Wavelet-Based Quantum Field Theory*

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Abstract. The Euclidean quantum field theory for the fields $\phi_{\Delta x}(x)$, which depend on both the position x and the resolution Δx, constructed in *SIGMA* **2** (2006), 046, on the base of the continuous wavelet transform, is considered. The Feynman diagrams in such a theory become finite under the assumption there should be no scales in internal lines smaller than the minimal of scales of external lines. This regularisation agrees with the existing calculations of radiative corrections to the electron magnetic moment. The transition from the newly constructed theory to a standard Euclidean field theory is achieved by integration over the scale arguments.

Key words: wavelets; quantum field theory; regularisation

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1 Introduction

The description of infinitedimensional nonlinear systems in quantum field theory and statistical physics always faces the problem of divergent loop integrals emerging in the Green functions. Different methods of regularisation have been applied to make the divergent integrals finite [1]. There are a few basic ideas connected with those regularisations. First, certain minimal scale $L = \frac{2\pi}{\Lambda}$, where Λ is the cut-off momentum, is introduced into the theory, with all the fields $\phi(x)$ being substituted by their Fourier transforms truncated at momentum Λ:

$$\phi(x) \to \phi_{(\frac{2\pi}{\Lambda})}(x) = \int_{|k|\le\Lambda} e^{-\imath kx}\tilde{\phi}(k)\frac{d^dk}{(2\pi)^d}. \tag{1}$$

The physical quantities are than demanded to be independent on the rescaling of the parameter Λ. The second thing is the Kadanoff blocking procedure [2], which averages the small-scale fluctuations up to a certain scale – this makes a kind of effective interaction.

These methods are related to the self-similarity assumption: blocks interact to each other similarly to the sub-blocks. Similarly, but not necessarily having the same interaction strength – the latter can be dependent on scale $g = g(a)$. It is the case for high energy physics, for the developed hydrodynamic turbulence, and for many other phenomena [3]. However there is no place for such dependence if the fields are described solely in terms of their Fourier transform – except for the cut-off momentum. The latter representation of the scale-dependence is rather restrictive: it determines the effective interaction of *all fluctuations up to a certain scale*, but says nothing about the interaction of the fluctuations at a given scale [4].

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We have to admit that the origin of divergences is not the singular behaviour of the interaction strength at small distance, but the inadequate choice of the functional space used to describe these interactions. Namely, the decomposition of the fields with respect to the representations of translation group, i.e. the Fourier transform

$$\phi(x) = \int e^{-\imath kx}\tilde{\phi}(k)\frac{d^dk}{(2\pi)^d},$$

is physically sound only for the problems that clearly manifest translational invariance. For more general cases one can use decompositions with respect to other Lie groups, different from translation group ($x \to x + b$), see e.g. [5]. The problem is what groups are physically relevant for a field theory? In physical settings, along with translation invariance, the other symmetry is observed quite often – the symmetry with respect to scale transformations $x \to \alpha x$. This suggests the affine group (5) may be more adequate for self-similar phenomena than the subgroup of translations. The discrete representation of the self-similarity idea can be found in the Kadanoff spin-blocking procedure, or in application of the discrete wavelet transform $\phi(x) = \sum d_k^j \psi_k^j(x)$ in field theory models, considered by Battle and Federbuch in lattice settings [6, 7].

The decomposition with respect to the representations of affine group may have a natural probabilistic interpretation. In (Euclidean) quantum field theory the L^2-norm of the field $\phi(x)$ determines the probability density of registering that particle in a certain region $\Omega \subset \mathbb{R}^d$:

$$P(\Omega) = \int_{x\in\Omega} |\phi(x)|^2 dx, \qquad P(\mathbb{R}^d) = 1, \tag{2}$$

i.e. defines a measure. The unit normalisation in (2) is understood as "the probability of registering a particle anywhere in space is exactly one". This tacitly assumes the existence of registration devices working at infinite coordinate resolution. There are no such devices in reality: even if particle is *there*, but its typical wavelength is much smaller or much bigger than the typical wavelength of the measuring device there is nonzero probability the particle will not be registered.

For this reason it seems beneficial for theoretical description to use wavefunctions, or fields, that are explicitly labelled by resolution of the measuring equipment: $\phi_a(x)$. The incorporation of an observation parameter a is in excellent agreement with the Copenhagen interpretation of quantum mechanics: $\phi_a(x)$ describes our perception of the object ϕ at resolution a, rather than an "object as it is", $\phi_{a\to 0}(x)$, the existence of which is at least questionable. Needless to say that infinitely small resolution ($a \to 0$) requires infinitely high energy ($E \to \infty$) and is therefore practically unreachable.

We suggest the normalisation for the resolution-dependent functions $\phi_a(x)$ should be

$$\int_{-\infty}^{\infty} dx \int_0^{\infty} d\mu(a)|\phi_a(x)|^2 = 1, \tag{3}$$

where $\mu(a)$ is a measure of the resolution of the equipment. The normalisation (3) will be read as "the probability to register the object ϕ anywhere in space tuning the resolution of the equipment from zero to infinity is exactly one".

In present paper we show how the quantum field theory of scale-dependent fields $\phi_a(x)$ can be constructed using continuous wavelet transform (CWT). The integration over all scales a of course will drive us back to the standard theory. The advantage is that the Green functions $\langle\phi_{a_1}(x_1)\cdots\phi_{a_n}(x_n)\rangle$, i.e. those really observed in experiment, are finite – no further renormalisation is required.

2 Continuous wavelet transform

Let us show how the field theory of scale-dependent fields $\phi_a(x)$ can be constructed using continuous wavelet transform [8, 4]. If $\mathcal{H}$ is the Hilbert space, with is a Lie group G acting transitively on that space, and there exists a vector $\psi \in \mathcal{H}$, called an *admissible vector*, such that

$$C_\psi = \frac{1}{\|\psi\|_2} \int_G |\langle \psi, U(g)\psi \rangle|^2 d\mu_L(g) < \infty,$$

where $U(g)$ is a representation of G in $\mathcal{H}$, and $d\mu_L(g)$ is the left-invariant measure, then for any $\phi \in \mathcal{H}$ the following decomposition holds [9, 10]:

$$|\phi\rangle = C_\psi^{-1} \int_G |U(g)\psi\rangle\langle\psi|U(g)\phi\rangle d\mu_L(g), \qquad \forall\, \phi \in \mathcal{H}. \tag{4}$$

The Lie group that comprises two required operations – change of scale and translations – is the affine group

$$x \to ax + b, \qquad \psi(x) \to U(a,b)\psi[x] = a^{-\frac{d}{2}} \psi\left(\frac{x-b}{a}\right), \tag{5}$$

where $x, b \in \mathbb{R}^d$, $a \in \mathbb{R}_+$. The decomposition (4) with respect to affine group (5) is known as *continuous wavelet transform.*

To keep the scale-dependent fields $\phi_a(x)$ the same physical dimension as the ordinary fields $\phi(x)$ we write the coordinate representation of wavelet transform (4) in L^1-norm [11, 12]:

$$\phi(x) = \frac{1}{C_\psi} \int \frac{1}{a^d} \psi\left(\frac{x-b}{a}\right) \phi_a(b) \frac{da d^d b}{a}, \tag{6}$$

$$\phi_a(b) = \int \frac{1}{a^d} \overline{\psi\left(\frac{x-b}{a}\right)} \phi(x) d^d x. \tag{7}$$

In the latter equations the field $\phi_a(b)$ – the wavelet coefficient – has a physical meaning of the amplitude of the field ϕ measured at point b using a device with an aperture ψ and a tunable spatial resolution a. For isotropic wavelets, which we assume in this paper, the normalisation constant C_ψ is readily evaluated using Fourier transform:

$$C_\psi = \int_0^\infty |\tilde{\psi}(ak)|^2 \frac{da}{a} = \int |\tilde{\psi}(k)|^2 \frac{d^d k}{S_d |k|} < \infty, \tag{8}$$

where $S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the area of unit sphere in d dimensions.

The idea of substituting CWT (7), (6) into quantum mechanics or field theory is not new [12, 13, 14, 15, 16]. However all attempts to substitute it into field theory models were aimed to take at the final end the inverse wavelet transform and calculate the Green functions for the "true" fields $\langle \phi(x_1) \cdots \phi(x_n) \rangle$, i.e. for the case of infinite resolution. Our claim is that this last step should be avoided because the infinite resolution can not be achieved experimentally. Instead we suggest to calculate the functions, which correspond to experimentally observable finite resolution correlations. The integration over all scales a_i of course will drive us back to the standard divergent theory. The advantage of our approach is that the Green functions $\langle \phi_{a_1}(x_1) \cdots \phi_{a_n}(x_n) \rangle$ become finite under certain causality assumptions.

3 Rules of the game

Let us start with the Euclidean field theory with the forth power interaction ϕ^4. The corresponding action functional can be written in the form

$$\begin{aligned} S_E[\phi(x)] &= \frac{1}{2}\int \phi(x_1)D(x_1-x_2)\phi(x_2)dx_1dx_2 \\ &+ \frac{\lambda}{4!}\int V(x_1,\ldots,x_4)\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)dx_1dx_2dx_3dx_4, \end{aligned} \tag{9}$$

where D is the inverse propagator. To calculate the n-point Green functions of such a theory the generation functional is constructed

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle = \left.\frac{\delta^n \ln W[J]}{\delta J^n}\right|_{J=0}, \qquad W[J] = \int e^{-S_E[\phi]+\int J(x)\phi(x)dx}\mathcal{D}\phi(x). \tag{10}$$

Similarly, to calculate the Green functions for scale-dependent fields $\langle\phi_{a_1}(x_1)\cdots\phi_{a_n}(x_n)\rangle$ we have to construct the generating functional for scale-dependent fields $\phi_a(x)$. This is readily done by substituting wavelet transform (6) into the action (9). This gives

$$W_W[J_a] = \int e^{-S_W[\phi_a]+\int J_a(x)\phi_a(x)\frac{dadx}{a}}\mathcal{D}\phi_a(x), \tag{11}$$

$$\begin{aligned} S_W[\phi_a] &= \frac{1}{2}\int \phi_{a_1}(x_1)D(a_1,a_2,x_1-x_2)\phi_{a_2}(x_2)\frac{da_1dx_1}{a_1}\frac{da_2dx_2}{a_2} \\ &+ \frac{\lambda}{4!}\int V^{a_1,\ldots,a_4}_{x_1,\ldots,x_4}\phi_{a_1}(x_1)\cdots\phi_{a_4}(x_4)\frac{da_1dx_1}{a_1}\frac{da_2dx_2}{a_2}\frac{da_3dx_3}{a_3}\frac{da_4dx_4}{a_4}, \end{aligned}$$

with $D(a_1,a_2,x_1-x_2)$ and $V^{a_1,\ldots,a_4}_{x_1,\ldots,x_4}$ denoting the wavelet images of the inverse propagator and that of the interaction potential, respectively.

The functional (11) keeps the same form as its counterpart (10) with the difference that the functional integration over the two-argument fields $\phi_a(x)$ requires their ordering in both the position x and the scale a, in case the fields are operator-valued. It is important that if the interaction in the original theory (9) is local, $V \sim \prod_{i=2}^4 \delta(x_1-x_i)$, its wavelet image $V^{a_1,\ldots,a_4}_{x_1,\ldots,x_4}$ may be nonlocal, and vice versa. Here the dependence of interaction on scale is only due to wavelet transform:

$$V(x_1,\ldots,x_n) \leftrightarrow V^{a_1,\ldots,a_n}_{x_1,\ldots,x_n}.$$

Generally speaking the explicit scale dependence of the coupling constant $\lambda=\lambda(a)$ is also allowed. In the framework of modern field theory such dependence can not be tested: the running coupling constant $\lambda=\lambda(2\pi/\Lambda)$, obtained by renormalisation group methods, accounts for the collective interaction of all modes up to the certain scale Λ, but says nothing about the interaction of modes precisely at the given scale.

The technical way to calculate the Green functions

$$\langle\phi_{a_1}(x_1)\cdots\phi_{a_n}(x_n)\rangle = \left.\frac{\delta^n \ln W_W[J_a]}{\delta J_a^n}\right|_{J_a=0}$$

is to apply the Fourier transform to the r.h.s. of wavelet transform (6) and then substitute the result

$$\phi(x) = \frac{1}{C_\psi}\int_0^\infty \frac{da}{a}\int \frac{d^dk}{(2\pi)^d}e^{-\imath kx}\tilde{\psi}(ak)\tilde{\phi}_a(k),$$

into the action (9). Doing so, we have the following modification of the Feynman diagram technique [8]:

- each field $\tilde{\phi}(k)$ will be substituted by the scale component $\tilde{\phi}_a(k) = \overline{\tilde{\psi}(ak)}\tilde{\phi}(k)$.
- each integration in momentum variable will be accompanied by integration in corresponding scale variable:

$$\frac{d^d k}{(2\pi)^d} \to \frac{d^d k}{(2\pi)^d}\frac{da}{a}.$$

- each vertex is substituted by its wavelet transform.

For instance, for the massive scalar field propagator we have the correspondence

$$D(k) = \frac{1}{k^2+m^2} \to D(a_1,a_2,k) = \frac{\tilde{\psi}(a_1 k)\tilde{\psi}(-a_2 k)}{k^2+m^2}.$$

Surely the integration over all scale arguments in infinite limits drive us back to the usual theory in $\mathbb{R}^d$, since

$$\frac{1}{C_\psi}\int_0^\infty \frac{da}{a}|\tilde{\psi}(ak)|^2 = 1.$$

In physical settings the integration should not be performed over *all* scales $0 \le a < \infty$. In fact, if the system is affected (prepared) at the point x with the resolution Δx and the response is measured at a point y with the resolution Δy, the modes that are essentially different from those two scales will hardly contribute to the result. In the simplest case of linear propagation the result will be proportional to the product of preparation and measuring filters

$$\int \frac{\tilde{\psi}(k\Delta x)\tilde{\psi}(-k\Delta y)}{k^2+m^2} e^{-\imath k(x-y)} \frac{d^d k}{(2\pi)^d},$$

with the maximum achieved when Δx and Δy are of the same order.

Because of the finite resolution of measurement the causality in wavelet-based quantum field theory (11) will be *the region causality* [17] in contrast to point causality of standard field theory. If two open balls have zero intersection $B_{\Delta x}(x) \cap B_{\Delta y}(y) = \varnothing$ the light-cone causality is applied, but if one of them is subset of another a new problem of how to commute the part and the whole wavefunctions arises [13]. Possible solution – “the coarse acts on vacuum first” – have been proposed in [16, 18]. In fact, *when we perform measurements on a quantum system of typical size a we ought use system of functions with resolution coarser or equal to a*: for knowing the finer details requires momentum higher than $1/a$. It may seem a trivial fact in Fourier representation: no details smaller than the radiation wavelength, used for the experiment, can be obtained since there is insufficient energy for that. However in wavelet representation this assumption should be made separately to ensure that we study any quantum system from *outside* and can use only outside scales for that.

A simplest assumption of this type formulated in the language of Feynman's diagrams is: *there should be no scales in internal lines smaller than the minimal scale of all external lines.* This means that there should be no virtual particles in internal lines unless there is sufficient energy in external lines to excite them.

4 Scalar field theory

Let us consider one-loop contribution to the two-point correlation function in ϕ^4-theory between two balls $B_{a_1}(x_1)$ and $B_{a_2}(x_2)$. According to the above made causality statement there should

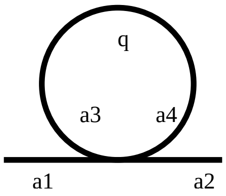

Figure 1. Tadpole diagram in scalar field scale-dependent theory with ϕ^4-interaction.

be no scales in internal loop smaller than the minimal scale of two external lines. The value for the amputated diagram, corresponding to that shown in Fig. 1, is

$$\frac{1}{C_\psi^2}\int_{a_3,a_4\ge A}|\tilde\psi(a_3q)|^2\frac{d^dq}{(2\pi)^d}\frac{1}{q^2+m^2}|\tilde\psi(-a_4q)|^2\frac{da_3}{a_3}\frac{da_4}{a_4}, \tag{12}$$

where $A=\min(a_1,a_2)$. In the limit of point events $A\to0$ the equation (12) recovers the divergent tadpole integral ($\int\frac{1}{q^2+m^2}\frac{d^dq}{(2\pi)^d}$) due to normalisation (8).

Let us see how the one-loop contribution (12) will look like for a particular types of wavelets. The basic wavelet ψ is just an analysing function to study the object ϕ, and the conditions imposed on it are rather loose: practically the requirement of normalisation (8) means the vanishing of the basic wavelet Fourier image in the infra-red limit $\tilde\psi(k=0)=0$ and good localisation properties. For simplicity, we assert the basic wavelet ψ to be isotropic and take it to be one of the derivatives of the Gaussian, i.e. in Fourier space

$$\tilde\psi_n(k)=(-\imath k)^n e^{-k^2/2}. \tag{13}$$

The normalisation constant (8) can be easily evaluated for the wavelets (13):

$$C_{\psi_n}=\int_0^\infty (ak)^{2n}e^{-a^2k^2}\frac{da}{a}=\frac{\Gamma(n)}{2}.$$

Since in each internal loop there is a wavelet factor $\tilde\psi(ak)$ from the vertex and that from the line, each internal connection to the vertex will contribute by a factor

$$f(n,x)=\frac{2}{\Gamma(n)}\int_x^\infty|\tilde\psi_n(ak)|^2\frac{da}{a}=\frac{2}{\Gamma(n)}\int_x^\infty a^{2n-1}e^{-a^2}da,$$

when integrating over the scales of internal loop. $x=Ak$ is the argument of the filtering function.

Let us present the filtering functions for the first four Gaussian wavelets (13) explicitly

$$\begin{aligned}
&f(1,x)=e^{-x^2},\\
&f(2,x)=(x^2+1)e^{-x^2},\\
&f(3,x)=(x^4+2x^2+2)e^{-x^2}/2,\\
&f(4,x)=(x^6+3x^4+6x^2+6)e^{-x^2}/6,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

Therefore, the equation (12), being rewritten in dimensionless momentum units, takes the form

$$T_n^d(A)=\frac{S_d}{(2\pi)^d}m^{d-2}\int_0^\infty f^2(n,Amk)\frac{k^{d-1}dk}{k^2+1}. \tag{14}$$

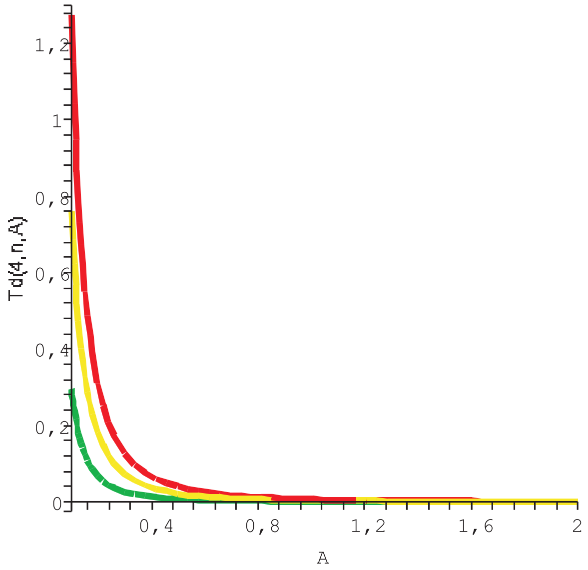

Figure 2. Scale dependence of the tadpole contributions calculated for the first three Gaussian wavelets of the family (13) in $d = 4$ dimensions.

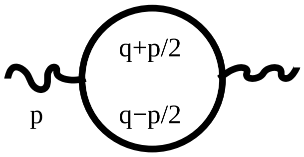

Figure 3. Polarisation operator with symmetric momenta in the loop.

The values of the integrals (14) for the special value of space dimension $d = 4$ and wavelet numbers $n = 1, 2, 3$ are presented below:

$$T_1^4 = \frac{-4a^4 e^{2a^2}\mathrm{Ei}(1,2a^2) + 2a^2}{64\pi^2 a^4} m^2, \tag{15}$$

$$T_2^4 = -\frac{\mathrm{Ei}(1,2a^2)e^{2a^2}a^2(4a^4 - 8a^2 + 4) + 5a^2 - 2a^4 - 5}{64\pi^2 a^2} m^2,$$

$$T_3^4 = -\frac{\mathrm{Ei}(1,2a^2)e^{2a^2}a^2(32+8a^8-32a^6+64a^4-64a^2) - 66 + 59a^2 - 42a^4 + 18a^6 - 4a^8}{512\pi^2 a^2} m^2,$$

with $a \equiv Am$ and $\mathrm{Ei}(1,z) = \int_1^\infty \frac{e^{-xz}}{x}dx$ being the exponential integral. The graphs of the dependence of the values (15) on the dimensionless scale a are shown in Fig. 2.

5 Theory with fermions

The example of massive scalar field presented above demonstrates that the wavelet-based field theory of scale-dependent functions $\phi_a(x)$ is determined by the ratio of two scales: the scale of observation A and the natural Compton scale of the theory $\frac{1}{m}$. There is a question, what will be the result for quantum electrodynamics, the theory that comprises massive fermions and massless boson. The answer is that localisation of photon in such a theory by any device of resolution A is possible only due to the finite electron mass $m_e > 0$. That is the Compton scale is the only natural scale in such theory.

To illustrate this fact let us present the calculation of the vacuum polarisation diagram in $d = 4$ quantum electrodynamics (QED). The vacuum polarisation diagram, shown in Fig. 3 in

(Euclidean) QED is given by the following integral:

$$\begin{aligned}\Pi_{\mu\nu} &= -e^2 \int \frac{d^4q}{(2\pi)^4} \frac{\mathrm{Tr}\,(\gamma_\mu(\hat q + \hat p/2 - m)\gamma_\nu(\hat q - \hat p/2 - m))}{[(q+p/2)^2+m^2]\,[(q-p/2)^2+m^2]}\\ &= -e^2 \int \frac{d^4q}{(2\pi)^4} \frac{8q_\mu q_\nu - 2p_\mu p_\nu - \delta_{\mu\nu}(4q^2-p^2+4m^2)}{[(q+p/2)^2+m^2]\,[(q-p/2)^2+m^2]}\end{aligned} \quad (16)$$

where the (Euclidean) identities for γ-matrices

$$\mathrm{Tr}\,(\gamma_\mu\gamma_\nu) = -4\delta_{\mu\nu}, \qquad \mathrm{Tr}\,(\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta) = 4(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\alpha\nu} - \delta_{\mu\nu}\delta_{\alpha\beta})$$

were used for the evaluation of trace in the numerator of the equation (16).

For definiteness, let us consider the first wavelet ψ_1 of the family (13). Each fermion line in the wavelet counterpart of the equation (16) after integration over internal scale variables contributes by wavelet factor $f^2(1,x)$, where $x = A(q \pm p/2)$, for upper and lower lines in the diagram Fig. 3, respectively. A is the minimal scale of two external lines.

The whole factor

$$F(A) = f^2(1, A(q+p/2))f^2(1, A(q-p/2)) = \exp\big(-A^2p^2 - 4A^2q^2\big)$$

is independent of the scalar product pq, and thus the resulting equation for the vacuum polarisation in ψ_1 wavelet-based theory can be casted in the form

$$\Pi^{(A)}_{\mu\nu} = -e^2 4 \int \frac{d^4q}{(2\pi)^4} \exp\big(-A^2p^2 - 4A^2q^2\big) \frac{2q_\mu q_\nu - \frac{1}{2}p_\mu p_\nu + \delta_{\mu\nu}\big(\frac{p^2}{4} - q^2 - m^2\big)}{[(q+p/2)^2+m^2]\,[(q-p/2)^2+m^2]}. \quad (17)$$

Evidently the limit of infinite resolution ($A \to 0$) taken in equation (17) gives the known divergent result (16).

The momentum integration in equation (17) is straightforward: having expressed all momenta in units of electron mass m, we express the loop momentum in terms of the photon momentum $\mathbf{q} = |\mathbf{p}|\mathbf{y}$ and perform the integration over the polar angle:

$$\begin{aligned}\Pi^{(A)}_{\mu\nu} &= -\frac{e^2}{\pi^3}(m^2p^2) \int_0^\infty dyy \exp\big(-A^2m^2p^2 - 4A^2m^2p^2y^2\big) \int_0^\pi d\theta \sin^2\theta \\ &\times \frac{2y_\mu y_\nu - \frac{1}{2}\frac{p_\mu p_\nu}{p^2} + \delta_{\mu\nu}(\frac{1}{4} - y^2 - \frac{1}{p^2})}{\left[\frac{\frac{1}{4}+y^2+\frac{1}{p^2}}{y} + \cos\theta\right]\left[\frac{\frac{1}{4}+y^2+\frac{1}{p^2}}{y} - \cos\theta\right]},\end{aligned}$$

where p is dimensionless, i.e. is expressed in units of m. Introducing the notation

$$\beta(y) \equiv \frac{\frac{1}{4} + y^2 + \frac{1}{p^2}}{y}$$

and using the substitution

$$y_\mu y_\nu \to Ay^2\delta_{\mu\nu} + By^2\frac{p_\mu p_\nu}{p^2},$$

under the angular integration we get

$$\begin{aligned}\Pi^{(A)}_{\mu\nu} &= -\frac{e^2}{\pi^3}\big(m^2p^2\big) \int_0^\infty dyy \exp\big(-A^2m^2p^2\big(1+4y^2\big)\big) \int_0^\pi d\theta \sin^2\theta \times \\ &\times \frac{\delta_{\mu\nu}\big((2A-1)y^2 + \frac{1}{4} - \frac{1}{p^2}\big) + \frac{p_\mu p_\nu}{p^2}\big(2By^2 - \frac{1}{2}\big)}{\beta^2(y) - \cos^2\theta},\end{aligned}$$

where A and B depend only on the modulus of y, but not on the direction, and can be expressed in terms of angle integrals

$$
\begin{aligned}
&I_k(y) \equiv \int_0^\pi d\theta \frac{\sin^2\theta\cos^{2k}\theta}{\beta^2(y)-\cos^2\theta},\\
&I_0(y) = \pi(1-\sqrt{1-\beta^{-2}(y)}),\\
&I_1(y) = -\frac{\pi}{2}+\beta^2(y)I_0(y),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

so that $4A+B=1, \quad A+B=I_1/I_0$, from where we get

$$
A=\frac{1}{3}+\frac{\pi}{6}I_0^{-1}(y)-\frac{1}{3}\beta^2(y), \qquad B=-\frac{1}{3}-\frac{2\pi}{3}I_0^{-1}(y)+\frac{4}{3}\beta^2(y).
$$

Finally, writing the polarisation operator as a sum of transversal and longitudinal parts, we have the equations

$$
\Pi^{(A)}_{\mu\nu} \equiv \delta_{\mu\nu}\pi^{(A)}_T + \frac{p_\mu p_\nu}{p^2}\pi^{(A)}_L,
$$

$$
\begin{aligned}
\pi^{(A)}_T &= -\frac{e^2}{3\pi^2}m^2p^2\int_0^\infty dy y \exp\left(-A^2m^2p^2(1+4y^2)\right) \qquad (18)\\
&\times\left[y^2+\left(1-\sqrt{\frac{\frac{1}{16}+y^4+\frac{1}{p^4}-\frac{y^2}{2}+\frac{1}{2p^2}+\frac{2y^2}{p^2}}{\left(\frac{1}{4}+y^2+\frac{1}{p^2}\right)^2}}\right)\left(\frac{5}{8}-\frac{4}{p^2}-\frac{2}{p^4}-2y^2\left(1+\frac{2}{p^2}\right)-2y^4\right)\right],\\
\pi^{(A)}_L &= -\frac{e^2}{3\pi^2}m^2p^2\int_0^\infty dy y \exp\left(-A^2m^2p^2(1+4y^2)\right) \qquad (19)\\
&\times\left[-4y^2+\left(1-\sqrt{\frac{\frac{1}{16}+y^4+\frac{1}{p^4}-\frac{y^2}{2}+\frac{1}{2p^2}+\frac{2y^2}{p^2}}{\left(\frac{1}{4}+y^2+\frac{1}{p^2}\right)^2}}\right)\left(8y^4+2y^2\left(1+\frac{8}{p^2}\right)+\frac{4}{p^2}+\frac{8}{p^4}-1\right)\right].
\end{aligned}
$$

The integrals (18), (19) can be evaluated in the limiting case $p^2 \gg 1$, when the external momentum is much greater the electron mass. In this case

$$
\begin{aligned}
\pi^{(A)}_T &= -\frac{e^2}{6\pi^2}m^2p^2\int_0^\infty dt \exp\left(-A^2m^2p^2(1+4t)\right)\left[t+\left(1-\sqrt{\frac{(\frac{1}{4}-t)^2}{(\frac{1}{4}+t)^2}}\right)\left(\frac{5}{8}-2t-2t^2\right)\right],\\
\pi^{(A)}_L &= -\frac{e^2}{6\pi^2}m^2p^2\int_0^\infty dt \exp\left(-A^2m^2p^2(1+4t)\right)\left[-4t+\left(1-\sqrt{\frac{(\frac{1}{4}-t)^2}{(\frac{1}{4}+t)^2}}\right)\left(8t^2+2t-1\right)\right],
\end{aligned}
$$

and can be evaluated as a sum $\int_0^\infty = \int_0^{1/4}+\int_{1/4}^\infty$. This gives

$$
\begin{aligned}
\pi^{(a)}_T &= -\frac{e^2}{6\pi^2}p^2\left\{\frac{e^{-a^2p^2}}{8a^6p^6}\left(4a^4p^4-a^2p^2-1\right)+\frac{e^{-2a^2p^2}}{8a^6p^6}\left(-4a^4p^4+2a^2p^2+1\right)\right.\\
&\left.-\frac{1}{2}\mathrm{Ei}\left(1,a^2p^2\right)+\mathrm{Ei}\left(1,2a^2p^2\right)\right\},\\
\pi^{(a)}_L &= -\frac{e^2}{6\pi^2}p^2\left\{\frac{e^{-a^2p^2}}{4a^6p^6}\left(-2a^4p^4-a^2p^2+2\right)+\frac{e^{-2a^2p^2}}{4a^6p^6}\left(2a^4p^4-a^2p^2-2\right)\right.\\
&\left.+\frac{1}{2}\mathrm{Ei}\left(1,a^2p^2\right)-\mathrm{Ei}\left(1,2a^2p^2\right)\right\}, \qquad a=Am.
\end{aligned}
$$

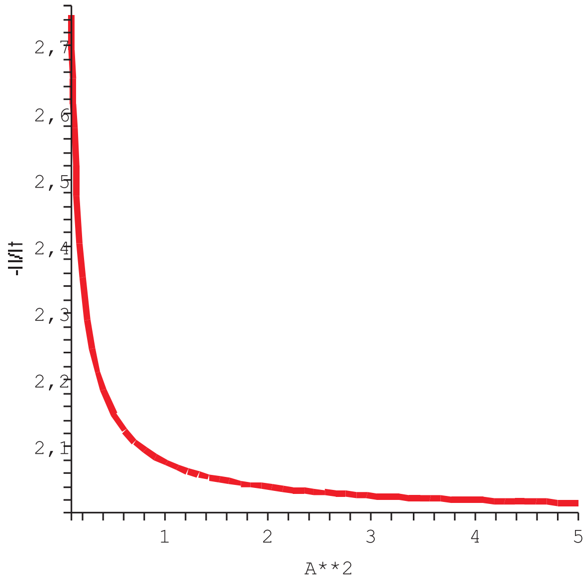

Figure 4. The ratio of the longitudinal part to the transversal part of the polarisation operator $-\pi_L^{(a)}/\pi_T^{(a)}$, shown for $p = 5.0$.

In the limiting case of $a^2 \to \infty$ the ratio of the longitudinal part to the transversal part $-\pi_L^{(a)}/\pi_T^{(a)} \to 2$, see Fig. 4.

6 Relation to the usual regularisations

The decomposition of wavefunctions with respect to representation of the affine group is of course a basis for certain regularisation, but is not identical to known regularisations, such as the Wilson RG procedure [21, 22], see [4] for more details. In the Wilson renormalisation group the integration over a thin shell in momentum space $[\Lambda e^{-\delta l}, \Lambda)$ averages the fast modes into the effective slow modes. The effective coupling constant $\tilde{g}(\Lambda)$ in such a theory stands for the effective interaction of modes with $k \le \Lambda$, rather than being a coupling constant describing the interaction strength at a given scale.

The renormalisation group (RG), that makes use of substitution of initial fields $\phi(x) \in \mathrm{L}^2(\mathbb{R}^d)$ by the scale-truncated fields (1) makes the coupling constants dependent on the cut-off momentum Λ, and requires that the final physical results should be independent of the introduced scale

$$\Lambda\partial_\Lambda(\text{Physical quantities}) = 0.$$

The standard regularisation schemes, the Wilson RG, the Pauli–Villars regularisation, etc., share an important common feature: if the studied process has a typical observation scale – the inverse momentum of external lines, – then the smaller scale contributions are effectively suppressed by a regularisation parameter (cut-off momentum, large mass, etc.), with their averaged effect being incorporated into the observable scale parameters.

Let us illustrate this using the example of vertex diagram in QED, and show that the wavelet transform with the above proposed causality assumption acts similarly.

The equation for the anomalous magnetic moment of the electron

$$\mu = \frac{e\hbar}{2mc}\left(1 + \frac{\alpha}{2\pi} - 0.328\frac{\alpha^2}{\pi^2}\right), \tag{20}$$

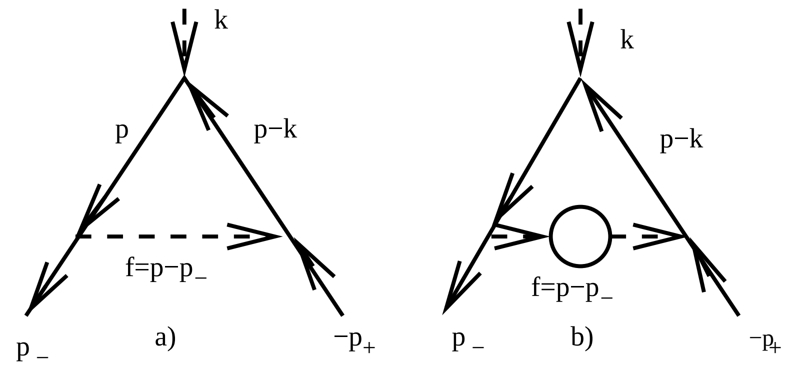

Figure 5. First and second radiation correction to the electron magnetic moment. For the second radiation corrections only one of the diagrams is shown.

where $\alpha = \frac{e^2}{\hbar c}$ is the fine structure constant, provides the basis for the most precise tests of quantum electrodynamics. The unit term in the equation (20) is just a magnetic moment of the electron, the second is the first radiation correction, corresponding to the diagram shown in Fig. 5a, first calculated by Schwinger, the second term, corresponding to the diagram shown in Fig. 5b, was first calculated by C. Sommerfield. The calculation is based on the evaluation of the electron formfactor

$$j^\mu = \bar{u}_2 \Gamma^\mu u_1, \qquad \Gamma^\mu = \gamma^\mu f(k^2) - \frac{1}{2m} g(k^2)\sigma^{\mu\nu}k_\nu$$

Following [23] we present the limitations on internal line momenta in the first radiation correction, Fig. 5a, – for the second one, shown in Fig. 5b, the procedure is the same.

The matrix element corresponding to the electron current shown in Fig. 5a is given by

$$-\imath e\bar{u}(p_-)\Gamma^\mu u(-p_+) = (-\imath e)^3 \bar{u}(p_-)\gamma^\nu \imath \int G(p)\gamma^\mu G(p-k)\gamma^\lambda D_{\lambda\nu}(f) u(-p_+)\frac{d^4p}{(2\pi)^4},$$

or explicitly

$$\bar{u}(p_-)\left(\gamma^\mu f(k^2) - \frac{1}{2m}g(k^2)\sigma^{\mu\nu}k_\nu\right)u(-p_+) = \imath \int \frac{\bar{u}(p_-)\phi^\mu(p)u(-p_+)d^4p}{(p^2-m^2)[(p-k)^2-m^2]},$$

where

$$\phi^\mu(p) = -e^2 \frac{\gamma^\nu(\hat{p}+m)\gamma^\mu(\hat{p}-\hat{k}+m)\gamma_\nu}{4\pi^3(p_- - p)^2}.$$

The loop integration is performed in momentum $f = p - p_-$ instead of p, so that

$$f^2 = (p-p_-)^2 = -2\mathbf{p}^2(1-\cos\theta) = -\frac{t-4m^2}{2}(1-\cos\theta), \qquad \theta = \angle(\mathbf{p}, \mathbf{p}_-)$$

and leads to the integrals

$$(I, I^\mu, I^{\mu\nu}) = \int \frac{(1, f^\mu, f^\mu f^\nu)}{1-\cos\theta}\frac{d\Omega_f}{2\pi}.$$

These integrals have infra-red divergences of the form

$$I = \int_0^{t-4m^2} \frac{d\mathbf{f}^2}{\mathbf{f}^2}, \tag{21}$$

where $t = k^2$. The regularisation is performed by introducing the small but finite photon mass ($\lambda \ll m$) and corresponding shift of the momentum $f^2 \to f^2 - \lambda^2$. Analogous consideration can be presented for the integrals I^μ and $I^{\mu\nu}$.

Thus, in the final limit of the large scale magnetic field ($k \to 0$) the integration in (21) is performed over the momenta less or equal than $k^2 - 4m^2$, i.e. in the scales larger than the scale of external lines. Similar consideration can be applied to other diagrams of radiation corrections, including that shown in Fig. 5b. This exactly corresponds to the idea presented above in this paper on page 5 in terms of continuous wavelet transform.

7 Conclusion

In this paper we sketched a way of constructing quantum field theory for the fields that depend on both the position and the scale using the continuous wavelet transform. The continuous wavelet transform has been already used for regularisation of field theory models [13, 6, 15]. The novelty of present approach (see also [4]), consists in understanding the scale-dependent fields $\phi_a(x)$ – the wavelet coefficients – as physical amplitudes of the fields, measured at a given resolution a. This seems to be advantageous if compared to mere regularisation, which is to be considered at the limit $a \to 0$ in the final results. The advantage is in explicit equations for the correlation between fields of different scales a_i, allowed at the same location x. Such correlations do really take place in the process of quantum measurement, when the system is initially measured at large scale, and then on a small scale, – say the measurement of the angular momentum of a molecule followed by a measurement of an electron angular momentum. Technically, the restriction of minimal scale of all internal lines in a (wavelet) Feynman diagram by the minimal scale of external lines provides the absence of processes with energies not supplied by the experimental device or the environment. This limitation makes the theory free of ultraviolet divergences.

Doing so we obtain a nonlocal field theory with region causality [17, 18] instead of point causality, accompanied by corresponding problems of nonlocal field theory [19, 20]. This makes the wavelet approach attractive for further applications in high energy physics and condensed matter field theoretic models. To go further in this direction we need to elucidate the effects of gauge invariance to the multiscale decomposition, but this will be the subject of the subsequent paper.

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