Symmetry, Integrability and Geometry: Methods and Applications SIGMA **3** (2007), 106, 9 pages

Biorthogonal Expansion of Non-Symmetric Jack Functions*

Siddhartha SAHI [†] *and Genkai ZHANG* [‡]

[†] *Department of Mathematics, Rutgers University, New Brunswick, New Jersey, USA*
E-mail: *sahi@math.rutgers.edu*

[‡] *Mathematical Sciences, Chalmers University of Technology and Mathematical Sciences, Göteborg University, Sweden*
E-mail: *genkai@math.chalmers.se*

Received August 08, 2007, in final form October 31, 2007; Published online November 15, 2007
Original article is available at http://www.emis.de/journals/SIGMA/2007/106/

Abstract. We find a biorthogonal expansion of the Cayley transform of the non-symmetric Jack functions in terms of the non-symmetric Jack polynomials, the coefficients being Meixner–Pollaczek type polynomials. This is done by computing the Cherednik–Opdam transform of the non-symmetric Jack polynomials multiplied by the exponential function.

Key words: non-symmetric Jack polynomials and functions; biorthogonal expansion; Laplace transform; Cherednik–Opdam transform

2000 Mathematics Subject Classification: 33C52; 33C67; 43A90

1 Introduction

In [10] Opdam studied the non-symmetric eigenfunctions of the Cherednik operators associated to a root system with general multiplicity and proved the Plancherel formula for the corresponding "Cherednik–Opdam" transform. For the root system of type A the polynomial eigenfunctions are also called the non-symmetric Jack polynomials and they have been extensively studied (see e.g. [11]). There are other related non-symmetric polynomials such as the Laguerre polynomials which are the eigenfunctions of the Hankel transform, which is basically the Fourier transform on the underlying space. The non-symmetric Laguerre polynomials form an orthogonal basis for the L^2-space and it is thus a natural problem to find their Cherednik–Opdam transforms. In this paper we prove that they are, apart from a factor of Gamma functions, the non-symmetric Meixner–Pollaczek (MP) polynomials and we find a formula for them in terms of binomial coefficients. As a corollary we find in Theorem 1 a biorthogonal expansion of the Cayley transform of the non-symmetric Jack functions in terms of the non-symmetric Jack polynomials, the coefficients being the MP polynomials. In the one variable case the Jack function is just the power function $x^{i\lambda}$, and Theorem 1 gives an expansion of the function $(1-t^2)^{\frac{b}{2}}(\frac{1-t}{1+t})^{i\lambda}$ of the MP polynomials (6); see [6, (1.7.11)].

There are basically three important families of polynomials associated with the root system of type A, namely, the Jack type polynomials, the Laguerre polynomials, and the MP type polynomials that are orthogonal with respect the Harish-Chandra measure $|c(\lambda)|^{-2}d\lambda$ multiplied with a certain Gamma factor, which is the Heckman–Opdam transform of an exponential function. Our results give a somewhat unified picture of the relation between these polynomials and provide a combinatorial formula for the MP type polynomials. In brief, the Laplace transform maps the Laguerre polynomials into Jack polynomials, and the Cherednik–Opdam transform maps

*This paper is a contribution to the Proceedings of the 2007 Midwest Geometry Conference in honor of Thomas P. Branson. The full collection is available at http://www.emis.de/journals/SIGMA/MGC2007.html the Laguerre polynomials into MP type polynomials. In the case when the root multiplicities correspond to that of a symmetric cone some results of this type have been obtained in [4, 14] and [3].

2 Non-symmetric Jack functions and the Opdam–Cherednik transform

In this section we recall the definition of the non-symmetric Jack polynomials and functions and the Plancherel formula for the Opdam–Cherednik transform, developed in [10].

We consider the root system of type A_{r-1} in $\mathbb{R}^r$. For the purpose of studying Laplace transform we make a change of variables $x_j = e^{2t_j}$ where $t = (t_1, \dots, t_r) \in \mathbb{R}^r$, and consider functions on $x \in \mathbb{R}^r_+$ instead. We fix an ordering of the roots so that (with some abuse of notation) the positive roots are $x_2 - x_1, x_3 - x_2, \dots, x_r - x_{r-1}$ with root multiplicity $a := \frac{2}{\alpha}$ and we will identify the roots as vectors in $\mathbb{R}^r$. Let ρ be the half sum of positive roots, so that $\rho = (\rho_1, \rho_2, \dots, \rho_r) = \frac{1}{\alpha}(-r+1, -r+3, \dots, r-1)$. We consider the measure

$$d\mu(x) = \frac{1}{2^r}(x_1 \cdots x_r)^{-\frac{1}{\alpha}(r-1)-1} \prod_{1 \le j < k \le r} |x_j - x_k|^a dx_1 \cdots dx_r$$

on $\mathbb{R}^r_+$ and the corresponding Hilbert space $L^2(\mathbb{R}^r_+, d\mu)$.

We consider the Dunkl operators

$$T_j = \partial_j + \frac{1}{\alpha} \sum_{i \ne j} \frac{1}{x_j - x_i}(1 - s_{ij})$$

and the Cherednik operators

$$U_j = U_j^A = x_j \partial_j + \frac{1}{\alpha} \sum_{i<j} \frac{x_j}{x_j - x_i}(1 - s_{ij}) + \frac{1}{\alpha} \sum_{j<k} \frac{x_k}{x_j - x_k}(1 - s_{jk}) - \frac{1}{2}\rho_j.$$

Here $\partial_j = \frac{\partial}{\partial x_j}$ and s_{ij} stands for the permutation (ij) that acts on functions $f(x_1, \dots, x_r)$ by interchanging the variables x_i and x_j. The operators $\{U_j\}$ can be expressed in terms of $\{T_j\}$ and the multiplication operators $\{x_j\}$, but we will not need this here. The Dunkl operators $\{T_j\}$ commute with each other, as do the Cherednik operators $\{U_j\}$.

The polynomial eigenfunctions of the operators $\{U_j\}$ are the non-symmetric Jack polynomials $E_\eta(x) = E(\eta, x)$, with $\eta = (\eta_1, \dots, \eta_r) \in \mathbb{N}^r$. They are characterized as the unique eigen-polynomials of $\{U_j\}$ with leading coefficients $x^\eta = x_1^{\eta_1} \cdots x_r^{\eta_r}$ in the sense that

$$E_\eta = x^\eta + \sum_{\zeta < \eta} c_{\eta\zeta} x^\zeta.$$

We recall that $\zeta < \eta$ here stands for the partial ordering defined by

$$\zeta < \eta \qquad \text{iff} \quad \begin{cases} \zeta^+ < \eta^+, & \zeta^+ \ne \eta^+, \\ \zeta < \eta, & \zeta^+ = \eta^+, \end{cases}$$

where η^+ is the unique partition obtained by permuting the entries of ζ and $<$ stands for the natural dominance ordering: $\zeta < \eta$ iff $\sum_{j=1}^p (\zeta_j - \eta_j) \ge 0$, $1 \le p \le r$.

The function E_η has holomorphic extension in the variable η. More precisely, there exists a function $G_\lambda(x) = G(\lambda, x)$ which we call the non-symmetric Jack function, real analytic in $x \in \mathbb{R}^r_+$ and holomorphic in $\lambda \in \mathbb{C}^r$, such that $G(\lambda, 1^r) = 1$, with $1^r = (1, \dots, 1)$, and

$$U_j G(\lambda, x) = \lambda_j G(\lambda, x).$$

The relation between $G_\lambda(x)$ and $E_\eta(x)$ is

$$G_{\eta+\rho}(x) = \mathcal{E}_\eta(x) := \frac{E_\eta(x)}{E_\eta(1^r)}. \tag{1}$$

The value $E_\eta(1^r)$ has been computed by Sahi [11],

$$E_\eta(1^r) = \frac{e_\eta}{d_\eta},$$

where e_η and d_η are defined in the next section. (See also [10] for general root systems.) Define

$$\mathcal{F}^w[f](\lambda) = \int_{\mathbb{R}^r_+} f(x)G(-\lambda, w^{-1}x)d\mu(x).$$

Then writing $d\omega(\lambda)$ for the Euclidean measure on the positive Weyl chamber $(\mathbb{R}^r)_+ = \{\lambda; \lambda_1 > \cdots > \lambda_r\}$, we have

$$\int_{\mathbb{R}^r_+} |f(x)|^2 d\mu(x) = \sum_{w\in S_r} \int_{i(\mathbb{R}^r)_+} \mathcal{F}^w[f](\lambda)\overline{\mathcal{F}^w(f)(\lambda)}d\tilde{\mu}(\lambda); \tag{2}$$

the Plancherel measure $d\tilde{\mu}$ is given by

$$d\tilde{\mu}(\lambda) = \frac{(2\pi)^{-r}\tilde{c}^2_{w_0}(\rho(k), k)}{\tilde{c}(\lambda)c(w_0\lambda)}d\omega(\lambda),$$

where $w_0 \in S_r$ is the longest Weyl group element, and $c(\lambda)$, $\tilde{c}_w(\lambda)$ are the Harish-Chandra c-functions [10].

3 Laplace transform

Adapting the notation in [2] we denote $q := 1 + \frac{1}{\alpha}(r-1)$. Recall that the non-symmetric Laplace transform is defined by

$$\tilde{\mathcal{L}}[f](t) = \int_{[0,\infty)^r} \mathcal{K}_A(-t, x)f(x)d\mu(x),$$

where

$$\mathcal{K}_A(t, y) = \sum_\eta \alpha^{|\eta|}\frac{1}{d'_\eta}\mathcal{E}_\eta(t)E_\eta(y)$$

is the non-symmetric analogue of the hypergeometric ${}_0F_0$ function. It satisfies $\mathcal{K}_A(cx; y) = \mathcal{K}_A(x; cy)$ and $\mathcal{K}_A(wx; y) = \mathcal{K}_A(x; wy)$ for any $w \in S_r$.

Here each tuple η will be identified with a diagram of nodes $s = (i, j)$, $1 \le j \le \eta_i$, $d_\eta = \prod_{s\in\eta} d(s)$, $d'_\eta = \prod_{s\in\eta} d'(s)$, $e'_\eta = \prod_{s\in\eta} e(s)$ with

$$d'(s) = \alpha(a(s)+1) + l(s), \qquad d(s) = d'(s) + 1, \qquad e(s) = \alpha(a'(s)+1) + r - l'(s))$$

and $a(s) = \eta_i - j$, $a'(s) = j - 1$ being the arm length, arm colength and

$$l(s) = \#\{k > i : j \le \eta_k \le \eta_i\} + \#\{k < i : j \le \eta_k + 1 \le \eta_i\},$$
$$l'(s) = \#\{k > i : \eta_k > \eta_i\} + \#\{k < i : \eta_k \le \eta_i\},$$

the leg length and leg colength, which were defined in [11] and [5].

Remark 1. Note that the Laplace transform $\tilde{\mathcal{L}}$ defined here differs from the Laplace transform $\mathcal{L}$ defined in [2] by a factor $(\prod_{j=1}^r x_j)^{-q}$ in the integration. More precisely our measure $d\mu$ is $(\prod_{j=1}^r x_j)^{-q}$ times the finite measure $\prod_{1\le j<k\le r} |x_j - x_k|^a dx_1 \cdots dx_r$ used in [2, (3.67)], so that

$$\widetilde{\mathcal{L}}[f(x)] = \mathcal{L}\Big[f(x)\Big(\prod_{j=1}^r x_j\Big)^{-q}\Big].$$

The advantage of $\widetilde{\mathcal{L}}$ is that there is no shift of q in Lemma 1 below, and that in the case of symmetric spaces of type A the measure $d\mu$ is precisely the radial part of the invariant measure [4].

The function $\mathcal{K}_A(x,y)$ generalizes the exponential function in the sense that

$$T_i^{(x)}\mathcal{K}_A(x,y) = y_i\mathcal{K}_A(x,y), \tag{3}$$

where $T_i^{(x)}$ is the Dunkl operator acting on the variable x.

Recall further the definition of generalized Gamma function

$$\Gamma_\alpha(\kappa) = \prod_{j=1}^r \Gamma\left(\kappa_j - \frac{1}{\alpha}(j-1)\right)$$

and the Pochammer symbol

$$[\nu]_\kappa = \frac{\Gamma_\alpha(\nu+\kappa)}{\Gamma_\alpha(\nu)}.$$

for $\nu, \kappa \in \mathbb{C}^r$, whenever it makes sense. A scalar $c \in \mathbb{C}$ will also be identified with $(c,\dots,c) \in \mathbb{C}$ in the text below. We will also use the abbreviation $x^c = x_1^c \cdots x_r^c$ and $1+x = (1+x_1,\dots,1+x_r)$ etc.

We recall further the binomial coefficients $\binom{\eta}{\nu}$ for $\eta, \nu \in \mathbb{N}^r$ are defined by the expansion

$$\mathcal{E}_\eta(1+t) = \sum_\nu \binom{\eta}{\nu}\mathcal{E}_\nu(t). \tag{4}$$

See also [11] and [12]. We make the following generalization.

Definition 1. The binomial coefficients $\binom{\eta}{\nu}$ for any $\eta \in \mathbb{C}^r$ and $\nu \in \mathbb{N}^r$ are defined by

$$G_{\eta+\rho}(1+t) = \sum_{\nu\in\mathbb{N}^r} \binom{\eta}{\nu} E_\nu(t).$$

Since $G_\lambda(1+t)$ is an analytic function near 1 and $E_\nu(t)$ form a basis for all polynomials the above definition makes sense, and it agrees with (4) in view of the relation (1). In particular $\binom{\eta}{\nu}$ is a polynomial of $\eta \in \mathbb{C}^r$. It follows from the definition and [2, Proposition 3.18] that

$$E_\nu(T)G_{\eta+\rho}(t)|_{t=1} = \frac{d'_\nu}{\alpha^{|\nu|}}\binom{\eta}{\nu}.$$

Here $E_\nu(T)$ stands for the differential-difference operator obtained from $E_\nu(x)$ replacing x_j by T_j. We will need a slight generalization of the binomial coefficient: If $w \in S_r$ we define $\binom{\eta}{\nu}_w$ by

$$E_\nu(T)G_{\eta+\rho}(wt)|_{t=1} = \frac{d'_\nu}{\alpha^{|\nu|}}\binom{\eta}{\nu}_w. \tag{5}$$

The following lemma is proved in [2, (4.38)]. (Note that there is a typo there: $E_\eta^{(L)}(\frac{1}{x})$ should be replaced by $E_\eta(\frac{1}{x})$. For symmetric case this was a conjecture of Macdonald [8] proved by e.g. in [1, (6.1)–(6.3)].)

Lemma 1. *Suppose $c > q-1$. The Laplace transform of the functions $x^c E_\eta(x)$ is given by*

$$\widetilde{\mathcal{L}}[x^c\mathcal{E}_\eta(x)](t) = \mathcal{N}_0^{(L)}[c]_\eta \left(\prod_{j=1}^r t_j^{-c}\right)\mathcal{E}_\eta\left(\frac{1}{t}\right).$$

The normalization constant $\mathcal{N}_0^{(L)}$ (depending on c) is computed in [2].

We fix in the rest of the text $b > q-1$. For simplicity we assume also that b is an even integer. Let

$$E_\kappa^{(L)}(x) = E_\kappa^{(L,b)}(x) = \frac{(-1)^{|\kappa|}[b]_\kappa e_\kappa}{d_\kappa}\sum_\sigma \frac{(-1)^{|\sigma|}}{[b]_\sigma}\binom{\kappa}{\sigma}\mathcal{E}_\sigma(x)$$

be the non-symmetric Laguerre polynomial. The next lemma follows from [2, Proposition 4.35] after a change of variable $2x = t^2$. Our parameter b is their $a+q$.

Lemma 2. *Suppose $b > (q-1) = \frac{1}{\alpha}(r-1)$. The Laguerre functions*

$$l_\kappa(x) := l_\kappa^b(x) := E_\kappa^{(L)}(2x)e^{-p_1(x)}(2x)^{\frac{b}{2}}, \qquad p_1(x) := \sum_{j=1}^r x_j,$$

form an orthogonal basis for the space $L^2(\mathbb{R}_+^r, d\mu)$.

The norm $\|l_\kappa\|^2$ has also been explicitly evaluated in [2].

We can now compute the Cherednik–Opdam transform of the Laguerre functions. For $\eta = (\eta_1, \eta_2, \dots, \eta_r)$, denote $\eta^* = (\eta_r, \eta_{r-1}, \dots, \eta_1)$.

Proposition 1. *Suppose $b > (q-1) = \frac{1}{\alpha}(r-1)$. The Cherednik–Opdam transform $\mathcal{F}^w[l_\kappa]$, $w \in S_r$, of the Laguerre function $l_\kappa(x)$, is*

$$\mathcal{F}^w[l_\kappa](\lambda) = 2^{\frac{rb}{2}}\mathcal{N}_0^{(L)}\frac{\Gamma_\alpha(\frac{b}{2}-\rho-\lambda)}{\Gamma_\alpha(\frac{b}{2})}M_\kappa^w(\lambda),$$

where

$$M_\kappa^w(\lambda) = \frac{(-1)^{|\kappa|}[b]_\kappa e_\kappa}{d_\kappa}\sum_\sigma \frac{1}{[b]_\sigma}\frac{d'_\sigma d_\sigma}{\alpha^{|\sigma|}e_\sigma}(-2)^{|\sigma|}\binom{\kappa}{\sigma}\binom{-\frac{b}{2}+\lambda^*+\rho^*}{\sigma}_w.$$

Proof. The function l_κ is a linear combinations of the functions $e^{-p_1(x)}E_\sigma(x)$ and we compute the Cherednik–Opdam transform of these functions. We write Lemma 1 as

$$\int_{\mathbb{R}_+^r}\mathcal{K}_A(-t,x)x^{\frac{b}{2}}\mathcal{E}_\eta(x)d\mu(x) = \mathcal{N}_0^{(L)}\frac{\Gamma_\alpha(\frac{b}{2}+\eta)}{\Gamma_\alpha(\frac{b}{2})}t^{-\frac{b}{2}}\mathcal{E}_{-\eta^*}(t) = \mathcal{N}_0^{(L)}\frac{\Gamma_\alpha(\frac{b}{2}+\eta)}{\Gamma_\alpha(\frac{b}{2})}\mathcal{E}_{-\frac{b}{2}-\eta^*}(t).$$

Replacing t by wt, and using $K(wt,x) = K(t,wx)$ [2, Theorem 3.8], we find

$$\int_{\mathbb{R}_+^r}\mathcal{K}_A(-t,x)(w^{-1}x)^{\frac{b}{2}}\mathcal{E}_\eta(w^{-1}x)d\mu(x) = \mathcal{N}_0^{(L)}\frac{\Gamma_\alpha(\frac{b}{2}+\eta)}{\Gamma_\alpha(\frac{b}{2})}\mathcal{E}_{-\frac{b}{2}-\eta^*}(wt).$$

Here we have used the relations $y^c\mathcal{E}_\sigma(y) = \mathcal{E}_{c+\sigma}(y)$. Let the operator $E_\sigma(T)$ act on it evaluated as $t = 1^r = (1,\dots,1)$. The resulting equality, by (3) and the fact that $\mathcal{K}_A(x,1^r) = e^{p_1(x)}$, and (5), is,

$$\int_{\mathbb{R}_+^r}e^{-p_1(x)}\mathcal{E}_\sigma(-x)(w^{-1}x)^{\frac{b}{2}}\mathcal{E}_\eta(w^{-1}x)d\mu(x) = \mathcal{N}_0^{(L)}\frac{\Gamma_\alpha(\frac{b}{2}+\eta)}{\Gamma_\alpha(\frac{b}{2})}\frac{d'_\sigma d_\sigma}{\alpha^{|\sigma|}e_\sigma}\binom{-\frac{b}{2}-\eta^*}{\sigma}_w.$$

We write $\eta = -\lambda - \rho$. Using the relation (1) we see that

$$\mathcal{F}^w[l_\kappa](\lambda) = \int_{[0,\infty)^r} l_\kappa(x) G(-\lambda, x) d\mu(x)$$
$$= 2^{\frac{rb}{2}} \mathcal{N}_0^{(L)} \frac{\Gamma_\alpha(\frac{b}{2} - \rho - \lambda)}{\Gamma_\alpha(\frac{b}{2})} \frac{(-1)^{|\kappa|}[b]_\kappa e_\kappa}{d_\kappa} \sum_\sigma \frac{1}{[b]_\sigma} \frac{d'_\sigma d_\sigma}{\alpha^{|\sigma|} e_\sigma} (-2)^{|\sigma|} \binom{\kappa}{\sigma} \binom{-\frac{b}{2} + \lambda^* + \rho^*}{\sigma}_w .$$

as claimed. Our result for general λ follows by analytic continuation. ■

The Plancherel formula (2) then gives an orthogonality relation for the polynomials $M_\kappa^w(\lambda)$:

$$\sum_w \int_{\lambda_1 > \cdots > \lambda_r} M_\kappa^w(\lambda) \overline{M_{\kappa'}^w(\lambda)} |\Gamma_\alpha(\frac{b}{2} - \rho - \lambda)|^2 d\widetilde{\mu}(\lambda) = C \delta_{\kappa,\kappa'} \|l_\kappa\|^2$$

for some constant C independent of κ, κ'.

In the next section we will find another formula for $M_\kappa(\lambda)$.

Remark 2. In the case of one variable $r = 1$ we have $E_l = x^l$, $d_l = d'_l e_l = l!$, our polynomial is

$$M_k(\lambda) = (b)_k (-1)^k \sum_l \frac{1}{(b)_l} l! (-2)^l \binom{k}{l} \binom{-\frac{b}{2} + \lambda}{l} = (b)_k (-1)^k {}_2F_1\left(-k, \frac{b}{2} - \lambda; b; 2\right)$$

which is the MP polynomial [6, (1.7.1)] $P_k^{(b/2)}(x; \frac{\pi}{2})$, more precisely

$$M_k(\lambda) = k! i^k P_k^{(b/2)}\left(-i\lambda; \frac{\pi}{2}\right). \tag{6}$$

The functions $\Gamma(\frac{b}{2} + i\lambda) M_k(i\lambda)$ are orthogonal in the space $L^2(\mathbb{R}, d\lambda)$, as a consequence of (2).

4 A binomial formula for $M_\kappa(\lambda)$

We recall first an expansion of the kernel $\mathcal{K}_A$ in [2, Proposition 4.13]. We adopt the shorthand notation $\frac{z}{1-z} = \prod_{j=1}^r \frac{z_j}{1-z_j}$, in particular $\frac{1-z}{1+z} = \prod_{j=1}^r \frac{1-z_j}{1+z_j}$ will be called the Cayley transform of $z = (z_1, \ldots, z_r)$.

Lemma 3. *The following expansion holds*

$$(1-z)^{-b} \mathcal{K}_A\left(-x; \frac{z}{1-z}\right) = \sum_\eta (-\alpha)^{|\eta|} \frac{1}{d'_\eta} E_\eta^{(L)}(x) \mathcal{E}_\eta(z).$$

We need another expansion of the $(1-z)^{-b} \mathcal{E}_\kappa(\frac{1-z}{1+z})$ in terms of the polynomials $E_\eta(z)$.

Lemma 4. *Consider the following expansion*

$$(1-z)^{-b} \mathcal{E}_\eta\left(\frac{1-z}{1+z}\right) = \sum_\kappa \mathcal{C}_\kappa(\eta) \mathcal{E}_\kappa(z),$$

for $\eta \in \mathbb{N}^r$, $\eta_j \geq \frac{b}{2}$. The coefficients are then given by

$$\mathcal{C}_\kappa(\eta) = \mathcal{E}_{\eta - b}(-1) \sum_\sigma (-2)^{|\sigma|} \binom{\eta - b}{\sigma} \binom{-\sigma^* - b}{\kappa}$$

and is a polynomial in η.

Proof. Note that it follows from the remark after Definition 1 that $\mathcal{C}_\kappa(\eta)$ is a polynomial in η. Change variables $y_j = 1 + z_j$, $\frac{1-z_j}{1+z_j} = \frac{2}{y_j} - 1$, $1 - z_j^2 = (1+z_j)^2 \frac{1-z_j}{1+z_j} = y_j^2(\frac{2}{y_j} - 1)$. We have

$$(1-z)^{-b}\mathcal{E}_\eta\left(\frac{1-z}{1+z}\right) = (1-z^2)^{-\frac{b}{2}}\mathcal{E}_{\eta-\frac{b}{2}}\left(\frac{1-z}{1+z}\right) = \left(y^2\left(\frac{2}{y}-1\right)\right)^{-\frac{b}{2}}\mathcal{E}_{\eta-\frac{b}{2}}\left(\frac{2}{y}-1\right)$$
$$= y^{-b}\mathcal{E}_{\eta-b}\left(\frac{2}{y}-1\right) = (-1)^{|\eta-b|}y^{-b}\mathcal{E}_{\eta-b}\left(1-\frac{2}{y}\right). \tag{7}$$

We expand $\mathcal{E}_{\eta-b}(1-\frac{2}{y})$ by using the binomial formula,

$$y^{-b}\mathcal{E}_{\eta-b}\left(1-\frac{2}{y}\right) = y^{-b}\sum_\sigma \binom{\eta-b}{\sigma}\mathcal{E}_\sigma\left(-\frac{2}{y}\right) = \sum_\sigma \binom{\eta-b}{\sigma}(-2)^{|\sigma|}\mathcal{E}_{-\sigma^*-b}(y).$$

Here we have used the relations $\mathcal{E}_\sigma(-\frac{2}{y}) = (-2)^{|\sigma|}\mathcal{E}_{-\sigma^*}(y)$ and $y^{-b}E_{-\sigma^*}(y) = E_{-\sigma^*-b}(y)$; see [2]. Now each term $\mathcal{E}_{-\sigma^*-b}(y) = \mathcal{E}_{-\sigma^*-b}(1+z)$ can again be expanded in terms of $\mathcal{E}_\kappa(z)$. Interchanging the order of summation we find that (7) is

$$\mathcal{E}_{\eta-b}(-1)\sum_\kappa\left(\sum_\sigma\binom{\eta-b}{\sigma}(-2)^{|\sigma|}\binom{-\sigma^*-b}{\kappa}\right)\mathcal{E}_\kappa(z).$$

This completes the proof. ■

In the statement of Lemma 4 we have written $(-1)^{|\eta-b|}$ as $\mathcal{E}_{\eta-b}(-1)$ since the latter has analytic continuation in η, while on the other hand $(-1)^{|\eta-b|} = e^{i\pi\sum_{j=1}^r(\eta_j-b)}$ is already analytic in η).

Theorem 1. *The coefficients $\mathcal{C}_\kappa$ in Lemma 4 are given by*

$$\mathcal{C}_\kappa(-\lambda-\rho) = (-\alpha)^{|\kappa|}\frac{1}{d'_\kappa}M_\kappa(\lambda).$$

Thus we have an expansion

$$(1-z)^{-b}G\left(-\lambda, \frac{1-z}{1+z}\right) = \sum_\kappa(-\alpha)^{|\kappa|}\frac{1}{d'_\kappa}M_\kappa(\lambda)\mathcal{E}_\kappa(z).$$

Proof. We replace x by $2x$ in Lemma 3,

$$\prod_{j=1}^r(1-z_j)^{-b}\mathcal{K}_A\left(-x;\frac{2z}{1-z}\right) = \sum_\kappa(-\alpha)^{|\kappa|}\frac{1}{d'_\kappa}E_\kappa^{(L)}(2x)\mathcal{E}_\kappa(z).$$

Multiplying both sides by $\prod_{j=1}^r(2x_j)^{\frac{b}{2}}e^{-p_1(x)}$ and since $\mathcal{K}_A(-x;y+1^r) = \mathcal{K}_A(-x;y)e^{-p_1(x)}$ we get

$$\prod_{j=1}^r(1-z_j)^{-b}\mathcal{K}_A\left(-x;\frac{1+z}{1-z}\right)\prod_{j=1}^r(2x_j)^{\frac{b}{2}} = \sum_\kappa(-\alpha)^{|\kappa|}\frac{1}{d'_\kappa}l_\kappa^{(\nu)}(x)\mathcal{E}_\kappa(z).$$

We fix an $\eta \in \mathbb{N}^r$, written also as $\eta = -\lambda-\rho$, and let z be in a small neiborhood of 0. Integrating both sides against $\mathcal{E}_\eta(x)d\mu(x)$ (we omit the routine estimates guaranteeing that the interchanging of the integration and summation is valid), we get, by Lemma 1 (and Proposition 1),

$$\mathcal{N}_0^{(L)}\left[\frac{b}{2}\right]_\eta\prod_{j=1}^r(1-z_j)^{-b}\mathcal{E}_\eta\left(\frac{1-z}{1+z}\right) = \mathcal{N}_0^{(L)}\left[\frac{b}{2}\right]_\eta\sum_\kappa(-\alpha)^{|\kappa|}M_\kappa(\lambda)\frac{1}{d'_\kappa}\mathcal{E}_\kappa(z).$$

Cancelling the common factor $\mathcal{N}_0^{(L)}[\frac{b}{2}]_\eta$ we get

$$(1-z)^{-b}G\left(-\lambda, \frac{1-z}{1+z}\right) = (1-z)^{-b}\mathcal{E}_\eta\left(\frac{1-z}{1+z}\right) = \sum_\kappa (-\alpha)^{|\kappa|}M_\kappa(\eta+\rho)\frac{1}{d'_\kappa}\mathcal{E}_\kappa(z).$$

Comparing this with Lemma 4 proves our claim for $\lambda = -(\eta+\rho)$. For a general λ we observe that $G(-\lambda, \frac{1-z}{1+z})$ is an analytic function of z in a neighborhood of $z=0$ and $\{\mathcal{E}_\kappa(z)\}$ for a basis for the polynomials, thus it has a power series expansion, with coefficients analytic in λ, which are in turn determined by their restriction to the values of the form $\lambda = -(\eta+\rho)$. ■

Note that we may define a Hilbert space (similar to the Fock space or Bergman space construction e.g. [10]) so that the polynomials $\mathcal{E}_\kappa(z)$ form an orthogonal basis. However the function $(1-z)^{-b}G(-\lambda, \frac{1-z}{1+z})$ is not in the Hilbert space and the above formula is not an expansion in the Hilbert space sense. However it might be interesting to find a formula for

$$\sum_\kappa t^{|\kappa|}(-\alpha)^{|\kappa|}M_\kappa(\eta+\rho)\frac{1}{d'_\kappa}\mathcal{E}_\kappa(z),$$

which is convergent in the Hilbert space for small t. This would be then an analogue of Mahler's formula for the Hermite functions.

Finally we can also consider the same problem for symmetric Jack functions, which are the Heckman–Opdam spherical function for a root system of Type A. The Laplace transform in the symmetric case has been studied earlier by Macdonald [8] and Baker–Forrester [1].

For a partition $\kappa = (\kappa_1, \dots, \kappa_r)$, let $\Omega_\kappa(x) = \Omega_\kappa^{(\alpha)}(x)$ be the Jack symmetric polynomials in r-variables, normalized so that $\Omega_\kappa(1^r) = 1$; see [7]. The corresponding function for $\kappa \in \mathbb{C}^r$, which we call the symmetric Jack function (and which for general root system is called the Heckman–Opdam hypergeometric function) will be denoted still by $\Omega_\kappa(x)$, $x \in (0,\infty)^r$.

Proposition 2. *Consider the following expansion*

$$\prod_{j=1}^r (1-z_j^2)^{-\frac{b}{2}}\Omega_{\eta-\frac{b}{2}}\left(\frac{1-z}{1+z}\right) = \sum_\kappa Q_\kappa(\eta)\Omega_\kappa(z).$$

The coefficient $Q_\kappa(\eta)$ are symmetric polynomials in η up to a ρ shift, and after a slight modification

$$f_\kappa(\lambda) = 2^{rb/2}\mathcal{N}_0^{(L)}\frac{d'_\kappa}{(-\alpha)^{|\kappa|}}Q_\kappa(-\lambda-\rho)$$

they form an orthogonal basis in the space

$$L^2\left((0,\infty)^r, \left|\frac{\Gamma_\alpha\left(\frac{b}{2}-\rho-\lambda\right)}{\Gamma_\alpha\left(\frac{b}{2}\right)}\right|^2 |c(\lambda)|^{-2}ds\right)^{S_r}$$

of symmetric L^2-functions.

Proof. The proof is identical to that for the non-symmetric case. Up to a constant, the function $\Gamma_\alpha(\frac{b}{2}-\rho-\lambda)Q_\kappa(-\lambda-\rho)$ is the Heckman–Opdam transform of the Laguerre function $e^{p_1(x)}L_\kappa(2x)(2x)^{b/2}$, where $L_\kappa(2x)(2x)^{b/2}$ is the symmetric Laguerre polynomial defined in [1]. This follows from the Laplace transform of the symmetric Jack polynomials studied in [1]. Thus the orthogonality of the functions f_κ is a consequence of the Plancherel formula (2) and the orthogonality of the Laguerre functions [1]. ■

We observe the Heckman–Opdam transform of the function $L_\kappa(2x)e^{p_1(x)}(2x)^{b/2}$, say $f_\kappa(\lambda)$, is of form $f_\kappa(\lambda) = p_\kappa(\lambda)f_0(\lambda)$ where $f_0(\lambda)$ is the transform of the function $e^{p_1(x)}(2x)^{b/2}$ and $p_\kappa(\lambda)$ is a symmetric polynomial which corresponds to a symmetric polynomial $p_\kappa(U_1, \dots, U_r)$ of the Cherednik operators under the Heckman–Opdam transform. In other words, there is a Rodrigues type formula expressing $L_\kappa(2x)e^{p_1(x)}(2x)^{b/2}$ as $p_\kappa(U_1, \dots, U_r)$ acting on the simple weight function $e^{p_1(x)}(2x)^{b/2}$. However it might be more interesting to reverse the procedure, and to find the polynomials p producing a Rodrigues type formula for $L_\kappa(2x)e^{p_1(x)}(2x)^{b/2}$, whose transform would then be an immediate consequence; see e.g. [9, 15] for the case of Wilson polynomials.

Acknowledgements

This work was done over a period of time when both authors were visiting the Newton Institute, Cambridge UK in July 2001, the Institute of Mathematical Sciences, Singapore National University in August 2002 and MPI/HIM (Bonn) July 2007. We would like to thank the institutes for their hospitality. S. Sahi was supported by a grant from the National Science Foundation (NSF) and G. Zhang by the Swedish Research Council (VR).

We dedicate this paper to the memory of our colleague and friend Tom Branson. We thank the five referees for helpful comments on an earlier version of this paper.

References

[1] Baker T.H., Forrester P.J., The Calogero–Sutherland model and generalized classical polynomials, *Comm. Math. Phys.* **188** (1997), 175–216, solv-int/9608004.

[2] Baker T.H., Forrester P.J., Non-symmetric Jack polynomials and intergral kernels, *Duke Math. J.* **95** (1998), 1–50, q-alg/9612003.

[3] Davidson M., Olafsson G., Zhang G., Segal–Bargmann transform on Hermitian symmetric spaces, *J. Funct. Anal.* **204** (2003), 157–195, math.RT/0206275.

[4] Faraut J., Koranyi A., Analysis on symmetric cones, Oxford University Press, Oxford, 1994.

[5] Knop F., Sahi S., A recursion and a combinatorial formula for Jack polynomials, *Invent. Math.* **128** (1997), 9–22, q-alg/9610016.

[6] Koekoek R., Swarttouw R.F., The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Math. Report 98-17, Delft Univ. of Technology, 1998, math.CA/9602214.

[7] Macdonald I.G., Symmetric functions and Hall polynomials, Clarendon Press, Oxford, 1995.

[8] Macdonald I.G., Hypergeometric functions, Lecture notes, unpublished.

[9] Peng L., Zhang G., Nonsymmetric Jacobi and Wilson-type polynomials, *Int. Math. Res. Not.* (2006), Art. ID. 21630, 13 pages, math.CA/0511709.

[10] Opdam E.M., Harmonic analysis for certain representations of graded Hecke algebras, *Acta Math.* **175** (1995), 75–121.

[11] Sahi S., A new scalar product for nonsymmetric Jack polynomials, *Int. Math. Res. Not.* (1996), no. 20, 997–1004, q-alg/9608013.

[12] Sahi S., The binomial formula for nonsymmetric Macdonald polynomials, *Duke Math. J.* **94** (1998), 465–477, q-alg/9703024.

[13] Sahi S., The spectrum of certain invariant differential operators associated to a Hermitian symmetric space, in Lie Theory and Geometry, *Progr. Math.*, Vol. 123, Birkhäuser, Boston MA, 1994, 569–576.

[14] Zhang G., Branching coefficients of holomorphic representations and Segal–Bargmann transform, *J. Funct. Anal.* **195** (2002), 306–349, math.RT/0110212.

[15] Zhang G., Spherical transform and Jacobi polynomials on root systems of type BC, *Int. Math. Res. Not.* (2005), no. 51, 3169–3190, math.RT/0503735.