locally conformally flat manifold, a metric $g_u \in [g]$ with unit volume is a critical point of the functional $\mathcal{F}_k$ on the space of metrics conformal to g and with unit volume if and only if $\sigma_k(g_u) = \mu_k$, for some constant μ_k.*

The assumptions in the preceding statement exclude the case $k = n/2$ which is precisely of main interest in our context. Consider first a four dimensional closed compact Riemannian manifold (M^4, g) (and thus $k = 2$), then the σ_2-Yamabe problem on 4-manifolds consists in finding a smooth function u_2 such that $\sigma_2(g_{u_2})$ is constant.

This highly non-linear problem has been studied and solved by S.-Y. Chang, M. Gursky and P. Yang in [6] (see also [11]) using various subtle methods in geometric analysis. One of the main difficulties comes from the conformal invariance of the total σ_2-curvature on 4-dimensional manifolds. Indeed, a simple calculation using (19) gives:

$$8\sigma_2(g) = \frac{1}{12}R^2 - |E|^2$$

and with the help of the Chern–Gauss–Bonnet formula (5) we get:

$$16\pi^2\chi(M^4) = \frac{1}{2}\int_M |W|^2 dv + 8\int_M \sigma_2(g) dv.$$

One can then apply formulae (6) to obtain:

$$4\int_M \sigma_2(g) dv = \int_M Q dv,$$

and thus the conformal invariance follows directly. In [6], to solve the σ_2-Yamabe problem, the authors first prove a nice result which is the key point in the solution of this problem. More precisely, they prove:

Theorem 5 ([6]). *Let (M^4, g) be a compact 4-dimensional manifold and denote by $\lambda_1(L)$ the first eigenvalue of the Yamabe operator. Assume that $\lambda_1(L) > 0$, and $\int_M \sigma_2(g)dv > 0$, then there exists a metric $\overline{g} \in [g]$ such that $\overline{g} \in \Gamma_2^+$.*

This theorem shows that one can conformally deform the σ_2-scalar curvature into a positive function f on M. Then to obtain solutions of the σ_2-Yamabe problem, one deforms this function into a constant using the continuity method together with a degree-theoretic argument.

An analogous problem on even-dimensional locally conformally flat manifolds (M^n, g) with $k = n/2$ has also been studied (see [10] or [18] for example). One of the difficulties in this setting also comes from the conformal invariance of the functional $\mathcal{F}_{n/2}$. Indeed as shown in [22], one can observe using the Chern–Gauss–Bonnet formula that on locally conformally flat manifolds:

$$\chi(M^n) = c_n \int_M \sigma_{n/2}(g) dv,$$

which gives the conformal invariance of $\mathcal{F}_{n/2}$.

Here we give a result of Wang [23] in which the spectrum of the Dirac operator is compared to the σ_2-scalar curvature. We have:

Theorem 6 ([23]). *Let (M^4, g) be a closed Riemannian spin manifold with $\lambda_1(L) > 0$ and $\int_M \sigma_2(g)dv > 0$. Then the first eigenvalue of the Dirac operator satisfies*

$$\lambda_1(D)^4 \geq \frac{32}{3}\frac{\int_M \sigma_2(g)dv}{\mathrm{Vol}(M^4, g)} = \frac{8}{3}\frac{\int_M Q dv}{\mathrm{Vol}(M^4, g)}. \tag{20}$$

Equality holds if and only if (M^4, g) is isometric to the round sphere $(\mathbb{S}^4, g_{st})$.

Proof. His proof is based on the Hijazi inequality (11) and on the results of Chang, Gursky and Yang [6] and [22]. Indeed, taking the square of Inequality (11) and using (18), one gets:

$$\lambda_1(D)^4 \geq \frac{n^2}{16(n-1)^2} \sup_u \inf_M \left(R_u^2 e^{4u}\right) = \frac{n^2}{4} \sup_u \inf_M \left(\sigma_1(g_u)^2 e^{4u}\right).$$

On the other hand, we can check that for n even, we have:

$$\sigma_1(g_u)^2 \geq \frac{2n}{n-1}\sigma_2(g_u),$$

so we finally get:

$$\lambda_1(D)^4 \geq \frac{n^3}{2(n-1)} \sup_u \inf_M \left(\sigma_2(g_u)e^{4u}\right).$$

For $n = 4$, the above inequality reads:

$$\lambda_1(D)^4 \geq \frac{32}{3} \sup_u \inf_M \left(\sigma_2(g_u)e^{4u}\right).$$

It remains to compare the right-hand side of the preceding estimate as in Section 3.3.1. First note that:

$$\inf_M \left(\sigma_2(g_u)e^{4u}\right) \leq \frac{1}{\mathrm{Vol}(M^4, g)} \int_M \sigma_2(g_u)e^{4u}dv = \frac{1}{\mathrm{Vol}(M^4, g)} \int_M \sigma_2(g_u)dv_u$$

and by the conformal invariance of the σ_2-scalar curvature in dimension four, we get:

$$\inf_M \left(\sigma_2(g_u)e^{4u}\right) \leq \frac{1}{\mathrm{Vol}(M^4, g)} \int_M \sigma_2(g)dv$$

for all $u \in C^\infty(M)$. On the other hand, under the hypothesis of Theorem 5, there exists a smooth function u_2 such that (see [24]):

$$\sigma_2(g_{u_2})e^{4u_2} = \mu_2.$$

Thus, we write:

$$\mu_2 = \frac{1}{\mathrm{Vol}(M^4, g)} \int_M \sigma_2(g_{u_2})e^{4u_2}dv = \frac{1}{\mathrm{Vol}(M^4, g)} \int_M \sigma_2(g)dv$$

which gives the result. Suppose now that equality is attained, then equality is also achieved in (11) which is a well-known result of Friedrich [8]. ■

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