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Some Sharp L^2 Inequalities for Dirac Type Operators*

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Abstract. We use the spectra of Dirac type operators on the sphere S^n to produce sharp L^2 inequalities on the sphere. These operators include the Dirac operator on S^n, the conformal Laplacian and Paenitz operator. We use the Cayley transform, or stereographic projection, to obtain similar inequalities for powers of the Dirac operator and their inverses in $\mathbb{R}^n$.

Key words: Dirac operator; Clifford algebra; conformal Laplacian; Paenitz operator

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This paper is dedicated to the memory of Tom Branson

1 Introduction

Sobolev and Hardy type inequalities play an important role in many areas of mathematics and mathematical physics. They have become standard tools in existence and regularity theories for solutions to partial differential equations, in calculus of variations, in geometric measure theory and in stability of matter. In analysis a number of inequalities like the Hardy–Littlewood–Sobolev inequality in $\mathbb{R}^n$ are obtained by first obtaining these inequalities on the compact manifold S^n and then using stereographic projections to $\mathbb{R}^n$ to obtain the analogous sharp inequality in that setting. See for instance [10]. This technique is also used in mathematical physics to obtain zero modes of Dirac equations in $\mathbb{R}^3$ (see [9]).

In fact the stereographic projection corresponds to the Cayley transformation from S^n minus the north pole to Euclidean space. Here we shall use this Cayley transformation to obtain some sharp L^2 inequalities on the sphere for a family of Dirac type operators. The main trick here is to employ a lowest eigenvalue for these operators and then use intertwining operators for the Dirac type operators to obtain analogous sharp inequalities in $\mathbb{R}^n$.

Our eventual hope is to extend the results presented here to obtain suitable L^p inequalities for the Dirac type operators appearing here, particularly the Dirac operator on $\mathbb{R}^n$.

2 Preliminaries

We shall consider $\mathbb{R}^n$ as embedded in the real, 2^n dimensional Clifford algebra Cl_n so that for each $x \in \mathbb{R}^n$ we have $x^2 = -\|x\|^2$. Consequently if $e_1, \dots, e_n$ is an orthonormal basis for $\mathbb{R}^n$

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then

$$e_ie_j + e_je_i = -2\delta_{ij}$$

and

$$1, e_1, \ldots, e_n, e_1e_2, \ldots, e_{n-1}e_n, \ldots, e_{j_1}, \ldots, e_{j_r}, \ldots, e_1, \ldots, e_n$$

is an orthonormal basis for Cl_n, with $1 \leq r \leq n$ and $j_1 < \cdots < j_r$.

Note that for each $x \in \mathbb{R}^n \backslash \{0\}$ we have that x is invertible, with multiplicative inverse $\frac{-x}{\|x\|^2}$. Here, up to a sign, x^{-1} is the Kelvin inverse of x. It follows that $\{A \in Cl_n : A = x_1 \cdots x_m$ with $m \in \mathbb{N}$ and $x_1, \ldots, x_m \in \mathbb{R}^n \backslash \{0\}\}$ is a subgroup of Cl_n. We shall denote this group by $GPin(n)$.

We shall need the following anti-automorphisms on Cl_n:

$$\sim: Cl_n \to Cl_n : e_{j_1} \cdots e_{j_r} \to e_{j_r} \cdots e_{j_1}$$

and

$$- : Cl_n \to Cl_n : e_{j_1} \cdots e_{j_r} \to (-1)^r e_{j_r} \cdots e_{j_1}.$$

For $A \in Cl_n$ we denote $\sim (A)$ by $\tilde{A}$ and we denote $-(A)$ by $\overline{A}$. Note that for $A = a_0 + \cdots + a_{1\ldots n}e_1 \cdots e_n$ the scalar part of $A\overline{A}$ is $a_0^2 + \cdots + a_{1\ldots n}^2 := \|A\|^2$.

Lemma 1. *If $A \in GPin(n)$ and $B \in Cl_n$ then $\|AB\| = \|A\|\|B\|$.*

Proof. $\overline{AB}AB = \overline{B}\ \overline{A}AB = \overline{B}\|A\|^2B = \|A\|^2\overline{B}B$. Therefore $Sc(\overline{AB}AB) = \|A\|^2Sc(\overline{B}B) = \|A\|^2\|B\|^2$, where $Sc(C)$ is the scalar part of C for any $C \in Cl_n$. The result follows. ■

In [1] it is shown that if $y = M(x)$ is a Möbius transformation then $M(x) = (ax+b)(cx+d)^{-1}$ where a, b, c and $d \in Cl_n$ and satisfy the conditions

(i) $a, b, c, d \in GPin(n)$.

(ii) $a\tilde{c}, \tilde{c}d, \tilde{d}b, \tilde{b}a \in \mathbb{R}^n$

(iii) $a\tilde{d} - c\tilde{c} \in \mathbb{R}\backslash\{0\}$.

In particular if we regard $\mathbb{R}^n$ as embedded in $\mathbb{R}^{n+1}$ in the usual way, then $y = (e_{n+1}x + 1)(x + e_{n+1})^{-1}$ is the Cayley transformation from $\mathbb{R}^n$ to the unit sphere S^n in $\mathbb{R}^{n+1}$. This map corresponds to the stereographic projection of $\mathbb{R}^n$ onto $S^n\backslash\{e_{n+1}\}$.

The Dirac operator in $\mathbb{R}^n$ is $\sum_{j=1}^n e_j \frac{\partial}{\partial x_j}$. Note that $D^2 = -\triangle_n$, where $\triangle_n$ is the Laplacian in $\mathbb{R}^n$, and D^4 is the bi-Laplacian $\triangle_n^2$.

3 Eigenvectors of the Dirac–Beltrami operator on S^n

We start with the Dirac operator $D_{n+1} = \sum_{j=1}^{n+1} e_j \frac{\partial}{\partial x_j}$ in $\mathbb{R}^{n+1}$. For each point in $x \in \mathbb{R}^{n+1}\backslash\{0\}$ this operator can be rewritten as $x^{-1}xD_{n+1}$. Now $xD_{n+1} = x \wedge D_{n+1} - x \cdot D_{n+1}$. Now $x \wedge D_{n+1} = \sum_{1 \leq j < k \leq n+1} e_ie_j(x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j})$ and $x \cdot D_{n+1}$ is the Euler operator $\sum_{j=1}^{n+1} x_j \frac{\partial}{\partial x_j} = r\frac{\partial}{\partial r}$ where $r = \|x\|$. It is well known and easily verified fact that if $p_m(x)$ is a polynomial homogeneous of degree $m \in \mathbb{N}$ then $x \cdot Dp_m(x) = mp_m(x)$. So in particular if $D_{n+1}p_m(x) = 0$ then $x \wedge D_{n+1}p_m(x) = -mp_m(x)$. So $p_m(x)$ is an eigenvector of the operator $x \wedge D_{n+1}$.

Further it is also easily verified that if $q_m(x)$ is homogeneous of degree $m \in -\mathbb{N}$ then $x \cdot D_{n+1}q_m(x) = mq_m(x)$. So if $D_{n+1}q = 0$ then $x \wedge D_{n+1}q = mq$ and q is an eigenvector of $x \wedge D_{n+1}$.

Now let us suppose that $p_m : \mathbb{R}^{n+1} \to Cl_{n+1}$ is a harmonic polynomial homogeneous of degree $m \in \mathbb{N}$. In [12] it is shown that $p_m(x) = p_{m,1}(x) + xp_{m-1,2}(x)$ where $D_{n+1}p_{m,1}(x) = D_{n+1}p_{m-1,2}(x) = 0$, with $p_{m,1}(x)$ homogeneous of degree m and $p_{m-1,2}(x)$ homogeneous of degree $m-1$.

Definition 1. Suppose U is a domain in $\mathbb{R}^{n+1}$ and $f : U \to Cl_{n+1}$ is a C^1 function satisfying $D_{n+1}f = 0$ then f is called a left monogenic function.

A similar definition can be given for right monogenic functions. See [4] for details.

In [14] it is shown that if U is a domain in $\mathbb{R}^{n+1}\backslash\{0\}$ and $f : U \to Cl_{n+1}$ is left monogenic then the function $G(x)f(x^{-1})$ is left monogenic on the domain $U^{-1} = \{x \in \mathbb{R}^{n+1} : x^{-1} \in U\}$ where $G(x) = \frac{x}{\|x\|^{n+1}}$. Note that on $S^n \cap U^{-1}$ for any function g defined on U the functions $G(x)g(x^{-1})$ and $xg(x^{-1})$ coincide.

Let H_m denote the restriction to S^n of the space of Cl_n valued harmonic polynomials homogeneous of degree $m \in \mathbb{N} \cup \{0\}$. This is the space of spherical harmonics homogeneous of degree m. Further let P_m denote the restriction to S^n of left monogenic polynomials homogeneous of degree $m \in \mathbb{N} \cup \{0\}$, and let Q_m denote the restriction to S^n of the space of left monogenic functions homogeneous of degree $-n-m$ where $m = 0, 1, 2, \ldots$. Then we have illustrated that $H_m = P_m \oplus Q_m$. This result was established in the quaternionic case in [15] and independently for all n in [14].

As $L^2(S^n) = \sum_{m=0}^{\infty} H_m$ then it follows that $L^2(S^n) = \sum_{m=0}^{\infty} P_m \oplus Q_m$ where $L^2(S^n)$ is the space of Cl_{n+1} valued square integrable functions on S^n. Further we have shown that if $p_m \in P_m$ then p_m is an eigenvector of the Dirac–Beltrami operator Γ_w, where Γ_w is the restriction to S^n of $x \wedge D_{n+1}$. Here $w \in S^n$. Further p_m has eigenvalue m. Also if $q_m \in Q_m$ is an eigenvector of Γ_w with eigenvalue $-n-m$. Consequently the spectrum, $\sigma(\Gamma_w)$ of the Dirac–Beltrami operator Γ_w is $\{0\} \cup \mathbb{N} \cup \{-n, -n-1, \ldots\}$. As $0 \in \sigma(\Gamma_w)$ the linear operator $\Gamma_w : L^2(S^n) \to L^2(S^n)$ is not invertible.

Further within our calculations we have also shown that if $h : S^n \to Cl_{n+1}$ is a C^1 function then $\Gamma_w wh(w) = -nwh(w) - w\Gamma_w h(w)$. By completeness this extends to all of $L^2(S^n)$.

4 Dirac type operators in $\mathbb{R}^n$ and S^n and conformal structure

The Dirac type operators that we shall consider here in $\mathbb{R}^n$ are integer powers of D. Namely D^m for $m \in \mathbb{N}$. In [3] it is shown that if $y = M(x) = (ax+b)(cx+d)^{-1}$ is a Möbius transformation and $f : U \to Cl_n$ is a C^k function then $D^k J_k(M,x)f(M(x)) = J_{-k}(M,x)D^k f(y)$, where $J_m(M,x) = \frac{\widetilde{cx+d}}{\|cx+d\|^{n+m}}$ for m an odd integer and $J_m(M,x) = \frac{1}{\|cx+d\|^{n+m}}$ for m an even integer. This describes intertwining operators for powers of the Dirac operator in $\mathbb{R}^n$ under actions of the conformal group.

In [13] the Cayley transformation $C(x) = (e_{n+1}x + 1)(x + e_{n+1})^{-1}$ is used to transform the euclidean Dirac operator, D, to a Dirac operator, D_S, over S^n. This Dirac operator is also described in [2, 5] and elsewhere. In [7] a simple geometric argument is used to show that $D_S = w(\Gamma_w + \frac{n}{2})$. Using the spectrum of Γ_w it can be seen that on $L^2(S^n)$ the operator D_S has spectrum $\sigma(D_S) = \sigma(\Gamma_w) + \frac{n}{2}$ which is always non-zero. In fact $\sigma(D_S) = \{\frac{n}{2} + m : m = 0, 1, 2, \ldots\} \cup \{-\frac{n}{2} - m : m = 0, 1, 2, 3, \ldots\}$. Consequently D_S has an inverse D_S^{-1} on $L^2(S^n)$ and following [6] the spherical Dirac operator has as fundamental solution $C_1(w, y) := D_S^{-1} \star \delta_y$ for each $y \in S^n$. Here δ_y is the Dirac delta function. In [13] it is shown that $C_1(w, y) = \frac{1}{\omega_n}\frac{y-w}{\|y-w\|^n}$ where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$. See also [11].

In fact one can for each $\alpha \in \mathbb{C}$ introduce the Dirac operator $D_\alpha := w(\Gamma + \alpha)$. Provided $-\alpha$ is not in $\sigma(\Gamma_w)$ then D_α is invertible and has fundamental solution $D_\alpha^{-1} \star \delta_y$. See [16] for further details. A main advantage that the Dirac operator D_S has over D_α for α not equal to $\frac{n}{2}$ is that D_S is conformally invariant. We shall use this fact to obtain our sharp inequalities in $\mathbb{R}^n$.

By applying D_S to $C_2(w,y) := \frac{1}{(n-2)\omega_n}\frac{1}{\|w-y\|^{n-2}}$ it may be determined [11] that $D_S C_2(w,y) = C_1(w,y) - wC_2(w,y)$. Consequently $D_S(D_S - w)C_2(w,y) = \delta_y$.

It is well known that in $\mathbb{R}^{n+1}$ the Laplacian in spherical co-ordinates is

$$\frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\triangle_w,$$

where $\triangle_w$ is the Laplace–Beltrami operator on S^n. It follows from arguments presented in [15] that $\triangle_w = ((1-n) - \Gamma_w)\Gamma_w$. Using this fact we can now simplify the expression $D_S(D_S - w)$ as follows:

$$D_S(D_S - w) = D_S^2 - D_S w.$$

But

$$D_S w = w\left(\Gamma_w + \frac{n}{2}\right)w = w^2\left(-\Gamma_w - n + \frac{n}{2}\right) = -wD_S.$$

So

$$\begin{aligned} D_S^2 - D_S w = D_S^2 + wD_S &= D_S w\left(\Gamma_w + \frac{n}{2}\right) + wD_S = -wD_S\left(\Gamma_w + \frac{n}{2}\right) + wD_S \\ &= \left(\Gamma_w + \frac{n}{2}\right)\left(\Gamma_w + \frac{n}{2}\right) - \left(\Gamma_w + \frac{n}{2}\right) = \Gamma_w^2 + n\Gamma_w - \Gamma_w + \frac{n^2}{4} - \frac{n}{2} \\ &= -\triangle_w + \frac{n^2 - 2n}{4} = -\triangle_w + \frac{n}{2}\left(\frac{n-2}{2}\right). \end{aligned}$$

This operator is the conformal Laplacian $\triangle_S$ on S^n described in [2, 5] and elsewhere.

One may also introduce generalized spherical Laplacians of the type $\triangle_{\alpha,\beta} = (\Gamma_w + \alpha)(\Gamma_w + \beta)$ where α and $\beta \in \mathbb{C}$. Provided $-\alpha$ and $-\beta$ do not belong to $\sigma(\Gamma_w)$ then the Laplacian is invertible with fundamental solution $\triangle_{\alpha,\beta}^{-1} \star \delta_y$. In [11] it is shown that $\triangle_{\alpha,-\alpha-n+1}$ is a scalar valued operator. This operator is invertible provided α does not belong to $\sigma(\Gamma_w)$. Further, explicit formulas for this operator are presented in [11].

Again a main advantage of the conformal Laplacian, $\triangle_S$ over the other choices of Laplacians presented here is its conformal covariance. We shall see the advantage of this in the next section.

In [11] we introduce the operators

$$D_S^{(k)} := D_S(D_S - w)\cdots\left(D_S - \frac{(k-1)}{2}w\right)$$

for k odd, $k > 0$, and

$$D_S^{(k)} := D_S(D_S - w)(D_S - w)\cdots\left(D_S - \frac{k}{2}w\right)$$

for k even and $k > 0$.

When $k = 1$ we obtain D_S, when $k = 2$ we obtain $\triangle_S$ and when $k = 4$ the operator $D_S^{(4)} = \triangle_S(D_S - w)(D_S - 2w)$. Moreover

$$(D_S - w)(D_S - 2w) = D_S^2 - wD_S - 2D_S w - 2 = D_S^2 + wD_S - 2 = -\triangle_S - 2.$$

Consequently $D_S^{(4)} = -\triangle_S(\triangle_S + 2)$. When $n = 4$ this operator becomes $-\triangle_S(\triangle_S + 2)$ is the Paenitz operator on S^4 described in [2] and elsewhere. As $2 \in \sigma(D_S)$ when $n = 4$ it may be seen that 0 is in the spectrum of $D_S - 2w$. Consequently when $n = 4$ zero is in the spectrum of the Paenitz operator and so this operator is not invertible on $L^2(S^4)$. It is easy to see that it is invertible in all other dimensions.

5 Some Sharp L^2 inequalities on S^n and $\mathbb{R}^n$

Theorem 1. *Suppose that $\phi : S^n \to Cl_{n+1}$ is a C^1 function. Then*

$$\|D_S\phi\|_{L^2} \geq \frac{n}{2}\|\phi\|_{L^2}.$$

Proof. As $\phi \in C^1(S^n)$ then $\phi \in L^2(S^n)$. It follows that

$$\phi = \sum_{m=0}^{\infty} \sum_{p_m \in P_m} p_m + \sum_{m=0}^{-\infty} \sum_{q_m \in Q_m} q_m,$$

where p_m and q_m are eigenvectors of Γ_w. Further the eigenvectors p_m can be chosen so that within P_m they are mutually orthogonal. The same can be done for the eigenvectors q_m. Moreover as $\phi \in C^1$ then $D_S\phi \in C^0(S^n)$ and so $D_S\phi \in L^2(S^n)$. Consequently

$$D_S\phi = w\left(\sum_{m=0}^{\infty}\left(m + \frac{n}{2}\right) \sum_{p_m \in P_m} p_m + \sum_{m=0}^{\infty}\left(-\frac{n}{2} - m\right) \sum_{q_m \in Q_m} q_m\right).$$

But $wp_m(w) \in Q_m$ and $wq_m(w) \in P_m$. Consequently

$$D_S\phi = \sum_{m=0}^{\infty}\left(m + \frac{n}{2}\right) \sum_{q_m \in Q_m} q_m + \sum_{m=0}^{\infty}\left(-\frac{n}{2} - m\right) \sum_{p_m \in P_m} p_m.$$

It follows that

$$\begin{aligned}\|D_S\phi\|_{L^2} &= \sum_{m=0}^{\infty}\left(m + \frac{n}{2}\right)^2 \sum_{q_m \in Q_m} \|q_m\|_{L^2}^2 + \sum_{m=0}^{\infty}\left(-\frac{n}{2} - m\right)^2 \sum_{p_m \in P_m} \|p_m\|_{L^2}^2 \\ &\geq \left(\frac{n}{2}\right)^2\left(\sum_{m=0}^{\infty} \sum_{p_m \in P_m} \|p_m\|_{L^2}^2 + \sum_{m=0}^{-\infty} \sum_{q_m \in Q_m} \|q_m\|_{L^2}^2\right)\end{aligned}$$

as $\pm\frac{n}{2}$ are the smallest eigenvalues of $\Gamma_w + \frac{n}{2}$. That is $\pm\frac{n}{2}$ are the eigenvalues closest to zero. Therefore

$$\|D_S\phi\|_{L^2}^2 \geq \left(\frac{n}{2}\right)^2 \|\phi\|_{L^2}^2.$$

The result follows. ■

It should be noted from the proof of Theorem 1 that this inequality is sharp.

In the proof of Theorem 1 it is noted that the operator D_S takes P_m to Q_m and it takes Q_m to P_m. This is also true of the operator $D_S + \alpha w$ for any $\alpha \in \mathbb{C}$. As $\triangle_S = D_S(D_S + w)$ it now follows that the spectrum, $\sigma(\triangle_S)$, of the conformal Laplacian, $\triangle_S$, is $\{-(\frac{n}{2} + m)(\frac{n}{2} + m + 1), -(\frac{n}{2} + m)(\frac{n}{2} + m - 1) : m \in \mathbb{N} \cup \{0\}\}$. So the smallest eigenvalue is $\frac{n(2-n)}{4}$. We therefore have the following sharp inequality:

Theorem 2. *Suppose $\phi : S^n \to Cl_{n+1}$ is a C^2 function. Then*

$$\|\triangle_S\phi\|_{L^2} \geq \frac{n(n-2)}{4}\|\phi\|_{L^2}.$$

We now proceed to generalize Theorems 1 and 2 for all operators $D_S^{(k)}$. We begin with:

Lemma 2. *(i) For k even the smallest eigenvalue of D_S^k is*

$$\frac{n(2-n)\cdots(n+k-2)(k-n)}{2^k}$$

and

(ii) for k odd

$$\frac{n(n+2)(2-n)\cdots(n+k-1)(k-1-n)}{2^k}.$$

Proof. Let us first assume that k even. As $D_S+\alpha w: P_m \to Q_m$ and $D_S+\alpha w: Q_m \to P_m$ for any $\alpha \in \mathbb{R}$ then

$$\frac{(n+2m)(2-n-2m)\cdots(n+k-2+2m)(k-n-2m)}{2^k}$$

and

$$\frac{(2m-n)(n+2+2m)\cdots(k-2-n-2m)(n+k+2m)}{2^k}$$

are eigenvalues of $D_S^{(k)}$ for $m=0,1,2,\ldots$. But for any positive even integer l the term $(n+l-2+2m)(l-n-2m)$ is closer to zero than $(l-2-n-2m)(n+l+2m)$. The result follows for k even. The case k is odd is proved similarly. ∎

It should be noted that when n is even and $k \geq n$ then 0 is an eigenvalue of $D_S^{(k)}$. Consequently in these cases $D_S^{(k)}$ is not an invertible operator on $L^2(S^n)$.

From Lemma 2 we have:

Theorem 3. *Suppose $\phi: S^n \to Cl_{n+1}$ is a C^k function. Then for k even*

$$\|D_S^{(k)}\phi\|_{L^2} \geq \frac{|n(2-n)\cdots(n+k-2)(k-n)|}{2^k}\|\phi\|_{L^2}$$

and for k odd

$$\|D_S^{(k)}\phi\|_{L^2} \geq \frac{|n(n+2)(2-n)\cdots(n+k-1)(k-1-n)|}{2^k}\|\phi\|_{L^2}.$$

Again these inequalities are sharp.

When n is odd then of course $\frac{n}{2}$ is not an integer. It follows that in odd dimensions zero is not an eigenvalue for the operator $D_S^{(k)}$. In the cases n even and $k \geq n$ the smallest eigenvalue is zero so for those cases the inequality in Theorem 3 is trivial. This includes the Paenitz operator on S^4. It follows that none of these operators have fundamental solutions. The fundamental solutions for $D_S^{(k)}$ for all k when n is odd and for $1 \leq k < n$ when n is even are given in [11]. We shall denote them by $C_k(w,y)$.

As $\frac{n(n+2)(2-n)\cdots(n+k-1)(nk-1-n)}{2^k}$ is the smallest eigenvalue for $D_S^{(k)}$ for k odd then

$$\frac{-2^k}{n(n+2)(2-n)\cdots(n+k-1)(k-1+n)}$$

is the largest eigenvalue of $D_S^{(k)-1}$ for n odd or for $1 \leq k \leq n-1$ when n is even.

Similarly for k even and n odd and k even with $1 < k < n-1$ for n even

$$\frac{2^k}{n(2-n)\cdots(n+k-2)(k-n)}$$

is the largest eigenvalue of $D_S^{(k)-1}$.

Similarly to Theorem 3 we now have the following sharp inequality:

Theorem 4. *Suppose $\phi : S^n \to Cl_{n+1}$ is a continuous function. Then for n odd and k even and for n even and k even with $1 < k < n$*

$$\|C_k(w,y) \star \phi(w)\|_{L^2} \leq \frac{2^k}{|n(2-n)\cdots(n+k-2)(k-n)|}\|\phi\|_{L^2}$$

and for n odd and k odd and n even and k odd with $1 \leq k \leq n-1$

$$\|C_k(w,y) \star \phi(w)\|_{L^2} \leq \frac{2^k}{|n(n+2)(2-n))\cdots(n+k-1)(k-1-n)|}\|\phi\|_{L^2}.$$

Let us now turn to $\mathbb{R}^n$ and retranslate Theorems 3 and 4 in this context. In [11] the Cayley transformation $C(x) = (e_{n+1}+1)(x+e_{n+1})^{-1}$ is used to show that

$$D_S^{(k)} = J_{-k}(C,x)^{-1} D^k J_k(C,x), \tag{1}$$

where $J_k(C,x) = \frac{2^{\frac{n-k}{2}}(x+e_{n+1})}{(\|1+\|x\|^2)^{\frac{n-k+1}{2}}}$ when k is odd and $J_k(C,x) = \frac{2^{\frac{n-k}{2}}}{(1+\|x\|^2)^{\frac{n-k}{2}}}$ when k is even. Note that $J_k(C,x) \in GPin(n+1)$. By applying Lemma 1 we now see that on $\mathbb{R}^n$ the Cayley transformation can be applied to Theorem 3 to give:

Theorem 5. *Suppose $\phi : \mathbb{R}^n \to Cl_{n+1}$ is a C^k function with compact support. Then for each $k \in \mathbb{N}$ for n odd and for $k = 1, \ldots, n-1$ for n even*

$$\left(\int_{\mathbb{R}^n} \|D^k\phi(x)\|^2(1+\|x\|^2)^k dx^n\right)^{\frac{1}{2}}$$
$$\geq |n(n+2)\cdots(n+k-1)(k-1-n)| \left(\int_{\mathbb{R}^n} \frac{\|\phi\|^2 2^k}{(1+\|x\|^2)^k} dx^n\right)^{\frac{1}{2}}$$

for k odd, and

$$\left(\int_{\mathbb{R}^n} \|\triangle_n^{\frac{k}{2}}\phi(x)\|^2(1+\|x\|^2)^k dx^n\right)^{\frac{1}{2}}$$
$$\geq |n(2-n)\cdots(n+k-2)(k-n)| \left(\int_{\mathbb{R}^n} \frac{\|\phi(x)\|^2 2k}{(1+\|x\|^2)^k} dx^n\right)^{\frac{1}{2}}$$

for k even.

Proof. For any Möbius transformation $M(x) = (ax+b)(cx+d)^{-1}$ the associated Jacobian over a domain in $\mathbb{R}^n$ is $\frac{2^n}{\|cx+d\|^{2n}}$. Consequently for $\psi : S^n \to Cl_{n+1}$ a C^k function the integral $\int_{S^n} \|D_S^{(k)}\psi(w)\|^2 d\sigma(w)$ by equation (1) becomes

$$\int_{\mathbb{R}^n} \|J_{-k}(C,x)^{-1} D^k J_k(C,x)\psi(C(x))\|^2 \frac{2^n dx^n}{(1+\|x\|^2)^n}.$$

By Lemma 1 this expression becomes

$$\frac{1}{2^k}\int_{\mathbb{R}^n} (1+\|x\|^2)^k \|D^k J_k(C,x)\psi(C(x))\|^2 dx^n.$$

Further

$$\int_{S^n} \|\psi\|^2 d\sigma(x) = \int_{\mathbb{R}^n} \|\psi(C(x))\|^2 \frac{2^n dx^n}{(1+\|x\|^2)^n}$$
$$= \int_{\mathbb{R}^n} \|J_k(C,x)^{-1} J_k(C,x)\psi(C(x))\| \frac{2^n dx^n}{(1+\|x\|^2)^n}.$$

By Lemma 1 this last expression becomes

$$2^k \int_{\mathbb{R}^n} \|J_k(C,x)\psi(x)\|^2(1+\|x\|^2)^{-k}dx^n.$$

On placing $J_k(C,x)\psi(C(x)) = \phi(x)$ Theorem 3 now gives the result. ∎

In [11] it is shown that the kernel $C_k(w,y)$ is conformally equivalent to the kernel $G_k(x-y)$ in $\mathbb{R}^n$, where $G_k(x-y) = \frac{C_k}{\omega_n}\frac{x-y}{\|x-y\|^{n+1-k}}$ when k is odd and $G_k(x-y) = \frac{C_k}{\omega_n}\frac{1}{\|x-y\|^{n-k}}$ when k is even. Here C_k is a real constant chosen so that $DG_k = G_{k-1}$ for $k>1$ and with $C_1 = 1$.

As $J_{-k}(C,x)^{-1}D_S^{(k)}J_k(C,x) = D^k$ then $D^{-k} = J_k(C,x)^{-1}D_S^{(k)-1}J_{-k}(C,x)$. Consequently:

Theorem 6. *Suppose $h:\mathbb{R}^n \to Cl_{n+1}$ is a continuous function with compact support. Then for n odd and k odd and for n even and any odd integer k satisfying $1 \le k < n$*

$$\left(\int_{\mathbb{R}^n}\left\|\int_{\mathbb{R}^n} G_k(x-y)h(x)dx^n)\right\|^2 \frac{1}{(1+\|y\|^2)^k}dy^n\right)^{\frac{1}{2}}$$
$$\le \frac{1}{|n(n+2)\cdots(n+k-1)(k-1-n)|}\left(\int_{\mathbb{R}^n}\|h(x)\|^2(1+\|x\|^2)^k dx^n\right)^{\frac{1}{2}}$$

and for n odd and k even and for n even and k an even integer satisfying $1<k<n$

$$\left(\int_{\mathbb{R}^n}\left\|\int_{\mathbb{R}^n} G_k(x-y)h(x)dx^n\right\|^2 \frac{1}{(1+\|y\|^2)^k}dy^n\right)^{\frac{1}{2}}$$
$$\le \frac{1}{|n(n+2)(2-n)\cdots(n+k-2)(k-n)|}\left(\int_{\mathbb{R}^n}\|h(x)\|^2(1+\|x\|^2)^k dx^n\right)^{\frac{1}{2}}.$$

6 Dirac type operators in $\mathbb{R}^n$

In this section we demonstrate a somewhat alternative approach to obtained Theorems 5 and 6.

We have previously seen that $D_S p_m = (m+\frac{n}{2})p_m$ for $p_m \in P_m$, that $D_S q_m = (-\frac{n}{2}-m)q_m$ for $q_m \in Q_m$ and $J_{-1}^{-1}(C,x)DJ_1(C,x) = D_S$. Consequently

$$DJ_1(C,x)p_m(C(x)) = \frac{2}{1+\|x\|^2}\left(m+\frac{n}{2}\right)J_1(C,x)p_m(C(x))$$

and

$$DJ_1(C,x)q_m(C(x)) = \frac{2}{1+\|x\|^2}\left(-\frac{n}{2}-m\right)J_1(C,x)q_m(C(x)).$$

Further:

Proposition 1. *$\psi(w) \in L^2(S^n)$ if and only if $\frac{1}{(1+\|x\|^2)^{\frac{1}{2}}}J_1(C,x)\psi(C(x)) \in L^2(\mathbb{R}^n)$. Further if $\psi'(x) = \frac{1}{(1+\|x\|^2)^{\frac{1}{2}}}J_1(C,x)\psi(C(x))$ and $\phi'(x) = \frac{1}{(1+\|x\|^2)^{\frac{1}{2}}}J_1(C,x)\phi(C(x))$ for ψ and $\phi \in L^2(S^n)$ then*

$$\int_{S^n}\overline{\phi}(w)\psi(w)d\sigma(w) = \int_{\mathbb{R}^n}\overline{\phi}'(x)\psi'(x)dx^n.$$

This leads us to:

Theorem 7. *Suppose $h : \mathbb{R}^n \to Cl_{n+1}$ is a smooth function with compact support. Then*

$$\left(\int_{\mathbb{R}^n} \|Dh(x)\|^2(1+\|x\|^2)dx^n\right)^{\frac{1}{2}} \geq n\left(\int_{\mathbb{R}^n} \frac{\|h(x)\|^2}{1+\|x\|^2}dx^n\right)^{\frac{1}{2}}.$$

In [11] it is shown that $J_{-2}(C,x)^{-1}\triangle_S J_2(C,x) = \triangle_n$. Proposition 1 can easily be adapted replacing $J_1(C,x)$ by $J_2(C,x)$ and $\frac{2}{1+\|x\|^2}$ by $\frac{4}{(1+\|x\|^2)^2}$. From Theorem 2 we now have:

Theorem 8. *Suppose that h is as in Theorem 7. Then*

$$\left(\int_{\mathbb{R}^n} \|\triangle_n h(x)\|^2(1+\|x\|^2)^2dx^n\right)^{\frac{1}{2}} \geq n(n-2)\left(\int_{\mathbb{R}^n} \frac{\|h(x)\|^2}{(1+\|x\|^2)^2}dx^n\right)^{\frac{1}{2}}.$$

Using Lemma 2 we also have

Theorem 9. *Suppose h is as in Theorem 7. Then for n odd and k even and for n even and k an even integer belonging to $\{1,\dots,n-1\}$*

$$\left(\int_{\mathbb{R}^n} \|\triangle_n^{\frac{k}{2}} h(x)\|^2(1+\|x\|^2)^k dx^n\right)^{\frac{1}{2}}$$
$$\geq |n(2-n)\cdots(n+k-2)(k-n)|\left(\int_{\mathbb{R}^n} \frac{\|h(x)\|^2}{(1+\|x\|^2)^k}dx^n\right)^{\frac{1}{2}}$$

and for n odd and k odd and for n even and k belonging to $\{1,\dots,n-1\}$

$$\left(\int_{\mathbb{R}^n} \|D^k h(x)\|^2(1+\|x\|^2)^k)dx^n\right)^{\frac{1}{2}}$$
$$\geq |n(n+2)(2-n)\cdots(n+k-1)(k-1-n)|\left(\int_{\mathbb{R}^n} \frac{\|h(x)\|^2}{(1+\|x\|^2)^k}dx^n\right)^{\frac{1}{2}}.$$

Theorem 10. *Suppose $h : \mathbb{R}^n \to Cl_{n+1}$ is a continuous function with compact support. Then for k odd and n odd and for n even and k odd and satisfying $1 \leq k \leq n-1$*

$$\left(\int_{\mathbb{R}^n} \frac{\|G_k \star h(x)\|^2}{(1+\|x\|^2)^k}dx^n\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{|n(n+2)(2-n)\cdots(n+k-1)(k-1-n)|}\left(\int_{\mathbb{R}^n} \|h(x)\|^2(1+\|x\|^2)^k dx^n\right)^{\frac{1}{2}}$$

and for n odd and k even and for n even and k even and satisfying $1 < k < n$

$$\left(\int_{\mathbb{R}^n} \frac{\|G_k \star h(x)\|^2}{(1+\|x\|^2)^k}dx^n\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{|n(n+2)(2-n)\cdots(n+k-2)(k-n)|}\left(\int_{\mathbb{R}^n} \|h(x)\|^2(1+\|x\|^2)^k dx^n\right)^{\frac{1}{2}}.$$

7 Concluding remarks

Let us consider the Paenitz operator on S^5. Via the Cayley transform this operator stereographically projects to the bi-Laplacian, $\triangle_5^2$ on $\mathbb{R}^5$. If we restrict attention to the equator, S^4, of S^5 we see that the restriction of the Paenitz operator in this context stereographically projects to the restriction of $\triangle_5^2$ to R^4. This operator is the bi-Laplacian $\triangle_4^2$ in $\mathbb{R}^4$, while the restriction of the Paenitz operator on S^5 to its equator, S^4, is the Paenitz operator on S^4. The Paenitz operator on S^4 has a zero eigenvalue. Consequently there is no real hope of obtaining inequalities of the type we have obtained here in $\mathbb{R}^n$ for the bi-Laplacian in $\mathbb{R}^4$. This should explain the breakdown of the Rellich inequality, described in [8], for the bi-Laplacian in $\mathbb{R}^4$. The same rationale also explains similar breakdowns of inequalities for D^k in $\mathbb{R}^n$ for n even and $k \geq n$.

It should be clear that similar sharp L^2 inequalities can be obtained for the operator $D_S + \alpha w$ provided $-\alpha$ is not in the spectrum of wD_S. These operators conformally transform to $D + \frac{\alpha}{1+\|x\|^2}$ in $\mathbb{R}^n$. When $-\alpha$ is in the spectrum of wD_S then we obtain a finite dimensional subspace of the weighted L^2 space $L^2(\mathbb{R}^n, (1+\|x\|^2)^{-2})$, with weight $(1+\|x\|^2)^{-2}$, consisting of solutions to the Dirac equation $Du + \frac{\alpha}{1+\|x\|^2} u = 0$.

All inequalities obtained here are L^2 inequalities. It would be nice to see similar inequalities for other suitable L^p spaces.

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