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Some Conformal Invariants from the Noncommutative Residue for Manifolds with Boundary*

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Abstract. We review previous work of Alain Connes, and its extension by the author, on some conformal invariants obtained from the noncommutative residue on even dimensional compact manifolds without boundary. Inspired by recent work of Yong Wang, we also address possible generalizations of these conformal invariants to the setting of compact manifolds with boundary.

Key words: manifolds with boundary; noncommutative residue; Fredholm module; conformal invariants

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1 Introduction

There is one particular aspect of noncommutative geometry that has historically received less attention than other of its subjects; the use of its machinery to obtain conformal invariants (associated to the underlying manifold). The motivating example in this venue, is a conformal invariant in dimension 4 (Connes [5]) and its extension to higher order even dimensional manifolds by the author [19]. The main idea lies on Theorem IV.4.2.c of Connes [6]. This theorem states that the oriented conformal structure of a compact even-dimensional smooth manifold is uniquely determined by the Fredholm module $(\mathcal{H}, F, \gamma)$ of Connes, Sullivan and Teleman [7]; via the noncommutative residue Res of Adler, Manin, Guillemin, and Wodzicki [1, 13, 12, 23].

In Section 2 of [5] Connes uses his quantized calculus to find a conformal invariant in the 4-dimensional case. A central part of the explicit computation of this conformal invariant is the study of a trilinear functional on smooth functions over the manifold M^4 given by the relation

$$\tau(f_0, f_1, f_2) = \mathrm{Res}(f_0[F, f_1][F, f_2]).$$

Here F is a pseudodifferential operator of order 0 acting on 2-forms over M^4. This conformal invariant computed by Connes in the 4-dimensional case is a natural bilinear differential functional of order 4 acting on $C^\infty(M^4)$. In [5] it is denoted Ω and in these notes it is denoted $B_4\,dx$.

This bilinear functional is symmetric, $B_4(f_1, f_2) = B_4(f_2, f_1)$, and conformally invariant, in the sense that $\widehat{B_4}(f_1, f_2) = e^{-4\eta} B_4(f_1, f_2)$ for a conformal change of the metric $\widehat{g} = e^{2\eta} g$. It is also uniquely determined by the relation:

$$\tau(f_0, f_1, f_2) = \int_M f_0 B_4(f_1, f_2)\,dx, \qquad \forall\, f_i \in C^\infty(M^4).$$

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Furthermore, in the 4-dimensional case, Connes has also shown that the Paneitz operator [14] (*critical* GJMS for $n = 4$ [10]), can be derived from B_4 by the relation

$$\int_M B_4(f_1, f_2)\,dx = \frac{1}{2}\int_M f_1 P_4(f_2)\,dx.$$

Aiming to extend the work of Connes to even dimensional manifolds, in [18] we have proved the following two results:

Theorem 1 of [18]. *Let M be an n-dimensional compact conformal manifold without boundary. Let S be a pseudodifferential operator of order 0 acting on sections of a vector bundle over M such that $S^2 f_1 = f_1 S^2$ and the pseudodifferential operator $P = [S, f_1][S, f_2]$ is conformally invariant for any $f_i \in C^\infty(M)$. Then there exists a unique, symmetric, bilinear, differential functional $B_{n,S}$ of order n conformally invariant in the sense that $\widehat{B_{n,S}}(f_1, f_2) = e^{-n\eta} B_{n,S}(f_1, f_2)$, for $\widehat{g} = e^{2\eta} g$, and such that*

$$\mathrm{Res}(f_0[S, f_1][S, f_2]) = \int_M f_0 B_{n,S}(f_1, f_2)\,dx$$

for all $f_i \in C^\infty(M)$.

A particular case of the above result occurs when one works with even-dimensional manifolds for then, it makes sense to consider the Fredholm module $(\mathcal{H}, F)$ associated to M. The operator F has the property $F^2 = 1$ and in general $[F, f] \neq 0$ for $f \in C^\infty(M)$. Taking S as F in the previous theorem one has:

Theorem 2 of [18]. *Let M be a compact conformal manifold without boundary of even dimension n and let $(\mathcal{H}, F)$ be the Fredholm module associated to M by A. Connes* [5]. *Then, by taking $S = F$ in Theorem 1 of* [18] *there is a unique, symmetric, and conformally invariant n-differential form $B_n = B_{n,F}$ such that*

$$\mathrm{Res}(f_0[F, f_1][F, f_2]) = \int_M f_0 B_n(f_1, f_2)\,dx$$

for all $f_i \in C^\infty(M)$.

These results are based on the study of the formula for the total symbol $\sigma(P_1 P_2)$ of the product of two pseudodifferential operators, in the particular case in which one of them is a multiplication operator. The following is the main result of [19]:

Theorem 1. *Let M be a compact conformal manifold without boundary of even dimension n and let $(\mathcal{H}, F)$ be the Fredholm module associated to M by Connes* [5]. *Let P_n be the differential operator given by the relation*

$$\int_M B_n(f, h)\,dx = \int_M f P_n(h)\,dx$$

for all $f, h \in C^\infty(M)$. Then,

i) P_n is formally selfadjoint;

ii) P_n is conformally invariant in the sense $\widehat{P_n}(h) = e^{-n\eta} P_n(h)$, if $\widehat{g} = e^{2\eta} g$;

iii) P_n is expressible universally as polynomial in the components of ∇ (the covariant derivative) and R (the curvature tensor) with coefficients rational in n.

iv) $P_n(h) = c_n \Delta^{n/2}(h) +$ "lower order terms", with c_n a universal constant;

v) *P_n has the form $\delta S_n d$ where S_n is an operator on 1-forms given as a constant multiple of $\Delta^{n/2-1}+$ "lower order terms" or $(d\delta)^{n/2-1}+$ "lower order terms";*

vi) *P_n and B_n are related by:*

$$P_n(fh) - fP_n(h) - hP_n(f) = -2B_n(f,h).$$

Because the critical GJMS operator and the operator P_n coincide in the flat case and share the same conformal behavior we have

Proposition 1. *In the even dimensional case, inside the conformally flat class of metrics, the critical* GJMS *operator and the operator P_n coincide up to a constant multiple.*

1.1 Yong Wang's work

Based on the work of [20], in [21] and [22] Y. Wang proposes to extend to the case of manifolds with boundary, the work of Connes in Section 2 of [5]. He is the first to suggest the replacement of the usual noncommutative residue by the noncommutative residue of Fedosov–Golse–Leichtnam–Schrohe [8], acting on Boutet de Monvel's algebra [3].

In Section 3 of [21], Y. Wang considers a compact n-dimensional manifold X with boundary Y and its double manifold $\widetilde{X} = X \cup_Y X$. For a vector bundle E over $\widetilde{X}$ and a pseudodifferential operator S with the transmission property and of order 0 acting on sections of E, the operator $\widetilde{P}$ is defined as the composition

$$\widetilde{P} := \begin{pmatrix} \pi^+ f_0 & 0 \\ 0 & 0 \end{pmatrix} \left[\begin{pmatrix} \pi^+ S & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \pi^+ f_1 & 0 \\ 0 & 0 \end{pmatrix} \right] \left[\begin{pmatrix} \pi^+ S & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \pi^+ f_2 & 0 \\ 0 & 0 \end{pmatrix} \right].$$

It is then observed that $\widetilde{P} = \pi^+(f_0[S,f_1][S,f_2]) + G$ for some singular Green operator G with singular Green symbol b. See [21] for the corresponding definitions. Based on this decomposition of $\widetilde{P}$ and the definition of Fedosov et al. of $\overline{\mathrm{Res}}$, Wang defines $\Omega_{n,S}$ and $\Omega_{n-1,S}$ via

$$\Omega_{n,S}(f_1,f_2) = \overline{\mathrm{res}}\big(([S,\overline{f}_1][S,\overline{f}_2])|_X\big) \qquad \text{and} \qquad f_0|_Y \Omega_{n-1,S}(f_1,f_2) = 2\pi \operatorname{res}_{x'} \operatorname{trace}(b),$$

with $\overline{f}_i$ a smooth extension of f_i to $\widetilde{X}$. Also $\overline{\mathrm{res}}$ is the density corresponding to $\overline{\mathrm{Res}}$ (similar to res for Res) and $\mathrm{res}_{x'}$ the noncommutative residue density for the manifold Y.

It is possible to verify that $\widetilde{P}$ satisfies the transmission property. In this way, forgetting about any conformal invariance property, in [21] Wang found a generalization of the relation

$$\mathrm{Res}(f_0[S,f_1][S,f_2]) = \int_M f_0 \Omega_{n,S}(f_1,f_2) = \int_M f_0 B_{n,S}(f_1,f_2)\,dx, \qquad \forall\, f_i \in C^\infty(X \cup Y)$$

in Theorem 1 of [18].

The main idea of Y. Wang relies on the use of the double manifold. Topologically, the double manifold of a given compact oriented manifold with boundary makes perfect sense. At the level of smooth manifold with a given Riemannian structure more work is needed to make sense of a double manifold.

In Section 4 of [21] the dimension is taken as even and the metric on X has a product structure near the boundary: $g^X = g^{\partial X} + dx^n$. On $\widetilde{X}$ the metric $\widetilde{g}$ is taken as $\widetilde{g} = g$ on both copies of X. Then Ω_n is defined as $\Omega_{n,F}(\overline{f_1},\overline{f_2})|_X$ where $(\mathcal{H},F)$ is the Fredholm module associated to $(\widetilde{X},\widetilde{g})$, and $\overline{f_i}$ is an extension of f_i to $\widetilde{M}$.

In [22] the even-dimensional Riemannian metric in consideration has the particular form $g^X = 1/(h(x_n))g^{\partial X} + dx_n^2$ on a collar neighborhood U of ∂X. Here h is the restriction to $[0,1)$ of a smooth function $\widetilde{h}$ on $(-\varepsilon,1)$ for some $\varepsilon > 0$ such that $h(0) = 1$ and $h(x_n) > 0$. A metric $\hat{g}$

is associated to the double manifold $\widetilde{X}$ in the following way. Given U and $\widetilde{h}$ as before, there is a metric $\hat{g}$ on $\widetilde{X}$ with the form $\hat{g}^{\widetilde{X}} = 1/(\widetilde{h}(x_n))g^{\partial X} + dx_n^2$ on $U \cup_{\partial X} \partial X \times (-\varepsilon, 0]$ and such that $\hat{g}|_X = g$.

Next, with $(\mathcal{H}, F_{\hat{g}})$ the Fredholm module associated to $(\widetilde{X}, \hat{g})$, Ω_n and Ω_{n-1} are defined via the relation

$$\overline{\mathrm{Res}}\big(\pi^+ f_0[\pi^+ F_{\hat{g}}, f_1][\pi^+ F_{\hat{g}}, f_2]\big) = \int_X f_0 \Omega_n(f_1, f_2)(\hat{g}) + \int_{\partial M} f_0|_{\partial X} \Omega_{n-1}(\hat{g}).$$

The described settings used in [21] and [22] have the following limitations: if we conformally rescale the metric in X then, the new metric $e^{2\eta} g$ is not anymore of the specific requested form near the boundary. How to define then the objects in question in terms of this new metric? That is to say, what is the definition of $\Omega_n(e^{2\eta} g)$ and how to compare it with $\Omega_n(g)$? Based on the idea of replacing Res with $\overline{\mathrm{Res}}$, and inspired by the work of Wang, we propose the approach in this work to the problem of extending the results in Section 2 of [5] to manifolds with boundary.

1.2 Contents

We first review the construction of the even Fredholm module $(\mathcal{H}, F, \gamma)$ over the commutative algebra $\mathcal{A} = C^\infty(M)$ (trivially an involutive algebra over $\mathbb{C}$) of smooth functions over a compact oriented manifold M without boundary. We give special attention to its conformal properties. Then we review the statement of Connes about recovering the conformal structure from this Fredholm module and a recent characterization on the subject by Bär. Next, we move to the setting of manifolds with boundary. Aiming to extend previous work, we briefly review the noncommutative residue for manifolds with boundary and Boutet de Monvel calculus according to our needs. Last, we present a couple of results that extend to the even dimensional case Theorems 1 and 2 in [18] to the following setting of manifolds with boundary: M is a compact manifold with boundary ∂M such that M is embedded in a compact oriented manifold $\widetilde{M}$ without boundary. Further we assume Riemannian structures $(\widetilde{M}, \widetilde{g})$ and (M, g) such that g coincides with $\widetilde{g}$ restricted to M. The results are Theorems 4 and 5 respectively.

1.3 Other possibilities

For a Riemann surface M, a map $f = (f^i)$ from M to $\mathbb{R}^2$ and metric $g_{ij}(x)$ on M, the 2-dimensional Polyakov action [15] is given by

$$I(f) = \frac{1}{2\pi} \int_M g_{ij}\, df^i \wedge \star df^j.$$

By considering instead of df its quantized version $[F, f]$, Connes [5] quantized the Polyakov action as a Dixmier trace:

$$\frac{1}{2\pi} \int_M g_{ij} df^i \wedge \star df^j = -\frac{1}{2} \operatorname{Tr}_\omega\big(g_{ij}[F, f^i][F, f^j]\big).$$

Connes' trace theorem [4] states that the Dixmier trace and the noncommutative residue of an elliptic pseudodifferential operator of order $-n$ on an n-dimensional manifold M are proportional by a factor of $n(2\pi)^n$. In the 2-dimensional case the factor is $8\pi^2$ and so, the quantized Polyakov action can be written as

$$-16\pi^2 I = \mathrm{Res}\big(g_{ij}[F, f^i][F, f^j]\big).$$

This quantized Polyakov action makes sense in the general even dimensional case.

Although this generalization of the Polyakov action motivates the particular form of the functional $\mathrm{Res}(f_0[F, f_1][F, f_2])$, the same Fredholm module yields other functionals in dimensions greater than 4. For instance, for dimension 6 one could also consider

$$\mathrm{Res}(f_0[F, f_1][F, f_2][F, f_3]) = \int_M f_0\, T(f_1, f_2, f_3)\, d^6x,$$

which is a Hochschild 3-cocycle and the trilinear expression $T(f_1, f_2, f_3)$ is conformally invariant. In greater $2l$ dimensions,

$$\mathrm{Res}(f_0[F, f_1]\cdots[F, f_l]) = \int_M f_0\, C(f_1, \dots, f_l)\, d^{2l}x,$$

invites to study the role of the conformal invariant $C(f_1, \dots, f_l)$.

1.4 Further directions

The possibility of obtaining from the expression

$$\begin{aligned}\int_M B_n(f_1, f_2)\, dx + 2\pi \int_{\partial M} \partial B_n(f_1|_{\partial M}, f_2|_{\partial M})\, dx' \\ = \int_M f_1 P_n f_2\, dx + \int_{\partial M} f_1|_{\partial M} P'_{n-1} f_2|_{\partial M}\, dx'\end{aligned}$$

conformally covariant differential operators P_n and P'_{n-1} acting on M and ∂M respectively (in a way similar to the boundaryless case), is the motivating force behind this project. We hope to report on that in the near future.

One more possibility is to study a Riemannian manifold with a particular metric structure near the boundary, in such a way that it makes sense to consider its double manifold and at the same time, study conformal variations of the metric. One can ask what sort of specific objects are to be found using the ideas presented here in such a particular situation. A seemingly promising case is that of manifolds with totally geodesic boundaries, for which the double manifold is natural to be considered.

2 The Fredholm module for a conformal manifold

Following Definition IV.4.1 [6], an even Fredholm module $(\mathcal{H}, F, \gamma)$ is given by

- An involutive algebra $\mathcal{A}$ (over $\mathbb{C}$) together with a Hilbert space $\mathcal{H}$ and an involutive representation π of $\mathcal{A}$ in $\mathcal{H}$.
- An operator F on $\mathcal{H}$ such that $F = F^*$, $F^2 = 1$, and $[F, \pi(a)]$ is a compact operator $\forall\, a \in \mathcal{A}$.
- A $\mathbb{Z}/2$ grading γ, $\gamma = \gamma^*$, $\gamma^2 = 1$ of $\mathcal{H}$ such that $\gamma\pi(a) = \pi(a)\gamma$, $\forall\, a \in \mathcal{A}$, and $\gamma F = -F\gamma$.

In the case of a manifold without boundary, the very first ingredient in $(\mathcal{H}, F, \gamma)$ is the involutive algebra $\mathcal{A} = C^\infty(M)$ where we allow complex values. The fact that, for an oriented Riemannian manifold M (with or without boundary) and of even dimension n, the restriction of the Hodge star operator to middle-dimension forms is conformally invariant, is central to what follows. We consider the vector bundle $\Omega^{n/2}_{\mathbb{C}}(M)$ of complex middle-dimension forms. We drop the subscript $\mathbb{C}$ from now on. In this way, for an n-dimensional oriented compact manifold, n even, the space $\Omega^{n/2}(M)$ of (complex) middle dimension forms has a (complexified) inner product

$$\langle \omega_1, \omega_2 \rangle = \int_M \overline{\omega_1} \wedge \star\omega_2.$$

This inner product is unchanged under a conformal change of the metric and so, its Hilbert space completion $\mathcal{H}_0 = L^2(M, \Omega^{n/2}(M))$ depends only on the conformal class of the metric. $\mathcal{H}_0$ is by construction a $C^\infty(M)$-module with $(f\omega)(p) = f(p)\omega(p)$ for all $f \in C^\infty(M)$, $\omega \in \mathcal{H}_0$, and $p \in M$.

If M is a compact manifold without boundary then, the harmonic forms (those in the kernel of Δ) are precisely those in $\operatorname{Ker} d \cap \operatorname{Ker} \delta$. If M is even dimensional with dimension n, the Hodge decomposition for middle-dimension forms looks like

$$\Omega^{n/2}(M) = \Delta(\Omega^{n/2}(M)) \oplus H^{n/2} = d(\Omega^{n/2-1}(M)) \oplus \delta(\Omega^{n/2+1}(M)) \oplus H^{n/2}.$$

Here $H^{n/2} = \operatorname{Ker}_{n/2} \Delta$. Thus $\mathcal{H}_0$ is the direct sum of $H^{n/2}$ and the images of d and δ.

The Hilbert space $\mathcal{H}$ is $\mathcal{H} = \mathcal{H}_0 \oplus H^{n/2}$, the direct sum of $\mathcal{H}_0$ with an extra copy of the finite dimensional Hilbert space of harmonic middle-dimension forms on M.

For each $f \in \mathcal{A} = C^\infty(M)$ we consider the multiplication operator on $\mathcal{H}_0$, $f : \omega \mapsto f\omega$. These multiplication operators on $\mathcal{H}_0$ do not preserve the subspace of harmonic forms. Thus the extra copy of $H^{n/2}$ in $\mathcal{H}$ is to preserve the notion of $\mathbb{Z}_2$-graded Hilbert space.

The Hilbert space representation of $C^\infty(M)$ in $\mathcal{H}$ is given by $f \mapsto \pi(f)$ with $\pi(f)(\omega+h) := f\omega$, for all $\omega \in \mathcal{H}_0$ and $h \in H^{n/2}$. Evidently, $\pi(f)$ is a bounded operator on $\mathcal{H}$. It is not difficult to verify that π is an involutive representation of $C^\infty(M)$ in $\mathcal{H}$. To simplify the notation we write f instead of $\pi(f)$.

Next we look at the $\mathbb{Z}/2$ grading. Because the Hodge star operator $\star$ acting on middle-dimension forms satisfies $\star^2 = (-1)^{n/2}$, the operator

$$\gamma_0 := (-1)^{\frac{(n/2)(n/2-1)}{2}} i^{n/2} \star$$

is of square one giving a $\mathbb{Z}_2$-grading on $\mathcal{H}_0$. Since $\star^* = (-1)^{n/2}\star$ when acting on middle forms we have

$$\gamma_0^* := (-1)^{\frac{(n/2)(n/2-1)}{2}} (-i)^{n/2} \star^* = (-1)^{\frac{(n/2)(n/2+3)}{2}} i^{n/2} \star = \gamma_0. \tag{1}$$

It follows from the definition of γ_0 that

Lemma 1. *The operator γ_0 exchanges the subspaces $d\big(\Omega^{n/2-1}\big)$ and $\delta\big(\Omega^{n/2+1}\big)$ and their closures. As a consequence and because of* (1), $\gamma_0(H^{n/2}) = H^{n/2}$.

We define $\gamma : \mathcal{H}_0 \oplus H^{n/2} \to \mathcal{H}_0 \oplus H^{n/2}$ by $\gamma(\omega + h) := \gamma_0(\omega) - \gamma_0 h$. In this way, the extra copy of $H^{n/2}$ is endowed with the opposite $\mathbb{Z}_2$-grading $-\gamma$: $H^{n/2\pm} = \{h \in H^{n/2} : \gamma h = \mp h\}$ and so $\mathcal{H}$ has the $\mathbb{Z}_2$-grading given by $\mathcal{H}^+ = \mathcal{H}_0^+ \oplus H^{n/2^+}$ and $\mathcal{H}^- = \mathcal{H}_0^- \oplus H^{n/2^-}$.

It is straightforward to verify that the operator γ defined on $\mathcal{H}$ satisfies $\gamma = \gamma^*$, $\gamma^2 = 1$, and $\gamma f = f\gamma$ for all $f \in C^\infty(M)$.

The first step to define the operator F is the following observation

Lemma 2. *For $\omega = d\beta + \delta\beta' \in d(\Omega^{n/2-1}(M)) \oplus \delta(\Omega^{n/2+1}(M))$, the operator $F_0 \colon \mathcal{H}_0 \to \mathcal{H}_0$ defined by $F_0(d\beta + \delta\beta') := d\beta - \delta\beta'$ and extended as zero over $H^{n/2}$ is a partial isometry such that $F_0^2 = 1$ on $\mathcal{H}_0 \ominus H^{n/2}$, and F_0 is its own formal adjoint operator. Furthermore, $1 - F_0^2$ is the orthogonal projection on the finite-dimensional Hilbert space of middle-dimension harmonic forms.*

The operator F is defined on $\mathcal{H} = d(\Omega^{n/2-1}(M)) \oplus \delta(\Omega^{n/2+1}(M)) \oplus H^{n/2} \oplus H^{n/2}$ by

$$F = \begin{pmatrix} F_0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2}$$

From the previous lemma $F^* = F$ and $F^2 = 1$.

For an even dimensional oriented compact manifold without boundary, both the Hilbert space $\mathcal{H}$ and the operator F are conformally invariant, thanks to the fact: for a k-form ρ,

$$\widehat{\delta}\rho = e^{-(n-2(k-1))\eta}\,\delta\, e^{(n-2k)\eta}\rho.$$

It is not difficult to verify that $\gamma F = -F\gamma$. Last, since each $[F, f]$ is a pseudodifferential operator of order -1 for all $f \in \mathcal{A} = C^\infty(M)$, the operator $[F, f]$ is a compact operator on $\mathcal{H}$ via Rellich's theorem [9, p. 306].

2.1 Recovering the conformal structure

Theorem IV.4.2.c of [6] states that the Fredholm module $(\mathcal{H}, F)$ uniquely determines the conformal structure of M. For that, Connes uses his trace theorem and the noncommutative residue to recover the L^n-norm for exterior 1-forms over the manifold.

The first step is to consider instead of df its quantized version $[F, f]$. Since F is a pseudodifferential operator of order 0, $[F, f]$ is a pseudodifferential operator of order -1 for all $f \in C^\infty(M^n)$, acting on the same vector bundle $\Omega^{n/2}M$ as F. The leading symbol of F is given by

$$\sigma_0(F)(x,\xi) = |\xi|^{-2}\left(\varepsilon_{\frac{n}{2}-1}(\xi)\iota_{\frac{n}{2}}(\xi) - \iota_{\frac{n}{2}+1}(\xi)\varepsilon_{\frac{n}{2}}(\xi)\right)$$

for all $(x, \xi) \in T^*M$, $\xi \neq 0$. Here $\varepsilon_k(\xi)$ and $\iota_k(\xi)$ represent the exterior and interior multiplication by the 1-form ξ on k-forms. Note how $\sigma_0(F)$ does not depend on $x \in M$. The principal symbol of $[F, f]$ is

$$\sigma_{-1}([F,f])(x,\xi) = -i\sum_{k=1}^{n}\partial_{x^k} f\,\partial_{\xi_k}(\sigma_0(F))$$

which by the expression for $\sigma_0(F)(x, \xi)$ depends only on the value on x of the 1-form $df = \sum \partial_{x^k} f\, dx^k$. The details of these statements can be seen for example in [20]. Next for $f_i \in C^\infty(M)$ the operator $(f_1[F, f_2])^n$ is a pseudodifferential operator of order $-n$.

What Theorem IV.4.2.c [6] shows is that $\mathrm{Res}(f_1[F, f_2])^n$ and $\int_M ||f_1 df_2||^n\, dx$, the L^n-norm for 1-forms, are proportional.

In the setting of spin Riemannian manifolds, for the algebra $C^\infty(M)$ of smooth complex valued functions, the Hilbert space is chosen to be $\mathcal{H} = L^2(M, \Sigma M)$, the square integrable complex spinor fields, and for F one considers the sign of the Dirac operator D. Recently, Bär [2] showed the following result.

Theorem 2. *Let M be a compact spin Riemannian manifold. Let g and g' be Riemannian metrics on M and let $(\mathcal{H}, \mathrm{sign}(D))$ and $(\mathcal{H}', \mathrm{sign}(D'))$ be the corresponding Fredholm modules of the algebra $C^\infty(M)$. Then g and g' are conformally equivalent if and only if $(\mathcal{H}, \mathrm{sign}(D))$ and $(\mathcal{H}', \mathrm{sign}(D'))$ are weakly unitarily equivalent. That is to say, there is a unitary isomorphism $U : \mathcal{H} \to \mathcal{H}'$ such that $D' - UDU^{-1}$ is a compact operator and for all $f \in C^\infty(M)$ and all $h \in \mathcal{H}$ one has $U(fh) = fU(h)$.*

The idea is based on the commutativity of U with the action of $C^\infty(M)$ which implies that U is induced by a (a.e. invertible) section Ψ of $L^\infty(M, \mathrm{Hom}(\Sigma M, \Sigma' M))$. The principal symbol of a Dirac operator is given by Clifford multiplication with respect to the metric g, $\sigma_D(\xi) = ic_g(\xi)$, for all $\xi \in T^*M$. Because of the relation $c_g(\xi)c_g(\eta) + c_g(\eta)c_g(\xi) = -2g(\xi, \eta)$, for all $\xi, \eta \in T^*M$, the principal symbol of $\mathrm{sign}(D)$ is

$$\sigma_{\mathrm{sign}(D)}(\xi) = \frac{ic_g(\xi)}{||\xi||_g}, \qquad \forall\, \xi \in T^*M \setminus \{0\}.$$

Since D' and UDU^{-1} differ by a compact operator, they have the same sign and thus

$$\frac{c_{g'}(\xi)}{||\xi||_{g'}} = \Psi(x)\frac{c_g(\xi)}{||\xi||_g}\Psi^{-1}(x)$$

for all nonzero $\xi \in T^*M$. Last

$$\begin{aligned}\frac{-2g'(\xi,\eta)}{||\xi||_{g'}||\eta||_{g'}} &= \frac{c_{g'}(\xi)c_{g'}(\eta)+c_{g'}(\eta)c_{g'}(\xi)}{||\xi||_{g'}||\eta||_{g'}}\\ &= \Psi(x)\left(\frac{c_g(\xi)c_g(\eta)+c_g(\eta)c_g(\xi)}{||\xi||_g||\eta||_g}\right)\Psi^{-1}(x) = \frac{-2g(\xi,\eta)}{||\xi||_g||\eta||_g}\end{aligned}$$

since the term in the middle is a scalar.

It is important to recall here the result of Connes (see for example [6, p. 544]) that says that one recovers the metric distance between points in a connected manifold (M,g) from the relation

$$d(x,y) = \sup\{f(x)-f(y) : f \in C^\infty(M) \text{ with } |||[D,f]||| \leq 1\}.$$

Note how with the stronger requirement $D' = UDU^{-1}$ (unitarily equivalent) then $|||[D,f]||| = |||[D',f]|||$ and thus $d = d'$.

If the conformal geometry of $(M,[g])$ is encoded in the Fredholm module $(\mathcal{H},F,\gamma)$ over the algebra $C^\infty(M)$, then how can one extract the conformal geometry from this Fredholm module?

One possibility is to use it to find conformal invariants associated to a given conformal manifold, for example, as in the introduction.

3 The noncommutative residue for manifolds with boundary

Remark 1. Wodzicki: (see e.g. [16]) There is no non-zero trace on the algebra of classical pseudodifferential operators mod the ideal of smoothing operators $\Psi^\infty(M)/\Psi^{-\infty}(M)$, whenever M is noncompact or has a boundary.

The noncommutative residue of Fedosov–Golse–Leichtnam–Schrohe [8] for manifolds with boundary is the unique (up to a constant multiple) continuous trace for the operators in Boutet de Monvel's algebra. Roughly speaking, this noncommutative residue acts on operators A that are described by pairs of symbols $\{a_i, a_b\}$ called interior and boundary symbol respectively. In case the manifold has empty boundary this noncommutative residue coincides with the usual noncommutative residue of Wodzicki, Guillemin, Adler, and Manin.

The setting for the noncommutative residue is given by a compact manifold M with boundary ∂M such that M is embedded in a compact manifold $\widetilde{M}$ without boundary, both M and $\widetilde{M}$ of dimension $n > 1$. For M we consider in a boundary chart local coordinates given by (x', x^n) with $x' = (x^1, \ldots, x^{n-1})$ coordinates for ∂M and x^n the geodesic distance to ∂M. It is important to mention that the geodesic coordinate chosen for x^n is only a technical tool since the noncommutative residue is independent of the metric and of local representations.

3.1 Boutet de Monvel's sub-algebra of diagonal symbols

In [3, 8, 11], and [16] one can find detailed introductions to Boutet de Monvel's calculus. The operators in Boutet de Monvel's algebra we are interested in are diagonal matrices of operators (endomorphisms) A acting on sections of vector bundles E over M and E' over ∂M :

$$A = \begin{pmatrix} r^M P e^M + G & 0 \\ 0 & S \end{pmatrix} : \begin{matrix} C^\infty(M,E) \\ \oplus \\ C^\infty(\partial M, E') \end{matrix} \to \begin{matrix} C^\infty(M,E) \\ \oplus \\ C^\infty(\partial M, E') \end{matrix}.$$

They are better described by a pair of symbols (a_i, a_b) where a_i is called the interior symbol and a_b is called the boundary symbol. According to our needs, the characterization of such an operator (or its symbol) of order m is as follows.

P. The operator P is a classical pseudodifferential operator of order m on $\widetilde{M}$. Furthermore, P has the so called transmission property. This guarantees that the composition of different elements remains inside the algebra. Analytically, in local coordinates near ∂M the transmission property is given by

$$\partial^k_{x^n}\partial^\alpha_{\xi'} p_j(x', 0, 0, +1) = (-1)^{j-|\alpha|}\partial^k_{x^n}\partial^\alpha_{\xi'} p_j(x', 0, 0, -1), \qquad \forall\, j, k, \alpha.$$

Here p_j is the homogeneous component of order j in the symbol expansion of the symbol p of P. Last, e^M is the extension by zero of functions (or sections) on M to functions (or sections) on $\widetilde{M}$ and r^M is the restriction from $\widetilde{M}$ to M. The interior symbol a_i of A is precisely p.

With $\mathcal{F}$ we denote the Fourier transform. Also

$$\begin{aligned} H^+ &= \{\mathcal{F}(\chi_{]0,\infty[}u) : u \text{ is a rapidly decreasing function on } \mathbb{R}\}, \\ H^-_0 &= \{\mathcal{F}((1-\chi_{]0,\infty[})u) : u \text{ is a rapidly decreasing function on } \mathbb{R}\}, \\ H^- &= H^-_0 \oplus \{\text{all polynomials}\}. \end{aligned}$$

The (diagonal) boundary symbol a_b is given by a pair of symbols b, s of operators G, S parametrized by $T^*\partial M \setminus \{0\}$ and the restriction of p to the boundary.

G. The operator G is given by a singular green operator-symbol $b(x', \xi', D_n)$ in the following way. For every l and fixed x', ξ',

$$b_l(x', \xi', \xi_n, \eta_n) \in H^+ \hat{\otimes}_\pi H^-.$$

With $\hat{\otimes}_\pi$ we denote Grothendieck's completion of the algebraic tensor product. The operator $b(x', \xi', D_n) : H^+ \to H^+$ is given by

$$[b(x', \xi', D_n)h](\xi_n) = \Pi'_{\eta_n}\big(b(x', \xi', \xi_n, \eta_n)h(\eta_n)\big) = \lim_{\eta_n \to 0^+} \mathcal{F}^{-1}(b(x', \xi', \xi_n, \cdot)h(\cdot))(\eta_n).$$

The operator G described by this operator-symbol $b(x', \xi', D_n)$ between functions on $[0, \infty[$ that are rapidly decreasing at ∞, defines a trace class operator on $L^2(R_+)$. The trace is given by

$$\operatorname{trace}(G)(x', \xi') = \frac{1}{2\pi}\int b(x', \xi', \xi_n, \xi_n)\, d\xi_n.$$

Note that this is actually a symbol itself.

S. The operator S is a classical pseudodifferential operator of order m along the boundary. It has values in $\mathcal{L}(\mathbb{C}^k)$ and each component s_j of its symbol expansion s acts by multiplication on $\mathbb{C}^k$.

The (diagonal) boundary symbol a_b is then

$$a_b(x', \xi', \xi_n, \eta_n) = \begin{pmatrix} p(x', 0, \xi', \xi_n) + b(x', \xi', \xi_n, \eta_n) & 0 \\ 0 & s(x', \xi') \end{pmatrix}$$

with

$$b(x',\xi',\xi_n,\eta_n)\sim\sum_{l=-\infty}^{m}b_l(x',\xi',\xi_n,\eta_n)\qquad\text{and}\qquad s(x',\xi')\sim\sum_{l=-\infty}^{m}s_l(x',\xi')$$

where for $\lambda>0$

$$b_l(x',\lambda\xi',\lambda\xi_n,\lambda\eta_n)=\lambda^l b_l(x',\xi',\xi_n,\eta_n),\qquad s_l(x',\lambda\xi')=\lambda^l s_l(x',\xi').$$

By $\mathcal{B}_D^m(M)$ we denote the collection of all operators of order m with diagonal boundary symbol and by $\mathcal{B}_D^\infty(M)$ the union of all the $\mathcal{B}_D^m(M)$. The intersection over all orders m of $\mathcal{B}_D^m(M)$ is denoted $\mathcal{B}_D^{-\infty}(M)$. Last $\mathcal{B}_D=\mathcal{B}_D^\infty(M)/\mathcal{B}_D^{-\infty}(M)$.

Given two operators A_1 and A_2 in $\mathcal{B}_D$ with symbols (a_{i1},a_{b1}) and (a_{i2},a_{b2}), with entries in the boundary symbols b_j, s_j, for $j=1,2$, the composition is again an operator in $\mathcal{B}_D$ with symbol (a_i,a_b) where a_i is the usual composition of symbols $a_i=a_{i1}\circ a_{i2}$. It also satisfies the transmission property.

The resulting boundary symbol is of the form

$$a_b=a_{b1}\circ' a_{b2}+\begin{pmatrix}L(p_{i1},p_{i2})+p_{i1}^+\circ' b_2+b_1\circ' p_{i2}^+ & 0\\ 0 & 0\end{pmatrix}.$$

The symbol $\circ'$ denotes the usual composition of pseudodifferential symbols on the variables (x',ξ'). The terms in the second summand represent the portion on the boundary symbol coming from the interior symbols. Here, we have hidden in $a_{b1}\circ' a_{b2}$ the part corresponding to the restriction to the boundary of the interior symbol.

The so called "left-over term" $L(p_{i1},p_{i2})$, reflects the particular way the pseudodifferential operators $P_M=r^M P e^M$ act on the manifold with boundary M. If P_1 and P_2 are two pseudodifferential operators on $\widetilde{M}$, the difference $(P_1P_2)_M-(P_1)_M(P_2)_M$ is a singular Green operator with associated singular Green operator-symbol $L(p_1,p_2)$. Since this left-over term need not be zero, we can not reduce the diagonal sub-algebra by requesting $G=0$ in all the operators.

As an example, and because they will be needed later on, let us look at $L(f,q)$ and $L(p,f)$ where p and q are the symbols of pseudodifferential operators P and Q on $\widetilde{M}$, and $f\in C^\infty(\widetilde{M})$, i.e. f represents the pseudodifferential operator on $\widetilde{M}$ given multiplication by f. Among all the possible formulae for $L(p,q)$ available in the literature we decided to use the one provided in [11].

In Section 3 of [11] one can read an explicit expression for $L(p,q)$ in which the effects of p and q are neatly separated. This expression uses singular Green operators $G^+(p)$ and $G^-(q)$ natural for the calculus in use (see Theorems 3.2 and 3.4 [11]).

We content ourselves by quoting a particular situation. By Theorem 3.4 [11], $G^-(f)=0$ and by (3.16) [11], $L(p,f)=G^+(p)G^-(f)$, thus $L(p,f)=0$.

Now, for $L(f,q)$ we must look at Theorem 3.5 [11]. In general,

$$L(p,q)=G^+(p)G^-(q)+\sum_{0\le m<\text{ order of }Q}K_m\gamma_m$$

where the K_m are operators obtained from symbols of a particular type known as Poisson symbols. By (3.35) [11], $K_m=0$ when $p=f$ since it depends on higher derivatives on ξ_n. Since by Theorem 3.2 [11], $G^+(f)=0$ we conclude that $L(f,p)=0$ as well.

Lemma 3. *For every $f\in C^\infty(\widetilde{M})$ and every pseudodifferential operator P on $\widetilde{M}$ with symbol p, both left-over terms $L(f,p)$ and $L(p,f)$ vanish.*

Last, the operator $p^+(x',\xi',D_n):H^+\to H^+$ is induced from the action (of the interior symbol) in the normal direction for fixed (x',ξ'). The only case we will be interested in are those of the form $f^+\circ' b_2$ where f is a smooth function on $\widetilde{M}$. We will address them in (3).

3.2 The noncommutative residue

On $\mathbb{R}^n$ with coordinates $\xi_1, \ldots, \xi_n$ we consider the $(n-1)$-form

$$\sigma = \sum_{j=1}^{n} (-1)^{j+1} \xi_j \, d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_n,$$

where the hat indicates this factor is omitted. Restricted to the unit sphere $\mathbb{S}^{n-1}$, σ gives the volume form on $\mathbb{S}^{n-1}$ and in general $d\sigma = n \, d\xi_1 \wedge \cdots \wedge d\xi_n$. For a coordinate chart U, the form $dx_1 \wedge \cdots \wedge dx_n$ defines an orientation on U and induces the orientation $d\xi_1 \wedge \cdots \wedge d\xi_n$ on $\mathbb{R}^n$.

For a closed compact manifold M without boundary, the noncommutative residue is defined as the unique trace (up to constant multiples) on the algebra $\Psi^\infty / \Psi^{-\infty}$ of classical pseudodifferential operators mod the ideal of smoothing operators.

The following is the main result of [8]:

Theorem 3 (Fedosov–Golse–Leichtnam–Schrohe). *Let M be a manifold of dimension n with smooth boundary ∂M, and let $M \cup \partial M$ be embedded in a connected manifold $\widetilde{M}$ of dimension n. Let*

$$A = \begin{pmatrix} r^M P e^M + G & K \\ T & S \end{pmatrix}$$

be an element in $\mathcal{B}^\infty(M)/\mathcal{B}^{-\infty}(M)$, with $\mathcal{B}^\infty(M)$ the algebra of all operators in Boutet de Monvel's calculus (with integral order), $\mathcal{B}^{-\infty}(M)$ the ideal of smoothing operators, and let p, b, and s denote the local symbols of P, G, and S respectively. Then

$$\begin{aligned} \overline{\mathrm{Res}} A = & \int_M \int_{S^{n-1}} \mathrm{Tr}_E \, p_{-n}(x,\xi) \sigma(\xi) \, dx \\ & + 2\pi \int_{\partial M} \int_{S^{n-2}} \left\{ \mathrm{Tr}_{E'} (\mathrm{trace} \, b_{-n})(x', \xi') + \mathrm{Tr}_{E'} \, s_{1-n}(x',\xi') \right\} \sigma'(\xi') \, dx', \end{aligned}$$

with σ' the $n-2$ analog of σ, is the unique continuous trace (up to constant multiples) on the algebra $\mathcal{B}^\infty(M)/\mathcal{B}^{-\infty}(M)$.

This trace reduces to the noncommutative residue (of Adler, Manin, Guillemin, and Wodzicki) in the case $\partial M = \varnothing$, and it is independent of the Riemannian metric (eventually) chosen on M.

4 On manifolds with boundary

In this section we present an extension of Theorem 1 in [18] to the setting of manifolds with boundary. Let M be a manifold with boundary ∂M. Assume that the compact manifold M is embedded in a compact manifold $\widetilde{M}$ without boundary. Further we assume $\widetilde{M}$ to be oriented which determines an orientation on M and thus on ∂M.

For P a pseudodifferential operator acting on a vector bundle E over $\widetilde{M}$ with symbol p having the transmission property up to the boundary, S a pseudodifferential operator acting on a vector bundle E' over ∂M with symbol s, and for $f \in C^\infty(\widetilde{M})$ we let $A(P,S)$ and $A(f)$ be the elements in Boutet de Monvel's algebra of diagonal elements given by

$$A(P,S) = \begin{pmatrix} r^M P e^M + 0 & 0 \\ 0 & S \end{pmatrix}, \qquad A(f) = \begin{pmatrix} r^M f e^M + 0 & 0 \\ 0 & f|_{\partial M} \end{pmatrix}.$$

We study $\mathrm{Res}(A(f_0)[A(P,S), A(f_1)][A(P,S), A(f_2)])$ for functions $f_i \in C^\infty(\widetilde{M})$.

First of all, we must check that this product operator remains inside the calculus in use. It follows from Proposition 2.7 of [17], which states that if two operators satisfy the transmission property then their product satisfies the transmission property as well.

Since $L(f,p)=0=L(p,f)$, for all $f\in C^\infty(\widetilde{M})$ it follows that

$$
\begin{aligned}
&A(f_0)[A(P,S),A(f_1)][A(P,S),A(f_2)] = \\
&\begin{pmatrix} r^M f_0[P,f_1][P,f_2]e^M + f_0^+\circ' L(\sigma([P,f_1]),\sigma([P,f_2])) & 0 \\ 0 & f_0|_{\partial M}\circ' [S,f_1|_{\partial M}]'\circ' [S,f_2|_{\partial M}]' \end{pmatrix}
\end{aligned}
$$

where $\circ'$ represents the symbol composition with respect to (x',ξ'). Here $\sigma([P,f_i])$ represents as usual the symbol of the operator $[P,f_i]$. Using the definition of $\overline{\mathrm{Res}}$ for manifolds with boundary we have

$$
\begin{aligned}
&\overline{\mathrm{Res}}\big(A(f_0)[A(P,S),A(f_1)][A(P,S),A(f_2)]\big)\\
&\quad = \int_M\int_{S^{n-1}} \mathrm{Tr}_E\left\{\sigma_{-n}\big(f_0[P,f_1][P,f_2](x,\xi)\big)\right\}\sigma(\xi)\,dx\\
&\qquad + 2\pi\int_{\partial M}\int_{S^{n-2}} \mathrm{Tr}_{E'}\left\{\sigma_{-(n-1)}\big(\mathrm{trace}\{f_0^+\circ' L(\sigma([P,f_1]),\sigma([P,f_2]))\}(x',\xi')\big)\right\}\\
&\qquad\qquad + \mathrm{Tr}_{E'}\left\{\sigma_{-(n-1)}\left(f_0|_{\partial M}\circ' [S,f_1|_{\partial M}]'\circ' [S,f_2|_{\partial M}]'(x',\xi')\right)\right\}\sigma'(\xi')\,dx'.
\end{aligned}
$$

4.1 A pair of bilinear functionals

Mimicking the boundaryless case and following [21] we define:

Definition 1.

$$
B_{n,P}(f_1,f_2) := \int_{S^{n-1}} \mathrm{Tr}_E\left\{\sigma_{-n}\big([P,f_1][P,f_2](x,\xi)\big)\right\}\sigma(\xi),
$$

and

$$
\begin{aligned}
\partial B_{n,P,S}(f_1,f_2) := \int_{S^{n-2}} &\mathrm{Tr}_{E'}\left\{\sigma_{-(n-1)}\big(\mathrm{trace}\{L(\sigma(([P,f_1]),\sigma(([P,f_2]))\}(x',\xi')\big)\right\}\\
&+ \mathrm{Tr}_{E'}\left\{\sigma_{-(n-1)}\big(([S,f_1|_{\partial M}]'\circ' [S,f_2|_{\partial M}])(x',\xi')\big)\right\}\sigma'(\xi'),
\end{aligned}
$$

for all $f_i\in C^\infty(\widetilde{M})$.

By definition, both $B_{n,P}$ and $\partial B_{n,P,S}$ are bilinear. Since f_0 is independent of ξ we have

$$
\int_{S^{n-1}} \mathrm{Tr}_E\left\{\sigma_{-n}\big(f_0[P,f_1][P,f_2](x,\xi)\big)\right\}\sigma(\xi) = f_0 B_{n,P}(f_1,f_2).
$$

The computations done in [19] with the symbol expansions for the case of empty boundary are also valid here. In particular we have in given local coordinates the explicit expression

$$
B_{n,P}(f_1,f_2) = \sum \frac{D_x^\beta(f_1)D_x^{\alpha''+\delta}(f_2)}{\alpha'!\alpha''!\beta!\delta!}\int_{S^{n-1}} \mathrm{Tr}\left\{\partial_\xi^{\alpha'+\alpha''+\beta}(\sigma_{k-i}^P)\partial_\xi^\delta(D_x^{\alpha'}(\sigma_{k-j}^P))\right\}\sigma(\xi)
$$

with the sum taken over $|\alpha'|+|\alpha''|+|\beta|+|\delta|+i+j=n+2k$, $|\beta|\geq 1$, and $|\delta|\geq 1$. It shows that $B_{n,P}(f_1,f_2)$ is differential in f_1 and f_2. Evidently it is possible to obtain a similar expression for the summand in $\partial B_{n,P,S}$ corresponding to S replacing n by $n-1$ and x by x'.

In p. 25 of [8] we can read an expression for the degree $-(n-1)$ component of the operator-symbol $\mathrm{trace}(c)$ with $c=p^+\circ' b$. It is given by

$$
\sigma_{-(n-1)}\big(\mathrm{trace}\,c(x',\xi')\big) \sim \sum_{j=0}^\infty \frac{i^j}{j!}\Pi'_{\xi_n}\left\{\sigma_{-n}\big(\partial_{\xi_n}^j[\partial_{x_n}^j p(x',0,\xi',\xi_n)\circ' b(x',\xi',\xi_n,\eta_n)]\big)|_{\eta_n=\xi_n}\right\}. \quad (3)
$$

Thus, since f_0 is independent of ξ,

$$\begin{aligned}&\operatorname{trace}\bigl\{\sigma_{-(n-1)}\bigl(f_0^+ \circ' L(\sigma([P,f_1]),\sigma([P,f_2]))\bigr)\bigr\}\\ &\qquad = f(x',0)\operatorname{trace}\{\sigma_{-(n-1)}(L(\sigma([P,f_1]),\sigma([P,f_2]))(x',\xi')\}.\end{aligned}$$

and it follows that

$$\begin{aligned}&\int_{S^{n-2}} \operatorname{Tr}\bigl\{\sigma_{-(n-1)}\bigl(\operatorname{trace}\bigl\{f_0^+ \circ' L(\sigma([P,f_1]),\sigma([P,f_2]))\bigr\}(x',\xi')\bigr)\bigr\}\,\sigma'(\xi')\\ &\qquad = f(x',0)\int_{S^{n-2}} \operatorname{Tr}\bigl\{\sigma_{-(n-1)}\bigl(\operatorname{trace}\bigl\{L(\sigma([P,f_1]),\sigma([P,f_2]))\bigr\}(x',\xi')\bigr)\bigr\}\,\sigma'(\xi'),\end{aligned}$$

for all $f_i \in C^\infty(\widetilde{M})$. In this way

$$\begin{aligned}&\overline{\operatorname{Res}}\bigl(A(f_0)[A(P,S),A(f_1)][A(P,S),A(f_2)]\bigr)\\ &\qquad = \int_M f_0 B_{n,P}(f_1,f_2)\,dx + 2\pi\int_{\partial M} f_0|_{\partial M}\partial B_{n,P,S}(f_1|_{\partial M}, f_2|_{\partial M})\,dx'. \end{aligned}\tag{4}$$

Lemma 4. *The functionals $B_{n,P}$ and $\partial B_{n,P,S}$ are bilinear and symmetric.*

Proof. The symmetry of both $B_{n,P}$ and $\partial B_{n,P,S}$ is not evident from the expressions above. For $B_{n,P}$ it was obtained in [20] in the boundaryless case from the trace property of Res. Because it shares the same local expression both for empty and non-empty boundary we have that $B_{n,P}$ is symmetric.

For $\partial B_{n,P,S}$ we are going to exploit the linearity and the trace property of the noncommutative residue. Denote $\overline{f} = A(f)$ and $\overline{P} = A(P,S)$. Using that $\overline{f_1}\,\overline{f_2} = \overline{f_2}\,\overline{f_1}$ for all $f_i \in C^\infty(\widetilde{M})$ and the trace property of the noncommutative residue we have that all of $\overline{\operatorname{Res}}\bigl(\overline{f_0}\,\overline{f_2}\,\overline{P}\,\overline{P}\,\overline{f_1} - \overline{f_0}\,\overline{f_1}\,\overline{P}\,\overline{P}\,\overline{f_2}\bigr)$, $\overline{\operatorname{Res}}\bigl(\overline{f_0}\,\overline{f_2}\,\overline{P}\,\overline{P}\,\overline{f_1} - \overline{f_0}\,\overline{f_1}\,\overline{P}\,\overline{P}\,\overline{f_2}\bigr)$, and $\overline{\operatorname{Res}}\bigl(\overline{f_0}\,\overline{P}\,\overline{f_2}\,\overline{P}\,\overline{f_1} - \overline{f_1}\,\overline{f_0}\,\overline{P}\,\overline{f_2}\,\overline{P}\bigr)$ vanish. In this way

$$\begin{aligned}&\overline{\operatorname{Res}}(\overline{f_0}[\overline{P},\overline{f_1}][\overline{P},\overline{f_2}] - \overline{f_0}[\overline{P},\overline{f_2}][\overline{P},\overline{f_1}])\\ &\qquad = \overline{\operatorname{Res}}(\overline{f_0}\,\overline{P}\,\overline{f_1}\,\overline{P}\,\overline{f_2} - \overline{f_0}\,\overline{P}\,\overline{f_1}\,\overline{f_2}\,\overline{P} - \overline{f_0}\,\overline{f_1}\,\overline{P}\,\overline{P}\,\overline{f_2} + \overline{f_0}\,\overline{f_1}\,\overline{P}\,\overline{f_2}\,\overline{P}\\ &\qquad\quad - \overline{f_0}\,\overline{P}\,\overline{f_2}\,\overline{P}\,\overline{f_1} + \overline{f_0}\,\overline{P}\,\overline{f_2}\,\overline{f_1}\,\overline{P} + \overline{f_0}\,\overline{f_2}\,\overline{P}\,\overline{P}\,\overline{f_1} - \overline{f_0}\,\overline{f_2}\,\overline{P}\,\overline{f_1}\,\overline{P})\\ &\qquad = \overline{\operatorname{Res}}(\overline{f_0}\,\overline{P}\,\overline{f_1}\,\overline{P}\,\overline{f}_2 + \overline{f_0}\,\overline{f_1}\,\overline{P}\,\overline{f_2}\,\overline{P} - \overline{f_0}\,\overline{P}\,\overline{f_2}\,\overline{P}\,\overline{f_1} - \overline{f_0}\,\overline{f_2}\,\overline{P}\,\overline{f_1}\,\overline{P}) = 0.\end{aligned}$$

Hence

$$\begin{aligned}&\int_M f_0 B_{n,P}(f_1,f_2) + 2\pi\int_{\partial M} f_0|_{\partial M}\partial B_{n,P,S}(f_1,f_2)\\ &\qquad = \int_M f_0 B_{n,P}(f_2,f_1) + 2\pi\int_{\partial M} f_0|_{\partial M}\partial B_{n,P,S}(f_2,f_1), \qquad \forall\, f_i \in C^\infty(\widetilde{M}).\end{aligned}$$

Since $B_{n,P}(f_1,f_2)$ is symmetric

$$\int_{\partial M} f_0|_{\partial M}\overline{B}_S(f_1,f_2) = \int_{\partial M} f_0|_{\partial M}\overline{B}_S(f_2,f_1), \quad \forall f_0 \in C^\infty(\widetilde{M})$$

and the result follows from the arbitrariness of f_0. ■

Lemma 5. *$\partial B_{n,P,S}(f_1,f_2)$ is differential on f_1 and f_2.*

Proof. We denote, to simplify the notation, $P_1 = [P, f_1]$ and $P_2 = [P, f_2]$ with symbols p_1 and p_2 respectively. In p. 27 of [8] we can read the following

$$\begin{aligned}&\operatorname{trace}\big\{L(p_1,p_2)\big\}(x',\xi')\\&\quad=\sum_{j,k=0}^{\infty}\frac{(-i)^{j+k+1}}{(j+k+1)!}\Pi'_{\xi_n}\big(\partial^j_{x_n}\partial^k_{\xi_n}\Pi^+_{\xi_n}(p_1)(x',0,\xi',\xi_n)\circ'\partial^{j+1}_{x_n}\partial^k_{\xi_n}\Pi^+_{\xi_n}(p_2)(x',0,\xi',\xi_n)\big),\end{aligned}$$

with $\Pi^+_{\xi_n}(s)(\cdot)$ the projection of the symbol s on H^+. The subscript in Π^+ indicates the variable it is acting on. From [20] we know

$$\sigma_{-k}([P,f])=\sum_{|\beta|=1}^{k}\frac{1}{\beta!}D^\beta_x(f)\partial^\beta_\xi(\sigma^P_{-(k-|b|)})$$

thus

$$\begin{aligned}\Pi^+_{\xi_n}(\sigma_{-k}([P,f]))(x',0,\xi',\xi_n)&=\sum_{|\beta|=1}^{k}\frac{1}{\beta!}\Pi^+_{\xi_n}\big(D^\beta_x(f)\partial^\beta_\xi(\sigma^P_{-(k-|b|)})\big)(x',0,\xi',\xi_n)\\&=\sum_{|\beta|=1}^{k}\frac{1}{\beta!}D^\beta_x(f)(x',0)\Pi^+_{\xi_n}\big(\partial^\beta_\xi(\sigma^P_{-(k-|b|)})\big)(x',0,\xi',\xi_n).\end{aligned}$$

Since any $\partial^j_{x_n}f$ factors out of Π'_{ξ_n} we conclude the result. ■

4.2 Conformal invariance of $B_{n,P}$ and $\partial B_{n,P,S}$

If we further assume Riemannian structures (M, g) and $(\widetilde{M}, \widetilde{g})$ such that g coincides with $\widetilde{g}$ restricted to M then, a conformal rescaling of g corresponds to a conformal rescaling of $\widetilde{g}$ (by an appropriate extension of the conformal factor) and a conformal rescaling of $\widetilde{g}$ can be restricted to a conformal rescaling of g. We obtain

Lemma 6. *Assume that P and S are such that $[P, f_1][P, f_2]$ and $[S, f_1|_{\partial M}][S, f_2|_{\partial M}]$ are conformally invariant for all $f_i \in C^\infty(\widetilde{M})$. Then*

$$\widehat{B_{n,P}}(f_1,f_2)(x)=e^{-2n\eta(x)}B_{n,P}(f_1,f_2)(x)$$

and

$$\widehat{\partial B_{n,P,S}}(f_1,f_2)(x')=e^{-2(n-1)\eta(x',0)}\partial B_{n,P,S}(f_1,f_2)(x').$$

Proof. We want to exploit the independence of $\overline{\operatorname{Res}}$ of local representations. We have

$$\begin{aligned}&\overline{\operatorname{Res}}(A(f_0)[A(P,S),A(f_1)][A(P,S),A(f_2)])\\&\quad=\int_M f_0B_{n,P}(f_1,f_2)\,dx+2\pi\int_{\partial M}f_0|_{\partial M}\partial B_{n,P,S}(f_1|_{\partial M},f_2|_{\partial M})\,dx'\\&\quad=\int_M f_0\widehat{B_{n,P}}(f_1,f_2)\,\widehat{dx}+2\pi\int_{\partial M}f_0|_{\partial M}\widehat{\partial B_{n,P,S}}(f_1|_{\partial M},f_2|_{\partial M})\,\widehat{dx'}\\&\quad=\int_M f_0e^{-2n\eta}\widehat{B_{n,P}}(f_1,f_2)\,dx+2\pi\int_{\partial M}f_0|_{\partial M}e^{-2(n-1)\eta|_{\partial M}}\widehat{\partial B_{n,P,S}}(f_1|_{\partial M},f_2|_{\partial M})\,dx',\end{aligned}$$

where we use $\widehat{\ }$ to represent quantities computed with respect to the conformal metric $\widehat{g} = e^{2\eta}g$. In particular

$$\int_M f_0(x)B_{n,P}(f_1,f_2)(x)\,dx=\int_M f_0(x)e^{-2n\eta(x)}\widehat{B_{n,P}}(f_1,f_2)(x)\,dx$$

for all $f_i \in C^\infty(\widetilde{M})$ with $f_0|_{\partial M} = 0$. Thus $B_{n,P}(f_1, f_2)(x) = e^{-2n\eta(x)}\widehat{B_{n,P}}(f_1, f_2)(x)$ for all $x \in M$. It follows that

$$\int_{\partial M} f_0(x', 0)\widehat{B_{n,P}}(f_1, f_2)(x')\,\widehat{dx'} = \int_{\partial M} f_0(x', 0)e^{-2n\eta(x',0)}\widehat{B_{n,P}}(f_1, f_2)(x')\,dx'$$

for all $f_i \in C^\infty(\widetilde{M})$. The result follows from the arbitrariness of f_0. ■

Remark 2. Note how the same reasoning in the proof above can be used to show the uniqueness of $B_{n,P}$ and $\partial B_{n,P,S}$ satisfying (4).

Summarizing this section we have

Theorem 4. *Let M be a compact manifold of dimension n and with boundary ∂M. Assume that M is embedded in a compact oriented manifold $\widetilde{M}$ without boundary. Further assume Riemannian structures $(\widetilde{M}, \widetilde{g})$ and (M, g) such that g coincides with $\widetilde{g}$ restricted to M. Let P be a pseudodifferential operator acting on a vector bundle E over $\widetilde{M}$ having the transmission property up to ∂M, let S be a pseudodifferential operator acting on a vector bundle E' over ∂M, such that $[P, f_1][P, f_2]$ and $[S, f_1|_{\partial M}][S, f_2|_{\partial M}]$ are conformally invariant for all $f_i \in C^\infty(\widetilde{M})$. Then $B_{n,P}$ and $\partial B_{n,P,S}$ given in Definition 1 are conformally invariant in the sense of Lemma 6. Furthermore, both $B_{n,P}$ and $\partial B_{n,P,S}$ are symmetric, bilinear differential functionals uniquely determined by the relation (4).*

5 On even-dimensional manifolds with boundary

Up to this point, we have a generalization of Theorem 1 in [20] to manifolds with boundary in the setting described above. Next, we want to state a generalization of Theorem 2 in [20] to this context. In order to do it, we consider the Fredholm module $(\mathcal{H}, F)$ now associated to the even-dimensional manifold without boundary $\widetilde{M}$.

5.1 The symbol of F and the transmission property

If $\omega = d\beta + \delta\beta' \in d(\Omega^{n/2-1}(\widetilde{M})) \oplus \delta(\Omega^{n/2+1}(\widetilde{M}))$ then

$$\Delta F_0(d\beta + \delta\beta') = \Delta(d\beta - \delta\beta') = d\delta d\beta - \delta d\delta\beta' = F_0(d(\delta d\beta) + \delta(d\delta\beta')) = F_0\Delta(d\beta + \delta\beta').$$

It follows

Lemma 7. *For an oriented compact manifold without boundary $\widetilde{M}$ and of even dimension n, the relation $F_0\Delta = \Delta F_0 = d\delta - \delta d$ holds on $d(\Omega^{n/2-1}(\widetilde{M})) \oplus \delta(\Omega^{n/2+1}(\widetilde{M}))$.*

To be able to use a given pseudodifferential operator in the machinery of the noncommutative residue for manifolds with boundary, it is essential for the operator to enjoy the transmission property up to the boundary of M.

Because we are interested in F acting on the orthogonal complement of the harmonic forms on $\widetilde{M}$, we abuse of the notation and use freely F for F_0. From the relation $\Delta F = d\delta - \delta d$ and the formula for the total symbol of the product of pseudodifferential operators we can compute the symbol expansion of F. First we note that F is a pseudodifferential operator of order 0.

We know $\sigma(\Delta F) = \sigma(d\delta - \delta d)$, thus the formula for the total symbol of the product of two pseudodifferential operators implies

$$\begin{aligned}\sigma_2^{d\delta-\delta d} + \sigma_1^{d\delta-\delta d} + \sigma_0^{d\delta-\delta d} = \sigma(d\delta - \delta d) = \sigma(\Delta F) &\sim \sum \frac{1}{\alpha!}\partial_\xi^\alpha \sigma(\Delta) D_x^\alpha(\sigma(F)) \\ &\sim \sum \frac{1}{\alpha!}\partial_\xi^\alpha(\sigma_2^\Delta + \sigma_1^\Delta + \sigma_0^\Delta) D_x^\alpha(\sigma_0^F + \sigma_{-1}^F + \sigma_{-2}^F + \cdots).\end{aligned}$$

Expanding the right hand side into sum of terms with the same homogeneity we conclude:

Lemma 8. *In any given system of local charts, we can express the total symbol of F, $\sigma(F) \sim \sigma_0^F + \sigma_{-1}^F + \cdots$ in a recursive way by the formulae:*

$$\sigma_0^F = (\sigma_2^\Delta)^{-1}\sigma_2^{d\delta-\delta d}, \qquad \sigma_{-1}^F = (\sigma_2^\Delta)^{-1}\Big(\sigma_1^{d\delta-\delta d} - \sigma_1^\Delta\sigma_0^F - \sum_{|\alpha|=1}\partial_\xi^\alpha(\sigma_2^\Delta)D_x^\alpha(\sigma_0^F)\Big),$$

$$\sigma_{-2}^F = (\sigma_2^\Delta)^{-1}\Big(\sigma_0^{d\delta-\delta d} - \sigma_1^\Delta\sigma_{-1}^F - \sigma_0^\Delta\sigma_0^F - \sum_{|\alpha|=1}\big(\partial_\xi^\alpha(\sigma_2^\Delta)D_x^\alpha(\sigma_{-1}^F) + \partial_\xi^\alpha(\sigma_1^\Delta)D_x^\alpha(\sigma_0^F)\big) - \sum_{|\alpha|=2}\frac{1}{\alpha!}\partial_\xi^\alpha(\sigma_2^\Delta)D_x^\alpha(\sigma_0^F)\Big),$$

$$\sigma_{-r}^F = -(\sigma_2^\Delta)^{-1}\Big(\sigma_1^\Delta\sigma_{-r+1}^F + \sigma_0^\Delta\sigma_{-r+2}^F + \sum_{|\alpha|=1}\partial_\xi^\alpha(\sigma_2^\Delta)D_x^\alpha(\sigma_{-r+1}^F) + \sum_{|\alpha|=1}\partial_\xi^\alpha(\sigma_1^\Delta)D_x^\alpha(\sigma_{-r+2}^F) + \sum_{|\alpha|=2}\frac{1}{\alpha!}\partial_\xi^\alpha(\sigma_2^\Delta)D_x^\alpha(\sigma_{-r+2}^F)\Big),$$

for every $r \geq 3$.

Lemma 2.4 of [17] states that all symbols which are polynomial in ξ have the transmission property. Thus both Δ and $d\delta - \delta d$ have the transmission property. Proposition 2.7 in the same reference states that if two operators satisfy the transmission property then their products, all their derivatives, and their parametrizes satisfy the transmission property as well. Furthermore, the same result also states that it is enough to check that each homogeneous component of the symbol expansion has the transmission property to conclude that the full symbol has the transmission property.

By Lemma 8, each homogeneous component σ_{-k}^F in the symbol expansion of F is given in terms of derivatives of the homogeneous components of Δ, $d\delta - \delta d$, $\sigma_0^F, \ldots, \sigma_{-k+1}^F$, and $\sigma_2(\Delta)^{-1}$. By Lemma 2.4 and Proposition 2.7 of [17] it follows that

Lemma 9. *The operator F satisfies the transmission property.*

5.2 $\overline{\mathrm{Res}}(A(f_0)[A(F,0), A(f_1)][A(F,0), A(f_2)])$

For F given in (2) now for the manifold $\widetilde{M}$, and for $f \in C^\infty(\widetilde{M})$ we let $A(F,0)$ and $A(f)$ be the elements in Boutet de Monvel's algebra of diagonal elements given by

$$\overline{F} = A(F,0) = \begin{pmatrix} r^M F e^M + 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \overline{f} = A(f) = \begin{pmatrix} r^M f e^M + 0 & 0 \\ 0 & f|_{\partial M} \end{pmatrix}.$$

Since $L(f, \sigma(F)) = 0 = L(\sigma(F), f)$, it follows that

$$\overline{f_0}[\overline{F}, \overline{f_1}][\overline{F}, \overline{f_2}] = \begin{pmatrix} r^M f_0[F, f_1][F, f_2]e^M + f_0^+ \circ' L(\sigma([F, f_1]), \sigma([F, f_2])) & 0 \\ 0 & 0 \end{pmatrix},$$

where $\circ'$ represents the symbol composition with respect to (x', ξ'). As before we define:

Definition 2.

$$B_n(f_1, f_2) := \int_{S^{n-1}} \mathrm{Tr}\left\{\sigma_{-n}\big([F, f_1][F, f_2](x, \xi)\big)\right\}\sigma(\xi),$$

and

$$\partial B_n(f_1, f_2) = \int_{S^{n-2}} \mathrm{Tr}\,\sigma_{-(n-1)}\left\{\big(\mathrm{trace}\big\{L(\sigma([F, f_1]), \sigma([F, f_2]))(x', \xi')\big\}\big)\right\}\sigma'(\xi'),$$

for all $f_i \in C^\infty(\widetilde{M})$.

As in Section 4

Theorem 5. *Both differential functionals B_n and ∂B_n are bilinear, symmetric, conformal invariant in the sense*

$$\widehat{B_n}(f_1, f_2)(x) = e^{-2n\eta(x)} B_n(f_1, f_2)(x)$$

and

$$\partial B_n\widehat{(f_1, f_2)}(x') = e^{-2(n-1)\eta(x',0)} \partial B_n(f_1, f_2)(x')$$

for a conformal change of the metric $\widehat{g} = e^{2\eta} g$, and are uniquely determined by the relation

$$\overline{\mathrm{Res}}\big(A(f_0)[A(F,0), A(f_1)][A(F,0), A(f_2)]\big)$$
$$= \int_M f_0 B_{n,P}(f_1, f_2)\, dx + 2\pi \int_{\partial M} f_0|_{\partial M} \partial B_{n,P,S}(f_1, f_2)\, dx'.$$

Remark 3. Even though both bilinear functionals B_n and ∂B_n are acting on $C^\infty(\widetilde{M})$, they depend on the particular embedding of the compact manifold M into $\widetilde{M}$, and thus, they can be defined on $C^\infty(M)$ by considering an extension of $f \in C^\infty(M)$ to $C^\infty(\widetilde{M})$.

Remark 4. In case M is odd dimensional, all results from the first part of these notes are valid on the compact even dimensional manifold without boundary ∂M. In this way, we can consider the commutative algebra $\mathcal{A} = C^\infty(\partial M)$ and the Fredholm module associated to the manifold ∂M. For F given in (2) on the manifold ∂M, we could look at

$$A(P, F) = \begin{pmatrix} P & 0 \\ 0 & F \end{pmatrix}$$

and try to study $\mathrm{Res}(A(f_0)[A(P,F), A(f_1)][A(P,F), A(f_2)])$ for functions $f_i \in C^\infty(\widetilde{M})$. The trivial choice $P = 0$ will produce $B_{n,P} = 0$ and $\partial B_{n,0,F} = B_{n-1}$. It is an open problem to search for a companion P for F that will produce more interesting results in the odd dimensional case.

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